AMALGAMS, GRAPHS, AND FREE GROUPS

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1. Amalgams

Definition 1.1. Let G_1, G_2, A be groups with embeddings $g_i : A \hookrightarrow G_i$ for $i = 1, 2$. If we let $G_1 = \langle \{a_i\} | \{R_j\} \rangle$ and $G_2 = \langle \{b_k\} | \{S_m\} \rangle$, then we define $G_1 *_A G_2$, the *amalgamation of* G_1 *and* G_2 *over* A to be the group

$$G_1 *_A G_2 = \langle \{a_i\} \cup \{b_k\} | \{R_j\} \cup \{S_m\} \cup \{g_1(a)g_2(a)^{-1} = e\}_{a \in A} \rangle$$

with generators the disjoint union of the generators of G_1 and G_2 and all the relations of G_1 and G_2 with the added relations $g_1(a)g_2(a)^{-1} = e$ for all $a \in A$.

This is written as $G_1 *_A G_2$ and can be thought of as taking the group generated by the disjoint union of G_1 and G_2 and associating the two copies of A with each other. In the case where A is the trivial group, this is just the free product on G_1, G_2, written as $G_1 * G_2$. It is also important to note that the are canonical embeddings f_1, f_2 and f of G_1, G_2 and A respectively into $G_1 *_A G_2$ such that $f(a) = (f_i \circ g_i)(a)\ \forall a \in A$.

Theorem 1.1. *The amalgamation has the following universal property: Given a group H and homomorphisms $h_1 : G_1 \to H$, $h_2 : G_2 \to H$ and $h : A \to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
G_1 \arrow[dr, "h_1"] & \\
A \arrow[u, "g_1"] \arrow[r, "h"] \arrow[d, "g_2"'] & H \\
G_2 \arrow[ur, "h_2"'] &
\end{tikzcd}$$

there exists a unique homomorphism $\phi : G_1 *_A G_2 \to H$ such that

$$\phi \circ f_i = h_i \text{ for } i = 1, 2 \text{ and } \phi \circ f = h$$

$G_1 *_A G_2$ is the unique group with the universal property and is therefore well-defined, independent of the presentation chosen.

Exercise 1.2. *Show that $D_\infty \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.*

Exercise 1.3. *Show that $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2 \simeq \mathbb{Z} \times F_2$ where $g_1 : n \mapsto (n, 0)$ and $g_2 : n \mapsto (n, 0)$. Generalize this to $\mathbb{Z}^n *_{\mathbb{Z}^k} \mathbb{Z}^m$, where $k \leq min\{n, m\}$.*

Let S_1 be a set of right coset representatives for G_1 modulo A and S_2 respectively for G_2. Assume $1 \in S_1$, $1 \in S_2$. Let $\hat{\imath} = (i_1, ..., i_n)$ for $n \geq 0$ be a sequence with $i_m \in \{1, 2\}$ and $i_m \neq i_{m+1}$ for $1 \leq m \leq n-1$. Then a *reduced word of type* $\hat{\imath}$ is a sequence

$$(a; s_1, ..., s_n) \text{ with } a \in A,\ s_1 \in (S_{i_1} \setminus \{1\}),\ \dots\ ,\ s_n \in (S_{i_n} \setminus \{1\})$$

Theorem 1.4 (Structure Theorem)**.** *Given $g \in G_1 *_A G_2$ there is a unique sequence $\hat{\imath} = (i_1, i_2, ..., i_n)$ and a unique reduced word $m = (a; s_1, s_2, ..., s_n)$ of type $\hat{\imath}$ such that*

$$g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n)$$

Exercise 1.5. *Prove the structure theorem: Let $X_{\hat{\imath}}$ be the set of reduced words of type $\hat{\imath}$ and let X be the union of all $X_{\hat{\imath}}$. For $i \in \{1, 2\}$ define an action of G_i on X such that its restriction to $f_i(A)$ is given by*

$$f_i(a') \cdot (a; s_1, ..., s_n) = (a'a; s_1, ..., s_n)$$

which is independent of i. Use this to find an inverse to the map

$$\beta : X \to G_1 *_A G_2$$

$$\beta : (a; s_1, ..., s_n) \mapsto f(a) f_{i_1}(s_1) \dots f_{i_n}(s_n)$$

and prove the structure theorem.

We now use the Structure Theorem to show that elements $G_1 *_A G_2$ can be thought of as sequences of elements of the G_i modulo moving elements of A between them. Let $G'_i = G_i \setminus A$ for $i \in 1, 2$. For a sequence $\hat{\imath} = (i_1, ..., i_n)$ let $G'_{\hat{\imath}}$ be $G'_{i_1} \times \cdots \times G'_{i_n}$ modulo the action of A^{n-1} defined by

$$(a_1, ..., a_{n-1})(g_1, ..., g_n) = (g_1 a_1^{-1}, a_1 g_2 a_2^{-1}, a_2 g_3 a_3^{-1}, ..., a_{n-1} g_n)$$

Let $f_{\hat{\imath}} : G'_{\hat{\imath}} \to G_1 *_A G_2$ be the map induced by $(g_1, ..., g_n) \mapsto f_{i_1}(g_1) \cdots f_{i_n}(g_n)$.

Exercise 1.6. *Use the Structure Theorem to show that f and the $f_{\hat{\imath}}$ give a bijection from the disjoint union of A and $G_{\hat{\imath}}$ onto $G_1 *_A G_2$.*

Definition 1.2. If $g \in G_1 *_A G_2$ is of type $\hat{\imath} = (i_1, ..., i_n)$, we define the *length* of g to be $l(g) = n$. We say g is *cyclically reduced* if $i_1 \neq i_n$.

Proposition 1.7. *Every element of $G_1 *_A G_2$ is conjugate to a cyclically reduced element or an element of one of the G_i. Show that every cyclically reduced element has infinite order. Deduce that if the G_i are torsion-free, so is $G_1 *_A G_2$.*

1.1. **Van Kampen's Theorem.**

Remark 1.8. The definition of amalgamation also applies to cases in which the g_i are not necessarily injective. In these cases the amalgamation still has the universal property, although the structure theorem no longer holds. In fact, G_1 and G_2 may not even embed in $G_1 *_A G_2$.

Van Kampen's Theorem allows us to determine the fundamental group of a topological space by covering it with open subsets and looking at the fundamental group of each open subset as well as their intersection.

Theorem 1.9. *(Van Kampen's Theorem) Let X be a topological space and let U_1, U_2 be path-connected open sets so that X is contained in $U_1 \cup U_2$, and let $U_1 \cap U_2$ be non-empty and path-connected. Fix x in $U_1 \cap U_2$. Then the fundamental group of*

X with respect to x is obtained by amalgamating the fundamental groups $\pi_1(U_1, x)$, $\pi_1(U_2, x)$ and $\pi_1(U_1 \cap U_2, x)$ over the following homomorphisms

$$\pi_1(U_1 \cap U_2, x) \to \pi_1(U_1, x) \qquad \textit{and} \qquad \pi_1(U_1 \cap U_2, x) \to \pi_1(U_2, x)$$

The proof of the Van Kampen Theorem is messy so we will not require it. However, it is important to see why the fundamental group is given by this amalgamation. Given a loop based at x, we can look at each time the loop passes from U_1 to U_2. Because the intersection is connected, we can "divert" the path so that it passes through x on its way through the intersection. Thus every loop at x can be associated with a sequence of loops at x, each contained entirely in U_1 or U_2. The intersection of $\pi_1(U_1)$ and $\pi_1(U_2)$ is precisely $\pi_1(U_1 \cap U_2)$, so instead of a free product of the fundamental groups, we have the amalgamation over $\pi_1(U_1 \cap U_2)$.

Exercise 1.10. *What is the fundamental group of two tori stacked on top of each other (so that the intersection is S^1)?*

2. Graphs

Definition 2.1. We may define a graph Γ as a set of verticies $X =$vert(Γ) and edges $Y =$edge(Γ). In addition to these two sets, we need a map to identify the vertex and edge sets:

$$Y \to X \times X,\ y \mapsto (o(y), t(y))$$

Here $o(y)$ is referred to as the origin of the edge y and $t(y)$ is called its terminus.

A geodesic in a tree is a path without backtracking. Between any two points P and Q in a tree there is a unique geodesic, with length $l(P, Q)$. The set of verticies can be thought of as a metric space under this distance.

Exercise 2.1. *Let Γ be a tree of finite diameter n. Show that all the geodesics of Γ of length n have the same middle vertex if n is odd and the same middle edge if n is even.*

Theorem 2.2. *A tree of finite even diameter (resp. of finite odd diameter) has a vertex (resp. edge), which is invariant under all automorphisms.*

Let Γ be a non-empty graph. The set of subgraphs of Γ which are trees has a maximal element called a *maximal tree* of Γ.

Proposition 2.3. *Show that a maximal tree of a connected non-empty graph Γ contains all the vertices of Γ.*

For a connected, non-empty graph Γ, let $\Lambda_i, i \in I$ be a family of trees and let Λ be their disjoint union. Since we work without orientation, we may consider any graph as a topological space with the obvious topology. We will refer to this topological space as real(Γ), or the realization of Γ.

Definition 2.2. We may then define the *quotient graph* Γ/Λ as the graph corresponding to the quotient space real(Γ/Λ) obtained by identification of each subspace real(Λ_i) to a point.

Theorem 2.4. *The canonical projection* real$(\Gamma) \to$ real(Γ/Λ) *is a homotopy equivalence.*

Theorem 2.5. *Let Γ be a connected non-empty graph. Then* real(Γ) *has the homotopy type of a bouquet of circles. Further, Γ is a tree if and only if* real(Γ) *is contractible. Given Λ as defined above, Γ is a tree if and only if Γ/Λ is one.*

3. Actions of Free Groups on Graphs

Let a group G act on a graph Γ. We say G acts without inversion if there does not exist $g \in G$ and $y \in \text{edge}(\Gamma)$ such that $o(gy) = t(y)$ and $t(gy) = o(y)$, i.e. it preserves an orientation on Γ. The group is said to act freely if it acts without inversion and there is no element $g \in G, g \neq 1$ that fixes a vertex of Γ.

Exercise 3.1. *Given a free group G with basis S show that G acts freely on $\Gamma(G, S)$.*

Theorem 3.2. *Let X be the Cayley graph $\Gamma(G, S)$ defined by a group G and a subset S of G. Then X is a tree if and only if G is a free group with basis S.*

This theorem hints at a correspondence between free groups and actions on trees. We will prove that *any* group which acts freely on *any* tree is a free group. For a group G acting on a graph Γ without inversion, we may define the quotient graph $G\backslash\Gamma$ to be the quotient of $\text{vert}(\Gamma)$ and $\text{edge}(\Gamma)$ under the action of G.

Definition 3.1. Let X be a subtree of $G\backslash\Gamma$ and $\phi : \Gamma \to G\backslash\Gamma$ be the natural map. A lift of X is a subgraph Y of Γ such that $\phi(Y) = X$ and the restriction of ϕ to Y is injective.

Theorem 3.3. *If Γ is connected and G acts on Γ without inversion, then for every subtree X of $G\backslash\Gamma$ there is a subtree Y of Γ that is a lift of X.*

Theorem 3.4. *Let G be a group acting freely on a tree X. Given a maximal tree $T' \subset G\backslash X$ let T be a lift of T'. vert(T) is a set of representatives of X mod G, that is*
(i) given $x \neq y \in \text{vert}(T), g(x) \neq y\ \forall g \in G$ and
(ii) for all $z \in X$, $\exists\ x \in$ vert(T) such that $z = g(x)$.
In addition, for $g_1 \neq g_2$, $g_1(T) \cap g_2(T) = \emptyset$.

Now choose an orientation of X preserved by G. Let

$$S = \{g \in G | g \neq 1 \text{ and } \exists\ y \text{ an edge with } o(y) \in T, t(y) \in gT\}$$

Let $X' = X/(G \cdot T)$, that is the graph associated with $\text{real}(X/(G \cdot T))$ where

$$G \cdot T = \coprod_{g \in G} g(T)$$

Theorem 3.5. *S is a basis for G and X' is isomorphic to $\Gamma(G, S)$.*

Corollary 3.6. *If G acts freely on a tree, then G is a free group.*

Corollary 3.7 (Schreier's Theorem)**.** *Every subgroup of a free group is free.*

However, the subgroup of a free group may be "larger" than the original free group:

Exercise 3.8. *Explicitly show that F_n is a subgroup of F_2 for all $n \geq 2$.*