Fast Phase Retrieval for High-Dimensions

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AMS Spring Central Sectional Meeting
Saturday, March 14^{th}, 2015

Joint work with

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Research supported in part by National Science Foundation grant DMS 1043034.

The Phase Retrieval Problem

$$\text{find} \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad |M\mathbf{x}| = \mathbf{b} \in \mathbb{R}^D,$$

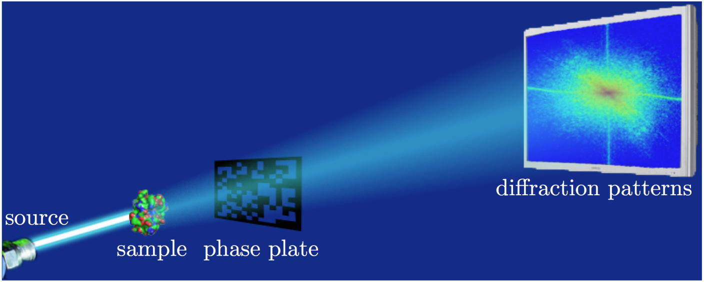
where

- $\mathbf{b} \in \mathbb{R}^D$ are the magnitude or intensity measurements.
- $M \in \mathbb{C}^{D \times d}$ is a measurement matrix associated with these measurements.

Let $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ denote the recovery method.

The phase retrieval problem involves designing measurement matrix and recovery method pairs.

Applications of Phase Retrieval



From "Phase Retrieval from Coded Diffraction Patterns" by E. J. Candes, X. Li, and M. Soltanolkotabi.

Important applications of Phase Retrieval

- X-ray crystallography
- Diffraction imaging
- Transmission Electron Microscopy (TEM)

In many molecular imaging applications, the detectors only capture intensity measurements.

Objectives

- Computational Efficiency – Can the recovery algorithm $\mathcal{A}$ be computed in $O(d \log^c d)$-time? Here, c is a small constant.

- Computational Robustness: The recovery algorithm, $\mathcal{A}$, should be robust to additive measurement errors (i.e., noise).

- Minimal Measurements: The number of linear measurements, D, should be minimized to the greatest extent possible.

Some Previous Approaches

- Alternating Projection Methods [Gerchberg and Saxton, 1972] and [Fienup, 1978]
    - Work well in practice, not well understood theoretically
- PhaseLift [Candes et. al., 2012]
    - Recovery guarantees for random measurements
    - Requires $\mathcal{O}(d)$ measurements
    - Requires solving a SDP – $\mathcal{O}\left(d^3\right)$-time
- Phase Retrieval with Polarization [Alexeev et. al. 2014]
    - Graph-theoretic frame-based approach
    - Requires $\mathcal{O}(d \log d)$ measurements
    - Error guarantee similar to PhaseLift

Overview of the Our Computational Framework

1. Use shifted compactly supported masks to obtain phase difference estimates.

$$|\mathtt{Circ}(\mathbf{w})\mathbf{x}|^2 \xrightarrow[\text{linear system}]{\text{solve}} x_j\overline{x}_{j+k}, \quad k = 0, \ldots, \delta$$

   - $\mathbf{w}$ is a mask, or window, with $\delta + 1$ non-zero entries.
   - $x_j\overline{x}_{j+k}$ gives us the (scaled) difference in phase between entries x_j and x_{j+k}.

2. Solve an angular synchronization problem on the phase differences to obtain the unknown signal.

$$x_j\overline{x}_{j+k} \xrightarrow[\text{synchronization}]{\text{angular}} x_j$$

Constraints on $\mathbf{x}$: We require $\mathbf{x}$ to be non-sparse.
(The number of consecutive zeros in $\mathbf{x}$ should be less than δ)

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Correlations with Support-Limited Functions

- Let $\mathbf{x} = [x_0 \; x_1 \; \ldots \; x_{d-1}]^T \in \mathbb{C}^d$ be the unknown signal.
- Let $\mathbf{w} = [w_0 \; w_1 \; \ldots \; w_\delta \; 0 \; \ldots 0]^T$ denote a support-limited mask. It has $\delta + 1$ non-zero entries.
- We are given the (squared) magnitude measurements

$$(b^m)^2 = \left|\mathtt{Circ}(\mathbf{w}^m)\mathbf{x}\right|^2, \qquad m = 0, \ldots, L$$

corresponding to $L + 1$ distinct masks.

Correlations with Support-Limited Functions

Explicitly writing out each measurement, we have

$$
\begin{aligned}
(b_k^m)^2 &= \left| \sum_{j=0}^{\delta} \overline{w}_j^m \cdot x_{k+j} \right|^2 \\
&= \sum_{i,j=0}^{\delta} w_i^m \overline{w}_j^m x_{k+j} \overline{x}_{k+i}
\end{aligned}
$$

We can also lift these equations to a set of **linear equations**!

Solving for Phase Differences

Ordering $x_n\overline{x}_{n+l}$ lexicographically, we obtain a linear system of equations for the phase differences.

Example: $\mathbf{x} \in \mathbb{R}^d, d = 4, \delta = 1$

$$
\underbrace{
\begin{array}{c}
\begin{array}{cccc} \quad M'_0 \quad & & \quad M'_1 \quad & \end{array} \\
\begin{pmatrix}
(w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 & 0 & 0 & 0 & 0 \\
(w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 & 0 & 0 \\
0 & 0 & (w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (w_0^0)^2 & 2w_0^0w_1^0 & (w_1^0)^2 & 0 \\
0 & 0 & 0 & 0 & (w_0^1)^2 & 2w_0^1w_1^1 & (w_1^1)^2 & 0 \\
(w_1^0)^2 & 0 & 0 & 0 & 0 & 0 & 2w_0^0w_1^0 & (w_1^0)^2 \\
(w_1^1)^2 & 0 & 0 & 0 & 0 & 0 & 2w_0^1w_1^1 & (w_1^1)^2
\end{pmatrix}
\end{array}
}_{M'}
\underbrace{
\begin{pmatrix}
x_0\overline{x}_0 \\ x_0\overline{x}_1 \\ x_1\overline{x}_1 \\ x_1\overline{x}_2 \\ x_2\overline{x}_2 \\ x_2\overline{x}_3 \\ x_3\overline{x}_3 \\ x_3\overline{x}_0
\end{pmatrix}
}_{\mathbf{y}}
=
\begin{pmatrix}
b_0^0 \\ b_0^1 \\ b_1^0 \\ b_1^1 \\ b_2^0 \\ b_2^1 \\ b_3^0 \\ b_3^1
\end{pmatrix}
$$

The system matrix M' is *block circulant*!

Some Entries of the Measurement Matrix

Two strategies

- Random entries (Gaussian, Uniform, Bernoulli, ...)
- Structured measurements, e.g.,

$$w_i^\ell = \begin{cases} \frac{\mathrm{e}^{-i/a}}{\sqrt[4]{2\delta+1}} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} \cdot i \cdot \ell}{2\delta+1}}, & i \leq \delta \\ 0, & i > \delta \end{cases} .$$

where $a := \max\left\{4, \frac{\delta-1}{2}\right\}$, and $0 \leq \ell \leq L$.

Structured Measurements

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $\mathbf{w}^m$ *as follows:*

$$w_i^\ell = \begin{cases} \frac{\mathbb{e}^{-i/a}}{\sqrt[4]{2\delta+1}} \cdot \mathbb{e}^{\frac{2\pi \mathbb{i} \cdot i \cdot \ell}{2\delta+1}}, & i \le \delta \\ 0, & i > \delta \end{cases}, \; a := \max\left\{4, \frac{\delta-1}{2}\right\}, \ell \in [0, L].$$

Then, the resulting system matrix for the phase differences, M'*, has condition number*

$$\kappa(M') < \max\left\{144\mathbb{e}^2, \frac{9\mathbb{e}^2}{4} \cdot (\delta-1)^2\right\}.$$

Note:

- w_i^ℓ are scaled entries of a DFT matrix.
- δ is typically chosen to be 6–12.

Angular Synchronization

The Angular Synchronization Problem

Estimate d unknown angles $\theta_1, \theta_2, \ldots, \theta_d \in [0, 2\pi)$ from $d(\delta + 1)$ noisy measurements of their differences
$\Delta\theta_{ij} := \angle x_i - \angle x_j = \angle \left(\frac{x_i \overline{x}_j}{\sqrt{x_i \overline{x}_i \cdot x_j \overline{x}_j}} \right)$ mod 2π.

Angular Synchronization

The unknown phases (modulo a global phase offset) may be obtained by solving a simple greedy algorithm.

1. Set the largest magnitude component to have zero phase angle; i.e.,

$$\angle x_k = 0, \qquad k := \operatorname*{argmax}_i x_i \overline{x}_i.$$

2. Use this entry to set the phase angles of the next δ entries; i.e.,

$$\angle x_j = \angle x_k - \Delta\theta_{kj}, \qquad j = 1, \ldots, \delta.$$

3. Use the next largest magnitude component from these δ entries and repeat the process.

Recovering Arbitrary Vectors

- Recall: Due to compact support of our masks, only "flat" vectors can be recovered
- Arbitrary vectors can be "flattened" by multiplication with a random unitary matrix such as $W = PFB$, where
  - $P \in \{0,1\}^{d \times d}$ is a permutation matrix selected uniformly at random from the set of all $d \times d$ permutation matrices
  - F is the unitary $d \times d$ discrete Fourier transform matrix
  - $B \in \{-1,0,1\}^{d \times d}$ is a random diagonal matrix with i.i.d. symmetric Bernoulli entries on its diagonal

A Noiseless Recovery Result

Theorem (Iwen, V., Wang 2015)

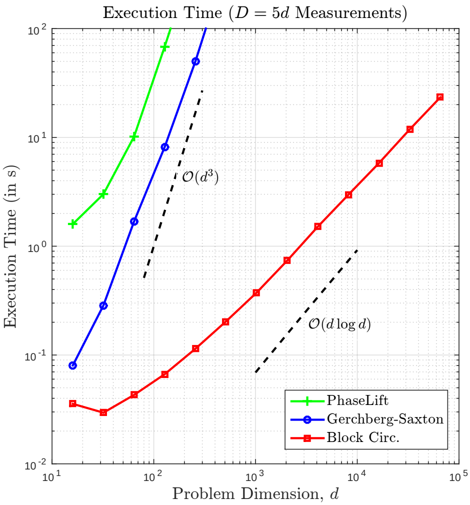
Let $\mathbf{x} \in \mathbb{C}^d$ with d sufficiently large. Then, one can select a random measurement matrix $\tilde{M} \in \mathbb{C}^{D \times d}$ such that the following holds with probability at least $1 - \frac{1}{c \cdot \ln^2(d) \cdot \ln^3(\ln d)}$: Our algorithm will recover an $\tilde{\mathbf{x}} \in \mathbb{C}^d$ with

$$\min_{\theta \in [0, 2\pi]} \left\| \mathbf{x} - \mathbb{e}^{\mathbb{i}\theta} \tilde{\mathbf{x}} \right\|_2 = 0$$

when given the noiseless magnitude measurements $|\tilde{M}\mathbf{x}|^2 \in \mathbb{R}^D$. Here D can be chosen to be $\mathcal{O}(d \cdot \ln^2(d) \cdot \ln^3(\ln d))$. Furthermore, the algorithm will run in $\mathcal{O}(d \cdot \ln^3(d) \cdot \ln^3(\ln d))$-time in that case.

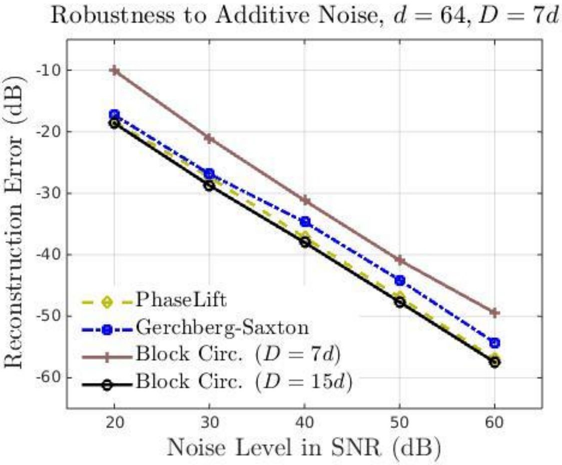
<u>To do:</u> Robustness to measurement noise...

Efficiency



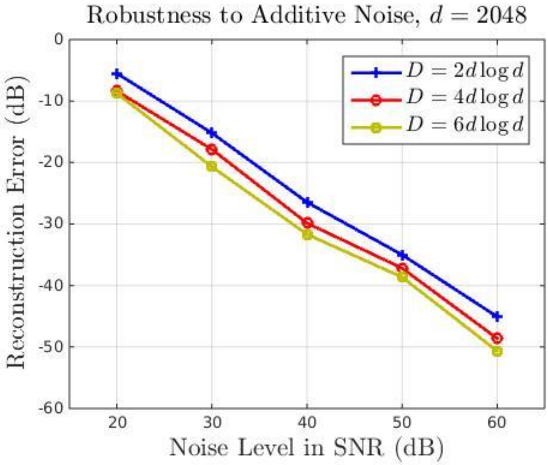
- iid Complex Gaussian signal
- High SNR applications
- $5d$ measurements
- $64k$ problem in ~ 20 s in Matlab!

Robustness



- iid complex Gaussian signal
- $d = 64$
- $7d$ measurements
- Deterministic (windowed Fourier-like) measurements

Robustness



- iid complex Gaussian signal
- $d = 2048$
- Not computationally feasible using PhaseLift (on a laptop in Matlab)
- Deterministic (windowed Fourier-like) measurements

The *Sparse* Phase Retrieval Problem

$$\text{find} \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad |\mathcal{M}\mathbf{x}| = \mathbf{b} \in \mathbb{R}^D,$$

where

- $\mathbf{x}$ is s-sparse, with $s \ll d$.
- $\mathbf{b} \in \mathbb{R}^D$ are the magnitude or intensity measurements.
- $\mathcal{M} \in \mathbb{C}^{D \times d}$ is a measurement matrix associated with these measurements.

Let $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ denote the recovery method.

The sparse phase retrieval problem involves designing measurement matrix and recovery method pairs.

Sublinear-time Results

Theorem (Iwen, V., Wang 2015)

There exists a deterministic algorithm $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ *for which the following holds: Let* $\epsilon \in (0,1]$, $\mathbf{x} \in \mathbb{C}^d$ *with* d *sufficiently large, and* $s \in [d]$. *Then, one can select a random measurement matrix* $\tilde{M} \in \mathbb{C}^{D\times d}$ *such that*

$$\min_{\theta\in[0,2\pi]} \left\| \mathbb{e}^{\mathbb{i}\theta}\mathbf{x} - \mathcal{A}\left(|\tilde{M}\mathbf{x}|^2\right) \right\|_2 \;\leq\; \left\|\mathbf{x} - \mathbf{x}_s{}^{\mathrm{opt}}\right\|_2 + \frac{22\epsilon \left\|\mathbf{x} - \mathbf{x}^{\mathrm{opt}}_{(s/\epsilon)}\right\|_1}{\sqrt{s}}$$

is true with probability at least $1 - \frac{1}{C\cdot\ln^2(d)\cdot\ln^3(\ln d)}$.[a] *Here* D *can be chosen to be* $\mathcal{O}\big(\frac{s}{\epsilon} \cdot \ln^3(\frac{s}{\epsilon}) \cdot \ln^3\left(\ln\frac{s}{\epsilon}\right) \cdot \ln d\big)$. *Furthermore, the algorithm will run in* $\mathcal{O}\left(\frac{s}{\epsilon} \cdot \ln^4(\frac{s}{\epsilon}) \cdot \ln^3\left(\ln\frac{s}{\epsilon}\right) \cdot \ln d\right)$*-time in that case.*[b]

[a]Here $C \in \mathbb{R}^+$ is a fixed absolute constant.

[b]For the sake of simplicity, we assume $s = \Omega(\log d)$ when stating the measurement and runtime bounds above.

References – Phase Retrieval

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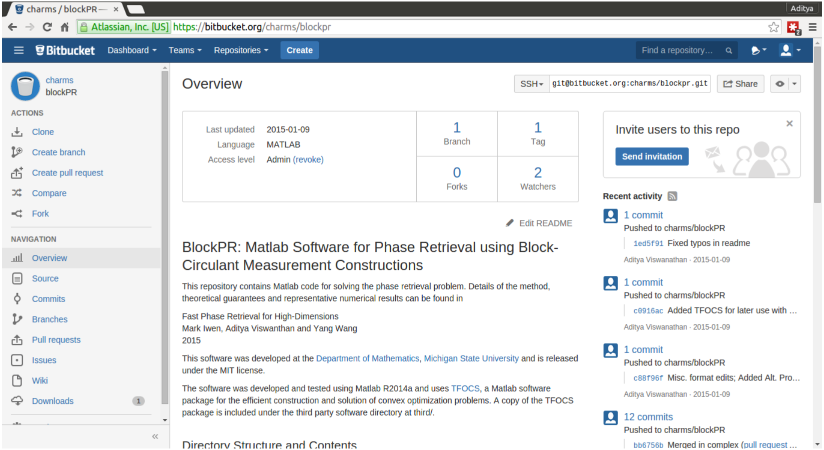
Software Repository



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