Definition 0.1. A triangle in a metric space consists of three points and a geodesic segment between each pair.

Definition 0.2. A metric space is said to be $\delta$-hyperbolic if it satisfies the thin triangle condition, namely, if there is some $\delta$ such that any triangle is contained within the $\delta$ radius of any two of its sides. A group is said to be $\delta$-hyperbolic if the Cayley graph is $\delta$-hyperbolic under the metric in which each edge has length 1.

Exercise 0.1. Trivially all finite groups are $\delta$-hyperbolic. Give an example of an infinite $\delta$-hyperbolic group. Show that $\mathbb{Z}_2$ is not one.

Remark 0.2. $\delta$-hyperbolicity is preserved under quasi-isometries. We have a proof for this and we were going to make a Fermat’s last theorem joke about it, but fortunately our group voted against it.

Definition 0.3. A K-local geodesic is a path with the property that every subpath of length $\leq K$ is a geodesic.

Proposition 0.3. If $G$ is $\delta$-hyperbolic there is some constant $K$ such that no $K$-local geodesic in $G$ is a loop. (Hint: Let $\gamma$ be a loop that is a $K$-local geodesic and pick a point $x$ on $\gamma$ whose distance from the basepoint is maximal. Show that for $K$ large enough there is a point on the geodesic containing $x$ that is further away from the basepoint than $x$.)

Definition 0.4. A Dehn presentation for a group $G$ is a presentation of the form $\langle X | u_1v_1^{-1}, u_2v_2^{-1}, \ldots, u_nv_n^{-1} \rangle$ where $l(u_i) < l(v_i)$ for all $1 \leq i \leq n$ and every word in $X$ that represents the identity contains at least one of the $u_i$.

Theorem 0.4. Every hyperbolic group can be given a Dehn presentation.

Corollary 0.5. Every hyperbolic group is finitely presented.

Definition 0.5. Let $G$ be a group with finite presentation $\langle X | R \rangle$. Let $w \in F(X)$ be a word representing $1 \in G$. $w$ can be expressed as a product of $n$ conjugates of elements of $R$ and their inverses. There are in general infinitely many such expressions; the least value of $n$ among these is the area of $w$.

This definition appears rather abstruse, but may be interpreted geometrically. For example the area of a word in $\mathbb{Z}_2$ corresponds to the actual area of the loop in the Cayley graph when it is isometrically embedded in $\mathbb{R}^2$.

Definition 0.6. A group has a linear isoperimetric inequality if there is some constant $K$ such that the area of any word $w$ representing the identity is $\leq Kl(w)$.

Theorem 0.6. Every hyperbolic group has a linear isoperimetric inequality.

In fact the converse is true as well. Any group satisfying a linear isoperimetric inequality is $\delta$-hyperbolic.

Definition 0.7. Given a group $G = \langle X | R \rangle$, the word problem in $G$ consists of trying to determine whether a given word in $X$ is the identity in $G$. A solution to the word problem is an algorithm that, given an arbitrary word in $X$, will determine whether or not that word is the identity in a finite amount of time.

It appears that the property of having a solvable word problem depends on the presentation for a group, but this is not the case.

Lemma 0.7. Let $\langle X | R \rangle$ and $\langle Y | S \rangle$ be finite presentations for a group $G$. Then $G$ has a solvable word problem with respect to $X$ and $R$ if and only if it has a solvable word problem with respect to $Y$ and $S$.

Proposition 0.8. Every hyperbolic group has a solvable word problem.