Elliptic Curves and the State of Survaillence

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Reference: Thomas C. Hales, *The NSA Back Door to NIST*, Notices of the AMS.

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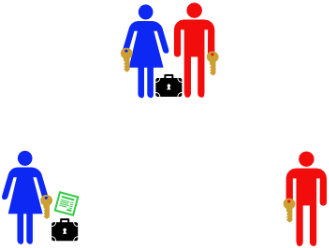

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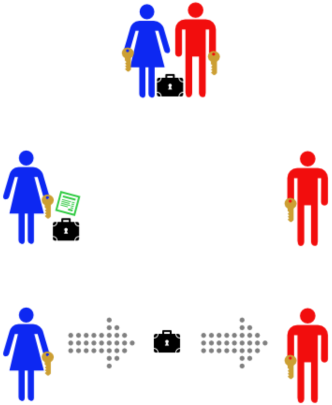

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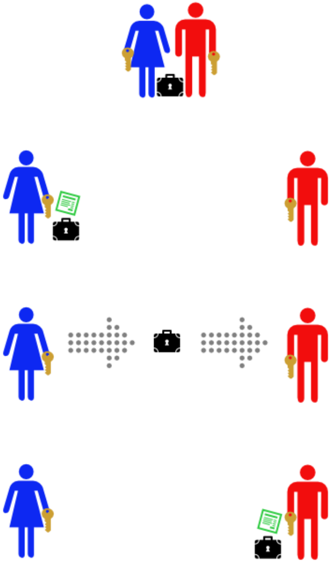

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One-time pad

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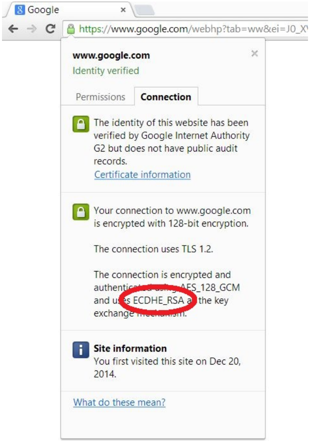
One method comes from the theory of elliptic curves, which are recently very common in cryptography.

Elliptic curves

Google Chrome:



Key exchange: ECDHE_RSA

EC = Elliptic Curve

Elliptic curves

Elliptic curves are a special kind of cubic curves on the plane.

Definition

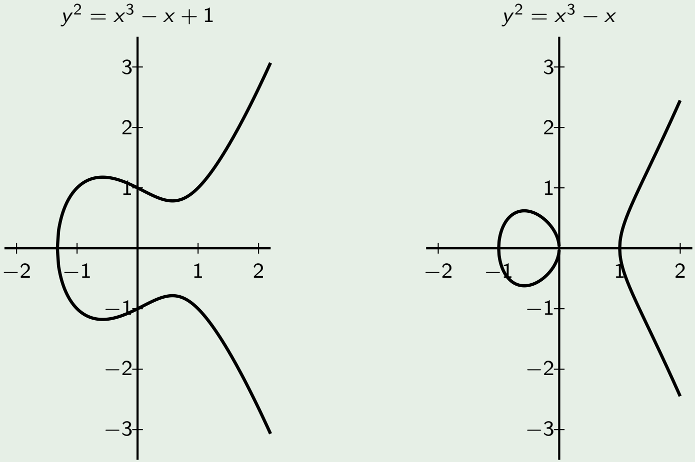
An *elliptic curve* over $\mathbb{R}$ is the set of solution $(x, y) \in \mathbb{R}^2$ of

$$y^2 = x^3 + ax + b$$

for $a, b \in \mathbb{R}$ such that $27b^2 + 4a^3 \neq 0$, together with a point O called the *point at infinity*.

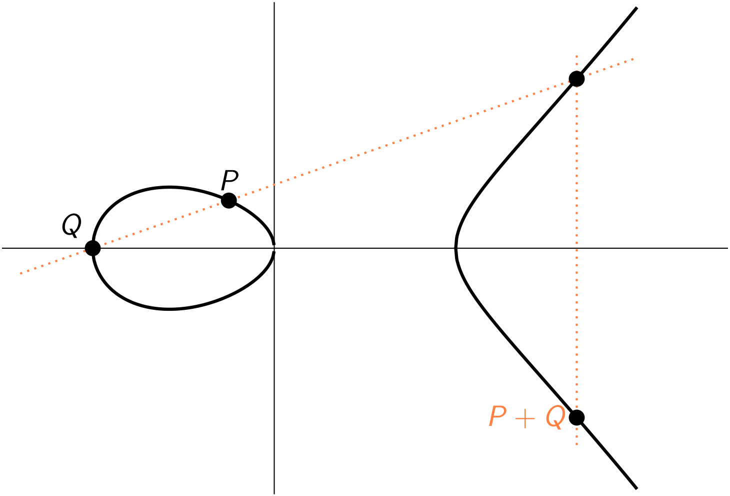
Elliptic curves

Examples



Addition on elliptic curves

Why are they so useful? You can define *addition* on them!



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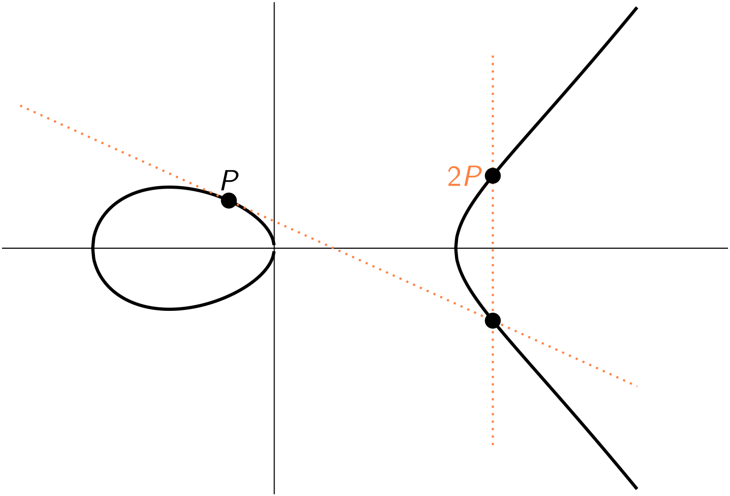

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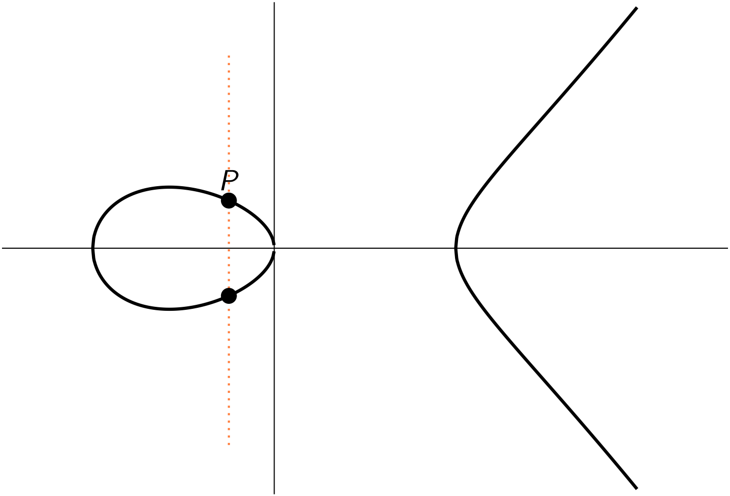

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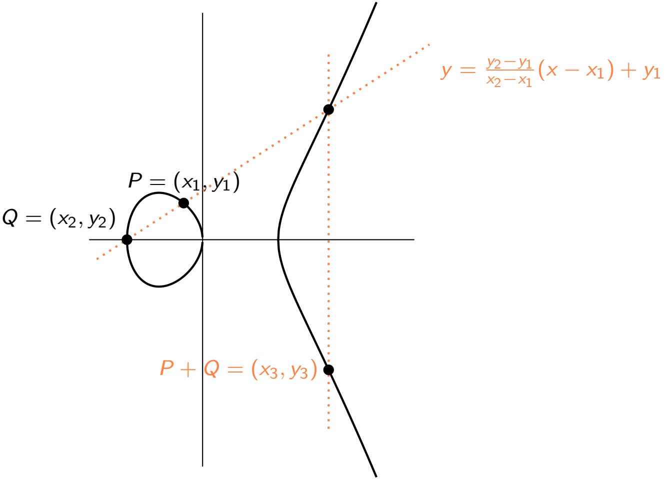

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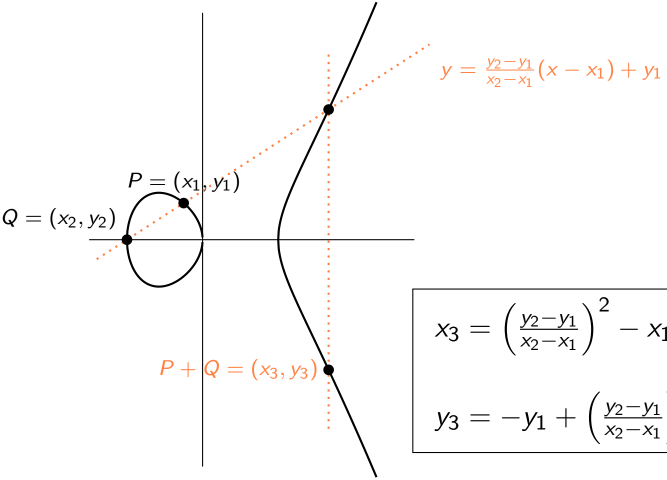

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2$$

$$y_3 = -y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3)$$

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We can make elliptic curves finite by reducing them modulo a prime number p:

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where $a, b \in \mathbb{F}_p = \{0, 1, \ldots, p-1\}$ and $27b^2 + 4a^3 \not\equiv 0 \mod p$.

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where $a, b \in \mathbb{F}_p = \{0, 1, \ldots, p-1\}$ and $27b^2 + 4a^3 \not\equiv 0 \mod p$. The addition formulas also reduce modulo p, because they only use $+$, $-$, $\times$, $\div$. We can do all of these in $\mathbb{F}_p$.

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3. Let s' be the x-coordinate of $rP = \underbrace{P + P + \ldots + P}_{r \text{ times}}$. This is the new *internal state*.

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There is a back door to this pseudo-random number generator; that is, a way to find the hidden state s and predict the "*random*" numbers.

The back door

For all the curves E listed by NIST, the number of points of $E[\mathbb{F}_p]$ is prime. Since $E[\mathbb{F}_p]$ is a group of prime order, every element (except O) is a generator, so $P = eQ$ for some integer e.

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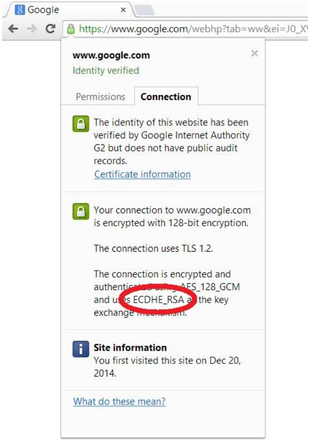
But the new internal state s' is the x-coordinate of rP. □

Diffie–Hellman Key Exchange

Let's go back to Google Chrome!



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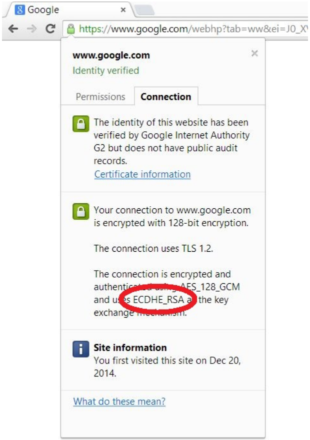

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What is that? A commonly used well-known key exchange. Every cryptographer knows it.

It is based on the same idea as the back door!