Fast and Robust Phase Retrieval

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The Phase Retrieval Problem

The finite dimensional phase retrieval problem may be written as:

$$\text{find} \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad |M\mathbf{x}| = \mathbf{b} \in \mathbb{R}^D,$$

where

- $\mathbf{b} \in \mathbb{R}^D$ are the magnitude or intensity measurements.
- $M \in \mathbb{C}^{D \times d}$ is a measurement matrix associated with these measurements.

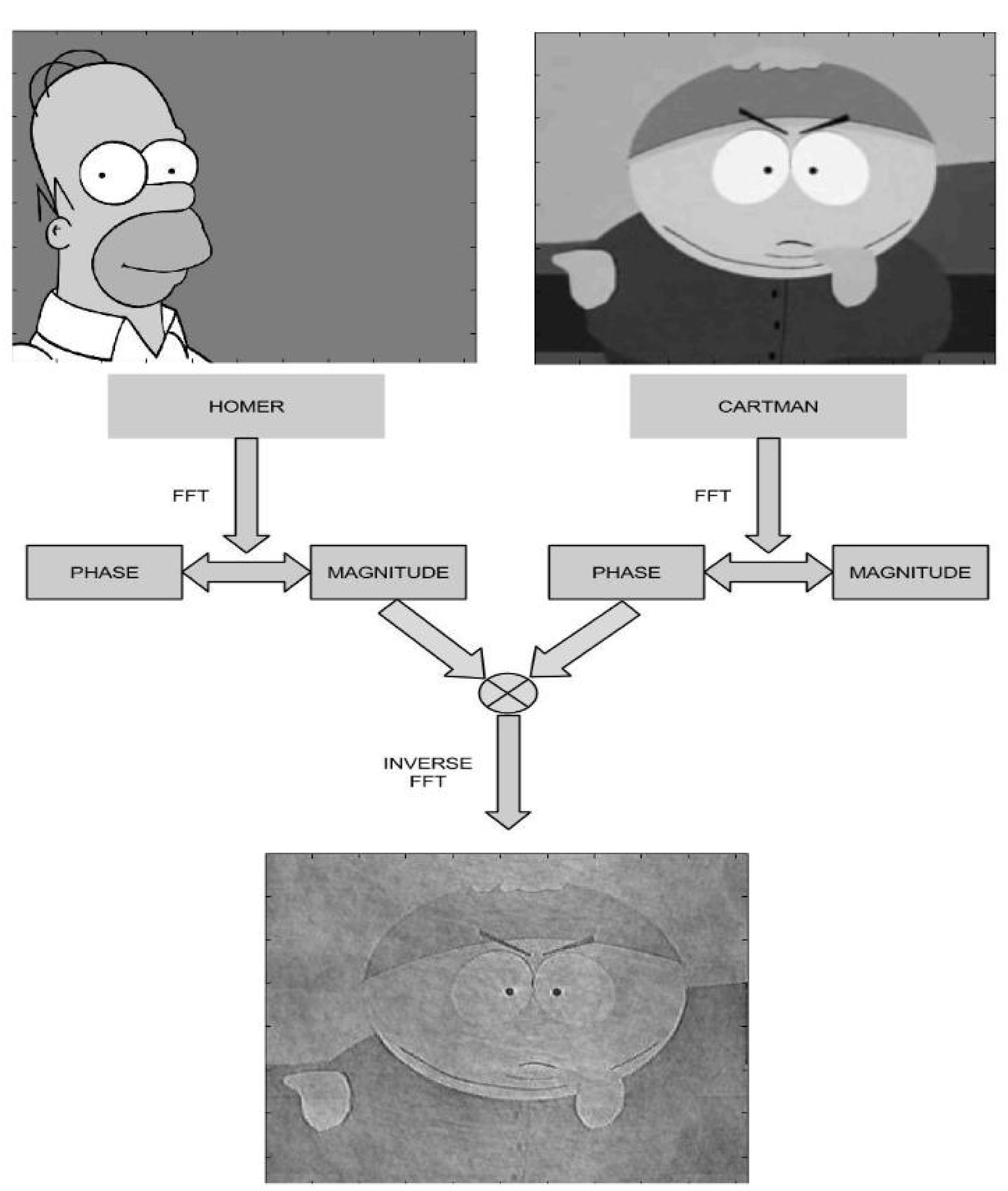
Let $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ denote the recovery method. The phase retrieval problem involves designing measurement matrix and recovery method pairs.

Typical objectives in designing phase retrieval algorithms:

- Computational Efficiency – Can the recovery algorithm $\mathcal{A}$ be computed in $O(d)$-time?
- Computational Robustness: The recovery algorithm, $\mathcal{A}$, should be robust to additive measurement errors (i.e., noise).
- Minimal Measurements: The number of linear measurements, D, should be minimized to the greatest extent possible.

Important applications of phase retrieval include X-ray crystallography, diffraction imaging and transmission electron microscopy (TEM).
In these (and many other molecular imaging applications), the underlying physics or instrumentation constraints mean that the detectors only capture intensity measurements.

Why is Phase Important?



Existing Methods

Two popular classes of methods for phase retrieval are

1. Greedy Alternating Projection Methods, [1]
   - Operate by alternately projecting the current iterate of the signal over two sets of constraints.
   - One of the constraints is the magnitude of the measurements.
   - The other constraint depends on the application – positivity, support constraints, ...
   - Efficient to implement, but convergence is slow.
2. Methods Employing Semi-Definite Programming (SDP), [2–3]
   - Representative example is the PhaseLift formulation.
   - Modify the problem to that of finding the rank-1 matrix $X = \mathbf{x}\mathbf{x}^*$
   - Use multiple random illuminations or masks; if $\mathbf{w}$ denotes a mask, measurements are of the form
   $$|\langle \mathbf{w}, \mathbf{x} \rangle|^2 = \operatorname{Tr}(\mathbf{x}^*\mathbf{w}\mathbf{w}^*\mathbf{x})$$
   - The phase recovery problem may be formulated as a trace minimization SDP.

Phase Retrieval Using Compactly–Supported Masks

1. Obtain phase differences using correlation measurements

$$|\mathtt{corr}(\mathbf{w}^m, \mathbf{x})| \longrightarrow x_j\overline{x}_{j+k}, \quad k = 0, \ldots, \delta, \quad m = 0, \ldots, L$$

- $\mathbf{w}$ is a mask or window function with $\delta + 1$ non-zero entries.
- $L + 1$ distinct masks are used.
- $x_j\overline{x}_{j+k}$ gives us the (scaled) difference in phase between entries x_j and x_{j+k}.
- Setting $Z_{n,l} := x_n\overline{x}_{n+l}, \quad -\delta \le l \le \delta$, we may write:
$$(b_k^m)^2 = \left|\langle \tau^k(\mathbf{w}^m), \mathbf{x} \rangle\right|^2 = \left|\Sigma_{j=0}^{\delta} \overline{w}_j^m \cdot x_{k+j}\right|^2 = \Sigma_{i,j=0}^{\delta} w_i \overline{w}_j Z_{k+j,i-j}$$
- Ordering $\{Z_{n,l}\}$ lexicographically, second index first, we obtain a linear system of equations.
- Example: for $\mathbf{x} \in \mathbb{R}^4, \delta = 1, L = 1$, we obtain:

$$\begin{pmatrix} (w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 & 0 & 0 & 0 & 0 \\ (w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 & 0 & 0 \\ 0 & 0 & (w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 \\ 0 & 0 & 0 & 0 & (w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 \\ (w_1^0)^2 & 0 & 0 & 0 & 0 & 0 & 2w_0^0w_1^0 & (w_1^0)^2 \\ (w_1^1)^2 & 0 & 0 & 0 & 0 & 0 & 2w_0^1w_1^1 & (w_1^1)^2 \end{pmatrix} \begin{pmatrix} Z_{0,0} \\ Z_{0,1} \\ Z_{1,0} \\ Z_{1,1} \\ Z_{2,0} \\ Z_{2,1} \\ Z_{3,0} \\ Z_{3,1} \end{pmatrix} = \begin{pmatrix} b_0^0 \\ b_0^1 \\ b_1^0 \\ b_1^1 \\ b_2^0 \\ b_2^1 \\ b_3^0 \\ b_3^1 \end{pmatrix}$$

- The system matrix is block circulant, with the blocks indicated by dashed lines.
- There are only $\delta + 1$ non-zero blocks ($2\delta + 1$ in the complex case).
- Block circulant structure allows for efficient FFT implementations.
- Deterministic (and random) prescriptions for masks available. For example
$$w_\ell^i = \begin{cases} \frac{\mathrm{e}^{-i/a}}{\sqrt[4]{2\delta+1}} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} \cdot i \cdot \ell}{2\delta+1}} & \text{if } i \le \delta \\ 0 & \text{if } i > \delta \end{cases} \qquad a \in [1, \infty), \qquad 0 \le \ell \le L$$
- System matrix can be shown to be well conditioned.

2. Solve an angular synchronization problem on the phase differences to obtain the unknown signal.

$$x_j\overline{x}_{j+k} \longrightarrow x_j$$

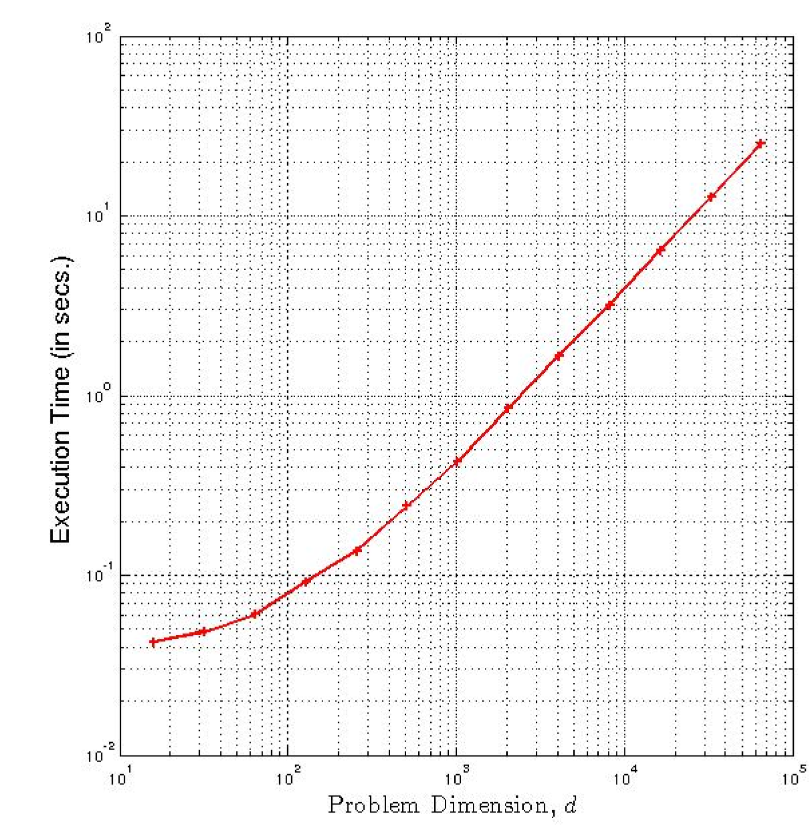
By definition, $|x_j|^2 = Z_{i,i},\ i = 0, \ldots, d-1$. The unknown phases (modulo a global phase offset) may be obtained by solving a simple greedy algorithm.

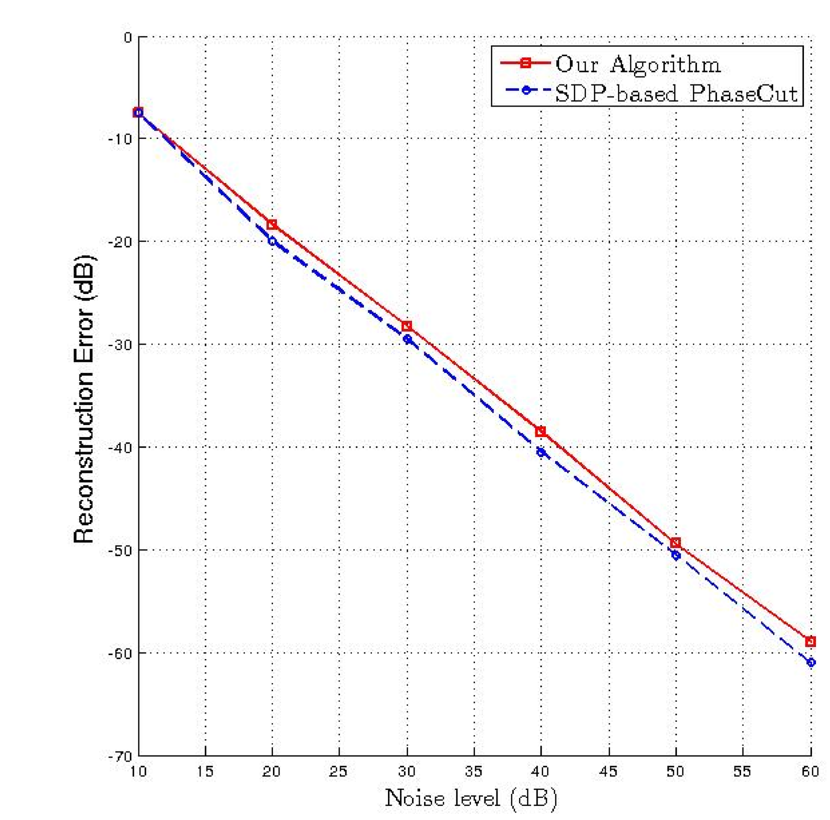
- Set the largest magnitude component to have zero phase angle; i.e.,
$$\angle x_j = 0, \qquad j = \operatorname*{argmax}_i Z_{i,i}.$$
- Use this entry to set the phase angles of the next δ entries; i.e.,
$$\angle x_k = \angle x_j - \angle Z_{j,k}, \qquad k = 1, \ldots, \delta.$$
- Use the largest magnitude component from these δ entries to repeat the process.

Finally, a few iterations of an alternating projections algorithm may be used to post-process the resulting solution.

Numerical Results and Discussion

- Left panel figure shows execution time as a function of problem dimension. The overall execution time is $\mathcal{O}(d \log d)$. This figure was generated using $\delta = 8, L = 17$, deterministic masks and no added noise.
- The right panel illustrates robustness in the presence of noise. Also plotted for comparison is the SDP-based PhaseCut ([3]) result. The problem size is $d = 64$.
- Our reconstruction algorithm requires a small number of additional measurements (2×–4×) while being several orders of magnitude faster than SDP-based methods.





References and Acknowledgement

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3. I. Waldspurger, A. d'Aspremont, and S. Mallat. *Phase recovery, MaxCut and complex semidefinite programming.* Math. Prog., 2013.

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