DETECTION OF SIGNAL DISCONTINUITIES FROM NOISY FOURIER DATA

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INTRODUCTION

- Detection of signal discontinuities is an important problem in many signal processing tasks
- In applications such as MRI, sampled Fourier data is provided and we are required to locate the signal discontinuities
- Presence of Gibbs oscillations in a partial Fourier sum reconstruction impedes accurate detection of these jumps
- Here, we present the design and analysis of a detector based on the *concentration method* which computes the location, sign and magnitudes of jump discontinuities, given a finite number of noisy Fourier coefficients

THE CONCENTRATION METHOD

Let f be a 2π-periodic piecewise-smooth function with well-defined right and left hand limits. Its *jump function* is then defined as

$$[f](x) := f(x^+) - f(x^-)$$

To show that jump information is contained in Fourier data, consider a function with a single jump at $x = \gamma$. We can show that

$$\hat{f}(k) = [f](\gamma)\frac{e^{-ik\gamma}}{2\pi i k} + O\left(\frac{1}{k^2}\right)$$

Now consider a partial sum of the form

$$S_L[f](x) = \sum_{k=-L}^{L}\left(\frac{i\pi k}{L}\right)\hat{f}(k)e^{ikx}$$

Substituting for the Fourier coefficients, we would obtain

$$S_L[f](x) = \underbrace{[f](\gamma)\frac{1}{2L}\sum_{k=-L}^{L} e^{ik(x-\gamma)}}_{\text{scaled Dirichlet kernel}} + \sum_{k=-L}^{L} O\left(\frac{1}{k}\right)e^{ik(x-\gamma)}$$

i.e., the jump approximation "concentrates" at the singular support of f. More generally, we have

$$S_L^{\sigma}[f](x) = i\sum_{k=-L}^{L}\hat{f}_k\, sgn(k)\,\sigma\left(\frac{|k|}{L}\right)e^{ikx}$$

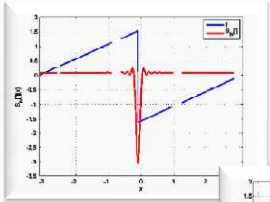
where $\sigma_{k,L}(\eta) = \sigma(|k|/L)$ are called concentration factors

The jump function approximation can be computed efficiently using a FFT.

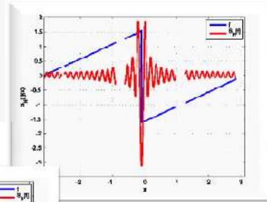
CONCENTRATION FACTORS

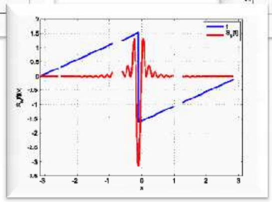
- Several concentration factors are available for use, including

TRIGONOMETRIC



POLYNOMIAL





- Choice of concentration factor dictates tradeoff between spurious values away from the discontinuity and width of the "mainlobe" at the discontinuity

DETECTOR DESIGN

- We model noise as additive white Gaussian with zero mean

$$\hat{\boldsymbol{g}}_k = \hat{f}_k + \hat{\boldsymbol{n}}_k \qquad \hat{\boldsymbol{n}}_k \sim \mathcal{N}[0,\sigma^2]$$

- The concentration method is linear and the noise component does not bias the jump function approximation
- Choosing a length R signal vector,

$$M = \left(S_L^{\sigma_1}[f](x_1), \ldots, S_L^{\sigma_R}[f](x_R)\right)^T$$

The detection problem becomes,

$$\mathcal{H}_0 : \mathbf{Y} = \mathbf{N} \quad \sim \mathcal{N}[0, \mathrm{C_N}]$$

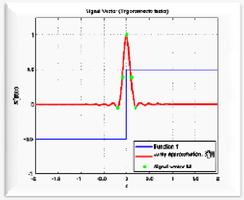
$$\mathcal{H}_1 : \mathbf{Y} = \mathrm{M} + \mathbf{N} \quad \sim \mathcal{N}[\mathrm{M}, \mathrm{C_N}]$$

- The Neyman-Pearson detector yields

$$M^T C_N^{-1}\mathbf{Y} > \gamma$$

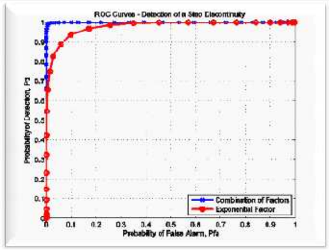
  - o γ is a threshold
  - o Performance metric is $M^T C_N^{-1} M$
  - o Performance curve is described by

$$P_d = Q\left(Q^{-1}(P_{fa}) - \sqrt{M^T C_N^{-1} M}\right)$$



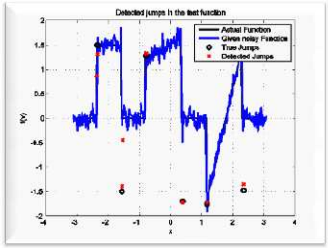
SIGNAL VECTOR *M*

RESULTS



DETECTION OF A STEP DISCONTINUITY – ROC CURVES

- *Specifications*: R = 3, L = 32, σ^2 = 7.5
- Red curve – Exponential concentration factor
- Blue curve – Combination of all factors
- SNR metrics: 9.028 dB (Red) and 13.434 dB (Blue)



DETECTION OF DISCONTINUITIES IN A TEST FUNCTION

- *Specifications*: R = 3, L = 128, σ^2 = 7.5
- Trigonometric concentration factor used

CURRENT DIRECTIONS

- Multi-dimensional edge detection
- Sidelobe effect mitigation and false-alarm reduction
- Jump detection from non-uniform Fourier data

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