# GROWTH OF SUBGROUPS OF $GL_n(\mathbb{R})$

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## 1. Tits' alternative

**Theorem 1.1** (Tits' alternative)**.** *Let $L$ be a Lie group with finitely many components and let $\Gamma \subset L$ be a finitely generated subgroup. Then either $\Gamma$ is virtually solvable or $\Gamma$ contains a free nonabelian subgroup.*

We will not develop a proof of Tits' alternative in full generality here. We focus on the case where $L$ is a linear group (to which the general case may be reduced). Since we are interested in the growth of subgroups of $GL_n(\mathbb{R})$, we will content ourselves with constructing a free nonabelian semigroup rather than a full group, as either of these is sufficient to establish exponential growth.

Note that we may assume for the purposes of Tits' alternative that $\Gamma$ is connected. If it is not, then the identity component has finite index and we will lose nothing by restricting our attention to this component.

### 1.1. Algebraic varieties and the Zariski topology.

Before we begin Tits' alternative in earnest, we will need some tools from algebraic geometry.

**Definition 1.1.** An algebraic variety $V$ over a field $K$ is the set of common zeroes to a system of polynomial equations $f_1, \ldots, f_n$ over $K$.

**Exercise 1.2.** *Why can we regard $SL_n(F)$ as an algebraic variety?*

We now introduce a particularly useful topology for studying algebraic varieties.

**Exercise 1.3.** *Show that specifying that closed sets are algebraic subvarieties (that is, subsets satisfying a system of polynomials) defines a topology on an algebraic variety $V$. (This topology is called the Zariski topology.)*

**Corollary 1.4.** *$SL_n(F)$ is a closed subgroup of $GL_n(F)$ in the Zariski topology.*

**Exercise 1.5.** *Show that the Zariski topology on $\mathbb{R}^n$ is $T_1$ and that with this topology $\mathbb{R}$ is compact.*

We will use these tools to prove a special case of Tits' alternative.

**Theorem 1.6** (Special case of Tits' alternative)**.** *Let $\Gamma$ be a finitely generated Zariski-dense subgroup of $SL_n(F)$. Then either $\Gamma$ is virtually solvable or $\Gamma$ contains a free nonabelian semigroup.*

### 1.2. Lie algebras and Levi decomposition.

Note: you are not expected to prove the 'facts' in this section.

The Lie algebra of the group can be used to decide the outcome of Tits' alternative.

**Definition 1.2.** A *Lie algebra* $V$ is a vector space together with a binary operation $[\cdot,\cdot]$, called the *bracket*, which is bilinear, anticommutative, and satisfies the *Jacobi identity*:

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$$

A *subalgebra* $S$ of a Lie algebra $V$ is a vector subspace which is closed under the bracket; that is, $[S,S]\subseteq S$.
A subalgebra $I$ is an *ideal* of $V$ if for all $v\in V$, $w\in I$, $[v,w]\in I$; that is, $I$ is closed under bracketing by *any* element of $V$.
A *Lie algebra isomorphism* is a vector space isomorphism which preserves the bracket.

Lie groups give rise to Lie algebras which can be used to examine their structure. These are constructed from the manifold structure of the Lie group along with its group structure.

**Definition 1.3.** Let $G$ be a Lie group. For $g\in G$, denote by $C_g^\infty(G)$ the set of germs of smooth functions on a neighborhood of $g$. (*Germs* are equivalence classes of functions, where two functions are equivalent if they agree on a neighborhood of $g$.) Denote by $C^\infty(G)$ the set of smooth functions $f:G\to\mathbb{R}$. Recall that a *tangent vector* at $g\in G$ is a map $v:C_g^\infty(G)\to\mathbb{R}$ satisfying:

(1) $v(\alpha f+\beta g)=\alpha v(f)+\beta v(g)$ for all $\alpha,\beta\in\mathbb{R}$, $f,g\in C^\infty(G)$.
(2) $v(fg)=v(f)g+fv(g)$ for all $f,g\in C^\infty(G)$.
(3) $v(f)=0$ if $f$ is a constant function.

A *smooth tangent vector field* $X$ is a map assigning to each point of $G$ a tangent vector at that point, such that if $f:G\to\mathbb{R}$ is a smooth function, $X(f):G\to\mathbb{R}$ is also smooth. (Note that we have identified $X$ as a map from $C^\infty(G)$ to itself. Why is this?)

To combine this with the group structure, we need to define an additional property for vector fields.

**Definition 1.4.** Let $v$ be a smooth tangent vector field on a Lie group $G$. $v$ is *left-invariant* if for all $g,h\in G$, $d\ell_g\circ v(h)=v(gh)$, where $\ell_g$ is the automorphism of $G$ induced by left multiplication by $g$.

**Proposition 1.7.** *Let $G$ be a Lie group. The set of all left-invariant vector fields on $G$, together with the bracket defined by $[X,Y](f)=X\circ Y(f)-Y\circ X(f)$, is a Lie algebra. This Lie algebra is called the Lie algebra of $G$ and generally denoted $\mathfrak{g}$. (Hint: Reduce the problem to a single tangent space by showing that a left-invariant vector field is uniquely determined by a tangent vector at the identity.)*

**Proposition 1.8.** *Denote by $\mathfrak{gl}_n(\mathbb{R})$ the set of all $n\times n$ real matrices. $\mathfrak{gl}_n(\mathbb{R})$, equipped with the commutator of matrices as a bracket, is the Lie algebra of $GL_n(\mathbb{R})$. (Hint: $GL_n(\mathbb{R})$ is an open subset of $\mathfrak{gl}_n(\mathbb{R})$. Use this to identify their tangent spaces at the identity matrix and show that the natural map between the Lie algebra of $GL_n(\mathbb{R}$ and $\mathfrak{gl}_n(\mathbb{R})$ is a Lie algebra isomorphism. Alternatively, write the action of a vector field on a function as the trace of a matrix and use this to explicitly compute the left-invariant vector fields.)*

The Lie group associated with a Lie algebra is not unique–a Lie algebra only determines uniquely a *simply connected* Lie group. Nonetheless, we can use properties of the Lie algebra to extract some properties of the Lie group. In particular, we want to extract the "part of $G$" responsible for the nonabelian free group.

**Definition 1.5.** Let $L$ be a Lie algebra. If the *derived series* $L^0 = L, L^{n+1} = [L^n, L^n]$ terminates at 0, $L$ is called *solvable.* A solvable Lie group is one with a solvable Lie algebra. (If the group is connected, this is equivalent to the usual definition of solvability.)
If $L$ contains no nontrivial solvable ideals, $L$ is called *semisimple.* A Lie group whose Lie algebra is semisimple is also called semisimple. A Lie group whose Lie algebra is the sum of a semisimple algebra and an abelian Lie algebra (i.e. one whose bracket is identically zero) is called *reductive.*

**Fact 1.** Aff(2), the group of affine transformations of $\mathbb{R}$, has as its Lie algebra the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

where the bracket is the commutator.
(Hint: Recall that a tangent vector acts on smooth functions like a directional derivative. Use this to write the action of a vector field on a smooth function as the trace of a matrix (involving this derivative) and apply the left invariance condition to obtain the matrices in the Lie algebra.)

**Corollary 1.9.** *Aff(2) is solvable.*

The following theorem, which you are not expected to prove, is an important statement about the structure of Lie algebras.

**Theorem 1.10** (Levi decomposition)**.** *Let L be a Lie algebra. L can be decomposed as the direct sum of vector spaces $V \oplus W$, where $V$ is a solvable Lie algebra and $W$ is a semisimple Lie algebra.*

The Levi decomposition translates to a semidirect product $G = S \rtimes H$ if $G$ is simply connected, where $S$ is solvable and $H$ is reductive. Otherwise, if $G$ is at least connected, we may still find subgroups of $G$ with these properties, satisfying $G = SH$ and $\dim S \cap H = 0$. This will be enough for our purposes (recall that we have assumed connectedness of $G$).

**Example 1.6.** Let

$$g = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Then $G = \langle g, h, 5I, 7I \rangle \subset GL_3(\mathbb{R})$ is virtually solvable. (Hint: Express $G$ as a semidirect product of a subgroup of $SL_3(\mathbb{R})$ and a subgroup of $\mathbb{R}^\times$.)

**Fact 2.** The Lie algebra $\mathfrak{sl}_n(F)$ of $SL_n(F)$ is given by the set of traceless $n \times n$ matrices, with the bracket defined as the commutator.

**Proposition 1.11.** $SL_n(F)$ *is semisimple.*

**Corollary 1.12.** $SL_n(F)$ *is not virtually solvable.*

Essentially, the size of the reductive part determines the outcome of Tits' alternative. If $\Gamma \subset SL_n(F)$ is not Zariski-dense, we set $G = \overline{\Gamma}$ (so that $\Gamma$ is Zariski-dense in $G$) and apply the Levi decomposition to obtain subgroups $S$ and $H$ as above. Then $\Gamma$ fits in a short exact sequence

$$1 \to S \cap \Gamma \to \Gamma \to Q \to 1$$

where $Q$ is a Zariski-dense subgroup of $H$. By examining $Q$ we can decide Tits' alternative.

We have the following cases:

(1) $Q$ is finite. Then $\Gamma$ is virtually solvable, since $S \cap \Gamma$ has finite index.
(2) $Q$ is unbounded. Then we can construct a free semigroup by methods such as the one in Section 1.3.
(3) $Q$ is infinite but bounded. When $F = \mathbb{R}$ this implies that $H$ is compact. The approach below to construct a free semigroup can be modified to work in this case, but this modification will not be covered in this sheet.

It turns out $SL_n(F)$ is in some ways a good model for semisimple algebraic groups in general. In the next section we will show how to handle case (2) when $\Gamma$ is Zariski-dense in $SL_n(F)$. In the case where we must replace $SL_n(F)$ by $G = \overline{\Gamma}$ as mentioned above, the proof follows a similar line of argument, but uses more complicated machinery (e.g. the *Cartan decomposition* instead of the singular value decomposition).

## 1.3. Playing 'Ping' in a subgroup of $SL_n(F)$.

This construction is based around the 'Ping' lemma, which is the semigroup equivalent of the 'Ping-Pong' lemma. We state it here for reference.

**Lemma 1.13** (Ping lemma)**.** *Let $G$ be a group generated by $s_1, \ldots, s_n$. Let $G$ act on a set $A$. Suppose there exists a collection of subsets of $A$,*

$$A_1^+, \ldots, A_n^+, A_1^-, \ldots, A_n^-$$

*satisfying the following:*

(1) *$A_i^+ \cap A_j^- = \emptyset$ for all $i \neq j$.*
(2) *$A_i^+ \cap A_j^- = \emptyset$ for all $i, j$.*
(3) *The set $\Phi = \bigcap_{i=1}^n (A \setminus (A_i^+ \cup A_i^-))$ is nonempty.*
(4) *For each $i$, $s_i(X \setminus A_i^-) \subseteq A_i^+$.*

*Then $G$ is generated freely by $s_1, \ldots, s_n$.*

We will ultimately apply the Ping lemma to prove the following.

**Theorem 1.14.** *Let $\Gamma$ be a Zariski-dense unbounded subgroup of $SL(n, F)$. Then $\Gamma$ contains a free semigroup on two generators.*

Fix a Zariski-dense unbounded subgroup $\Gamma \subset SL_n(F)$.

To apply the Ping lemma we will consider the action of $\Gamma$ on projective space, and construct sequences of transformations which send points in the projective space away from a 'source' set toward a 'sink' point. With two such sequences, by choosing sufficiently late terms, we can meet the hypotheses of the Ping lemma. In particular, we will find two divergent sequences, two subsets of a projective space (sources), and two distinct points (sinks) such that the sequences eventually send everything outside the sources to neighborhoods of the sinks. The next few lemmas establish the source and sink for a sequence of diagonal matrices.

**Definition 1.7.** Let $v, w$ in $F^n$. Denote by $v \wedge w$ the oriented area of a parallelogram spanned by $v$ and $w$. This may be computed by placing $v$ and $w$ in a $2 \times n$ matrix and summing all $2 \times 2$ minors.

**Definition 1.8.** Let $\mathbb{P}^{n-1}$ denote the projective space of 1-dimensional subspaces of $F^n$. If $v \in F^n$, denote by $[v]$ the projection to $\mathbb{P}^{n-1}$; similarly if $V$ is a nonzero subspace of $F^n$, denote by $[V]$ the projection. If $g \in SL(n, F)$, denote by $[g] \in PSL(n, F)$ the corresponding projective transformation. Define the metric

$$d([v], [w]) \equiv \frac{|v \wedge w|}{|v| \cdot |w|},$$

i.e. the sine of the angle between the lines $[v]$ and $[w]$. Clearly, then, $d$ is invariant under orthogonal (resp. unitary) transformations if $F = \mathbb{R}$ (resp. $\mathbb{C}$).

Let $g_i \in SL(n, F)$ be an unbounded sequence of diagonal matrices with diagonal elements $\lambda_1(g_i), \ldots, \lambda_n(g_i)$, where $\lambda_k$ is nonincreasing with $k$.

**Exercise 1.15.** *There exists* $1 \leq k < n$ *such that*

$$\lim_{i\to\infty} \frac{\lambda_k(g_i)}{\lambda_{k+1}(g_i)} = \infty$$

Define the subspaces $V_+ \equiv \operatorname{span}\{e_1, \ldots, e_k\}$, $V_- \equiv \operatorname{span}\{e_{k+1}, \ldots, e_n\}$. The following shows that elements of the sequence $[g_i]$ send a fixed $[v]$ progressively further from $[V_-]$ toward $[V_+]$.

**Lemma 1.16.** *Fix* $[v] \in \mathbb{P}^{n-1} \setminus [V_-]$. *The sequence*

$$d_i([v]) = d([g_i]([v]), [V_+])$$

*converges to 0.*

**Lemma 1.17.** *The above convergence is uniform on compact subsets of* $\mathbb{P}^{n-1} \setminus [V_-]$.

**Definition 1.9.** Let $F$ be a field. The *Grassmanian* denoted $Gr(n, k, F)$ is the collection of $k$-dimensional subspaces of $F^n$.

**Example 1.10.** $Gr(2, 1, \mathbb{R})$ is the familiar projective space of lines passing through the origin in the plane. We may identify this space with the unit circle with antipodal points identified.

**Definition 1.11.** Let $g_i$ be as above. Denote by $\omega \in Gr(n, k, F)$ the $k$-dimensional subspace $V_+ \subset F^n$. For concision we will henceforth write $Gr = Gr(n, k, F)$. Denote by $E \subset Gr$ the set of $k$-planes intersecting $V_-$ nontrivially. Clearly $\omega \notin E$.

**Lemma 1.18.** *$Gr$ is an algebraic variety and $E$ is a proper algebraic subvariety.*

As the next lemma shows, we use $E$ and $\omega$ to obtain the 'source' and 'sink,' respectively, for the Ping lemma.

**Lemma 1.19.** *The sequence* $[g_i]([v])$ *converges uniformly to* $\omega$ *on compact subsets of* $Gr \setminus E$.

To extend this to arbitrary (i.e. non-diagonal) matrices in $SL(n, F)$ we use the following construction.

**Definition 1.12.** If $g \in SL(n, F)$, $g$ has the *singular value decomposition*

$$g = U \cdot a(g) \cdot U'$$

where $U, U' \in K$ (where $K = SO(n)$ (resp. $SU(n)$) if $F = \mathbb{R}$ (resp. $\mathbb{C}$)) and $a(g)$ is a diagonal matrix with elements $\lambda_1 \geq \ldots \geq \lambda_n > 0$ which are the singular values of $g$.

**Lemma 1.20.** *Let $g_i \in SL(n,F)$. There exists a subsequence $g_{i_j}$ and a number $1 \le k < n$ such that*

$$\lim_{j\to\infty} \frac{\lambda_k(g_{i_j})}{\lambda_{k+1}(g_{i_j})} = \infty$$

**Corollary 1.21.** *There exists $\omega(g_i) \in Gr$ and $E(g_i) \subset Gr$ such that the sequence $g_i([v])$ converges to $\omega$ uniformly on compact subsets of $Gr \setminus E$.*

Note that in the general case $\omega$ may belong to $E$, unlike in the diagonal case.

**Exercise 1.22.** *Let $F^n = \mathbb{R}^2$. Identify $\mathbb{P}^{n-1}$ with the unit circle as before. Let*

$$g_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

*Then $E = \{(-1,0)\}$ and $\omega = (1,0)$.*

**Lemma 1.23.** *Let $\Gamma$ be a Zariski-dense subgroup of $SL(n,F)$. Let $g_i$ be a divergent sequence with $\omega, E$ as above. Then there exists $h \in \Gamma$ such that*

$$\omega \notin \{h(\omega)\} \cup h(E) \cup h^{-1}(E).$$

**Lemma 1.24.** *Let $\Gamma$ be a Zariski-dense unbounded subgroup of $SL(n,F)$. Then there exist divergent sequences $\beta_i, \gamma_i \in \Gamma$ such that*

(1) $\{\omega(\beta_i), \omega(\gamma_i)\} \cap (E(\beta_i) \cup E(\gamma_i)) = \emptyset$.
(2) $\omega(\beta_i) \neq \omega(\gamma_i)$.

Now apply the Ping lemma, and arrive at Theorem 1.12.

**Example 1.13.** Let

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The group $\langle g, h\rangle \subset GL_2(\mathbb{R})$ has exponential growth.

## 2. Growth

2.1. **The growth function.** Let $G$ be a group with a finite generating set $S$ (for simplicity we assume $S = S^{-1}$). We may then define a metric on $G$ by letting $d_G(g,h)$ be the least integer such that $gh^{-1}$ can be written as an $S$-word of length $d_G(g,h)$.

**Exercise 2.1.** *Show that this is actually a metric.*

We define the growth function of $G$ with respect to $S$ to be the function $g_{G,S} : \mathbb{N} \to \mathbb{R}$ defined by $g_{G,S}(n) = |B_n|$ where $B_n$ is a ball of radius $n$ centered at the identity element. This function gives us a method for determining the size of infinite groups by looking at the rates at which they grow.

**Definition 2.1.** A function $f : \mathbb{N} \to \mathbb{N}$ has *polynomial growth of degree $d$* if there exist $C_1, C_2 \in \mathbb{R}_+$ such that $f(n) \ge C_1 n^{C_2 d}$. It has *exponential growth* if there exist $C_1, C_2 \in \mathbb{R}_+$ such that $f(n) \ge C_1 e^{C_2 n}$. It has *intermediate growth* if it does not have polynomial or exponential growth.

**Definition 2.2.** We say a group has *polynomial (resp. exponential, intermediate) growth* with respect to the generating set $S$ if$g_{G,S}$ has polynomial (resp. exponential, intemediate) growth.

The following proposition shows that the property of having polynomial growth or exponential growth does not actually depend on the choice of generating set.

**Proposition 2.2.** *Let $S, T$ be finite generating sets of the group $G$. There exist $C, C' \in \mathbb{N}$ such that $g_{G,S}(n) \leq g_{G,T}(Cn)$ and $g_{G,T}(n) \leq g_{G,S}(C'n)$. Thus $g_{G,S}$ has polynomial (resp. exponential) growth if and only if $g_{G,T}$ has polynomial (resp. exponential) growth.*

**Exercise 2.3.** *Let $F_n$ be the free group on $n$ generators. Show that $F_n$ has exponential growth for $n \geq 2$. Show that every abelian group has polynomial growth.*

Thus we can see that abelian groups are relatively small whereas free groups, which are far from abelian are very large.

2.2. **Solvable, polycyclic, and nilpotent groups.** We first define several series of subgroups:

**Definition 2.3.** Recall the *derived series* of a group $G$

$$G^0 = G \supset G^1 = [G^0, G^0] \supset G^2 = [G^1, G^1] \supset \cdots \supset G^i = [G^{i-1}, G^{i-1}] \supset \cdots .$$

The *lower central series* is the series

$$G_0 = G \supset G_1 = [G_0, G] \supset G_2 = [G_1, G] \supset \cdots \supset G_i = [G_{i-1}, G] \supset \cdots .$$

The *upper central series* is the series

$$Z_0(G) = \{1\} \subset Z_1(G) = Z(G) \subset \cdots \subset Z_i(G) = \{g \in G | [g, G] \subset Z_{i-1}(G)\} \subset \cdots .$$

**Exercise 2.4.** *Show that each $G^i$, $G_i$ and $Z_i(G)$ is charceristic in $G$.*

**Exercise 2.5.** *$Z_{i+1}/Z_i$ is contained in the center of $Z_{i+2}/Z_i$.*

**Definition 2.4.** Recall that a group $G$ is *solvable* if $G^n$ is trivial for some $n$. A group $G$ is *polycyclic* if it is solvable and each $G^i/G^{i+1}$ is finitely generated for every $0 \leq i \leq n-1$. A group $G$ is *nilpotent* if $G_s$ is trivial for some $s$. and in this case we say $G$ is $s$-step nilpotent.

**Exercise 2.6.** *Show that $G$ is nilpotent if and only if $Z_l(G) = G$ for some $l$.*

**Example 2.5.** Show that $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ (where $\mathbb{Z}/\mathbb{Z}$ acts on $\mathbb{Z}$ by negation) is solvable.

**Example 2.6.** Let $BS(2,1) = \langle a, b | aba^{-1} = b^2 \rangle$. Show that $BS(2,1)$ is solvable, but not polycyclic (Hint: The commutator subgroup is isomorphic to the diadic rationals).

**Example 2.7.** Let $H(n)$ be the group of strictly upper triangular $n$ by $n$ matrices with integer coefficients. Show that $H(n)$ is nilpotent.

**Exercise 2.7.** *Let*

$$G = H_0 \supset H_1 \supset \cdots \supset H_l \supset H_{l+1} = \{1\}$$

*be a sequence of subgroups in $G$ such that $H_{i+1}$ is normal in $H_i$. Show that if each $H_i/H_{i+1}$ is abelian then $G$ is solvable. Show that if each $H_i/H_{i+1}$ is finitely generated abelian then $G$ in polycyclic.*

**Exercise 2.8.** *If $G$ is finitely generated milpotent, then $G_i/G_{i+1}$ is finitely generated abelian of all $i$. Thus we have the following containments of sets of groups:*

$$\{\text{abelian}\} \subset \{\text{nilpotent}\} \subset \{\text{polycyclic}\} \subset \{\text{solvable}\}.$$

**Exercise 2.9.** *Every polycyclic group has finite presentation.*

Our main goal is to completely classify the growth of solvable groups. This will be done by showing that all solvable groups have either polynomial or exponential growth and specifying exactly when each happens. We begin with nilpotent groups.

2.3. **Growth of nilpotent groups.** We begin by proving that if $\Gamma$ is a nilpotent group then $\Gamma$ has polynomial growth. This is proved by induction on the length of the lower central series and we have already taken care of the base case ($\Gamma$ is 1-step nilpotent). However, we first need to introduce the idea of distortion:

**Definition 2.8.** Let $H$ be a finitely generated subgroup of $\Gamma$ and let $S$ be a generating set for $\Gamma$ which contains the set $T$ which generates $H$. The distortion function is defined by $\delta(\Gamma : H, n) = \max\{d_H(e, h) : h \in B_\Gamma(e, n) \cap H\}$.

**Example 2.9.** Let $\Gamma = BS(2, 1)$ and $H$ be the subgroup generated by $b$. Show that $\delta(\Gamma : H, n)$ has exponential growth.

**Fact 3.** If $\Gamma$ is nilpotent, then every subgroup $H$ of $\Gamma$ has a distortion function satisfying $\delta(\Gamma : H, n) \leq cn^d$ for some constants $c, d$.

Use this to prove:

**Theorem 2.10.** *Every nilpotent group has polynomial growth.*

Hint: Assume the result holds for all $s - 1$ step nilpotent groups. Then if $G$ is $s$-step nilpotent, $G_1$ is $s - 1$-step nilpotent, and it's distortion is bounded by a polynomial equation.

**Corollary 2.11.** *If a group $\Gamma$ is virtually nilpotent, then $\Gamma$ has polynomial growth.*

However, not all solvable groups have polynomial growth.

2.4. **Solvable groups which are not polycyclic.**

**Example 2.10.** We define $L$, *the lamplighter group*, by

$$L = \langle a, t | a^2, [t^n a t^{-n}, t^m a t^{-m}]\, \forall n, m \in \mathbb{Z}\rangle.$$

$L$ is called the lamplighter group because if we imagine $\mathbb{Z}$ with a lamp at each integer and a lamplighter at the origin, then $t$ acts by moving the lamplighter up one and $a$ acts by hoving him switch the light he is on. Defined more rigorously, $L$ acts on $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ by $t : (x, y) \mapsto (x + 1, y)$, $a : (x, y) \mapsto (x, 1 - y)$.

**Exercise 2.12.** *Show that $L$ is solvable and has exponential growth. (Hint: Look at elements of the form $ta^{e_1}ta^{e_2}t \cdots ta^{e_n}t^{-n}$ where each $e_i$ is either 0 or 1).*

Notices that one of the subgroups in the derived series for $L$ is infinitely generated and therefore $L$ is not polycyclic. We have show that virtual nilpotence is a sufficient condition for a solvable group to have polynomial growth. As it turns out, this condition is also necessary. We continue by showing that all solvable groups which are not virtually nilpotent have exponential growth, but first we show that all groups which are not polycyclic have exponential growth.

**Exercise 2.13.** *A group $\Gamma$ is polycyclic if and only if there is a chain of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_s = 1$ such that $\Gamma_i/\Gamma_{i-1}$ is cyclic.*

Once again we prove the result by induction, but this time on the length of the derived series and use the following lemma (which you will prove later):

**Lemma 2.14.** *Suppose we have a short exact sequence*

$$A \to B \to C$$

*such that $C$ is polycyclic, $B$ is finitely generated and does not have exponential growth, and $A$ is abelian, but not finitely generated. Then $B$ is polycyclic.*

Show the lemma implies:

**Theorem 2.15.** *Let $\Gamma$ be a solvable group which is not polycyclic. Then $\Gamma$ has exponential growth.*

You will find the following propositions helpful in proving the lemma (in all the propositions, $A, B, C$ are as above):

**Proposition 2.16.** *Given $b \in B$ and $x \in A$ let $x_i = b^i x b^{-i}$ for all $i \in \mathbb{Z}$. The subgroup generated by all the $x_i$ is finitely generated.*

Hint: Look at elements of the form $bx^{e_1}bx^{e_2}b \cdots x^{e_n}b^{-n}$ and use the fact that $B$ does not have exponential growth to get a relation. Notice how this is similar to the argument used in showing that the lamplighter group has exponential growth.

**Proposition 2.17.** *There exist generaters $c_1, \dots, c_n$ such that every element of $C$ can be written in the form $c_1^{i_1} \cdots c_n^{i_n}$ with each $i_j \in \mathbb{Z}$.*

**Proposition 2.18.** *There exist $a_1, \dots, a_k \in A$ such that every element of $A$ is a product of conjugates of the $a_i$ (by elements in $B$). (Hint: This hinges on the fact that polycyclic groups are finitely presented).*

**Exercise 2.19.** *Prove Lemma 2.14 (Hint: Let $b_1, \dots, b_l \in B$ where $b_j$ maps to $c_j$ let $A_0$ be the subgroup of $A$ generated by the $a_i$. Then let $A_1$ be the subgroup of $A$ genetated by elements of the form $b_1^{i_1} a_j b_1^{-i_1}$ and continue to get a sequence $A_0 \subset A_1 \subset \dots \subset A_n = A$ where $A_l$ is generated by the set*

$$\{b_l^{i_l} b_{l-1}^{i_{l-1}} \cdots b_1^{i_1} a_j b_1^{-i_1} \cdots b_{l-1}^{i_{l-1}} b_l^{i_l} | \forall (i_1, \dots i_l) \in \mathbb{Z}^l,\ 1 \le j \le k\}.$$

*Use this to show $A$ is polycyclic and therefore $B$ is polycyclic.*

**Corollary 2.20.** *$BS(2,1)$ has exponential growth.*

**Exercise 2.21.** *Given $p \in \mathbb{Z}$, let $BS(p,1) = \langle a, b | aba^{-1} = b^p \rangle$. Directly show that $BS(p,1)$ has exponential growth by using a method similar to that used for the Lamplighter group.*

2.5. **Polycyclic groups.** Now we must take care of the case in which $\Gamma$ is polycyclic, but first let us examine some more examples.

**Example 2.11.** Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$ where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by multiplication by $A$. Show that $\Gamma$ is polycyclic and has exponential growth. (Hint: We once again want to look at elements of the form $ba^{e_1}b \cdots ba^{e_n}b^{-n}$, but what are $a$ and $b$ in this case?)

**Example 2.12.** In general, we may construct a group just as in Example 2.11 but with $\mathbb{Z}^n$ for any $n$ and any $n$ by $n$ invertible matrix instead of $A$. Such a group with $n$ by $n$ matrix $M$ is denoted by $BS(M, n)$. In fact, the matrix need only have nonzero determinant, so $BS(2, 1)$ is just a case of this.

**Exercise 2.22.** *Show that $BS(M, n)$ is polycyclic if and only if* $\det(M) = \pm 1$.

Now we need some propositions about polycyclic groups. In these propositions, $\Gamma$ is a polycyclic group with derived series

$$\Gamma = \Gamma^0 \supset \dots \Gamma^d \supset \Gamma^{d+1} = \{1\}.$$

**Proposition 2.23.** *Every subgroup of $\Gamma$ is finitely generated.*

Hint: First show that $\Gamma$ satisfies the ascending chain condition, that is every chain $H_0 \subset H_1 \subset \dots$ of subgroups of $\Gamma$ is eventually constant (This is trivial for $\Gamma^d$ and $\Gamma^{d-1}/\Gamma^d$ and you can use induction to go up the chain).

The following proposition gives us an exact sequence $H \to \Gamma^* \to F$ (with $\Gamma^*$ finite index in $\Gamma$) and our main strategy will be to show that if $\Gamma$ is not virtually nilpotent, then the elements of $F$ act on $H$ in such a way that we can construct distinct elements as we did for the lamplighter group, $BS(A, 2)$, $BS(p, 1)$ and in Proposition 2.16. The statements following the proposition will guide you through the proof.

**Proposition 2.24.** *There is a short exact sequence $H \to \Gamma^* \to F$ such that $H$ is finitely generated nilpotent, $F$ is finitely generated free abelian and $\Gamma^*$ is a torsion free finite index subgroup of $\Gamma$.*

$\Gamma^*$ is constructed as follows: First show that there exists a chain of subgroups (all normal in $\Gamma$) $\Gamma = A_0 \supset A_1 \supset \dots \supset A_{n+1} = 1$ such that each quotient $A_i/A_{i+1}$ is either finitely generated torsion-free abelian or finite. Now define a descending chain $B_i$ inductively, beginning with $B_0 = \Gamma$. Given $B_i$, we let it act on $A_i$ by conjugation and this induces an action of $B_i$ on $A_i/A_{i+1}$. If $A_i/A_{i+1}$ is finite, we let $B_{i+1}$ be the subgroup that acts trivially. If $A_i/A_{i+1}$ is finitely generated torsion-free abelian, then the action of $B_i$ can be represented by matrices over $\mathbb{Z}$. We now use the following fact:

**Fact 4.** Any solvable subgroup of $GL_n(\mathbb{C})$ has a finite index subgroup which can be simultaneously reduced to triangular form.

Using this lemma we set $B_{i+1}$ to be the preimage of this finite index subgroup in $B_i$. We then set $\Gamma^* = B_{n+1}$. Letting $H = [G^*, G^*]$ and $F = H$ it suffices to show that $H$ is nilpotent (why?).

We proceed by refining the chain $A_i$:

**Exercise 2.25.** *Show that for each $i$ such that $A_i/A_{i+1}$ is not finite, there exists a chain of subgroups*

$$A_i/A_{i+1} = C_{i,0}/A_{i+1} \supset C_{i,1}/A_{i+1} \supset \dots \supset C_{i,n}/A_{i+1} \supset C_{i,n+1}/A_{i+1} = A_{i+1}/A_{i+1}$$

*invariant under the action of $H$ such that $C_{i,j}/C_{i,j+1}$ has rank 1 and the action of $H$ on $C_{i,j}/C_{i,j+1}$ is trivial.*

Now let the chain $\Gamma = D_0 \supset D_1 \supset \dots \supset D_{l+1} = \{1\}$ be the refinement of the chain $A_i$ that is acheived by including the $C_{i,j}$.

**Exercise 2.26.** *Show that the sequence $H = N_0 \supset N_1 \supset \ldots \supset N_{l+1} = \{1\}$ where $N_i = H \cap D_i$ is a lower central series for $H$ and thus $H$ is nilpotent.*

This is fine except that we have not show that $\Gamma^*$ is torsion free. The following facts allow us to assume $\Gamma^*$ is torsion free:

**Fact 5.** Every polycyclic group is isomorphic to a subgroup of $GL_n(\mathbb{Z})$ for some $n$.

**Fact 6.** Any finitely generated group of matrices over a field of characteristic 0 has a finite index torsion free normal subgroup.

This completes the proof of Proposition 2.24.

To finish our proof that polycyclic groups do not have intermediate growth, we must view them as subgroups of a Lie group. We fix the sequence $H \to \Gamma^* \to F$ which was proven to exist in Proposition 2.24. As in many our earlier examples, we will have $F$ act on $H$, but unlike the case of $BS(2,1)$ or $BS(A,2)$ we will not be able to directly show directly that all elements of the form $xy^{e_1}xy^{e_2}\cdots xy^{e_n}x^{-n}$ are distinct when $\Gamma^*$ is not virtually nilpotent. To find $x$ and $y$ and show that each of these elements are distinct, we must look at the Lie group where $H$ lives and let $F$ act on the Lie group. Because the Lie algebra is a vector space, we can talk about the eigenvalues of this action and this will allow us to choose $x$ and $y$ with convenient eigenvalues. Because every polycyclic group lives in some $GL_n(\mathbb{Z})$, we have the following:

**Fact 7.** Every polycyclic group $\Gamma$ is isomorphic to a discrete subgroup of a connected solvable Lie group $G$. The Lie group may be assumed to be nilpotent if $\Gamma$ is nilpotent and if $\Gamma$ is torsionfree there is a unique such $G$ under the added condition that $G/\Gamma$ is compact and $G$ is simply connected. In this case every automorphism of the $\Gamma$ extends to a unique automorphism of $G$.

Let $G$ be the unique connected simply connected nilpotent Lie group which contains $H$ as a discrete subgroup and has $G/H$ compact. For all $g \in \Gamma^*$, we have an automorphism of $H$ defined by $h \mapsto ghg^{-1}$ and we let $\zeta(g)$ be the unique extension of this automorphism to all of $G$. In addition, we define $\zeta_*(g)$ to be the induced automorphism of $\mathfrak{g}$, the Lie algebra of $G$. We now examine the connection between the eigenvalues of this action and the virtual nilpotence of $\Gamma^*$:

Let $N$ be a finite index nilpotent subgroup of $\Gamma$. First we can take the intersection of $N$ with all its conjugates and assume $N$ is normal. This implies a$\Gamma^* \cap N$ is a finite index nilpotent normal subgroup of $\Gamma^*$ (why?). Let $\Gamma^* \cap N = L_0 \supset L_1 \supset \ldots \supset L_k \supset L_{k+1} = \{1\}$ be the upper central series for $\Gamma^* \cap N$ and show that for $g \in \Gamma^* \cap N$ and $h \in L_i$ we have $ghg^{-1} = hh'$ for some $h' \in L_{i+1}$. If we let $H_i = H \cap L_i$ then there is a unique (sufficiently nice) subgroup $G_i$ of $G$ such that $G_i/H_i$ is compact. Then for all $g \in \Gamma^* \cap N$ and $d \in G_i$, $\zeta(g)d = dd'$ for some $d' \in G_{i+1}$ and thus $\zeta_*(g)$ preserves each $\mathfrak{g}_i$ and is the identity on each $\mathfrak{g}_i/\mathfrak{g}_{i-1}$. Thus we may conclude:

**Lemma 2.27.** *If $N$ is a finite index nilpotent subgroup of $\Gamma$, then for every $g \in G^*$, all the eigenvalues of $\zeta_*(g)$ have absolute value 1.*

It turns out that if we let $U$ be the set of $g \in \Gamma^*$ such that every eigenvalue of $\zeta_*(g)$ is 1, then $U$ is nilpotent. This can be proved by induction on $\dim G$ using the follwing facts: If $\dim G = 0$, then $H$ is trivial. If $\dim G = 1$ then $H$ is in the center of $U$. Otherwise we may refine the the lower central series of $G$ such that each subgroup is $\zeta(U)$ stable and each quotient is one dimensional.

**Proposition 2.28.** *Let $U = \{g \in \Gamma^* :$ every eigenvalue of $\zeta_*(g)$ is 1$\}$ is nilpotent.*

**Exercise 2.29.** *Use the Jordan canonical form to show that for all $g \in \Gamma^*$ there exists linear transformations $A_g$ and $U_g$ which are given by polynomials in $\zeta_*(g)$ such that $A_g$ is diagonalizable over $\mathbb{C}$, all the eigenvalues of $U_g$ are 1, and $A_g U_g = \zeta_*(g) = U_g A_g$.*

Assuming $\zeta_*(\Gamma^*)$ is simultaneously diagonalizable, prove the following:

**Lemma 2.30.** *The map defined by $g \mapsto A_g$ is a homomorphism from $\Gamma^*$ into the automorphisms of $\mathfrak{g}$ and its kernel is $U$. In addition, $A = \Gamma^*/U$ is a discrete subgroup of the automorphism group of $\mathfrak{g}$.*

**Exercise 2.31.** *Use the lemma to prove that if for all $g \in \Gamma^*$ all the eigenvalues of $\zeta_*(g)$ have absolute value 1, then $A$ is finite.*

This gives us:

**Proposition 2.32.** *$\Gamma$ has a finite index nilpotent subgroup if and only if for every $g \in \Gamma^*$ all the eigenvalues of $\zeta_*(g)$ have absolute value 1.*

We can now complete our analysis of the growth of solvable groups. The rest of the sheet will prove:

**Theorem 2.33.** *If $\Gamma$ is a polycyclic group that is not virtually nilpotent, then $\Gamma$ has exponetial growth.*

In this case there is some $g \in \Gamma$ with an eigenvalue $\lambda$ such that $|\lambda| \neq 1$. If we let $G = G_0 \supset G_1 \supset \ldots \supset G_s \supset G_{s+1} = \{1\}$ be the lower central series for $G$, then the induced automorphism on one of the $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ has $\lambda$ as an eigenvalue. Letting $\beta$ denote the action of $\zeta_*(g)$ on the Lie algebra $\mathfrak{h}$ of $\Sigma := (H \cap \Gamma_i)/(H \cap \Gamma_{i+1})$ (which is a vector space group), we may assume that $|\lambda| > 1$ and if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of $\beta$ then $|\lambda| \geq |\lambda_i|$ for all $i$ (why?). Let $\mathfrak{h}^{\mathbb{C}}$ denote the complex vector space with the same basis as $\mathfrak{h}$. Let $V = \{v_1, \ldots, v_k\}$ be a minimal generating set for $\Sigma$ and $\{Z_1, \ldots, Z_k\}$ be the basis of $\mathfrak{h}$ such that $\exp(Z_i) = v_i$.

**Exercise 2.34.** *Let $L \in \mathfrak{h}^{\mathbb{C}}$ be an eigenvector of $\beta$ with eigenvalue $\lambda$. If $\lambda$ is real then we may assume $L \in \mathfrak{h}$. Otherwise, $\lambda = re^{i\theta}$ and $L + \bar{L}$ and $i(L - \bar{L})$ generate a two dimensional subspace of $\mathfrak{h}$ on which $\beta$ is the product of a rotation by angle $\theta$ and scaling by $r$.*

If $L \in \mathfrak{h}$ we take an inner product on $\mathfrak{h}$ such that $||Z_j|| \geq 1$ for all $1 \leq j \leq k$ and let $X_0 = L$. If $L \notin \mathfrak{h}$ we take an inner product on $\mathfrak{h}$ such that $||Z_j|| \geq 1$ for all $1 \leq j \leq k$, but we also require that $\langle L + \bar{L}, i(L - \bar{L})\rangle = 0$ and in this case set $X_0 = L + \bar{L}$. Let $b = \max\{||Z_j||\}$ and replace $g$ by some power of itself such that $\lambda = re^{i\theta}$ with $r > 12b$ and $|\theta| < 10^{-3}$.

**Exercise 2.35.** *Show $\beta(X_0) = \rho_0 X_0 + Y_0$ where $\langle X_0, Y_0\rangle = 0$, $\rho_0 > 11b$, and $||Y_0|| < 10^{-2}\rho_0||X_0||$.*

Let $X$ be an approximation of $X_0$ by a rational linear combination of $Z_j$ and let $n$ be an integer such that $||nX|| \geq b$ and let $\sigma = \exp(nX)$.

**Exercise 2.36.** *Show $\beta(X) = \rho X + Y$ with $\rho > 11b$, $\langle X, Y\rangle = 0$, and $||Y|| < 10^{-1}\rho||X||$. Show $||\beta^i X|| > 10b||\beta^{i-1}X||$ for every $i \geq 1$.*

Let $\Psi = \langle g \rangle \ltimes \Sigma$ where $g$ acts on $\Sigma$ by conjugation. We will find the growth of $\Psi$ with respect to the generating set $\{g, v_1, \ldots, v_k\}$. Given an integer $p$, for all integers $q$ with decimal expansion of the form $k_0 + k_1 10 + \ldots + k_p 10^p$ we associate the element

$$\sigma_q = \sigma^{k_0}(g\sigma^{k_1}g^{-1}) \cdots (g^p \sigma^{k_p} g^{-p}).$$

**Exercise 2.37.** *Show that each of the $\sigma_q$ are distinct. Letting $l$ be the length of $\sigma$ with respect to the generating set given, show that $\sigma_q$ is a word of length $\leq l(k_0 + \ldots + k_p) + 2p$.*

Now complete the proof of Theorem 2.33.