Phase Retrieval from Local Measurements: Deterministic Measurement Constructions and Efficient Recovery Algorithms

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Collaborators

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Motivating Application
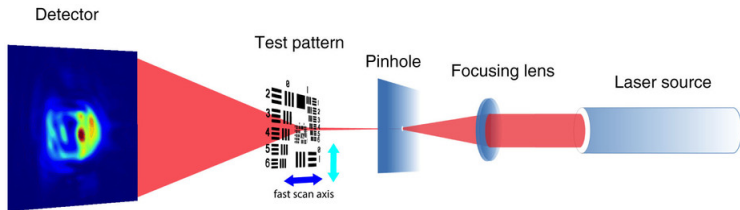
From Huang, Xiaojing, et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

Other applications can be found in optics, astronomy and speech processing.

Mathematical Model

$$\text{find}^1 \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + \eta_i \qquad i \in 1, \ldots, D,$$

where

- $y_i \in \mathbb{R}$ denotes the phaseless (or magnitude-only) measurements (D measurements acquired),

- $\mathbf{a}_i \in \mathbb{C}^d$ are known (by design or estimation) measurement vectors, and

- $\eta_i \in \mathbb{R}$ is measurement noise.

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[1](upto a global phase offset)

Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Marchesini et al., 2006], [Fannjiang, Liao, 2012]
  and many others...

- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], ...

- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Stochastic) gradient descent (*Wirtinger Flow*) [Candes et al., 2014]

Most methods (with provable recovery guarantees) require impractical
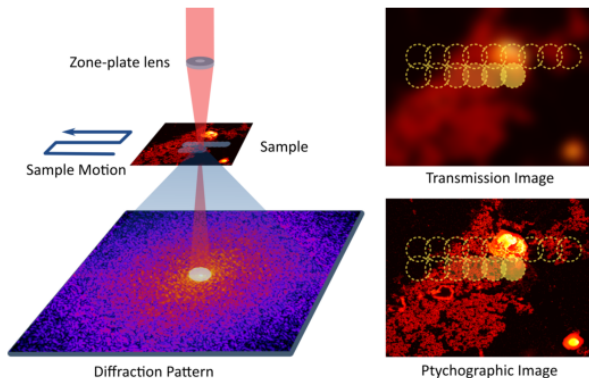(global, random) measurement constructions.

Today...

- We discuss a recently introduced fast (essentially linear-time) phase retrieval algorithm based on realistic (deterministic)[2] local measurement constructions.

- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.

- (*Time Permitting*) extensions to 2D and compressive phase retrieval.

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[2]for a large class of real-world signals

Outline

Local Correlation Measurements

From Qian, Jianliang, et al. "Efficient algorithms for ptychographic phase retrieval." *Inverse Problems Appl., Contemp. Math* 615 (2014).

Each $\mathbf{a}_i$ is a **shift** of a **locally-supported** vector (*mask or window*)
$$\mathbf{m}^{(j)} \in \mathbb{C}^d, \quad \text{supp}(\mathbf{m}^{(j)}) = [\delta] \subset [d], \quad j = 1, \ldots, K$$

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Define the discrete circular shift operator

$$S_\ell : \mathbb{C}^d \to \mathbb{C}^d \quad \text{with} \quad (S_\ell \mathbf{x})_j = x_{\ell+j}.$$

Our measurements are then

$$(\mathbf{y}_\ell)_j = |\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 + \eta_{j,\ell}, \quad (j, \ell) \in [K] \times P, \quad P \subset \{0, ..., d-1\}$$

We will consider $K \approx \delta$ and $P = [d]_0 := \{0, ..., d-1\}$

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We will consider $K \approx \delta$ and $P = [d]_0 := \{0, \ldots, d-1\}$

What are we measuring?

Lifted System:
$$|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_\ell \rangle.$$

Example: (6×6 system, $\delta = 2$, blue denotes non-zero entries)

$$|\langle \mathbf{x}\mathbf{x}^*, S_0^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_0 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_1^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_1 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

What are we measuring?

Lifted System: $\qquad |\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{xx}^*, S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell \rangle.$

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$$|\langle \mathbf{xx}^*, S_2^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_2 \rangle| = \left\langle \mathbf{xx}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{xx}^*, S_3^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_3 \rangle| = \left\langle \mathbf{xx}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{xx}^*, S_4^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_4 \rangle| = \left\langle \mathbf{xx}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \right\rangle$$

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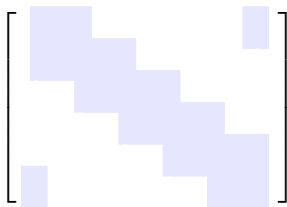
$$|\langle \mathbf{x}\mathbf{x}^*, S_5^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_5 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} \blacksquare & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \blacksquare & 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \right\rangle$$

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Observation: The only entries of $\mathbf{x}\mathbf{x}^*$ we can hope to recover (via linear inversion) are supported on a (circulant) band

Useful Observations (I)

$T_\delta(\mathbb{C}^{d \times d})$:   Let

$$T_k : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d}$$

$$T_k(A)_{ij} = \begin{cases} A_{ij}, & |i-j| \mod d < k \\ 0, & \textbf{otherwise}. \end{cases}$$

Lifted System Revisited: $|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle T_\delta(\mathbf{x}\mathbf{x}^*), S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_\ell \rangle.$

Bottom Line: If we can find $\mathbf{m}^{(j)}$ such that

$$\mathrm{Span} \left\{ S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)*} S_\ell \right\}_{\ell, j} = T_\delta(\mathbb{C}^{d \times d}),$$

then we can recover $T_\delta(\mathbf{x}\mathbf{x}^*)$.

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Useful Observations (II)

Why this is useful:

(a) Diagonal entries of $T_\delta(\mathbf{x}\mathbf{x}^*)$ are $|x_i|^2$.

(b) Off-diagonals give the relative phases

$$\widetilde{X} := \frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}$$

$$T_\delta(\widetilde{X})_{(j,k)} = \mathrm{e}^{\mathrm{i}(\arg(x_j)-\arg(x_k))}, \quad |j-k| \mod d < \delta$$

Phase Synchronization:

(a) The leading eigenvector (appropriately normalized) of

$$T_\delta(\widetilde{X}) = \mathrm{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) T_\delta(\mathbb{1}\mathbb{1}^*) \mathrm{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right)$$

$$= \mathrm{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) F\Lambda F^* \mathrm{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right)$$

is the vector of phases of $\mathbf{x}$.

<u>Note</u>: $\frac{\mathbf{x}}{|\mathbf{x}|} = [\mathrm{e}^{\mathrm{i}\phi_1}\ \mathrm{e}^{\mathrm{i}\phi_2}\ \ldots \mathrm{e}^{\mathrm{i}\phi_d}]^T$ is the (unknown) phase vector!

$F \in \mathbb{C}^{d\times d}$ is the discrete Fourier transform (DFT) matrix

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Recovery Algorithm

Define the map $\mathcal{A} : \mathbb{C}^{d \times d} \to \mathbb{C}^D$

$$\mathcal{A}(Z)_{(\ell,j)} = \langle Z, S_\ell^* m^{(j)} m^{(j)^*} S_\ell \rangle_{(\ell,j)}.$$

and its restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d \times d})}$ to our subspace.

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In the **noisy** setting:

Step 1: Estimate $T_\delta(\mathbf{x}\mathbf{x}^*)$ by the banded matrix

$$Z = T_\delta(Z) := \left(\mathcal{A}|_{T_\delta(\mathbb{C}^{d \times d})}^{-1} \frac{y}{2} \right) + \left(\mathcal{A}|_{T_\delta(\mathbb{C}^{d \times d})}^{-1} \frac{y}{2} \right)^*.$$

Step 2: Estimate the phase by computing the leading eigenvector of $T_\delta \left(\frac{Z}{|Z|} \right)$.

Step 3: Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$.

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In the **noisy** setting:

Step 1:   Estimate $T_\delta(\mathbf{x}\mathbf{x}^*)$ by   <u>Cost</u>: $\mathcal{O}(d \cdot \delta^3 + \delta \cdot d \log d)$ flops

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Step 2:   Estimate the phase by computing the leading eigenvector of $T_\delta \left(\frac{Z}{|Z|} \right).$ <u>Cost</u>: $\mathcal{O}(\delta^2 \cdot d \log d)$ flops

Step 3:   Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$.   <u>Total Cost</u>: $\mathcal{O}(\delta^2 \cdot d \log d + d \cdot \delta^3)$ flops

Outline

Well-Conditioned Linear Systems

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $(\mathbf{m}^{(i)})$ as follows:

$$(\mathbf{m}^{(i)})_\ell = \begin{cases} \frac{\mathbb{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathbb{e}^{\frac{2\pi \mathbf{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\ 0, & \ell > \delta \end{cases}, \qquad \begin{array}{l} a := \max\left\{4, \frac{\delta-1}{2}\right\}, \\ i = 1, 2, \ldots, N. \end{array}$$

Then, the resulting system matrix for the phase differences (step 1), $\mathcal{A}|_{T_\delta}$, has condition number

$$\kappa(\mathcal{A}|_{T_\delta}) < \max\left\{144\mathbb{e}^2, \frac{9\mathbb{e}^2}{4} \cdot (\delta-1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- δ is typically chosen to be $c\log_2 d$ with c small (2–3).
- Extensions: oversampling, random masks . . . .

Recovery Guarantee

Theorem (Iwen, Preskitt, Saab, V. 2016)

Let $x_{\min} := \min_j |x_j|$ be the smallest magnitude of any entry in $\mathbf{x}$. Then, the estimate $\mathbf{z}$ produced by the proposed algorithm satisfies

$$\min_{\theta \in [0,2\pi]} \left\| \mathbf{x} - e^{i\theta} \mathbf{z} \right\|_2 \leq C \left(\frac{\|\mathbf{x}\|_\infty}{x_{\min}^2} \right) \left(\frac{d}{\delta} \right)^2 \kappa \|\eta\|_2 + C d^{\frac{1}{4}} \sqrt{\kappa \|\eta\|_2},$$

where $C \in \mathbb{R}^+$ is an absolute universal constant.

- This result yields a *deterministic* recovery result for any signal $\mathbf{x}$ which contains no zero entries.

- A randomized result can be derived for arbitrary $\mathbf{x}$ by right multiplying the signal $\mathbf{x}$ with a random "flattening" matrix. (this is also useful for performing *sparse* phase retrieval!)
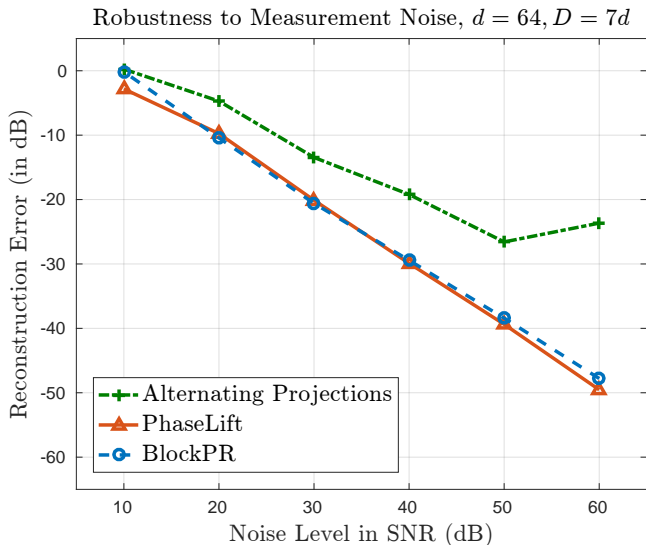
Outline

Efficiency – FFT–time phase retrieval

Execution Time ($D = \lceil 2d \log_2(d) \rceil$ measurements)
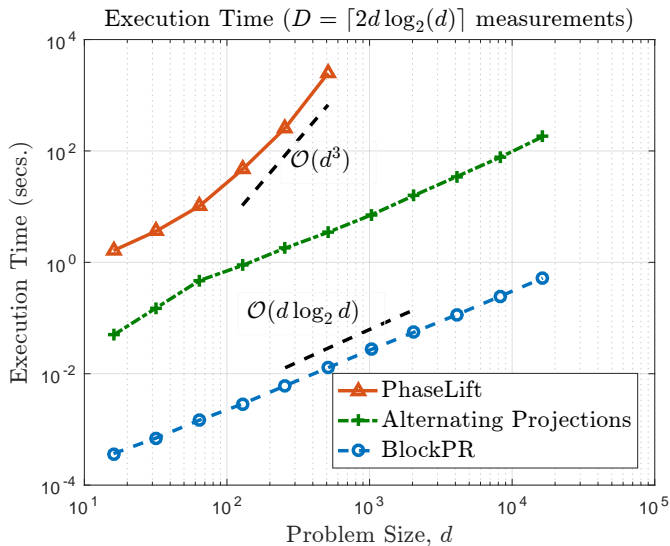
Robustness to Measurement Errors

Robustness to Measurement Noise, $d = 64, D = 7d$

- Alternating Projections
- PhaseLift
- BlockPR

Reconstruction Error vs. No. of Measurements

Reconstruction Accuracy vs No. of Measurements, $d = 64$

Legend:
- Alternating Projections
- BlockPR
- PhaseLift

Reconstruction at 20dB SNR

Reconstruction at 40dB SNR

x-axis: No. of Measurements (in multiples of d)

y-axis: Reconstruction Error (in dB)
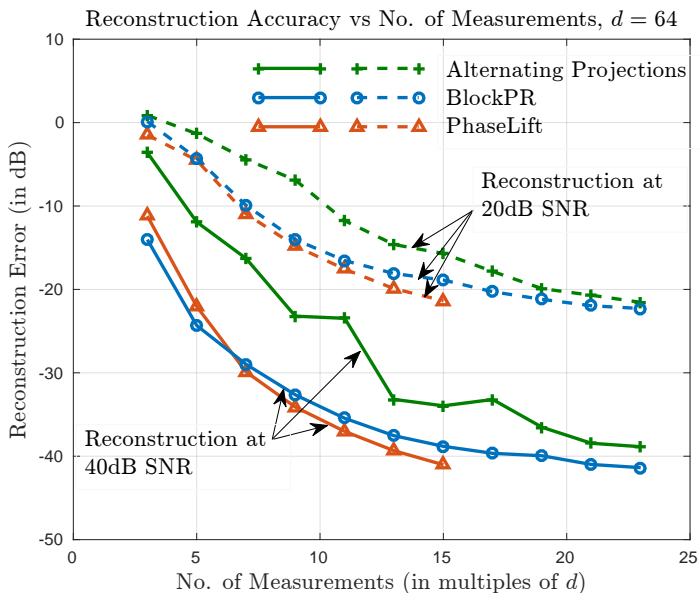
Summary and Current/Future Research Directions

<u>Today</u>

- Phase retrieval is an immensely challenging problem seen in important applications such as x-ray crystallography.

- Proposed mathematical framework: Essentially linear-time robust phase retrieval from deterministic local correlation measurement constructions with rigorous **recovery guarantee**.

<u>Current and Future Directions</u>

- More robust measurement constructions

- Compressive phase retrieval

- Extensions to 2D and Ptychographic datasets

- Continuous problem formulation

Outline

Extension – 2D Phase Retrieval

- Preliminary results for 2D masks with tensor product structure

- Results from 1D extend to 2D; 2D linear system is a tensor product of the 1D linear system (up to row permutations)

- Eigenvector-based phase synchronization also works – calculation of spectral gap and error analysis pending

Test Image (256×256 pixels)

Recon. (Rel. error 2.857×10^{-16})

Extension – Compressive Phase Retrieval

<u>Model</u> find $\mathbf{x} \in \mathbb{C}^d$ given $|\mathcal{M}\mathbf{x}|^2 + \mathbf{n} = \mathbf{y} \in \mathbb{R}^D$

where $\mathbf{x}$ is k-sparse, with $k \ll d$,

$|\cdot|$ is entry-wise absolute value, and

$\mathcal{M}$ is a measurement matrix.

<u>Measurement Design</u> Assume $\mathcal{M} = \mathcal{PC}$ where

$\mathcal{P} \in \mathbb{C}^{D \times \tilde{d}}$ is an admissible phase retrieval matrix with an associated recovery algorithm $\Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^{\tilde{d}}$, and

$\mathcal{C} \in \mathbb{C}^{\tilde{d} \times d}$ is an admissible compressive sensing matrix with an associated recovery algorithm $\Delta_{\mathcal{C}} : \mathbb{C}^{\tilde{d}} \to \mathbb{C}^d$.

<u>Recovery Algorithm</u> (Two-stage) $\Delta_{\mathcal{C}} \circ \Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^d$

<u>Performance Metrics</u> No. of measurements required is $\mathcal{O}(k \ln(d/k))$

Computational cost (sub-linear) is $\mathcal{O}(k \ln^c k \ln d)$

Pubs./Preprints/Code (see www-personal.umich.edu/~adityavv)

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Compressive Phase Retrieval
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Code https://bitbucket.org/charms/{blockpr,sparsepr}

Questions?