Overview talk

Wednesday, May 6, 2020 12:15 PM

SHIMURA = SHimura Internet Michigan University Reading Adventures

TODAY:
- Overview talk
- Assigning talks
- Announcements:
  - notes?
  - Discord for discussions/questions

Shimura varieties: overview.

G = red. alg. gp / $\mathbb{Q}$ (Rep. theory? Geometry? Langlands program?)

E.g. $G = SL_n, GL_n, GSp_n, O(n), \ldots$

Consider $G(\mathbb{R}), G(\mathbb{Q}), G(\mathbb{Q}_p), \ldots$

$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} \rightsquigarrow G(\mathbb{A}) = G(\mathbb{A}^{fin}) \times G(\mathbb{R}) \hookleftarrow G(\mathbb{Q})$

For $G(\mathbb{R})$, $K \leq G(\mathbb{R})$ max'l compact

$\rightsquigarrow K \backslash G(\mathbb{R})$ symmetric space ass. to G

For an open compact $\mathcal{U} \leq G(\mathbb{A}^f)$ ("level structure")

$\rightsquigarrow G(\mathbb{Q}) \backslash G(\mathbb{A}) / K\mathcal{U}$ locally symm. space

$$= G(\mathbb{Q}) \backslash (G(\mathbb{A}^{fin}) \times G(\mathbb{R})) / K \cdot \mathcal{U}$$

Matsushima's Thm. Automorphic forms/reps live in the cohomology of these loc. sym. sp.

Example. $G = SL_2$, $K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$

$K \backslash SL_2(\mathbb{R}) \simeq \mathfrak{h} = \{ z \in \mathbb{C} \mid \operatorname{Im} z \neq 0 \}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{ai+b}{ci+d}$ (K fixes i)

$\mathfrak{h}^+$, $\mathfrak{h}^-$

$\mathcal{D} = \mathfrak{h}^{\pm} \supseteq \mathfrak{h}^+ = \mathcal{D}^+$

$GL_2(\mathbb{R})/K \cong \mathfrak{h}^{\pm}$

$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) / K \cdot \underbrace{SL_2(\hat{\mathbb{Z}})}_{\mathcal{U}} \simeq \mathfrak{h}^+ / SL_2(\mathbb{Z})$ modular curve

$SL_2(\hat{\mathbb{Z}}) = \prod_p SL_2(\mathbb{Z}_p)$

$\rightsquigarrow$ modular forms

In general, find $\mathcal{D}$ "Hermitian symm. domain"

w/ $G(\mathbb{R}) \curvearrowright \mathcal{D}$ & some axioms...

$\rightsquigarrow (G, \mathcal{D})$ Shimura datum

Sometimes, you can find! E.g. $G = Sp_n, O(n), \ldots$ ✓ not for $G = GL_n$, $n \geq 3$.

$\forall \mathcal{U} \subseteq G(\mathbb{A}^{fin})$ open compact

$X_{\mathcal{U}} = G(\mathbb{Q}) \backslash (\mathcal{D} \times G(\mathbb{A}^{fin})) / \mathcal{U}$ Shimura variety

$\simeq \coprod_{i \in I \text{ finite}} \Gamma_i \backslash \mathcal{D}^+$

MOTIVATIONS.

(1) Langlands program

G = red. alg. gp / $\mathbb{Q}$ $\rightsquigarrow G(\mathbb{R}), G(\mathbb{A}), \ldots$

$\rightsquigarrow {}^L G$ Langlands dual group

E.g. $G = GL_n \rightsquigarrow {}^L G = GL_n(\mathbb{C})$

$G = Sp_{2n}$ $Sp_{2n}(\mathbb{R}) = \{ g \in GL_{2n}(\mathbb{R}) \mid {}^t g \cdot J_n \cdot g = J_n \}$ fixes a sympl. form

$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$

$\Rightarrow {}^L Sp_{2n} = SO(2n+1)(\mathbb{C})$ fixes an orth. form

Langlands corresp. (vaguely).

$\left\{ \begin{matrix} \text{aut. cusp. reps} \\ \text{of } G(\mathbb{A}^{fin}) \end{matrix} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{matrix} \text{Galois reps} \\ \Gamma_{\mathbb{Q}} \to {}^L G \end{matrix} \right\}$

$\Gamma_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$G = GL_n$: comp. system of reps: $\Gamma_{\mathbb{Q}} \to GL_n(\mathbb{Q}_p)$ cts

Idea. Realize this by finding a v. sp. V w/ action of both $G(\mathbb{A}^{fin})$ & $\Gamma_{\mathbb{Q}}$.

If have Sh. var. $\{X_{\mathcal{U}}\}_{\mathcal{U}}$ ass. to G

$\rightsquigarrow \varinjlim_{\mathcal{U}} H^i_{ét}(X_{\mathcal{U}}, \mathbb{Q}_p) \circlearrowleft G(\mathbb{A}^{fin})$

To get a Galois action, need to know $X_{\mathcal{U}}$ is defined / $\mathbb{Q}$ (or / # field).

Even before: is $X_{\mathcal{U}}$ a variety?

(2) Moduli of Hodge structures

f = deg. 4 homog. poly in 4 vars $\rightsquigarrow X_f \subseteq \mathbb{P}^3$ quartic surface $\rightsquigarrow \operatorname{rk} H^2(X_f, \mathbb{Z}) = 22$

$\operatorname{rk} \operatorname{Prim} H^2(X_f, \mathbb{Z}) = 21$

& $\operatorname{Prim} H^2(X_f, \mathbb{Z})$ is a HS with type $h^{20} = 1 = h^{02}$, $h^{11} = 19$

dim 19 $\left\{ \begin{matrix} \text{moduli space} \\ \text{of smooth quartic} \\ \text{surfaces } X_f \end{matrix} \right\}$

$\exists$ 34-dim'l space of quartics w/ action of PGL_4 by change of vars

15-dim'l

$X_f \rightsquigarrow \Lambda = \operatorname{Prim} H^2(X_f, \mathbb{Z})$

open immersion w/ complements $\cong$ smaller-dim'l sub Shimura variety.

$\mathcal{S} = \left\{ \begin{matrix} \text{polarizes wt 2} \\ \text{Hodge str.} \\ \text{of type } (1, 1, 19) \end{matrix} \right\}$

$\cong$

$O(19) \times O(2) \backslash SO(19,2) / \Gamma$

SHIMURA VARIETY

$(G, \mathcal{D})$ Shimura datum

$\mathcal{U} \subseteq G(\mathbb{A}^{fin})$ open cpct "neat" $\rightsquigarrow X_{\mathcal{U}} = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}^{fin}) / \mathcal{U}$

THM 1. The family $\{X_{\mathcal{U}}\}_{\mathcal{U}}$ of complex manifolds is the analytification of alg. quasi-proj. $\{\mathcal{X}_{\mathcal{U}}\} / \mathbb{C}$.

Idea: Express these as moduli spaces of abelian varieties.

E.g. $\mathfrak{h} / SL_2(\mathbb{Z}) \cong \left\{ \begin{matrix} \text{moduli space} \\ \text{of elliptic curves} \end{matrix} \right\}$

THM 2 (Canonical models).

$(G, \mathcal{D})$ Sh. datum $\rightsquigarrow \{\mathcal{X}_{\mathcal{U}}\} / \mathbb{C}$

$\exists F_0$ = number field (reflex field)

s.t. $\{\mathcal{X}_{\mathcal{U}}\}_{\mathcal{U}}$ is defined / F_0.

This family is "canonical" if we require $\{\mathcal{X}_{\mathcal{U}}\}_{\mathcal{U}}$ to be "canonical models".

← identify special points (abelian vars w/ CM) on which $\exists$ prescribed Galois action.

Rmk. $\{x^2 + y^2 = 1\} / \mathbb{C} \leftarrow \{x^2 + y^2 = 1\} / \mathbb{Q}$

$\nwarrow \{x^2 + 2y^2 = 1\} / \mathbb{Q}$

$\vdots$