Phase Retrieval from Local Measurements: Deterministic Measurement Constructions and Efficient Recovery Algorithms

Aditya Viswanathan
Department of Mathematics and Statistics
adityavv@umich.edu
www-personal.umich.edu/∼adityavv

UNIVERSITY OF MICHIGAN-DEARBORN

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Collaborators

Mark Iwen

Rayan Saab

Brian Preskitt

Yang Wang

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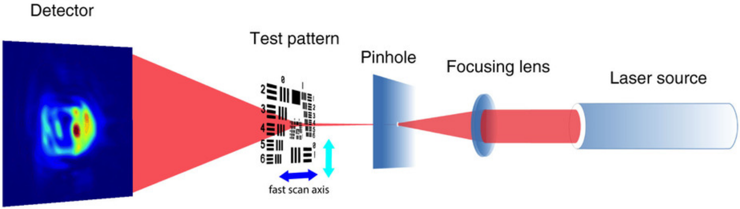
Motivating Application



From Huang, Xiaojing, et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

Other applications can be found in optics, astronomy and speech processing.

Mathematical Model

$$\text{find}^1 \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + \eta_i \qquad i \in 1, \ldots, D,$$

where

- $y_i \in \mathbb{R}$ denotes the phaseless (or magnitude-only) measurements (D measurements acquired),
- $\mathbf{a}_i \in \mathbb{C}^d$ are known (by design or estimation) measurement vectors, and
- $\eta_i \in \mathbb{R}$ is measurement noise.

[1](upto a global phase offset)

Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Marchesini et al., 2006], [Fannjiang, Liao, 2012]
  and many others...
- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], ...
- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Spectral) initialization + gradient descent (*Wirtinger Flow*) [Candes et al., 2014]

Most methods (with provable recovery guarantees) require impractical (global, random) measurement constructions.

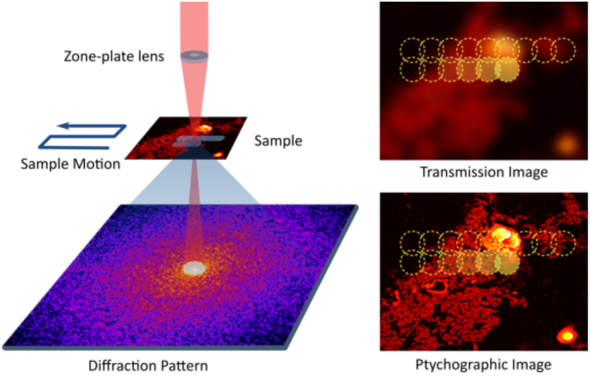
Today...

- We discuss a recently introduced fast (essentially linear-time) phase retrieval algorithm based on realistic (deterministic)[2] local measurement constructions.
- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.
- (*Time Permitting*) extensions to 2D and compressive phase retrieval.

[2]for a large class of real-world signals

Outline

Local Correlation Measurements



From Qian, Jianliang, et al. "Efficient algorithms for ptychographic phase retrieval." *Inverse Problems Appl., Contemp. Math* 615 (2014).

Each $\mathbf{a}_i$ is a **shift** of a **locally-supported** vector (*mask or window*)

$$\mathbf{m}^{(j)} \in \mathbb{C}^d, \quad \mathrm{supp}(\mathbf{m}^{(j)}) = [\delta] \subset [d], \quad j = 1, \ldots, K$$

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Define the discrete circular shift operator

$$S_\ell : \mathbb{C}^d \to \mathbb{C}^d \quad \textbf{with} \quad (S_\ell \mathbf{x})_j = x_{\ell+j}.$$

Our measurements are then

$$(\mathbf{y}_\ell)_j = |\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 + \eta_{j,\ell}, \quad (j,\ell) \in [K] \times P, \quad P \subset \{0, ..., d-1\}$$

We will consider $K \approx \delta$ and $P = [d]_0 := \{0, ..., d-1\}$

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What are we measuring?

Lifted System: $$|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_\ell \rangle.$$

Example: (6×6 system, $\delta = 2$, blue denotes non-zero entries)

$$|\langle \mathbf{x}\mathbf{x}^*, S_{0}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{0} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_{2}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{2} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_{\color{red}3}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{\color{red}3} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

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Example: $(6 \times 6$ system, $\delta = 2$, blue denotes non-zero entries)

$$|\langle \mathbf{x}\mathbf{x}^*, S_4^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_4 \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \right\rangle$$

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$$|\langle \mathbf{x}\mathbf{x}^*, S_{5}^* \mathbf{m}^{(j)} {\mathbf{m}^{(j)}}^* S_{5} \rangle| = \left\langle \mathbf{x}\mathbf{x}^*, \begin{bmatrix} \blacksquare & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \blacksquare & 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \right\rangle$$

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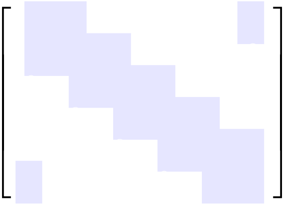
Observation: The only entries of $\mathbf{x}\mathbf{x}^*$ we can hope to recover (via linear inversion) are supported on a (circulant) band



Useful Observations (I)

$T_\delta(\mathbb{C}^{d\times d})$: Let

$$T_k : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$$

$$T_k(A)_{ij} = \begin{cases} A_{ij}, & |i-j| \mod d < k \\ 0, & \textbf{otherwise}. \end{cases}$$

Lifted System Revisited: $|\langle \mathbf{x}, S_\ell^* \mathbf{m}^{(j)} \rangle|^2 = \langle T_\delta(\mathbf{x}\mathbf{x}^*), S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell \rangle$.

Bottom Line: If we can find $\mathbf{m}^{(j)}$ such that

$$\mathrm{Span}\left\{ S_\ell^* \mathbf{m}^{(j)} \mathbf{m}^{(j)^*} S_\ell \right\}_{\ell, j} = T_\delta(\mathbb{C}^{d\times d}),$$

then we can recover $T_\delta(\mathbf{x}\mathbf{x}^*)$.

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Useful Observations (II)

Why this is useful:

(a) Diagonal entries of $T_\delta(\mathbf{x}\mathbf{x}^*)$ are $|x_i|^2$.

(b) Off-diagonals give the relative phases

$$\widetilde{X} := \frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}$$

$$T_\delta(\widetilde{X})_{(j,k)} = \mathbb{e}^{\mathbb{i}(\arg(x_j) - \arg(x_k))}, \quad |j-k| \mod d < \delta$$

Phase Synchronization:

(a) The leading eigenvector (appropriately normalized) of

$$\begin{aligned} T_\delta(\widetilde{X}) &= \operatorname{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \; T_\delta(\mathbb{1}\mathbb{1}^*) \operatorname{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \\ &= \operatorname{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \; F\Lambda F^* \operatorname{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \end{aligned}$$

is the vector of phases of $\mathbf{x}$.

Note: $\frac{\mathbf{x}}{|\mathbf{x}|} = [\mathbb{e}^{\mathbb{i}\phi_1} \;\; \mathbb{e}^{\mathbb{i}\phi_2} \;\; \ldots \mathbb{e}^{\mathbb{i}\phi_d}]^T$ is the (unknown) phase vector!
$F \in \mathbb{C}^{d \times d}$ is the discrete Fourier transform (DFT) matrix

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Recovery Algorithm

Define the map $\mathcal{A} : \mathbb{C}^{d\times d} \to \mathbb{C}^{D}$

$$\mathcal{A}(Z)_{(\ell,j)} = \langle Z, S_\ell^* m^{(j)} m^{(j)^*} S_\ell \rangle_{(\ell,j)}.$$

and its restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d\times d})}$ to our subspace.

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In the **noisy** setting:

Step 1: Estimate $T_\delta(\mathbf{x}\mathbf{x}^*)$ by the banded matrix

$$Z = T_\delta(Z) := \left(\mathcal{A}|^{-1}_{T_\delta(\mathbb{C}^{d\times d})} \frac{y}{2}\right) + \left(\mathcal{A}|^{-1}_{T_\delta(\mathbb{C}^{d\times d})} \frac{y}{2}\right)^*.$$

Step 2: Estimate the phase by computing the leading eigenvector of $T_\delta\left(\frac{Z}{|Z|}\right)$.

Step 3: Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$.

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Step 2: Estimate the phase by computing the leading eigenvector of $T_\delta\left(\frac{Z}{|Z|}\right)$. *Cost*: $\mathcal{O}(\delta^2 \cdot d \log d)$ flops

Step 3: Combine phase with $\sqrt{\cdot}$ of diagonal entries of $T_\delta(Z)$ to estimate $\mathbf{x}$. *Total Cost*: $\mathcal{O}(\delta^2 \cdot d \log d + d \cdot \delta^3)$ flops

Outline

Well-Conditioned Linear Systems

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $(\mathbf{m}^{(i)})$ *as follows:*

$$(\mathbf{m}^{(i)})_\ell = \begin{cases} \frac{\mathrm{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\ 0, & \ell > \delta \end{cases}, \qquad a := \max\left\{4, \tfrac{\delta-1}{2}\right\}, \quad i = 1, 2, \ldots, N.$$

Then, the resulting system matrix for the phase differences (step 1), $\mathcal{A}|_{T_\delta}$*, has condition number*

$$\kappa(\mathcal{A}|_{T_\delta}) < \max\left\{144\mathrm{e}^2, \frac{9\mathrm{e}^2}{4} \cdot (\delta-1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- δ is typically chosen to be $c \log_2 d$ with c small (2–3).
- Extensions: oversampling, random masks ....

Well-Conditioned Linear Systems

Mask Construction II (Iwen, Preskitt, Saab, V. 2016)

Choose entries of the measurement mask $(\mathbf{m}_i)$ as follows:
For $i = 1, 2, \ldots, \delta - 1$

$$\begin{aligned} \mathbf{m}_1 &= \mathbf{e}_1 \\ \mathbf{m}_{2i} &= \mathbf{e}_1 + \mathbf{e}_{i+1} \\ \mathbf{m}_{2i+1} &= \mathbf{e}_1 - \mathbb{i}\mathbf{e}_{i+1} \end{aligned}$$

Then, the resulting system matrix for the phase differences, M', has condition number

$$\kappa(M') < c\delta.$$

Recovery Guarantee

Theorem (Iwen, Preskitt, Saab, V. 2016)

Let $x_{\min} := \min_j |x_j|$ *be the smallest magnitude of any entry in* $\mathbf{x}$*. Then, the estimate* $\mathbf{z}$ *produced by the proposed algorithm satisfies*

$$\min_{\theta \in [0,2\pi]} \left\| \mathbf{x} - \mathbb{e}^{\mathbb{i}\theta} \mathbf{z} \right\|_2 \leq C \left(\frac{\|\mathbf{x}\|_\infty}{x_{\min}^2} \right) \left(\frac{d}{\delta} \right)^2 \kappa \|\eta\|_2 + C d^{\frac{1}{4}} \sqrt{\kappa \|\eta\|_2},$$

where $C \in \mathbb{R}^+$ *is an absolute universal constant.*

- This result yields a *deterministic* recovery result for any signal $\mathbf{x}$ which contains no zero entries.
- A randomized result can be derived for arbitrary $\mathbf{x}$ by right multiplying the signal $\mathbf{x}$ with a random "flattening" matrix. (this is also useful for performing *sparse* phase retrieval!)

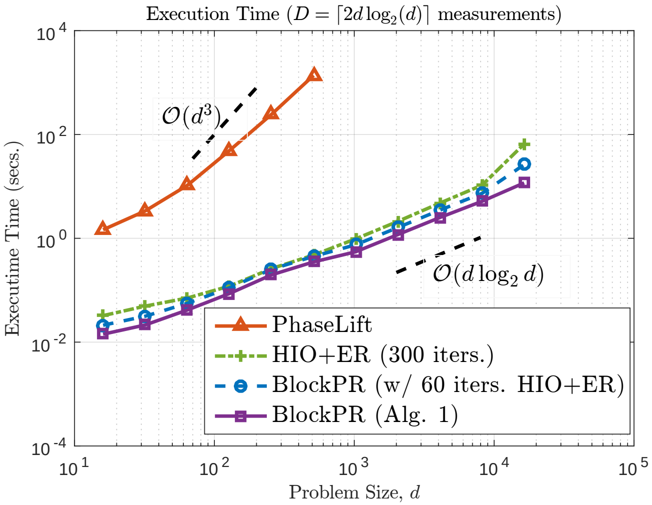
Main Elements of the Proof

1. Well-conditioned measurements:
   - Linear system for the lifted variables is block-circulant
   - Bound condition number of each block to find κ.
2. (Reconstruction error) $\approx$ (Phase error) $+$ (Magnitude error)
   - Magnitude error (second term in error guarantee) – follows from error in inverting linear system for lifted variables
   - Phase error (first term in error guarantee) – evaluate eigenvalue gap $+$ Cheeger inequality of [Bandeira et al. 2013] $+$ adaptation of proof method from [Alexeev et al. 2014]

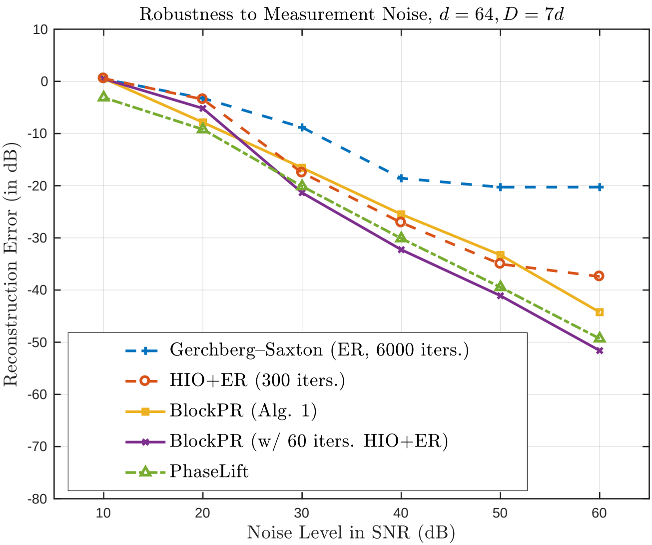
Note: Bound not optimized; for example, magnitude estimation can be improved!

Outline

Efficiency – FFT–time phase retrieval



Robustness to Measurement Errors



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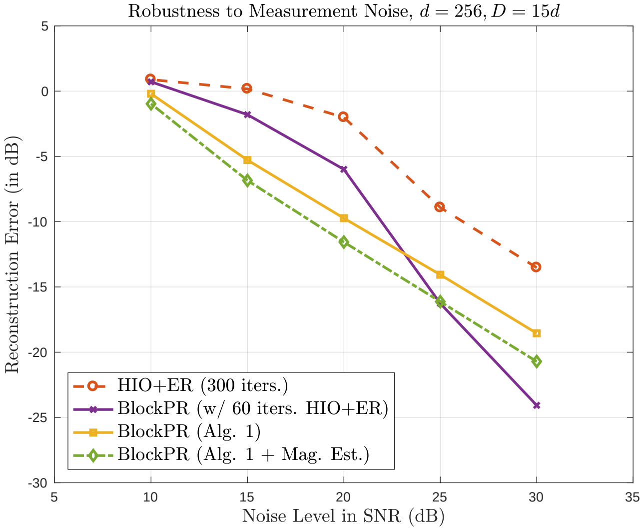
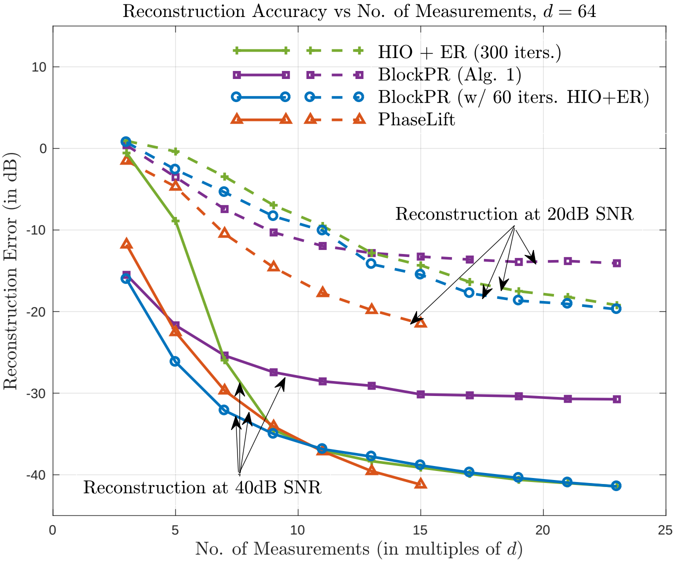
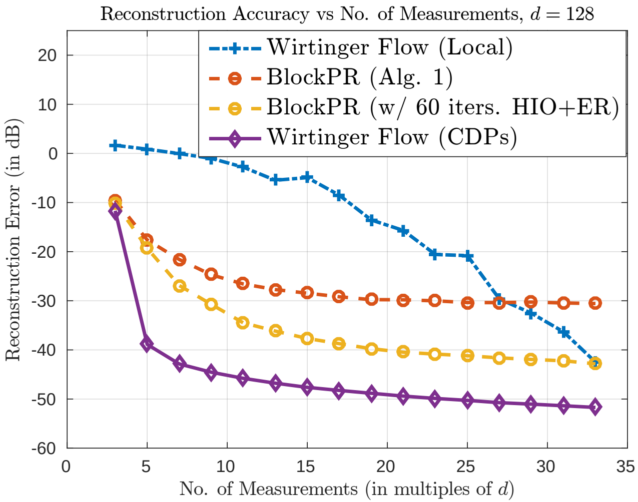


Reconstruction Error vs. No. of Measurements



Local vs Global Measurements



Summary and Current/Future Research Directions

Today

- Phase retrieval is an immensely challenging problem seen in important applications such as x-ray crystallography.
- Proposed mathematical framework: Essentially linear-time robust phase retrieval from deterministic local correlation measurement constructions with rigorous **recovery guarantee**.

Current and Future Directions

- More robust measurement constructions
- Compressive phase retrieval
- Extensions to 2D and Ptychographic datasets
- Continuous problem formulation

Outline

Extension – 2D Phase Retrieval

- Preliminary results for 2D masks with tensor product structure
- Results from 1D extend to 2D; 2D linear system is a tensor product of the 1D linear system (up to row permutations)
- Eigenvector-based phase synchronization also works – calculation of spectral gap and error analysis pending

Test Image (256×256 pixels)

Recon. (Rel. error 2.857×10^{-16})

Extension – Compressive Phase Retrieval

Model find $\mathbf{x} \in \mathbb{C}^d$ given $|\mathcal{M}\mathbf{x}|^2 + \mathbf{n} = \mathbf{y} \in \mathbb{R}^D$

where $\mathbf{x}$ is k-sparse, with $k \ll d$,

$|\cdot|$ is entry-wise absolute value, and

$\mathcal{M}$ is a measurement matrix.

Measurement Design Assume $\mathcal{M} = \mathcal{P}\mathcal{C}$ where

$\mathcal{P} \in \mathbb{C}^{D\times\tilde{d}}$ is an admissible phase retrieval matrix with an associated recovery algorithm $\Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^{\tilde{d}}$, and

$\mathcal{C} \in \mathbb{C}^{\tilde{d}\times d}$ is an admissible compressive sensing matrix with an associated recovery algorithm $\Delta_{\mathcal{C}} : \mathbb{C}^{\tilde{d}} \to \mathbb{C}^d$.

Recovery Algorithm (Two-stage) $\Delta_{\mathcal{C}} \circ \Phi_{\mathcal{P}} : \mathbb{R}^D \to \mathbb{C}^d$

Performance Metrics No. of measurements required is $\mathcal{O}(k \ln(d/k))$

Computational cost (sub-linear) is $\mathcal{O}(k \ln^c k \ln d)$

Pubs./Preprints/Code (see `www-personal.umich.edu/~adityavv`)

M. Iwen, B. Preskitt, R. Saab and A. Viswanathan. "Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector-Based Angular Synchronization." arXiv:1612.01182, 2016.

M. Iwen, A. Viswanathan, and Y. Wang. "Fast Phase Retrieval from Local Correlation Measurements." SIAM J. Imag. Sci., Vol. 9(4), pp. 1655–1688, Oct. 2016.

Compressive Phase Retrieval

M. Iwen, A. Viswanathan, and Y. Wang. "Robust Sparse Phase Retrieval Made Easy." Appl. Comput. Harmon. Anal., Vol. 42(1), pp. 135–142, Jan. 2017.

2D Phase Retrieval

Mark Iwen, Brian Preskitt, Rayan Saab and A. Viswanathan. "Phase Retrieval from Local Measurements in Two Dimensions.", Proc. SPIE 10394, Wavelets and Sparsity XVII, 103940X, Aug. 2017.

Code https://bitbucket.org/charms/{blockpr,sparsepr}

Questions?