Phase Retrieval from Local Measurements: Deterministic Measurement Constructions and Efficient Recovery Algorithms

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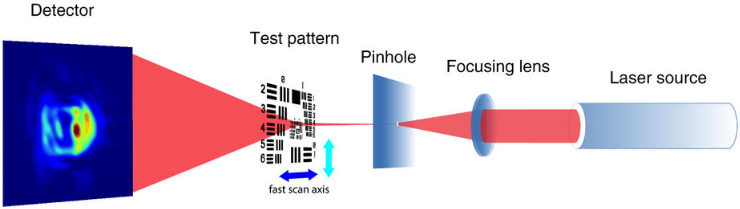
Motivating Application



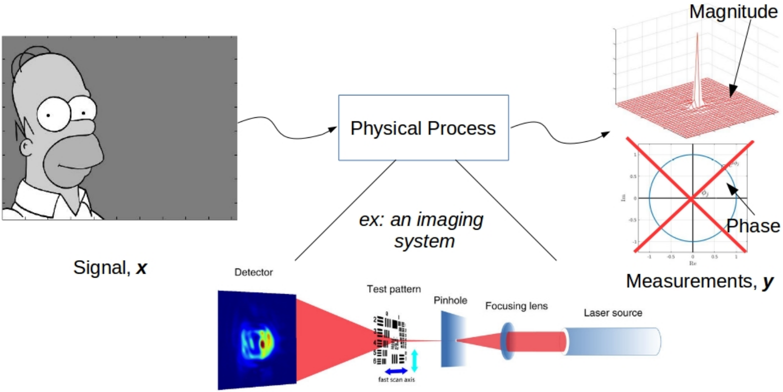
From Huang, Xiaojing, et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

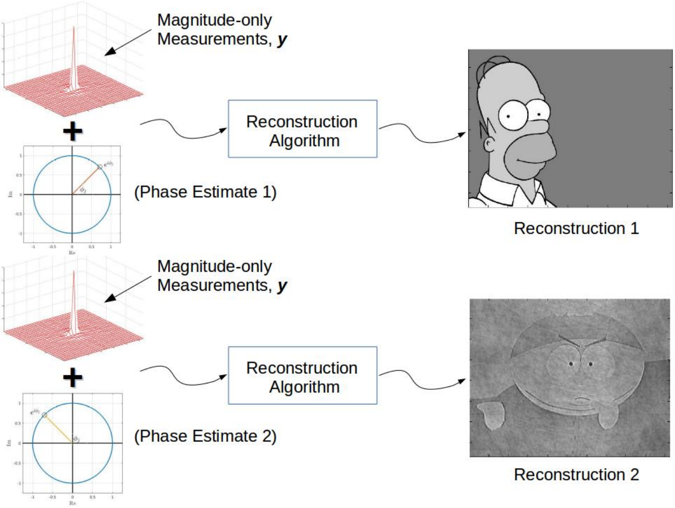
Other applications can be found in optics, astronomy and speech processing.

Motivation – Inverse Problem



Given $\mathbf{y}$, can we recover $\mathbf{x}$?

Motivation – Ill-Posed Inverse Problem



Which is the correct reconstruction?

Mathematical Model

$$\text{find}^1 \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + \eta_i \qquad i \in 1, \ldots, D,$$

where

- $y_i \in \mathbb{R}$ denotes the phaseless (or magnitude-only) measurements (D measurements acquired),
- $\mathbf{a}_i \in \mathbb{C}^d$ are known (by design or estimation) measurement vectors, and
- $\eta_i \in \mathbb{R}$ is measurement noise.

[1](upto a global phase offset)

Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Marchesini et al., 2006], [Fannjiang, Liao, 2012]
  and many others. . .
- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], . . .
- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Spectral) initialization + gradient descent (*Wirtinger Flow*) [Candes et al., 2014]

Most methods (with provable recovery guarantees) require impractical (global, random) measurement constructions.

Today...

- We discuss a recently introduced fast (essentially linear-time) phase retrieval algorithm based on realistic (deterministic)[2] local measurement constructions.

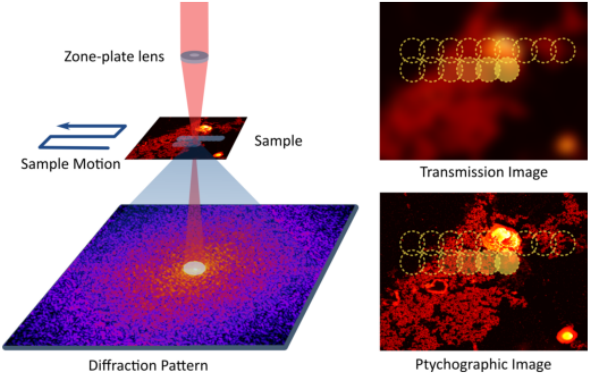
- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.

- (*Time Permitting*) extensions to 2D and compressive phase retrieval.

[2] for a large class of real-world signals

Outline

Local Correlation Measurements



From Qian, Jianliang, et al. "Efficient algorithms for ptychographic phase retrieval." *Inverse Problems Appl., Contemp. Math* 615 (2014).

Each $\mathbf{a}_i$ is a **shift** of a **locally-supported** vector (*mask or window*)

$$\mathbf{m}^{(j)} \in \mathbb{C}^d, \quad \mathrm{supp}(\mathbf{m}^{(j)}) = [\delta] \subset [d], \quad j = 1, \ldots, K$$

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Define the discrete circular shift operator

$$S_\ell : \mathbb{C}^d \to \mathbb{C}^d \quad \textbf{with} \quad (S_\ell \mathbf{x})_j = x_{\ell+j}.$$

Our measurements are then

$$(\mathbf{y}_\ell)_j = |\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 + \eta_{j,\ell}, \quad (j,\ell) \in [K] \times P, \quad P \subset \{0, ..., d-1\}$$

We will consider $K \approx \delta$ and $P = [d]_0 := \{0, ..., d-1\}$

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What are we measuring?

Lifted System: $|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_\ell \rangle.$

Example: $(6 \times 6$ system, $\delta = 2$, blue denotes non-zero entries$)$

$$|\langle \mathbf{x}\mathbf{x}^*, S_0^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_0 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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Example: (6×6 system, $\delta = 2$, blue denotes non-zero entries)

$$|\langle \mathbf{x}\mathbf{x}^*, S_1^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_1 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_{\color{red}2}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{\color{red}2} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_3^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_3 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_{4}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{4} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \right\rangle$$

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Lifted System: $$|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell \rangle.$$

Example: $(6 \times 6$ system, $\delta = 2$, blue denotes non-zero entries)

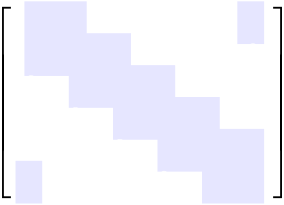
$$|\langle \mathbf{x}\mathbf{x}^*, S_5^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_5 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & \end{bmatrix} \right\rangle$$

What are we measuring?

Lifted System: $\quad |\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_\ell \rangle.$

Example: $\quad$ (6×6 system, $\delta = 2$, blue denotes non-zero entries)

Observation: $\quad$ The only entries of $\mathbf{x}\mathbf{x}^*$ we can hope to recover (via linear inversion) are supported on a (circulant) band



Useful Observations (I)

$T_\delta(\mathbb{C}^{d\times d})$: Let

$$T_k : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$$

$$T_k(A)_{ij} = \begin{cases} A_{ij}, & |i-j| \mod d < k \\ 0, & \textbf{otherwise}. \end{cases}$$

Lifted System Revisited: $|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle T_\delta(\mathbf{x}\mathbf{x}^*), S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell \rangle.$

Bottom Line: If we can find $\mathbf{m}^{(j)}$ such that

$$\mathrm{Span}\left\{S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell\right\}_{\ell,j} = T_\delta(\mathbb{C}^{d\times d}),$$

then we can recover $T_\delta(\mathbf{x}\mathbf{x}^*)$.

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A Highly Structured Linear System!

Example linear system

$$M' \, \mathbf{q} = \widetilde{\mathbf{y}},$$

where

- $\mathbf{q}$ denotes the vectorized (non-zero) entries of $T_\delta(\mathbf{x}\mathbf{x}^*)$
- $\widetilde{\mathbf{y}}$ denotes the (interleaved) measurements

$$\mathbf{q} = \left[|x_1|^2 \;\; x_1 x_2^* \;\; x_2 x_1^* \;\; |x_2|^2 \;\; x_2 x_3^* \;\; x_3 x_2^* \;\; |x_3|^2 \;\; x_3 x_4^* \;\; x_4 x_3^* \;\; |x_4|^2 \;\; x_4 x_1^* \;\; x_1 x_4^* \right]^T,$$

$$\widetilde{\mathbf{y}} = \left[(y_1)_1 \;\; (y_2)_1 \;\; (y_3)_1 \;\; (y_1)_2 \;\; (y_2)_2 \;\; (y_3)_2 \;\; (y_1)_3 \;\; (y_2)_3 \;\; (y_3)_3 \;\; (y_1)_4 \;\; (y_2)_4 \;\; (y_3)_4 \right]^T,$$

$$M' = \begin{bmatrix}
(\mathbf{m}^{(1)})_{1,1} & (\mathbf{m}^{(1)})_{1,2} & (\mathbf{m}^{(1)})_{2,1} & (\mathbf{m}^{(1)})_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}^{(2)})_{1,1} & (\mathbf{m}^{(2)})_{1,2} & (\mathbf{m}^{(2)})_{2,1} & (\mathbf{m}^{(2)})_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}^{(3)})_{1,1} & (\mathbf{m}^{(3)})_{1,2} & (\mathbf{m}^{(3)})_{2,1} & (\mathbf{m}^{(3)})_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}^{(1)})_{1,1} & (\mathbf{m}^{(1)})_{1,2} & (\mathbf{m}^{(1)})_{2,1} & (\mathbf{m}^{(1)})_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}^{(2)})_{1,1} & (\mathbf{m}^{(2)})_{1,2} & (\mathbf{m}^{(2)})_{2,1} & (\mathbf{m}^{(2)})_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}^{(3)})_{1,1} & (\mathbf{m}^{(3)})_{1,2} & (\mathbf{m}^{(3)})_{2,1} & (\mathbf{m}^{(3)})_{2,2} & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}^{(2)})_{1,1} & (\mathbf{m}^{(2)})_{1,2} & (\mathbf{m}^{(2)})_{2,1} & (\mathbf{m}^{(2)})_{2,2} & 0 & 0 \\
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(\mathbf{m}^{(1)})_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}^{(1)})_{1,1} & (\mathbf{m}^{(1)})_{1,2} & (\mathbf{m}^{(1)})_{2,1} \\
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\end{bmatrix}$$

Useful Observations (II)

Why this is useful:

(a) Diagonal entries of $T_\delta(\mathbf{x}\mathbf{x}^*)$ are $|x_i|^2$.

(b) Off-diagonals give the relative phases

$$\widetilde{X} := \frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}$$

$$T_\delta(\widetilde{X})_{(j,k)} = \mathbb{e}^{\mathbb{i}(\arg(x_j) - \arg(x_k))}, \quad |j-k| \mod d < \delta$$

Phase Synchronization:

(a) The leading eigenvector (appropriately normalized) of

$$\begin{aligned} T_\delta(\widetilde{X}) &= \operatorname{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \, T_\delta(\mathbb{1}\mathbb{1}^*) \operatorname{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \\ &= \operatorname{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \, F\Lambda F^* \operatorname{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \end{aligned}$$

is the vector of phases of x.

Note: $\frac{\mathbf{x}}{|\mathbf{x}|} = [\mathbb{e}^{\mathbb{i}\phi_1} \ \mathbb{e}^{\mathbb{i}\phi_2} \ \ldots \mathbb{e}^{\mathbb{i}\phi_d}]^T$ is the (unknown) phase vector!
$F \in \mathbb{C}^{d\times d}$ is the discrete Fourier transform (DFT) matrix

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Leading Eigenvector $\leftrightarrow$ Phase Vector

$$\left[|x_1|^2\ x_1x_2^*\ x_2x_1^*\ |x_2|^2\ x_2x_3^*\ x_3x_2^*\ |x_3|^2\ x_3x_4^*\ x_4x_3^*\ |x_4|^2\ x_4x_1^*\ x_1x_4^*\right]^T$$

$$\Big\downarrow \text{(re-arrange)}$$

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix} \quad (T_\delta(\mathbf{x}\mathbf{x}^*), 2\delta{-}1 \text{ entries in band})$$

$$\Big\downarrow \text{(normalize)}$$

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

$$\Big\downarrow \text{(top eigenvector)}$$

$$\left[\mathrm{e}^{\mathbb{i}\phi_1}\ \mathrm{e}^{\mathbb{i}\phi_2}\ \mathrm{e}^{\mathbb{i}\phi_3}\ \mathrm{e}^{\mathbb{i}\phi_4}\right]^T$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1}\quad |x_2|e^{\mathbb{i}\,\phi_2}\quad |x_3|e^{\mathbb{i}\,\phi_3}\quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

Leading Eigenvector $\leftrightarrow$ Phase Vector

$$\left[|x_1|^2\ x_1x_2^*\ x_2x_1^*\ |x_2|^2\ x_2x_3^*\ x_3x_2^*\ |x_3|^2\ x_3x_4^*\ x_4x_3^*\ |x_4|^2\ x_4x_1^*\ x_1x_4^*\right]^T$$

$\downarrow$ (re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix} \quad (T_\delta(\mathbf{xx}^*), 2\delta-1 \text{ entries in band})$$

$\downarrow$ (normalize)

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

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$$\left[\mathbb{e}^{\mathbb{i}\phi_1}\ \mathbb{e}^{\mathbb{i}\phi_2}\ \mathbb{e}^{\mathbb{i}\phi_3}\ \mathbb{e}^{\mathbb{i}\phi_4}\right]^T$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1}\quad |x_2|e^{\mathbb{i}\,\phi_2}\quad |x_3|e^{\mathbb{i}\,\phi_3}\quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

Leading Eigenvector $\leftrightarrow$ Phase Vector

$$\left[|x_1|^2\ x_1x_2^*\ x_2x_1^*\ |x_2|^2\ x_2x_3^*\ x_3x_2^*\ |x_3|^2\ x_3x_4^*\ x_4x_3^*\ |x_4|^2\ x_4x_1^*\ x_1x_4^*\right]^T$$

$\downarrow$(re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix}$$ ($T_\delta(\mathbf{x}\mathbf{x}^*), 2\delta-1$ entries in band)

$\downarrow$(normalize)

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

$\downarrow$(top eigenvector)

$$\left[\mathbb{e}^{\mathbb{i}\phi_1}\ \mathbb{e}^{\mathbb{i}\phi_2}\ \mathbb{e}^{\mathbb{i}\phi_3}\ \mathbb{e}^{\mathbb{i}\phi_4}\right]^T$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1}\quad |x_2|e^{\mathbb{i}\,\phi_2}\quad |x_3|e^{\mathbb{i}\,\phi_3}\quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

Leading Eigenvector $\leftrightarrow$ Phase Vector

$$\left[|x_1|^2\ x_1x_2^*\ x_2x_1^*\ |x_2|^2\ x_2x_3^*\ x_3x_2^*\ |x_3|^2\ x_3x_4^*\ x_4x_3^*\ |x_4|^2\ x_4x_1^*\ x_1x_4^*\right]^T$$

$$\Big\downarrow \text{(re-arrange)}$$

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix} \quad (T_\delta(\mathbf{x}\mathbf{x}^*), 2\delta-1 \text{ entries in band})$$

$$\Big\downarrow \text{(normalize)}$$

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$$\Big\downarrow \text{(leading eigenvector)}$$

$$\left[e^{\mathbb{i}\phi_1}\ e^{\mathbb{i}\phi_2}\ e^{\mathbb{i}\phi_3}\ e^{\mathbb{i}\phi_4}\right]^T$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1}\quad |x_2|e^{\mathbb{i}\,\phi_2}\quad |x_3|e^{\mathbb{i}\,\phi_3}\quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

Leading Eigenvector $\leftrightarrow$ Phase Vector

$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

$\downarrow$ (re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix} \quad (T_\delta(\mathbf{x}\mathbf{x}^*), 2\delta-1 \text{ entries in band})$$

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$\downarrow$ (leading eigenvector)

$$\left[e^{\mathbb{i}\phi_1} \; e^{\mathbb{i}\phi_2} \; e^{\mathbb{i}\phi_3} \; e^{\mathbb{i}\phi_4}\right]^T$$

(Signal Reconstruction) $\quad \left[|x_1|e^{\mathbb{i}\,\phi_1} \quad |x_2|e^{\mathbb{i}\,\phi_2} \quad |x_3|e^{\mathbb{i}\,\phi_3} \quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

Recovery Algorithm

Define the map $\mathcal{A} : \mathbb{C}^{d\times d} \to \mathbb{C}^D$

$$\mathcal{A}(Z)_{(\ell,j)} = \langle Z, S_\ell^* m^{(j)} m^{(j)^*} S_\ell \rangle_{(\ell,j)}.$$

and its restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}$ to our subspace.

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and its restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}$ to our subspace.

In the **noisy** setting:

Step 1: Estimate $T_\delta(\mathbf{x}\mathbf{x}^*)$ by the banded matrix

$$Z = T_\delta(Z) := \left(\mathcal{A}|^{-1}_{T_\delta(\mathbb{C}^{d\times d})} \frac{y}{2} \right) + \left(\mathcal{A}|^{-1}_{T_\delta(\mathbb{C}^{d\times d})} \frac{y}{2} \right)^*.$$

Step 2: Estimate the phase by computing the leading eigenvector of $T_\delta\left(\frac{Z}{|Z|}\right)$.

Step 3: Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$.

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Recovery Algorithm

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$$\mathcal{A}(Z)_{(\ell,j)} = \langle Z, S_\ell^* m^{(j)} m^{(j)^*} S_\ell \rangle_{(\ell,j)}.$$

and its restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}$ to our subspace.

In the **noisy** setting:

Step 1: Estimate $T_\delta(\mathbf{x}\mathbf{x}^*)$ by *Cost*: $\mathcal{O}(d \cdot \delta^3 + \delta \cdot d \log d)$ flops

$$Z = T_\delta(Z) := \left(\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}^{-1} \frac{y}{2} \right) + \left(\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}^{-1} \frac{y}{2} \right)^*.$$

Step 2: Estimate the phase by computing the leading eigenvector of $T_\delta\left(\frac{Z}{|Z|}\right)$. *Cost*: $\mathcal{O}(\delta^2 \cdot d \log d)$ flops

Step 3: Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$. *Total Cost*: $\mathcal{O}(\delta^2 \cdot d \log d + d \cdot \delta^3)$ flops

Outline

Well-Conditioned Linear Systems

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $(\mathbf{m}^{(i)})$ *as follows:*

$$(\mathbf{m}^{(i)})_\ell = \begin{cases} \frac{\mathbb{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathbb{e}^{\frac{2\pi \mathbb{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\ 0, & \ell > \delta \end{cases}, \qquad \begin{array}{l} a := \max\left\{4, \frac{\delta-1}{2}\right\}, \\ i = 1, 2, \ldots, N. \end{array}$$

Then, the resulting system matrix for the phase differences (step 1), $\mathcal{A}|_{T_\delta}$*, has condition number*

$$\kappa(\mathcal{A}|_{T_\delta}) < \max\left\{144\mathbb{e}^2, \frac{9\mathbb{e}^2}{4} \cdot (\delta-1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- δ is typically chosen to be $c \log_2 d$ with c small (2–3).
- Extensions: oversampling, random masks ....

Recovery Guarantee

Theorem (Iwen, Preskitt, Saab, V. 2016)

Let $x_{\min} := \min_j |x_j|$ *be the smallest magnitude of any entry in* $\mathbf{x}$. *Then, the estimate* $\mathbf{z}$ *produced by the proposed algorithm satisfies*

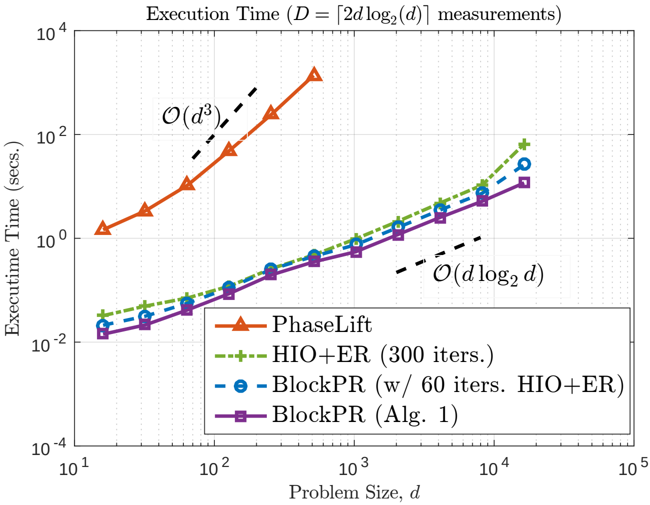
$$\min_{\theta \in [0, 2\pi]} \left\| \mathbf{x} - \mathbb{e}^{\mathbb{i}\theta} \mathbf{z} \right\|_2 \leq C \left(\frac{\|\mathbf{x}\|_\infty}{x_{\min}^2} \right) \left(\frac{d}{\delta} \right)^2 \kappa \|\eta\|_2 + C d^{\frac{1}{4}} \sqrt{\kappa \|\eta\|_2},$$

where $C \in \mathbb{R}^+$ *is an absolute universal constant.*

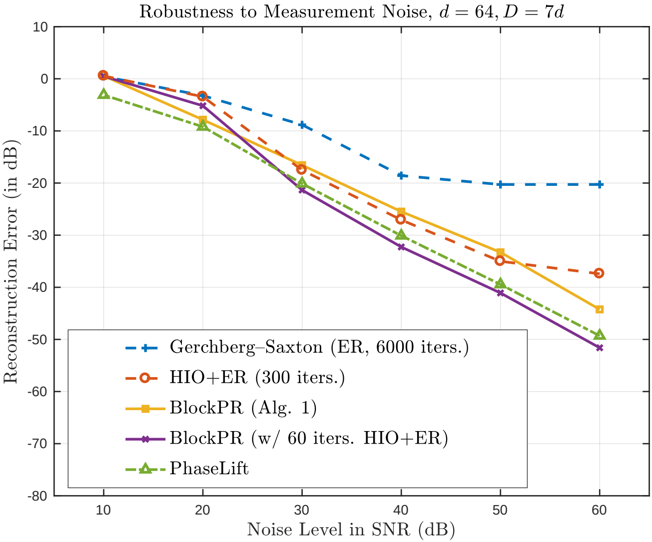
- This result yields a *deterministic* recovery result for any signal $\mathbf{x}$ which contains no zero entries.
- A randomized result can be derived for arbitrary $\mathbf{x}$ by right multiplying the signal $\mathbf{x}$ with a random "flattening" matrix. (this is also useful for performing *sparse* phase retrieval!)

Outline

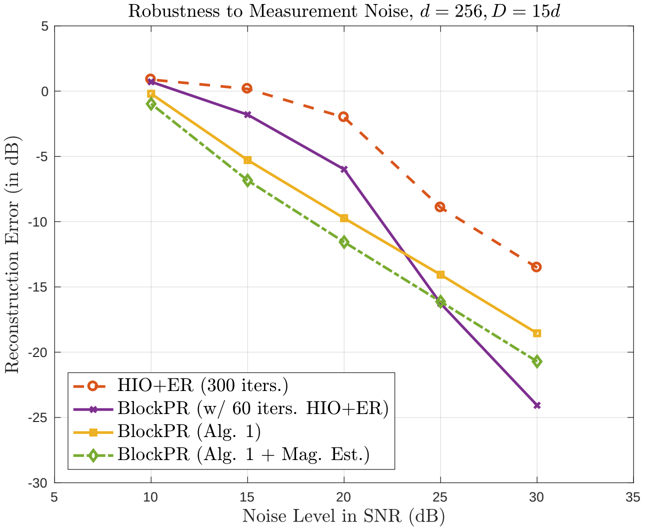
Efficiency – FFT–time phase retrieval



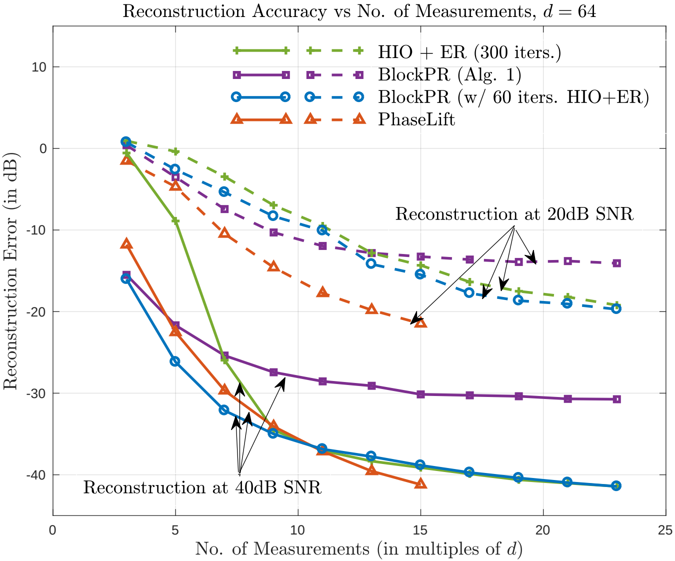
Robustness to Measurement Errors



Robustness to Measurement Errors



Reconstruction Error vs. No. of Measurements



Summary and Current/Future Research Directions

Today

- Phase retrieval is an immensely challenging problem seen in important applications such as x-ray crystallography.
- Proposed mathematical framework: Essentially linear-time robust phase retrieval from deterministic local correlation measurement constructions with rigorous **recovery guarantee**.

Current and Future Directions

- More robust measurement constructions
- Compressive phase retrieval
- Extensions to 2D and Ptychographic datasets
- Continuous problem formulation

Outline

Extension – 2D Phase Retrieval

- Preliminary results for 2D masks with tensor product structure
- Results from 1D extend to 2D; 2D linear system is a tensor product of the 1D linear system (up to row permutations)
- Eigenvector-based phase synchronization also works – calculation of spectral gap and error analysis pending

Test Image (256×256 pixels)

Recon. (Rel. error 2.857×10^{-16})

Extension – Compressive Phase Retrieval

Model find $\mathbf{x} \in \mathbb{C}^d$ given $|\mathcal{M}\mathbf{x}|^2 + \mathbf{n} = \mathbf{y} \in \mathbb{R}^D$

where $\mathbf{x}$ is k-sparse, with $k \ll d$,

$|\cdot|$ is entry-wise absolute value, and

$\mathcal{M}$ is a measurement matrix.

Measurement Design Assume $\mathcal{M} = \mathcal{P}\mathcal{C}$ where

$\mathcal{P} \in \mathbb{C}^{D \times \tilde{d}}$ is an admissible phase retrieval matrix with an associated recovery algorithm $\Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^{\tilde{d}}$, and

$\mathcal{C} \in \mathbb{C}^{\tilde{d} \times d}$ is an admissible compressive sensing matrix with an associated recovery algorithm $\Delta_{\mathcal{C}} : \mathbb{C}^{\tilde{d}} \to \mathbb{C}^d$.

Recovery Algorithm (Two-stage) $\Delta_{\mathcal{C}} \circ \Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^d$

Performance Metrics No. of measurements required is $\mathcal{O}(k \ln(d/k))$

Computational cost (sub-linear) is $\mathcal{O}(k \ln^c k \ln d)$

Pubs./Preprints/Code (see `www-personal.umich.edu/~adityavv`)

M. Iwen, B. Preskitt, R. Saab and A. Viswanathan. "Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector-Based Angular Synchronization." arXiv:1612.01182, 2016.

M. Iwen, A. Viswanathan, and Y. Wang. "Fast Phase Retrieval from Local Correlation Measurements." SIAM J. Imag. Sci., Vol. 9(4), pp. 1655–1688, Oct. 2016.

<u>Compressive Phase Retrieval</u>
M. Iwen, A. Viswanathan, and Y. Wang. "Robust Sparse Phase Retrieval Made Easy." Appl. Comput. Harmon. Anal., Vol. 42(1), pp. 135–142, Jan. 2017.

<u>2D Phase Retrieval</u>
Mark Iwen, Brian Preskitt, Rayan Saab and A. Viswanathan. "Phase Retrieval from Local Measurements in Two Dimensions.", Proc. SPIE 10394, Wavelets and Sparsity XVII, 103940X, Aug. 2017.

<u>Code</u> https://bitbucket.org/charms/{blockpr,sparsepr}

Questions?