A Proof of the Generalized Newman Conjecture

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Definitions

Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \operatorname{Re} s > 1$$

Riemann xi function:

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

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The Riemann xi function is entire and satisfies the functional equation

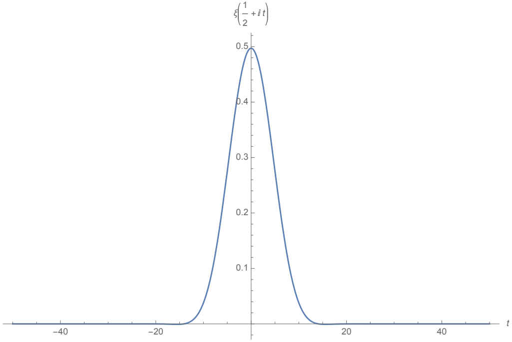
$$\xi(1-s) = \xi(s)$$

Riemann Hypothesis (RH)

All the zeros of ξ have real part $\frac{1}{2}$.

Riemann xi function

From the functional equation we get that $\xi\left(\frac{1}{2}+it\right)$ is real and even



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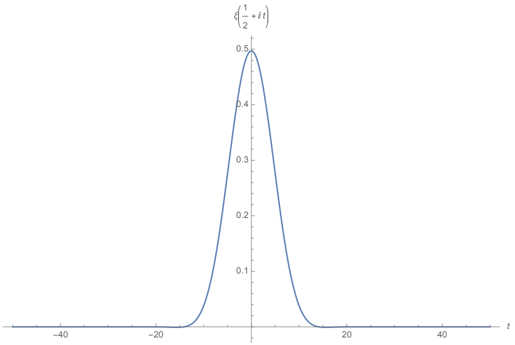
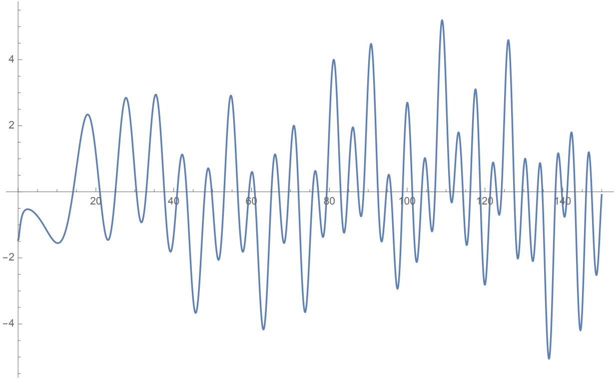


Hard to make out any of the zeros due to the damping effect of the gamma function.

Riemann xi function (cont.)

Removing the damping, we see $\xi(1/2 + it)$ is oscillatory:



...and has many zeros

Fourier representation of the Riemann xi function

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Shift the "critical line" to the real axis:

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The properties of $\xi(s)$ carry over to $H_0 \colon \mathbb{C} \to \mathbb{C} \ldots$

- H_0 is entire
- $H_0(-z) = H_0(z)$
- H_0 takes real values on the real axis

Riemann Hypothesis

All the zeros of H_0 lie on the real line.

Fourier representation of the Riemann xi function

Fourier representation of H_0

$$H_0(z) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi(u) e^{izu}\, du$$

where

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) \exp\left(-\pi n^2 e^{4u}\right)$$

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Pólya studied functions F which have the property that $\int_{-\infty}^{\infty} F(u) e^{izu}\, du$ has only real zeros. He considered a few approximations of Φ and showed that the resulting "approximate" H_0 functions satisfy RH.

Defining H_t

Theorem (Pólya 1927)

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a measurable function such that $F(u) = \overline{F(-u)}$ and $|F(u)| \leq Ae^{-|u|^b}$ for some $b > 2, A > 0$. If all the zeros of

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are real, then all the zeros of $\int_{-\infty}^{\infty} e^{tu^2} F(u)e^{izu}\,du$ are real for any $t > 0$.

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Definition

$$H_t(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{tu^2}\Phi(u)e^{izu}\,du$$

Can think of t as a time variable. $H_t(z)$ obeys the (backwards) heat equation

The de Bruijn-Newman Constant

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Definition

The *de Bruijn-Newman constant* is the unique real number Λ with the property that for all $t \geq \Lambda$ all the zeros of H_t are real, and for all $t < \Lambda$ there is a nonreal zero of H_t.

- De Bruijn 1950: $\Lambda \leq \frac{1}{2}$
- Newman 1976: $\Lambda > -\infty$

Newman's Conjecture

Fact

$$\text{RH} \iff \text{all zeros of } H_0 \text{ are real} \iff \Lambda \leq 0$$

Newman's Conjecture

$$\Lambda \geq 0$$

$$(\iff H_t \text{ has zeros off the real line for all } t < 0)$$

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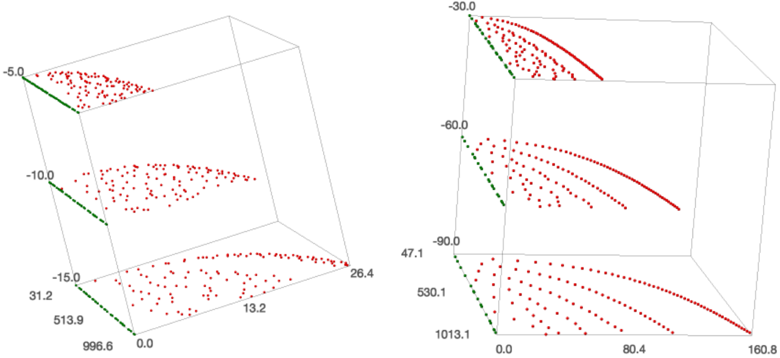
Newman's Conjecture was proved by Rodgers & Tao in 2018:

- Suppose for contradiction that $\Lambda < 0$.
- Show that zeros of H_t become (locally) evenly spaced for times $t > \Lambda$.
- Zeros of the zeta function (equivalently H_0) are known to have spacing irregularities. Contradiction.

The zeros of H_t for $t < 0$



(plots by rudolph01 of the Polymath 15 project)

Questions

- Can we prove zeros lie on/near the curves in the picture?
- If so does this lead to a simpler proof of Newman's conjecture?
- To what extent can Newman's conjecture be generalized to other L-functions? (and proved!)
- How about other Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$?

Restating Newman's conjecture

Recall $H_0(z) = \frac{1}{8}\xi\left(\frac{1+iz}{2}\right)$. Let's remove the rotation from the H_t's:

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Let $\xi_t \colon \mathbb{C} \to \mathbb{C}$ be the entire function such that $H_t(z) = \frac{1}{8}\xi_t\left(\frac{1+iz}{2}\right)$

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After some manipulations we get that for any $t < 0$:

$$\xi_t(s) = \frac{-i}{\sqrt{\pi|t|}} \int_{c-i\infty}^{c+i\infty} \xi(w) e^{\frac{1}{|t|}(w-s)^2}\, dw$$

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Newman's Conjecture (reformulated)

For every $t < 0$, the function ξ_t has a zero off the line $\mathrm{Re}(s) = \frac{1}{2}$.

Relating ξ_t to a Dirichlet series

Recall that

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \text{(gamma factor)(Dirichlet series)}$$

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$$\xi_t(\widetilde{s}) = G_t(\widetilde{s})(\zeta_t(s) + o_{y\to\infty}(1))$$

where

$$\zeta_t(s) = \sum_{n=1}^{\infty} \exp\left(-\frac{|t|}{4}(\log n)^2\right) n^{-s}$$

and G_t is analytic and nonzero in the region of interest.

Applying the theorem

So

$$\xi_t(\widetilde{s}) = G_t(\widetilde{s})(\zeta_t(s) + o_{y\to\infty}(1))$$
$$= (\text{gamma-like factor})(\zeta_t(s) + \text{error})$$

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Strategy for proving Newman's Conjecture:

- Find zeros of $\zeta_t(s)$ in vertical strips (in the s-plane)
- Make sure $\zeta_t(s) + o_{y\to\infty}(1)$ also has zeros in these strips
- Then $\xi_t(\widetilde{s})$ must have zeros in corresponding "curved strips" in the $\widetilde{s}$-plane

Warm-up: large negative t

Let's examine the behavior of the Dirichlet series
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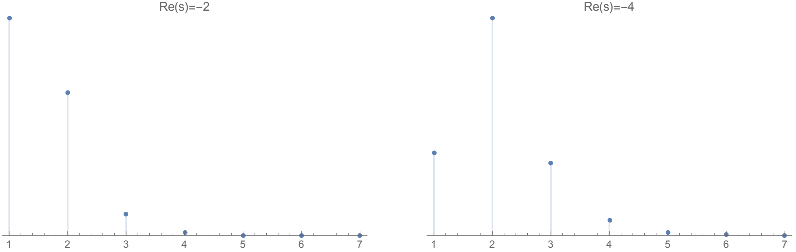
Relative sizes of the terms of ζ_t on two vertical lines (for t=-15):



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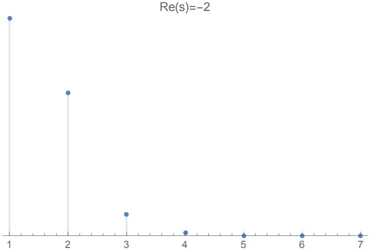

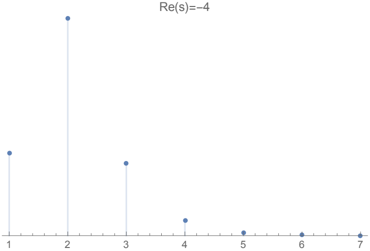

$$\left(\text{winding speed of } \zeta_t \text{ along line } \{\mathrm{Re}(s) = -2\}\right) = 0$$

$$\left(\text{winding speed of } \zeta_t \text{ along line } \{\mathrm{Re}(s) = -4\}\right) = \frac{\log 2}{2\pi}$$

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$$\left|n\text{th term}\right| = \left|(n+1)\text{th term}\right| \iff \operatorname{Re}(s) = -\frac{|t|}{4}\log(n(n+1))$$

To the right nth term is bigger: winding speed $= \frac{\log n}{2\pi}$
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$\implies$ ζ_t will have $\sim \left(\frac{1}{2\pi}\log\frac{n+1}{n}\right)H$ zeros near this line up to height H

Small negative t (i.e. Newman's conjecture)

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Lemma

For any $t < 0$, ζ_t has a zero.

Proof.

- Suppose ζ_t has no zeros.
- ζ_t is an entire function of order 2 so by Hadamard factorization (or Borel-Carathéodory) we have $\zeta_t(s) = e^{P(s)}$ where $\deg P \leq 2$.
- ζ_t is bounded in right half plane and nonconstant $\implies \zeta_t(s) = e^{-\lambda s}$ for some nonnegative λ.
- Impossible by uniqueness of coefficients of Dirichlet series. □

Small negative t (i.e. Newman's conjecture)

How do we deduce that $\zeta_t(s) + o_{y\to\infty}(1)$ has zeros?

- Let w be a zero of ζ_t.
- Let C_w be a small circle around w. Then $\zeta_t(s)$ winds around zero at least once as s traverses C_w. Let $\delta = \min_{s\in C_w} |\zeta_t(s)|$.
- We can find shifts $T_1, T_2, \ldots \in \mathbb{R}$ which get arbitrarily big such that $|\zeta_t(s + T_n i) - \zeta_t(s)| < \delta/3$ for all $s \in C_w$, $n \in \mathbb{N}$.
- For large enough n, the $o_{y\to\infty}(1)$ term is smaller than $\delta/3$.
- $\implies \zeta_t(s) + o_{y\to\infty}(1)$ winds around 0 as s traverses $C_{w+T_n i}$.

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- $\implies \zeta_t(s) + o_{y\to\infty}(1)$ winds around 0 as s traverses $C_{w+T_n i}$.

So $\xi_t(\tilde{s})$ has zeros inside the "circles" in the $\tilde{s}$-plane corresponding to the circles $C_{w+T_n i}$ in the s-plane. Hence ξ_t has zeros off the critical line. □

Proof of theorem

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where $\xi(w) = \frac{w(w-1)}{2}\pi^{-w/2}\Gamma\left(\frac{w}{2}\right)\zeta(w)$

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Expanding everything out we get...

$$\xi_t(\widetilde{s}) = \frac{-2i}{\sqrt{\pi|t|}} \sum_{n=1}^{\infty} \int_{1-i\infty}^{1+i\infty} \Gamma\left(\frac{w}{2}+2\right) (\sqrt{\pi}n)^{-w} e^{\frac{1}{|t|}(w-\widetilde{s})^2} \, dw$$
$$+ \frac{3i}{\sqrt{\pi|t|}} \sum_{n=1}^{\infty} \int_{1-i\infty}^{1+i\infty} \Gamma\left(\frac{w}{2}+1\right) (\sqrt{\pi}n)^{-w} e^{\frac{1}{|t|}(w-\widetilde{s})^2} \, dw$$

Method of steepest descent

Reduce to estimating integrals like:

$$\int_{1-i\infty}^{1+i\infty} \Gamma(w) e^{\frac{1}{2\sigma}(w-\eta)^2}\, dw$$

where $\eta \in \mathbb{C}, \sigma > 0$.

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Note:

- The integrand decays very rapidly $\implies$ its mass is concentrated. This is good!
- The integrand is often oscillatory. This is bad!

Method: shift the contour to somewhere where the mass is localized but there's no oscillation

Method of steepest descent

First rewrite the integrand using Stirling's approximation:

$$\Gamma(w)e^{\frac{1}{2\sigma}(w-\eta)^2}\,dw \approx \sqrt{\frac{2\pi}{w}}\exp(S(w))$$

where $S(w) = w\,\mathrm{Log}\,w - w + \frac{1}{2\sigma}(w-\eta)^2$.

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where $S(w) = w \operatorname{Log} w - w + \frac{1}{2\sigma}(w-\eta)^2$.

Taylor expansion near w_0 gives:

$$\approx \sqrt{\frac{2\pi}{w_0}}\exp\left(S(w_0) + S'(w_0)(w-w_0) + \frac{1}{2}S''(w_0)(w-w_0)^2\right)$$

Very good approximation for $|w_0|$ large compared to $|w-w_0|$ because $S^{(k)}(w_0) = O(|w_0|^{-k+1})$ for $k > 2$

Pick w_0 so that $S'(w_0) = \operatorname{Log} w_0 + \frac{1}{\sigma}(w_0 - \eta) = 0$. Then,

$$\approx \sqrt{\frac{2\pi}{w_0}} \exp\left(S(w_0) + \frac{1}{2\sigma}(w - w_0)^2 \right)$$

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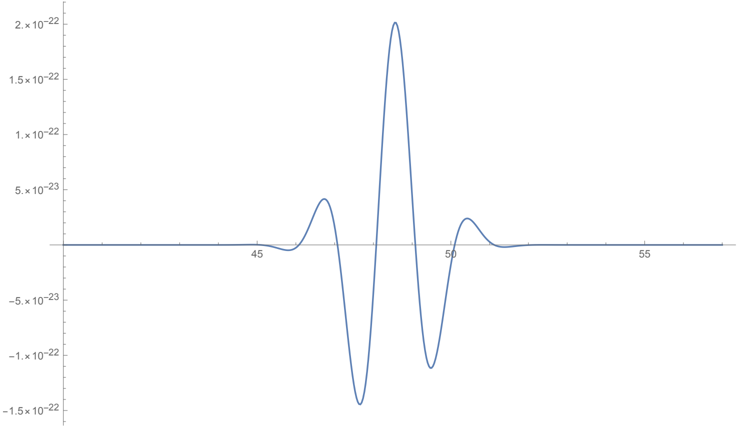
$$\approx \sqrt{\frac{2\pi}{w_0}} \exp\left(S(w_0) + \frac{1}{2\sigma}(w - w_0)^2 \right)$$

On a vertical line contour through w_0, this integrand is a Gaussian. Integrating gives:

$$\int_L \Gamma(w) e^{\frac{1}{2\sigma}(w-\eta)^2}\, dw \approx 2\pi i \sqrt{\frac{\sigma}{w_0}} \exp(S(w_0))$$

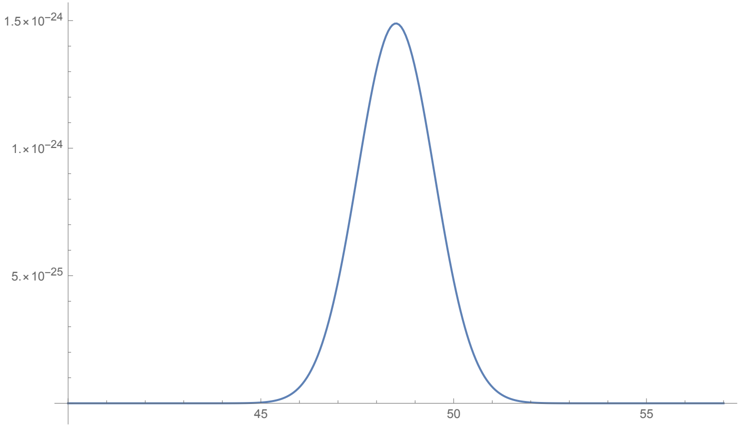
Example

$\mathrm{Re}\left(\Gamma(w)e^{\frac{1}{2}(w-(8+50i))^2}\right)$ on the original contour



Example

$\mathrm{Re}\left(\Gamma(w)e^{\frac{1}{2}(w-(8+50i))^2}\right)$ after shifting to the line $\mathrm{Re}\, w = 4.115$



Lemma

Fix $C > 0$. Let $0 < \sigma \leq C$, $r \in \mathbb{R}$, $\eta = a + bi$ with $|a|, |r| \leq Cb^{3/5}$ and $b > 0$. Then for sufficiently large b we have the estimate:

$$\int_{1-i\infty}^{1+i\infty} \Gamma(w) e^{-rw} e^{\frac{1}{2\sigma}(w-\eta)^2} \, dw = $$

$$2\pi i \sqrt{\sigma} \exp(\mathcal{R} + \mathcal{I} i)\left(1 + O\left(b^{-1/5}\right)\right)$$

where

$$\mathcal{R} = -\frac{\pi b}{2} + a \log(b) - \frac{\log(b)}{2} + \frac{\sigma \pi^2}{8} - ra - \frac{\sigma}{2}(\log(b) - r)^2$$

$$\mathcal{I} = b \log(b) - b + \frac{\pi(a + \sigma r)}{2} - \frac{(a + \sigma r)^2}{2b} - \frac{\sigma \pi \log(b)}{2} - \frac{\pi}{4} - rb$$

General Dirichlet series

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be convergent in some half plane.

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Let $\widetilde{F}(s) = Q^s \Gamma(\omega s + \mu) F(s)$ where $Q > 0, \omega > 0, \mu \in \mathbb{R}$.
For all $t < 0$, define

$$\widetilde{F}_t(s) = \frac{-i}{\sqrt{\pi |t|}} \int_{c-i\infty}^{c+i\infty} \widetilde{F}(w) e^{\frac{1}{|t|}(w-s)^2} \, dw$$

where c is large enough that this makes sense.

Then we can get the same theorem as before. . .

Theorem for general Dirichlet series

Theorem (D.)

Let $t < 0, C > 0$. *Let* $s = x + iy$ *and* $\widetilde{s} = x + \frac{\omega|t|}{2}\log\left(\omega Q^{1/\omega}y\right) + i\left(y + \frac{\pi\omega|t|}{4}\right)$ *where* $|x| \leq C$ *and* y *sufficiently large (depending on* C*). Then we have*

$$\widetilde{F}_t(\widetilde{s}) = G_t(\widetilde{s})(F_t(s) + o_{y\to\infty}(1))$$

where

$$F_t(s) = \sum_{n=1}^{\infty} \exp\left(-\frac{|t|}{4}(\log n)^2\right)a_n n^{-s}$$

and G_t *is analytic and nonzero in the region of interest.*

Generalized Newman's conjecture

Suppose

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

is a Dirichlet series which has a meromorphic continuation to the whole complex plane with the only possible pole at $s = 1$ and for which we have a "completed" F:

$$\widetilde{F}(s) = \begin{cases} Q^s \Gamma(\omega s + \mu) F(s) & \text{if } F \text{ has no poles} \\ s(s-1) Q^s \Gamma(\omega s + \mu) F(s) & \text{if } F \text{ has a pole at } s = 1 \end{cases}$$

which is entire and satisfies the functional equation

$$\widetilde{F}(s) = \alpha \overline{\widetilde{F}(1 - \overline{s})}$$

Then there is a generalized de Bruijn-Newman constant Λ_F associated to F which is nonnegative!

Thanks for listening!