TALK 1: MINIMAL (SATAKE) COMPACTIFICATIONS

Recall: (G, X) Shimura datum
$K \subseteq G(\mathbb{A}_f)$ compact open $\rightsquigarrow$ $Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$
Shimura variety

What we know so far...

1. Baily–Borel Thm: • $Sh_K(G,X)$ are quasi-projective varieties $/ \mathbb{C}$
   • for $K \subseteq K'$, $Sh_K \longrightarrow Sh_{K'}$ are regular maps (finite étale)

2. Modular interpretations: In many cases, $Sh_K(G,X)$ represents a moduli problem involving abelian varieties + pol. + lvl structure.

3. Canonical Models: $Sh_K(G,X)$ have canonical models $/ E(G,X)$ reflex field

So we're almost ready to do algebraic geometry with Shimura var., but ... it's a problem that they're not projective!

Goal: $Sh_K(G,X) \overset{\text{open immersion}}{\hookrightarrow} Sh_K(G,X)^*$ (projective) $\hookrightarrow$ minimal / Satake compactification

compatible with $K \subseteq K'$ etc.

Historical/logical remark. Baily & Borel constructed $Sh_K(G,X)^*$ first and proved:

$$\underset{\text{quasi-projective}}{Sh_K(G,X)} \hookrightarrow \underset{\text{projective}}{Sh_K(G,X)^*} \cong \mathrm{Proj}\left(\begin{array}{c}\text{graded space}\\ \text{of automorphic forms}\end{array}\right)$$

Recall, $\exists X^+$ (ass. to connected Sh. datum) s.t.

$$Sh_K(G,X) = \coprod_{i\in I} \underbrace{\Gamma_i \backslash X^+}_{\text{connected components}}$$

(David's talk)

finite

Goal. Compactify $\Gamma\backslash X^+$.

We'll see how to do this through examples ...

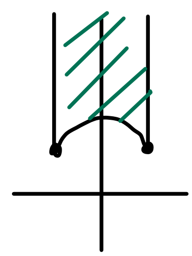
Example 1. Modular curves.

$\mathcal{H} = \{\operatorname{Im} z > 0\}$

Recall: $G = SL_2$, $X^+ = \mathcal{H}$

$\mathbb{C}$

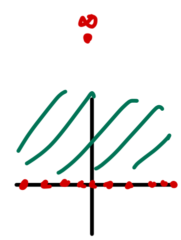
For $\Gamma = SL_2(\mathbb{Z}) \rightsquigarrow SL_2(\mathbb{Z})\backslash\mathcal{H}$



$\Rightarrow (\Gamma\backslash\mathcal{H})^* = \Gamma\backslash\mathcal{H} \cup \{\infty\}$

Note: $\operatorname{Stab}_{SL_2(\mathbb{Q})}(\infty) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} = B$.

In general, look at $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$
$= \mathcal{H} \cup (G(\mathbb{Q})\cdot\infty)$



Side note: Could look at $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{P}^1(\mathbb{R})$ but that's not necessary!

Note that $SL_2(\mathbb{Q}) \curvearrowright \mathbb{P}^1(\mathbb{Q})$

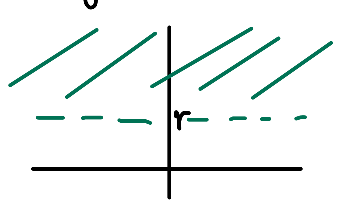
$\Rightarrow \forall\, \Gamma \leq SL_2(\mathbb{Q}),\quad (\Gamma\backslash\mathcal{H})^* := \Gamma\backslash\mathcal{H}^* = \Gamma\backslash\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$.

When $\Gamma = SL_2(\mathbb{Z})$: $\forall\, \frac{a}{c} \in \mathbb{Q}\quad \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{a}{c}$

$\Rightarrow \Gamma\backslash\mathcal{H}^* = \Gamma\backslash\mathcal{H} \cup \{\infty\}$.

In general, $\Gamma\backslash\mathcal{H}^* \cong \Gamma\backslash\mathcal{H} \cup \underbrace{\Gamma\backslash\mathbb{P}^1(\mathbb{Q})}_{\text{finite}}$

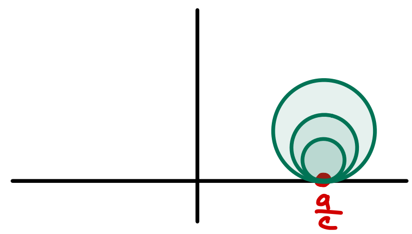
Topology? Open neighborhoods of ∞: $W_r = \{z \mid \operatorname{Im} z > r\} \cup \{\infty\}$ for $r > 0$



W_r

Open nbhd of cusp $\frac{a}{c}$: Pick $\gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma\infty = \frac{a}{c}$

$\rightsquigarrow \quad \gamma W_r \quad$ for $r > 0$



HOROCYCLE TOPOLOGY (SATAKE TOP.)

Example 2. Two examples with $X^+ = \mathcal{H} \times \mathcal{H}$.

$G = SL_2 \times SL_2$

We take:

$$(\mathcal{H}\times\mathcal{H})^* = \mathcal{H}^* \times \mathcal{H}^*$$
$$= \mathcal{H}\times\mathcal{H} \cup (G(\mathbb{Q})\cdot(\mathcal{H}\times\infty)) \quad \text{1-dim'l}$$
$$\cup \ (G(\mathbb{Q})\cdot(\infty\times\mathcal{H})) \quad \text{1-dim'l}$$
$$\cup \ (G(\mathbb{Q})\cdot(\infty\times\infty)) \quad \text{0-dim'l}$$

$$(\Gamma\backslash\mathcal{H}\times\mathcal{H})^* = \Gamma\backslash\mathcal{H}^*\times\mathcal{H}^*$$

Note:

$$\mathrm{Stab}_G(\mathcal{H}\times\infty) = SL_2(\mathbb{Q})\times B(\mathbb{Q})$$
$$\mathrm{Stab}_G(\gamma(\mathcal{H}\times\infty)) = SL_2(\mathbb{Q})\times\gamma B(\mathbb{Q})\gamma^{-1}$$
$$\mathrm{Stab}_G(\infty\times\mathcal{H}) = B(\mathbb{Q})\times SL_2(\mathbb{Q})$$
$$\mathrm{Stab}_G(\infty\times\infty) = B(\mathbb{Q})\times B(\mathbb{Q})$$

etc.

Hilbert modular variety

$G = \mathrm{Res}_{F/\mathbb{Q}} SL_{2,F}$ $\quad F/\mathbb{Q}$ real quadratic

Recall: $G(\mathbb{Q}) = SL_2(F)$
$G(\mathbb{R}) = SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ etc.

$SL_2(F) \circlearrowright \mathcal{H}\times\mathcal{H}$ via

$\gamma\in SL_2(F)$, $\gamma(z_1,z_2) = (\sigma_1(\gamma)z_1, \sigma_2(\gamma)z_2)$

$\sigma_1,\sigma_2 : F \hookrightarrow \mathbb{R}$
$a+b\sqrt{d} \mapsto a \pm b\sqrt{d}$

We take:

$$(\mathcal{H}\times\mathcal{H})^* = (\mathcal{H}\times\mathcal{H}) \cup G(\mathbb{Q})\cdot(\infty,\infty)$$
0-dim'l

with similar "Satake" topology

$$(\Gamma\backslash\mathcal{H}\times\mathcal{H})^* = \Gamma\backslash(\mathcal{H}\times\mathcal{H})^*$$

Note: $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$\mathrm{Stab}_G(\infty\times\infty) = B(\mathbb{Q}) \subseteq SL_2(F)$$
$$\mathrm{Stab}_G(\sigma_1(\gamma)\infty\times\sigma_2(\gamma)\infty) = \gamma B(\mathbb{Q})\gamma^{-1}$$

Take-away.

$$\left\{ \begin{array}{c} \text{RATIONAL BOUNDARY} \\ \text{COMPONENTS} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{RATIONAL PARABOLIC SUBGROUPS} \\ \qquad P \subseteq G \\ \text{S.T. PULLBACKS OF } P \text{ ONTO} \\ \text{SIMPLE FACTORS } G' \text{ OF } G \text{ IS} \\ \text{EITHER } G' \text{ OR A MAX'L PARABOLIC} \end{array} \right\}$$

$$C \longmapsto P = \mathrm{Stab}_G(C)$$

$$P = \mathrm{Stab}(V_1, \ldots, V_n)$$

for G defined by non-deg. pairings ...

$$\left\{ \begin{array}{l} \text{INCREASING SEQUENCE} \\ \qquad V_1 \subseteq \cdots \subseteq V_n \\ \text{OF TOTALLY ANISOTROPIC SUBSPACES} \end{array} \right\}$$

$W \subseteq V$ anisotr. if $\langle w, w \rangle \neq 0 \ \forall w \in W$

(more in Lan!)

Final example: Siegel threefolds

$$G = Sp_{2n} = \left\{ M \in M_{2n} \;\middle|\; M^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = 0 \right\}$$

$$\rightsquigarrow \mathcal{H}_n = \left\{ Z \in \mathrm{Sym}_{n \times n}(\mathbb{C}) \;\middle|\; \mathrm{Im}\, Z > 0 \right\}$$

$$\text{w/ } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B) \cdot (CZ + D)^{-1}$$

We consider $n = 2$. There are 3 orbits of parabolics.

(1) Borel

$$B(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix} \right\} \longleftrightarrow \left\{ \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \subseteq \left\{ \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix} \right\}$$

Note $\left\{ \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}^{\perp} = \left\{ \begin{pmatrix} * \\ * \\ 0 \\ * \end{pmatrix} \right\}$

$\Longrightarrow$ weird shape (maybe)

(2) Klingen parabolic

$$P^{(1)}(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix} \right\} \longleftrightarrow \left\{ \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

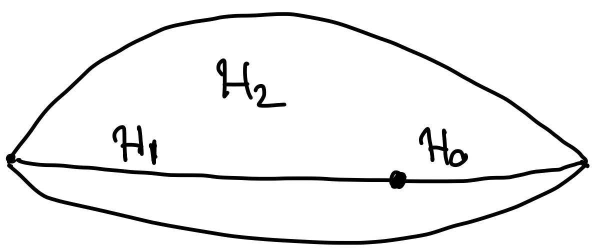
(3) Siegel parabolic

$$P^{(2)}(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\} \longleftrightarrow \left\{ \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix} \right\}$$

Note: $B(\mathbb{Q}) = P^{(1)}(\mathbb{Q}) \cap P^{(2)}(\mathbb{Q}) \rightsquigarrow$ not maximal !

Finally:

$$\mathcal{H}_2^* = \mathcal{H}_2 \cup \underset{\displaystyle P^{(1)}(\mathbb{Q})}{\underset{\updownarrow}{\overset{\text{1-dim'l}}{G(\mathbb{Q}) \cdot \begin{pmatrix} \infty & \\ & \mathcal{H}_1 \end{pmatrix}}}} \cup \underset{\displaystyle P^{(2)}(\mathbb{Q})}{\underset{\updownarrow}{\overset{\text{0-dim'l}}{G(\mathbb{Q}) \cdot \begin{pmatrix} \infty & \\ & \infty \end{pmatrix}}}}$$



Upshot. A general recipe for finding

$$Sh_K(G,X) \hookrightarrow Sh_K(G,X)^*$$

is now clear (although the combinatorics may get tricky).

Problem. $Sh_K(G,X)^*$ has high codimension boundary components which are highly singular !

Of course, we could use Hironaka's resolution of sing.

but how can we do this without messing up the arithmetic?

To be continued ... (mostly in the Hilbert case)