Fast 2D Phase Retrieval using Bandlimited Masks

Cyril Cordor
Department of Mathematics
University of Michigan, Ann Arbor
Ann Arbor, Michigan, USA
ccordor@umich.edu

Brendan Williams, Yulia Hristova, Aditya Viswanathan
Department of Mathematics & Statistics
University of Michigan – Dearborn
Dearborn, Michigan, USA
{brendwil,yuliagh,adityavv}@umich.edu

***Abstract*—We propose a new phase retrieval algorithm for recovering 2D discrete signals from the squared magnitudes of their short-time Fourier transform measurements. The algorithm works by efficiently inverting (in FFT–time) a Fourier-based, physics-driven, and highly structured linear system to obtain relative phase information. The missing phases are subsequently recovered through the use of an eigenvector-based angular synchronization procedure. In addition to providing a deterministic measurement mask construction, the efficiency and robustness of the proposed method are demonstrated through numerical experiments.**

***Index Terms*—phaseless imaging, phase retrieval, bandlimited masks, ptychography, angular synchronization.**

I. Introduction

We consider the problem of recovering an unknown signal $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ given the (phaseless) squared magnitude short-time Fourier transform (STFT) measurements

$$\mathcal{Y}_{j_1,j_2,\ell_1,\ell_2} := \left| \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} (\mathcal{X})_{n_1,n_2} (\mathcal{M})_{n_1-\ell_1,n_2-\ell_2} e^{-2\pi i \left(\frac{n_1 j_1}{N_1} + \frac{n_2 j_2}{N_2} \right)} \right|^2. \tag{1}$$

Here $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$ is a known *mask*, and $j_1, \ell_1 \in \mathbb{Z}_{N_1}$, $j_2, \ell_2 \in \mathbb{Z}_{N_2}$. Such *phase retrieval* problems (see [1], [2]) arise in molecular imaging applications such as ptychography [3], where overlapping regions of an object $\mathcal{X}$ are illuminated, usually by placing a pinhole between the light source and object, and sequentially moving the pinhole (modeled by shifts of $\mathcal{M}$; by ℓ_1 pixels in the vertical direction and ℓ_2 pixels in the horizontal direction). The resulting diffraction patterns are then sampled and used to approximate the unknown object. State of the art methods used by physicists, e.g. [4], offer no (global) recovery guarantees, while recent mathematically rigorous methods, e.g. [5], often require the use of highly random and physically impractical measurement constructions. In this paper, we propose a 2D phase retrieval framework motivated by the work of Chapman [6], extending the 1D results and methods of Perlmutter et al. [7]. In particular, we show in §II that through careful application of 2D Fourier transform properties, the nonlinear problem in (1) can be linearized (see Theorem 1), and that in the case of *bandlimited* masks $\mathcal{M}$ and careful selection of the number of shifts, this linear system reduces to a diagonal system. This improves upon the results in [8] by providing a more direct, explicit, and efficient linear system construction (for STFT measurements), thereby facilitating the solution of significantly larger problems. Furthermore, a prescription for a *deterministic* mask which allows for such an inversion is provided in §II-D, along with a recovery algorithm. Numerical results verifying the efficiency and robustness of the proposed method are provided in §III, while §IV offers some directions for future research.

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II. Proposed 2D Phase Retrieval Framework

A. Preliminaries and Notation

We represent discrete, 2-dimensional signals as matrices $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ and use lowercase letters with bold vector indexing to denote the components of such a signal. Thus, for $\mathbf{n} = [n_1 \ n_2]^T$, the component $x_{\mathbf{n}}$ denotes the entry at the n_1th horizontal row and n_2th vertical column of a signal $\mathcal{X}$ where $n_1 \in \mathbb{Z}_{N_1}$ and $n_2 \in \mathbb{Z}_{N_2}$. All indexing is implied to be modulo N_1 along columns and modulo N_2 along rows. Occasionally (and with slight abuse of notation), we may refer to the index $\mathbf{n}$ as an ordered pair (i.e., $\mathbf{n} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$). Some elementary definitions and operations on signals are shown in Table I.

TABLE I
Elementary Operations and Definitions

Operation	Definition
(Rectangular) Periodicity Matrix N	$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$
Region of Summation R_N	$R_N := \{\mathbf{n} = (n_1, n_2) : n_1 \in \mathbb{Z}_{N_1}, n_2 \in \mathbb{Z}_{N_2}\}$
Conjugate, Transpose, & Conjugate Transpose	$\overline{\mathcal{X}}, \mathcal{X}^T, \mathcal{X}^*$
Time Reversal	$\widetilde{x}_{\mathbf{n}} := x_{-\mathbf{n}}$
Circular Shift in Time Operator $S_{\boldsymbol{\ell}} : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(S_{\boldsymbol{\ell}} \mathcal{X})_{\mathbf{n}} := x_{\mathbf{n}-\boldsymbol{\ell}}$
Modulation Operator $W_{\mathbf{k}} : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(W_{\mathbf{k}} \mathcal{X})_{\mathbf{n}} := x_{\mathbf{n}} e^{-2\pi i \mathbf{k}^T N^{-1} \mathbf{n}}$
Hadamard (Element-wise) product	$(\mathcal{X} \circ \mathcal{H})_{\mathbf{n}} := x_{\mathbf{n}} h_{\mathbf{n}}$
Circular Convolution $\circledast_N$: $\mathbb{C}^{N_1 \times N_2} \times \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(\mathcal{X} \circledast_N \mathcal{H})_{\mathbf{n}} := \sum_{\mathbf{k} \in R_N} x_{\mathbf{k}} h_{\mathbf{n}-\mathbf{k}}$

Definition 1. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$. The Discrete Fourier Transform (DFT) $F_N : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$ and Inverse Discrete Fourier Transform (IDFT) $F_N^{-1} : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$ are defined as follows (with $\mathbf{j}, \mathbf{n} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$):*

$$\widehat{x}_{\mathbf{j}} = (F_N \mathcal{X})_{\mathbf{j}} := \sum_{\mathbf{n} \in R_N} x_{\mathbf{n}} e^{-2\pi i \mathbf{n}^T N^{-1} \mathbf{j}}$$

$$x_{\mathbf{n}} = (F_N^{-1} \widehat{\mathcal{X}})_{\mathbf{n}} := \frac{1}{\det N} \sum_{\mathbf{j} \in R_N} \widehat{x}_{\mathbf{j}} e^{2\pi i \mathbf{j}^T N^{-1} \mathbf{n}}.$$

B. Measurement Model and Preliminary Lemmas

Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ be the unknown signal and let $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$ denote a *known* mask. The phaseless Fourier measurements populate a fourth-order tensor $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$, where L_1, L_2 represent the total number of vertical and horizontal shifts of the mask, respectively. Thus, $\mathcal{Y}_{j_1, j_2, \ell_1, \ell_2}$ denotes the measurement corresponding to the $\ell_1{}^{\text{th}}$ and $\ell_2{}^{\text{th}}$ vertical and horizontal shifts, and the $j_1{}^{\text{th}}$ and $j_2{}^{\text{th}}$ Fourier modes, respectively. Letting $\mathbf{j}, \mathbf{n}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ and letting $\mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]}$ correspond to the measurement $\mathcal{Y}_{j_1, j_2, \ell_1, \ell_2} \in \mathbb{R}$, then we may write

$$\begin{aligned} \mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]} &= \left| \sum_{\mathbf{n} \in R_N} x_{\mathbf{n}} m_{\mathbf{n}-\boldsymbol{\ell}} e^{-2\pi i \mathbf{n}^T N^{-1} \mathbf{j}} \right|^2 \\ &= \left| (F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}))_{\mathbf{j}} \right|^2. \end{aligned} \tag{2}$$

We next present some preliminary lemmas which follow from elementary Fourier transform properties.

Properties. *Let $\mathcal{X}, \mathcal{H} \in \mathbb{C}^{N_1 \times N_2}$ and $\alpha, \beta \in \mathbb{C}$. Then for all $\mathbf{n}, \mathbf{j}, \mathbf{k}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, the following properties hold:*

i. $(F_N(\alpha \mathcal{X} + \beta \mathcal{H}))_{\mathbf{j}} = \alpha \widehat{x}_{\mathbf{j}} + \beta \widehat{h}_{\mathbf{j}}$
ii. $(F_N(S_{\boldsymbol{\ell}} \mathcal{X}))_{\mathbf{j}} = (W_{\boldsymbol{\ell}} \widehat{\mathcal{X}})_{\mathbf{j}}$
iii. $(F_N \widetilde{\mathcal{X}})_{\mathbf{j}} = \widetilde{\widehat{x}}_{\mathbf{j}} = \widehat{\widetilde{x}}_{\mathbf{j}}$
iv. $(\widetilde{S_{\boldsymbol{\ell}} \mathcal{X}})_{\mathbf{n}} = (S_{-\boldsymbol{\ell}} \widetilde{\mathcal{X}})_{\mathbf{n}}$
v. $(F_N(\mathcal{H} \circledast_N \mathcal{X}))_{\mathbf{j}} = (\widehat{\mathcal{H}} \circ \widehat{\mathcal{X}})_{\mathbf{j}}$
vi. $(F_N(\mathcal{H} \circ \mathcal{X}))_{\mathbf{j}} = \frac{1}{\det N} (\widehat{\mathcal{H}} \circledast_N \widehat{\mathcal{X}})_{\mathbf{j}}$
vii. $(F_N \overline{\mathcal{X}})_{\mathbf{j}} = \widehat{\overline{x}}_{\mathbf{j}} = \overline{\widetilde{\widehat{x}}}_{\mathbf{j}}$
viii. $(F_N(F_N \mathcal{X}))_{\mathbf{j}} = (F_N \widehat{\mathcal{X}})_{\mathbf{j}} = (\det N)\ \widetilde{x}_{\mathbf{j}}$.

Lemma 1. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ and $\mathbf{j}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, then*

$$\left(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{X}})\right)_{\mathbf{j}} = \frac{1}{\det N} e^{-2\pi i \mathbf{j}^T N^{-1} \boldsymbol{\ell}} \left(F_N(\widehat{\mathcal{X}} \circ S_{\mathbf{j}} \overline{\widehat{\mathcal{X}}})\right)_{-\boldsymbol{\ell}}.$$

Proof. Using Properties (vi.), (ii.), and (vii.) we obtain,

$$\begin{aligned} \left(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{X}})\right)_{\mathbf{j}} &= \frac{1}{\det N} (\widehat{\mathcal{X}} \circledast_N F_N(S_{\boldsymbol{\ell}} \overline{\mathcal{X}}))_{\mathbf{j}} \\ &= \frac{1}{\det N} (\widehat{\mathcal{X}} \circledast_N W_{\boldsymbol{\ell}} \widehat{\overline{\mathcal{X}}})_{\mathbf{j}} = \frac{1}{\det N} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} (W_{\boldsymbol{\ell}} \widehat{\overline{\mathcal{X}}})_{\mathbf{j}-\mathbf{n}} \\ &= \frac{1}{\det N} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} \widehat{\overline{x}}_{\mathbf{j}-\mathbf{n}} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} (\mathbf{j}-\mathbf{n})} \\ &= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{j}} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} \widehat{\overline{x}}_{\mathbf{j}-\mathbf{n}} e^{2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{n}} \\ &= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{j}} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} \widetilde{\widehat{\overline{x}}}_{\mathbf{n}-\mathbf{j}} e^{2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{n}}. \end{aligned}$$

Using (vii.), $\widetilde{\widehat{\overline{x}}}_{\mathbf{n}-\mathbf{j}} = \overline{\widehat{x}}_{\mathbf{n}-\mathbf{j}}$, so the last expression equals

$$\begin{aligned} &\frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{j}} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} \overline{\widehat{x}}_{\mathbf{n}-\mathbf{j}} e^{2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{n}} \\ &= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{j}} \sum_{\mathbf{n} \in R_N} \left(\widehat{\mathcal{X}} \circ S_{\mathbf{j}} \overline{\widehat{\mathcal{X}}}\right)_{\mathbf{n}} e^{-2\pi i (-\boldsymbol{\ell})^T N^{-1} \mathbf{n}} \\ &= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} \mathbf{j}} \left(F_N(\widehat{\mathcal{X}} \circ S_{\mathbf{j}} \overline{\widehat{\mathcal{X}}})\right)_{-\boldsymbol{\ell}}. \qquad \square \end{aligned}$$

For notational convenience, we define $\mathcal{Y}_{[:\ \boldsymbol{\ell}]} \in \mathbb{R}^{N_1 \times N_2}$ as the 2-dimensional slice of $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$ for a fixed vector index $\boldsymbol{\ell} = [\ell_1\ \ell_2]^T \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$. And thus, the notation $(\mathcal{Y}_{[:\ \boldsymbol{\ell}]})_{\mathbf{k}}$ refers to the (k_1, k_2) entry of $\mathcal{Y}_{[:\ \boldsymbol{\ell}]}$.

Lemma 2. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$, $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$, and $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$ denote the signal to be recovered, the mask, and the measurements as given by* (2)*, respectively. Then for a fixed $\boldsymbol{\ell} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$ and any $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$,*

$$\left(F_N(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]})\right)_{\mathbf{k}} = (\det N) \left[F_N(\mathcal{X} \circ S_{\mathbf{k}} \overline{\mathcal{X}}) \circ F_N(\widetilde{\mathcal{M}} \circ S_{-\mathbf{k}} \overline{\widetilde{\mathcal{M}}})\right]_{\boldsymbol{\ell}}.$$

Proof. From (2), we have that

$$\begin{aligned} \mathcal{Y}_{[:\ \boldsymbol{\ell}]} &= [F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})] \circ \overline{[F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})]} \\ &= \left[(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})) \circ (\det N) \left(F_N^{-1}(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})\right)\right]. \end{aligned}$$

Therefore, by properties (vi.), (viii.), and (iv.),

$$\begin{aligned} F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]} &= F_N[F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}) \circ (\det N) F_N^{-1}(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})] \\ &= \frac{1}{\det N} \left[F_N(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})) \circledast_N (\det N)(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})\right] \\ &= \frac{1}{\det N} \left[(\det N)(\widetilde{\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}}) \circledast_N (\det N)(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})\right] \\ &= (\det N)\ \left[(\widetilde{\mathcal{X}} \circ S_{-\boldsymbol{\ell}} \widetilde{\mathcal{M}}) \circledast_N (\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})\right]. \end{aligned}$$

Then,

$$\begin{aligned} \left(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]}\right)_{\mathbf{k}} &= (\det N) \sum_{\mathbf{n} \in R_N} \widetilde{x}_{\mathbf{n}} (S_{-\boldsymbol{\ell}} \widetilde{\mathcal{M}})_{\mathbf{n}}\ \overline{x}_{\mathbf{k}-\mathbf{n}} (S_{\boldsymbol{\ell}} \overline{\mathcal{M}})_{\mathbf{k}-\mathbf{n}} \\ &= (\det N) \sum_{\mathbf{n} \in R_N} x_{-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}}\ \overline{x}_{\mathbf{k}-\mathbf{n}} \overline{m}_{\mathbf{k}-\mathbf{n}-\boldsymbol{\ell}} \\ &= (\det N) \sum_{\mathbf{n} \in R_N} x_{-\mathbf{n}} \overline{x}_{\mathbf{k}-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}} \overline{\widetilde{m}}_{(\mathbf{n}+\boldsymbol{\ell})-\mathbf{k}}. \end{aligned}$$

After the change $\mathbf{n} = -\mathbf{p}$,

$$\begin{aligned} \left(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]}\right)_{\mathbf{k}} &= (\det N) \sum_{\mathbf{p} \in R_N} x_{\mathbf{p}} \overline{x}_{\mathbf{p}+\mathbf{k}}\ \widetilde{m}_{\boldsymbol{\ell}-\mathbf{p}} \overline{\widetilde{m}}_{(\boldsymbol{\ell}-\mathbf{p})-\mathbf{k}} \\ &= (\det N) \sum_{\mathbf{p} \in R_N} (\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}})_{\mathbf{p}} (\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})_{\boldsymbol{\ell}-\mathbf{p}} \\ &= (\det N) \left[(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}}) \circledast_N \left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}}\right)\right]_{\boldsymbol{\ell}}. \end{aligned} \tag{3}$$

Finally, applying Property (v.) yields

$$\left(F_N(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]})\right)_{\mathbf{k}} = (\det N) \left[F_N(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}}) \circ F_N\left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}}\right)\right]_{\boldsymbol{\ell}}. \qquad \square$$

Definition 2. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ and $L_1, L_2, M_1, M_2 \in \mathbb{N}$ be such that $L_1 M_1 = N_1$ and $L_2 M_2 = N_2$. Then, we define the uniform sub-sampling operator $Z_L \colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{M_1 \times M_2}$ as*

$$(Z_L \mathcal{X})_{\mathbf{n}} := x_{L\mathbf{n}}, \quad \forall \mathbf{n} \in \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2},$$

where the sub-sampling periodicity matrix is

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

We will also make use of the matrix $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$.

Lemma 3 (Aliasing in 2D)**.** *For any $\mathbf{j} \in \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2}$,*

$$(F_M(Z_L \mathcal{X}))_{\mathbf{j}} = \frac{1}{\det L} \sum_{\mathbf{p} \in R_L} \widehat{x}_{\mathbf{j} - M\mathbf{p}}.$$

Proof. By the definition of DFT of signals with periodicity M, we have

$$\begin{aligned}
(F_M(Z_L \mathcal{X}))_{\mathbf{j}} &= \sum_{\mathbf{n} \in R_M} x_{L\mathbf{n}} e^{-2\pi i \mathbf{n}^T M^{-1} \mathbf{j}} \\
&= \sum_{\mathbf{n} \in R_M} \left(\frac{1}{\det N} \sum_{\mathbf{k} \in R_N} \widehat{x}_{\mathbf{k}} e^{2\pi i \mathbf{k}^T M^{-1} \mathbf{n}} \right) e^{-2\pi i \mathbf{n}^T M^{-1} \mathbf{j}} \\
&= \frac{1}{\det N} \sum_{\mathbf{k} \in R_N} \widehat{x}_{\mathbf{k}} \sum_{\mathbf{n} \in R_M} e^{2\pi i \mathbf{k}^T M^{-1} \mathbf{n}} e^{-2\pi i \mathbf{n}^T M^{-1} \mathbf{j}} \\
&= \frac{1}{\det N} \sum_{\mathbf{k} \in R_N} \widehat{x}_{\mathbf{k}} \sum_{\mathbf{n} \in R_M} e^{2\pi i \mathbf{n}^T M^{-1} (\mathbf{k} - \mathbf{j})} \\
&= \frac{1}{\det L} \sum_{\mathbf{k} \in R_N} \widehat{x}_{\mathbf{k}} \delta_{\mathbf{k} - \mathbf{j} \bmod M},
\end{aligned}$$

where mod M denotes modulo M_1 vertically (first index component) and modulo M_2 horizontally (second index component). We have used the Kronecker delta notation, here $\mathbf{p} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$,

$$\delta_{\mathbf{k} - \mathbf{j} \bmod M} = \begin{cases} 1, & \mathbf{k} = \mathbf{j} - M\mathbf{p} \\ 0, & \text{otherwise} \end{cases}.$$

More explicitly, $\delta_{\mathbf{k} - \mathbf{j} \bmod M} = 1$ exactly when

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} j_1 - p_1 N_1 / L_1 \\ j_2 - p_2 N_2 / L_2 \end{bmatrix}.$$

Since this happens for exactly L_1 values of p_1 and L_2 values of p_2, we have

$$(F_M(Z_L \mathcal{X}))_{\mathbf{j}} = \frac{1}{\det L} \sum_{\mathbf{p} \in R_L} \widehat{x}_{\mathbf{j} - M\mathbf{p}}. \qquad \square$$

C. Main Result

Before presenting our main result, we define the following transposition of a 4−tensor $\mathcal{Y}$: $(\mathcal{Y}^T)_{ijkl} := \mathcal{Y}_{klij}$, which reads as $(\mathcal{Y}^T)_{[\boldsymbol{\ell}\ \mathbf{j}]} = \mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]}$ in bracket notation.

Theorem 1. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$, $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$, and $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$ denote the signal to be recovered, the mask, and the measurements, respectively. Suppose $N_1 = L_1 M_1$ and $N_2 = L_2 M_2$, where $L_1, L_2, M_1, M_2 \in \mathbb{N}$ (i.e., the shifts L_1, L_2 divide the signal dimensions N_1, N_2 respectively). Then, for any $\boldsymbol{\omega} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$ and any $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$,*

$$\begin{aligned}
&\frac{(\det N)^2}{\det L} (F_L(F_N \mathcal{Y})^T)_{[\boldsymbol{\omega}\ \mathbf{k}]} = \\
&\sum_{\mathbf{p} \in R_M} \Big(F_N \left(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - L\mathbf{p}} \overline{\widehat{\mathcal{X}}} \right) \Big)_{\mathbf{k}} \Big(F_N \Big(\widehat{\mathcal{M}} \circ S_{L\mathbf{p} - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}} \Big) \Big)_{\mathbf{k}}.
\end{aligned} \tag{4}$$

Moreover, if $\operatorname{supp}(\widehat{\mathcal{M}}) = \{\mathbf{n} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} : 0 \le n_1 \le \delta_1 - 1, 0 \le n_2 \le \delta_2 - 1\}$ *for some $\delta_1, \delta_2 \in \mathbb{N}$ such that $L_1 = 2\delta_1 - 1, L_2 = 2\delta_2 - 1$, then the sum above collapses to exactly one of the four terms:*

$$\begin{aligned}
&i. \Big(F_N(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega}} \overline{\widehat{\mathcal{X}}}) \Big)_{\mathbf{k}} \Big(F_N(\widehat{\mathcal{M}} \circ S_{-\boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}) \Big)_{\mathbf{k}} \\
&ii. \Big(F_N(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [0\ L_2]^T} \overline{\widehat{\mathcal{X}}}) \Big)_{\mathbf{k}} \Big(F_N(\widehat{\mathcal{M}} \circ S_{[0\ L_2]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}) \Big)_{\mathbf{k}} \\
&iii. \Big(F_N(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [L_1\ 0]^T} \overline{\widehat{\mathcal{X}}}) \Big)_{\mathbf{k}} \Big(F_N(\widehat{\mathcal{M}} \circ S_{[L_1\ 0]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}) \Big)_{\mathbf{k}} \\
&iv. \Big(F_N(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [L_1\ L_2]^T} \overline{\widehat{\mathcal{X}}}) \Big)_{\mathbf{k}} \Big(F_N(\widehat{\mathcal{M}} \circ S_{[L_1\ L_2]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}) \Big)_{\mathbf{k}}
\end{aligned} \tag{5}$$

where i. if $\boldsymbol{\omega} \in \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2}$, ii. if $\boldsymbol{\omega} \in \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{L_2} \backslash \mathbb{Z}_{\delta_2}$, iii. if $\boldsymbol{\omega} \in \mathbb{Z}_{L_1} \backslash \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2}$, and iv. if $\boldsymbol{\omega} \in \mathbb{Z}_{L_1} \backslash \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{L_2} \backslash \mathbb{Z}_{\delta_2}$.

Proof. In (3) we saw that

$$(F_N \mathcal{Y})_{k_1, k_2, \ell_1, \ell_2} = (\det N) \Big[\big(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}} \big) \circledast_N \Big(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}} \Big) \Big]_{\boldsymbol{\ell}}.$$

Equivalently,

$$\big((F_N \mathcal{Y})^T \big)_{[\boldsymbol{\ell}\ \mathbf{k}]} = (\det N) \Big[\big(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}} \big) \circledast_N \Big(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}} \Big) \Big]_{\boldsymbol{\ell}}$$

for $\boldsymbol{\ell} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$.

Define $\mathcal{U} := (\det N)[(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}}) \circledast_N (\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})], \in \mathbb{C}^{N_1 \times N_2}$. Taking $\ell_1 \in \mathbb{Z}_{N_1}$ and $\ell_2 \in \mathbb{Z}_{N_2}$ at equally spaced L_1 vertical and L_2 horizontal shifts, respectively, $u_{\boldsymbol{\ell}}$ corresponds to sub-sampled elements of $\mathcal{U}$ for shifts $\boldsymbol{\ell}$ such that

$$\boldsymbol{\ell} \in \left\{ 0, \frac{N_1}{L_1}, \dots, \frac{(L_1 - 1) N_1}{L_1} \right\} \times \left\{ 0, \frac{N_2}{L_2}, \dots, \frac{(L_2 - 1) N_2}{L_2} \right\}.$$

Now, we take the 2D L-periodic DFT of 2-dimensional slices of $(F_N \mathcal{Y})^T \in \mathbb{C}^{L_1 \times L_2 \times N_1 \times N_2}$ for fixed modes of $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. Then for $\boldsymbol{\omega} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$, we have by Definition 2,

$$\big(F_L (F_N \mathcal{Y})^T \big)_{[\boldsymbol{\omega}\ \mathbf{k}]} = (F_L(Z_M \mathcal{U}))_{\boldsymbol{\omega}}.$$

Thus, by Lemma 3 where M and L are exchanged,

$$\begin{aligned}
&\big(F_L (F_N \mathcal{Y})^T \big)_{[\boldsymbol{\omega}\ \mathbf{k}]} = \\
&\frac{1}{\det M} \sum_{\mathbf{p} \in R_M} \widehat{u}_{\boldsymbol{\omega} - N M^{-1} \mathbf{p}} = \frac{\det L}{\det N} \sum_{\mathbf{p} \in R_M} \widehat{u}_{\boldsymbol{\omega} - L\mathbf{p}}.
\end{aligned}$$

Applying Lemma 2, we get

$$\begin{aligned}
&\big(F_L (F_N \mathcal{Y})^T \big)_{[\boldsymbol{\omega}\ \mathbf{k}]} = \\
&(\det L) \sum_{\mathbf{p} \in R_M} [F_N(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}})]_{\boldsymbol{\omega} - L\mathbf{p}} [F_N(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})]_{\boldsymbol{\omega} - L\mathbf{p}}.
\end{aligned}$$

Now, we use the fact that time reversal is an involution to rewrite the second term in the sum.

$$(F_L(F_N\mathcal{Y})^T)_{[\boldsymbol{\omega}\ \mathbf{k}]} =$$
$$(\det L) \sum_{\mathbf{p}\in R_M} [F_N(\mathcal{X}\circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega}-L\mathbf{p}} [F_N(\widetilde{\widetilde{\mathcal{M}}\circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}}})]_{L\mathbf{p}-\boldsymbol{\omega}}.$$

Then, by Properties (iii.) and (iv.),

$$(F_L(F_N\mathcal{Y})^T)_{[\boldsymbol{\omega}\ \mathbf{k}]} =$$
$$(\det L) \sum_{\mathbf{p}\in R_M} [F_N(\mathcal{X}\circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega}-L\mathbf{p}} [F_N(\mathcal{M}\circ S_{-\mathbf{k}}\overline{\mathcal{M}})]_{L\mathbf{p}-\boldsymbol{\omega}}.$$

Therefore, by Lemma 1, we have

$$\frac{(\det N)^2}{\det L}\left(F_L(F_N\mathcal{Y})^T\right)_{[\boldsymbol{\omega}\ \mathbf{k}]} =$$
$$\sum_{\mathbf{p}\in R_M} e^{2\pi i(\boldsymbol{\omega}-L\mathbf{p})^T N^{-1}\mathbf{k}} \left[F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-L\mathbf{p}}\overline{\widehat{\mathcal{X}}}\right)\right]_{\mathbf{k}} \cdot$$
$$e^{2\pi i(L\mathbf{p}-\boldsymbol{\omega})^T N^{-1}\mathbf{k}} \left[F_N\left(\widehat{\mathcal{M}}\circ S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\right]_{\mathbf{k}} =$$
$$\sum_{\mathbf{p}\in R_M} \left(F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-L\mathbf{p}}\overline{\widehat{\mathcal{X}}}\right)\right)_{\mathbf{k}} \left(F_N\left(\widehat{\mathcal{M}}\circ S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\right)_{\mathbf{k}},$$

as desired.

To show the second part of Theorem 1, we want to find conditions where

$$\operatorname{supp}\left(\widehat{\mathcal{M}}\right) \cap \operatorname{supp}\left(S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right) \neq \emptyset. \tag{6}$$

We consider *bandlimited* masks[1] $\mathcal{M}$, whose Fourier transform satisfies the following support constraint:

$$\operatorname{supp}\left(\widehat{\mathcal{M}}\right) = \{\mathbf{n}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2} : 0\le n_1\le \delta_1 - 1, \quad 0\le n_2\le\delta_2-1\}. \tag{7}$$

Under this constraint, (6) is satisfied if and only if

$$|p_1L_1-\omega_1|\le\delta_1-1 \quad \text{and} \quad |p_2L_2-\omega_2|\le\delta_2-1.$$

It is now easy to verify that these two inequalities are satisfied only when $p_1 = 0$ for $\omega_1\in\mathbb{Z}_{\delta_1}$ and $p_1 = 1$ for $\omega_1\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}$, and similarly only when $p_2 = 0$ for $\omega_2\in\mathbb{Z}_{\delta_2}$ and $p_2 = 1$ for $\omega_2\in\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}$. This leads to the following piecewise definition of $\mathbf{p} = [p_1\ p_2]^T$:

$$\mathbf{p} = \begin{cases} [0\ \ 0]^T, & \text{if } \boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2} \\ [0\ \ 1]^T, & \text{if } \boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2} \\ [1\ \ 0]^T, & \text{if } \boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2} \\ [1\ \ 1]^T, & \text{if } \boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}. \end{cases}$$

Substituting in (4) collapses the sum to yield the four terms enumerated in (5) as desired. □

Intuitively, this result establishes the decoupling of the unknown signal from the known mask; furthermore, this decoupling is achieved by means of a highly structured (and efficient to invert) linear system. Indeed, this linear system can be inverted efficiently by computing a small (proportional to L_1L_2) number of FFTs.

D. Mask Design and Angular Synchronization

From Theorem 1, we see that recovery of $F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}}\right)$ requires componentwise (for all $\mathbf{k}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$) division by $F_N(\widehat{\mathcal{M}}\circ S_{-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})$. This requires the second terms in each of the four cases in (5) to be nonzero for all $\mathbf{k}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$. The design of masks $\mathcal{M}$ which satisfy this criteria (in addition to the bandlimitedness constraint imposed by (7)) is an open research problem. Here, we present one such mask – motivated by the 1D mask prescription in [9] – which has been empirically verified to satisfy these conditions.

Let $a\in[4,\infty)$ and define the support-constrained 1D vectors ${}^x\widehat{m}\in\mathbb{R}^{N_1}$ and ${}^y\widehat{m}\in\mathbb{R}^{N_2}$ by

$$({}^x\widehat{m})_k = \frac{e^{-k/a}}{\sqrt[4]{2\delta_1-1}}\cdot\mathbb{1}_{k<\delta_1}, ({}^y\widehat{m})_k = \frac{e^{-k/a}}{\sqrt[4]{2\delta_2-1}}\cdot\mathbb{1}_{k<\delta_2}.$$

Then, using $\otimes$ to denote the tensor product,

$$\mathcal{M} = F_N^{-1}({}^x\widehat{m}\otimes{}^y\widehat{m}) \tag{8}$$

defines an admissible 2D mask.

Having recovered $F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}}\right)$ using Theorem 1, computing a series of IDFTs allows for the estimation of $\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}}\right)_{\mathbf{k}}$ for all $\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}$ and all $\mathbf{k}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$. These can be interpreted as (scaled) relative phase estimates (since each of the entries is of the form $\widehat{x}_{\mathbf{n}}\overline{\widehat{x}}_{\mathbf{n}-\boldsymbol{\omega}}$). Recovering individual phase information ($\widehat{x}_{\mathbf{n}}$) amounts to solving an angular synchronization problem [10]; this can be done efficiently and robustly by computing the (normalized) eigenvector corresponding to the largest eigenvalue (see [11] and [8] for a detailed discussion). We note that the leading eigenvector can be computed in essentially linear time using the power method (or one of its variants) due to the highly structured (and sparse) nature of the matrices involved. The complete pseudocode for this phase recovery method is provided in Alg. 1.

III. Numerical Evaluation

We now present numerical results verifying the efficiency and noise robustness of the proposed method. Fig. 1 shows the reconstruction of a synthetic 230×230 pixel complex test image $\mathcal{X} = \Psi e^{i\Theta}$ (with Ψ and Θ chosen to be the `cameraman` and `circuits` test images respectively), using phaseless measurements generated as per (2), $L_1 = L_2 = 23$ shifts, and the *deterministic* mask prescription from (8). The relative (Frobenius norm) error in recovering $\mathcal{X}$ was 1.834×10^{-11}, with an execution time[2] of 19.55s. We next investigate the robustness of Alg. 1 to measurement errors by adding zero mean i.i.d. random Gaussian noise to the measurements in (2) at various signal to noise ratios (SNR, measured in dB). The errors in the reconstruction (up to a global phase factor,

[1] Such masks have applications in coded diffraction pattern (CDP) based imaging setups and Fourier ptychography. Equivalent results can be developed for support-limited masks – we leave this extension to future research.

[2] Implemented in Matlab R2018b on a laptop computer with an Intel Core M-5Y10c CPU, 8 GB RAM, and running Ubuntu 18.04 (64-bit).

Algorithm 1 2D Phase Retrieval for Bandlimited Masks

Inputs

1) Tensor $\left(\mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]}\right)_{\mathbf{j}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\boldsymbol{\ell}\in\mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}}$ of measurements as in (2).
2) Known bandlimited mask $\mathcal{M}$ (see, for example, (8)).

Steps

1) Estimate $\left(F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}}\right)\right)_{\mathbf{k}}$ for $\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}$ and $\mathbf{k}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ from (5).
2) Invert the Fourier transforms above to recover estimates of the $L_1\cdot L_2$ matrices $\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}}$.
3) Vectorize the recovered matrices to $N_1\cdot N_2$ length vectors.
4) Form a banded matrix $\mathbf{X}$ by populating the vectorized estimates from Step 3 on its L_1L_2 diagonals, each diagonal corresponding to a single distinct $\boldsymbol{\omega}$ value.
5) Hermitianize by setting $\mathbf{X}\hookleftarrow(\mathbf{X}+\mathbf{X}^*)/2$.
6) Estimate $\left|\left(\text{vec}(\widehat{\mathcal{X}})\right)_j\right|\approx\eta_j=\sqrt{|\mathbf{X}_{j,j}|}$.
7) Normalize (the non-zero entries of) $\mathbf{X}$ component-wise to form the relative phase matrix $\mathring{\mathbf{X}}$.
8) Compute the leading normalized eigenvector of $\mathring{\mathbf{X}}$, $\mathbf{u}$.

Output

Estimate of $\mathcal{X}$ (converted from a vector to matrix form), $\mathcal{X}_{\text{rec}}:=F_N^{-1}\widehat{\mathcal{X}}_{\text{rec}}$, where $\widehat{\mathcal{X}}_{\text{rec}}$ derives from

$$\left(\text{vec}\left(\widehat{\mathcal{X}}_{\text{rec}}\right)\right)_j:=\eta_j u_j.$$

(a) Recovered Magnitude (b) Recovered Phase

Fig. 1. Recovered magnitude and phase of a test image using Alg. 1.

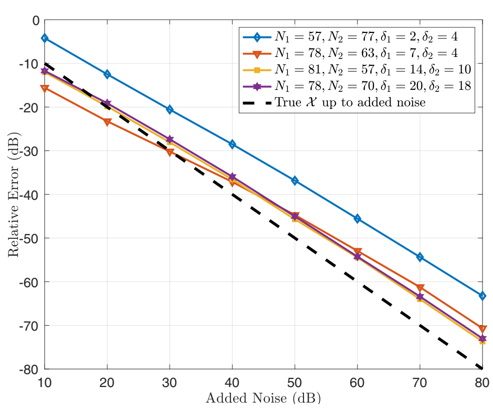

Fig. 2. Robustness to Additive Measurement (Gaussian) Noise

measured in dB, averaged over 100 trials) are plotted in Fig. 2. Here, we define

$$\text{SNR (dB)}=10\log_{10}\left(\frac{\|\text{vec}(\mathcal{Y})\|_2^2}{(N_1N_2L_1L_2)\sigma^2}\right),\text{ and}$$

$$\text{Error (dB)}=10\log_{10}\left(\frac{\min_\theta\left\|e^{i\theta}\mathcal{X}-\mathcal{Q}\right\|_F^2}{\|\mathcal{X}\|_F^2}\right),$$

where vec($\cdot$) denotes tensor vectorization (consistent with Matlab's colon operator), σ^2 is the variance of added noise, and $\mathcal{Q}$, $\mathcal{X}$ denote the recovered and true images, respectively. As expected, the reconstruction accuracy improves with increase in SNR, with improved accuracy offered by masks with larger support (larger δ_1,δ_2 values). Finally, Table II lists execution times for Alg. 1 (in sec., averaged over 100 trials) for various problem sizes. The execution time grows approximately log-linearly (in N_1N_2), with room for improvement through code optimization and use of compiled code.

TABLE II
EXECUTION TIME (IN SEC.) VS. PROBLEM SIZE

(N_1,N_2)	$(22,18)$	$(75,45)$	$(152,102)$	$(209,119)$
Exec. Time (s)	0.06	0.64	3.48	5.85

IV. Future Directions

Several compelling avenues for further research exist, including developing error guarantees in the presence of noise, designing more (practicable) families of measurement masks, performing more elaborate empirical evaluations, and implementing this algorithm with real-world datasets.

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