COMPLEX LIE ALGEBRAS

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Abstract. We prove that every Lie algebra can be decomposed into a solvable Lie algebra and a semisimple Lie algebra. Then we show that every complex semisimple Lie algebra is a direct sum of simple Lie algebras. Finally, we give a complete classification of simple complex Lie algebras.

Contents

1. Lie algebras

Definition 1.1. A *Lie algebra* L over a field F is a finite dimensional vector space over F with a bracket operation $[-,-] : L \times L \to L$ with the following properties for all $x, y, z \in L$, $a \in F$:
i) bilinearity: $[ax, y] = a[x, y] = [x, ay]$, $[x + y, z] = [x, z] + [y, z]$, $[x, y + z] = [x, y] + [x, z]$
ii) anti-symmetry: $[x, x] = 0$
iii) the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Lemma 1.1. *For all $x, y \in L$, $[x, y] = -[y, x]$ and $[x, 0] = 0$.*

Example 1.2. Let F be a field and let $\mathfrak{gl}(n, F)$ be the algebra of n by n matrices over F. If we define $[-,-] : \mathfrak{gl}(n, F) \times \mathfrak{gl}(n, F) \to \mathfrak{gl}(n, F)$ by $[x, y] \mapsto xy - yx$ then this makes $\mathfrak{gl}(n, F)$ a Lie algebra over F. Given any vector space V over F we may similarly define the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of V.

Definition 1.3. A *subalgebra* of a Lie algebra L is a linear subspace $K \subset L$ such that $[x, y] \in K$ for all $x, y \in K$.

Definition 1.4. An *ideal* of a Lie algebra L is a linear subspace $I \subset L$ such that $[x, y] \in I$ for all $x \in L, y \in I$.

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Definition 1.5. The *center* of a Lie algebra is the set

$$Z(L) := \{x \in L : [x, y] = 0, \forall y \in L\}.$$

L is *abelian* if $Z(L) = L$.

Definition 1.6. A Lie algebra is *simple* if it has no non-trivial subalgebras and is not abelian.

Example 1.7. Let $\mathfrak{sl}(n, F)$ be the subset of $\mathfrak{gl}(n, F)$ consisting of matrices with trace 0. This is an ideal of $\mathfrak{gl}(n, F)$.

Proof. Clearly $\mathfrak{sl}(n, F)$ is a linear subspace. Let $a = (a_{ij}), b = (b_{ij}) \in \mathfrak{gl}(n, F)$. Then

$$\operatorname{tr} ab = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}b_{lk} = \sum_{l=1}^{n}\sum_{k=1}^{n} b_{lk}a_{kl} = \operatorname{tr} ba.$$

Thus $\operatorname{tr}(ab - ba) = 0$ for all $a, b \in \mathfrak{gl}(n, F)$ so in particular $[a, b] \in \mathfrak{sl}(n, F)$ for all $a \in \mathfrak{gl}(n, F), b \in \mathfrak{sl}(n, F)$. $\square$

Definition 1.8. Given Lie algebras L_1, L_2, a homomorphism of Lie algebras from $L_1 \to L_2$ is a linear transformation $\phi : L_1 \to L_2$ such that $[\phi(x), \phi(y)] = \phi([x, y])$ for all $x, y \in L_1$.

Definition 1.9. A *representation* of L consists of a vector space V over F and a homomorphism of Lie algebras $\phi : L \to \mathfrak{gl}(V)$.

Definition 1.10. The *adjoint representation* of L is the representation $\operatorname{ad} : L \to \mathfrak{gl}(L)$ defined by $\operatorname{ad} a : b \mapsto [a, b]$.

Definition 1.11. The *derived series* of a Lie algebra L is the series

$$L \supset L' = L^{(1)} = [L, L] \supset L^{(2)} = [L^{(1)}, L^{(1)}] \supset \cdots \supset L^{(n)} = [L^{(n-1)}, L^{(n-1)}] \supset \cdots .$$

The *lower central series* is the series

$$L \supset L' = L^1 = [L, L] \supset L^2 = [L, L^1] \supset \cdots \supset L^n = [L, L^{n-1}] \cdots .$$

Definition 1.12. A Lie algebra L is *solvable* if $L^{(n)} = 0$ for some n. L is *nilpotent* if $L^n = 0$ for some n.

Proposition 1.2. *A Lie algebra L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_n = 0$ such that L_i/L_{i+1} is abelian for all $0 \leq i \leq n$.*

Proof. Since $[L^{(i)}, L^{(i)}] \in L^{(i+1)}$ we have $L^{(i)}/L^{(i+1)}$ abelian and thus if L is solvable we may take the derived series as our chain. If such a chain exists, then we have $L^{(1)} \subset L_1$ because $L_0/L_1 = L/L_1$ abelian implies $[L, L] \subset L_1$. Similarly, if $L^{(i)} \subset L_i$ then we must have $L^{(i+1)} \subset L_{i+1}$. Thus by induction $L^{(i)} \subset L_i$ for all $i \geq 1$ which implies $L^{(n)} = 0$ therefore L is solvable. $\square$

Corollary 1.3. *If a Lie algebra L is nilpotent then L is solvable.*

Proposition 1.4. *There exists a unique solvable ideal I of L such that any solvable ideal $J \subset L$ is contained in I.*

Proof. Let I be a solvable ideal of L such that $\dim J \leq \dim I$ for any solvable ideal J. Suppose J is a solvable ideal of L not contained in I. Then $I+J$ is an ideal with dimension strictly greater than $\dim I$. $(I+J)/I \simeq J/(I \cap J)$ thus the solvability of J implies $(I+J)/I$ is solvable. Thus we have

$$I + J = K_0 \supset K_1 \supset \cdots \supset K_n = I$$

such that K_i/K_{i+1} is abelian for all i. Setting $K_i = I^{(i-n)}$ for $i > n$ then gives us a sequence

$$I + J = K_0 \supset K_1 \supset \cdots \supset K_l = 0$$

with K_i/K_{i+1} abelian for all i. Thus $I+J$ is a solvable ideal with $\dim I+J > \dim I$ and we have a contradiction. □

Definition 1.13. The unique solvable ideal of L containing all other solvable ideals is called the *radical* of L and denoted by $\operatorname{rad} L$.

Definition 1.14. A Lie algebra L is *semisimple* if $\operatorname{rad} L = 0$.

Thus every Lie algebra L fits into an exact sequence

$$0 \to I \to L \to K \to 0$$

where I is solvable and K is semisimple. Thus we may focus on studying solvable and semisimple Lie algebras. We will use this to classify complex Lie algebras, that is, Lie algebras over $\mathbb{C}$, so for the rest of the paper L will denote a complex Lie algebra.

2. The Killing Form and Cartan's Criterion

The Killing form is a symmetric bilinear form on Lie algebras that will allow us to determine when Lie algebras are semisimple or solvable.

Definition 2.1. The *Killing form* of a Lie algebra L is the symmetric bilinear form $\kappa_L(-,-) : L \times L \to F$ defined by

$$\kappa_L(a, b) = \operatorname{tr}(\operatorname{ad} a \circ \operatorname{ad} b).$$

Proposition 2.1. *If I is an ideal of L and $a, b \in I$ then $\kappa_I(a, b) = \kappa_L(a, b)$.*

We will usually denote the Killing form by simply κ when the algebra is not ambiguous.

The Killing form allows us to determine when a Lie algebra is semisimple or solvable:

Theorem 2.2 (Cartan's Criterion)**.** *Let L be a complex Lie algebra. L is a semisimple Lie algebra if and only if the Killing form is non-degenerate. L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$, $y \in L'$.*

First we will use the semisimple part of Cartan's Criterion to prove that any complex semisimple Lie algebra is a direct sum of simple Lie algebras.

Definition 2.2. Given a subset $S \subset L$, the *perpendicular space of S* is the subset

$$S^\perp := \{x \in L : \kappa(s, x) = 0, \forall s \in S\}.$$

Lemma 2.3. *Given an ideal $I \subset L$, $I^\perp$ is an ideal.*

Proof. Let $a \in I^\perp, x \in L$. We need to show that $[x,a] \in I^\perp$, that is $\kappa(b,[x,a]) = 0$ for all $b \in I$. But

$$\kappa(b,[x,a]) = \mathrm{tr}(\mathrm{ad}\, b \circ \mathrm{ad}[x,a]) = \mathrm{tr}(\mathrm{ad}\, b \circ [\mathrm{ad}\, x, \mathrm{ad}\, a]) =$$
$$\mathrm{tr}([\mathrm{ad}\, b, \mathrm{ad}\, x] \circ \mathrm{ad}\, a) = \mathrm{tr}(\mathrm{ad}[b,x] \circ \mathrm{ad}\, a) = \kappa([b,x],a) = 0$$

because $[b,x] \in I$. □

Theorem 2.4. *A complex Lie algebra L is semisimple if and only if $L = L_1 \oplus \cdots \oplus L_n$ where L_i is a simple ideal of L for all i.*

Proof. Suppose L is semisimple. We induct on the dimension of L to get a decomposition into simple Lie algebras. Let $I \subset L$ be an ideal of the smallest positive dimension. If $I = L$ then L is simple and we are done. Otherwise we note that the restriction of the Killing form to $I \cap I^\perp$ is zero, so $I \cap I^\perp$ is solvable. Since L is semisimple this implies $I \cap I^\perp = 0$. Since $\dim I + \dim I^\perp = \dim L$ we have $I \oplus I^\perp = L$. By the induction hypothesis, $I = L_1 \oplus \cdots \oplus L_n$ and $I^\perp = L_{n+1} \oplus \cdots \oplus L_m$ where each of the L_i are simple ideals of I for $1 \le i \le n$ and the L_j are simple ideals of $I^\perp$ for $n+1 \le j \le m$. Since $[a,b] = 0$ for all $a \in I$, $b \in I^\perp$, L_i is an ideal of L for all $1 \le i \le m$. Thus $L = L_1 \oplus \cdots \oplus L_m$, a direct sum of simple ideals.

If L is a direct sum of simple ideals $L_1 \oplus \cdots \oplus L_n$ then we let $I = \mathrm{rad}\, L$. For all i, $[I, L_i]$ is contained in I so it is solvable ideal of L_i. Since L_i is simple, $[I, L_i] = 0$ or $[I, L_i] = L_i$. But in the later case, L_i would be solvable which would imply $L_i' = 0$ and this is impossible because simple Lie algebras were defined to be non-abelian. Thus $[I, L_1] \oplus \cdots \oplus [I, L_n] = 0$ so $[I, L] = 0$ and therefore $I \subset Z(L)$. However $Z(L) = Z(L_1) \oplus \cdots \oplus Z(L_n) = 0$ so $\mathrm{rad}\, L = 0$. □

3. Root Space Decomposition

Theorem 3.1 (Jordan Decomposition). *Given $x \in L$, x can be uniquely written as $x = d + n$ with $\mathrm{ad}\, d$ diagonalizable, $\mathrm{ad}\, n$ nilpotent and $[d,n] = 0$.*

Definition 3.1. An element $x \in L$ is *semisimple* if $\mathrm{ad}\, x$ is diagonalizable.

Definition 3.2. A *Cartan subalgebra* of L is an abelian subalgebra H consisting of semisimple elements and H is not properly contained in an abelian subalgebra of semisimple elements.

Example 3.3. The diagonal matrices form a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

Definition 3.4. Given a subalgebra $A \subset L$, a *weight* is an element $\alpha \in A^*$ such that

$$L_\alpha := \{x \in L : [a,x] = \alpha(a)x, \forall a \in A\}$$

is nonzero. L_α is called the corresponding *weight space.*

If H is a Cartan subalgebra of L then since the elements of H are semisimple and H is abelian, the elements of $\mathrm{ad}\, H$ are simultaneously diagonalisable. Thus we have:

Proposition 3.2. *If H is a Cartan subalgebra of L, then $L = L_{\alpha_1} \oplus \cdots \oplus L_{\alpha_n}$ where $\alpha_1, \ldots, \alpha_n$ are the roots of H.*

This is called the *root space decomposition* of L.

Lemma 3.3. *If L is a complex semisimple Lie algebra, then L contains a non-zero Cartan subalgebra.*

Thus we can study semisimple Lie algebras by looking at root space decompositions. Note that the weight space L_0 (corresponding to the zero map $H \to \mathbb{C}$) is just the set of elements $x \in L$ such that $[h,x] = 0$ for all $h \in H$. Since H is abelian, $H \subset L_0$ and in fact it can be shown that $H = L_0$. Thus if we let Φ be the set of nonzero roots our root space decomposition takes the form:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

It turns out that Φ spans H^*, so we may choose a basis $\{\alpha_1, \ldots, \alpha_n\}$ for H^* consisting of roots.

Proposition 3.4. *If β is a root, then $\beta = \sum a_i \alpha_i$ where $a_i \in \mathbb{Q}$ for all i.*

Thus although H^* is a vector space over $\mathbb{C}$, the real vector space spanned by $\{\alpha_1, \ldots, \alpha_n\}$ is independent of the choice of basis. This vector space is called E. We now define an inner product on E as follows:

Definition 3.5. Given $\theta \in E$ there is a unique $t_\theta \in H$ such that $\kappa(t_\theta, h) = \theta(h)$ for all $h \in H$. The inner product on E is defined by

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta).$$

Proposition 3.5. *Φ satisfies the following properties:*
i) Φ spans E and does not contain 0.
ii) Given $\alpha \in \Phi$, $c \in \mathbb{C}$, $c\alpha \in \Phi$ if and only if $c = \pm 1$.
iii) Given $\alpha \in \Phi$ the map $s_\alpha : E \to E$ defined by

$$s_\alpha : x \mapsto x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$$

permutes the elements of Φ.
iv) Given $\alpha, \beta \in \Phi$, $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Definition 3.6. Any real inner product space E with a subset Φ satisfying properties i,ii,iii and iv is called a *root system*.

Definition 3.7. Given root systems Φ and Φ' of vector spaces E and E' respectively, a *homomorphism* of root systems is an isometry $\phi : E \to E'$ such that $\phi(\Phi) \in \Phi'$.

Theorem 3.6. *If two complex Lie algebras have isomorphic root systems, then they are isomorphic.*

Thus we may classify complex Lie algebras by classifying root systems.

4. Classifying Root Systems

For this section, E will be a real inner product space and Φ will be a root system.

Lemma 4.1. *If $\alpha, \beta \in \Phi$ and $\alpha \neq \pm\beta$ then*

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \{0,1,2,3\}.$$

Proof. Let θ be the angle between α and β. Then $(\alpha,\beta) = \sqrt{(\alpha,\alpha)(\beta,\beta)}\cos\theta$, so

$$\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta < 4$$

because $\alpha \neq \pm\beta$. Since $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ and $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ are both integers (property iv) with the same sign, their product can only be 0, 1, 2, or 3. $\square$

As a special case of this, if the angle between α and β is strictly obtuse and $(\beta,\beta) \geq (\alpha,\alpha)$ then we must have $\frac{2(\alpha,\beta)}{(\beta,\beta)} = -1$, so property iii implies $\alpha+\beta \in \Phi$.

Definition 4.1. A root system Φ is irreducible if it cannot be written as the disjoint union of two sets Φ_1, Φ_2 where $(\alpha,\beta) = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$.

Lemma 4.2. *Every root system Φ is a disjoint union of finitely many sets $\Phi_1,\dots,\Phi_n$ where each Φ_i is a root system of the subspace it spans.*

We now define a base of a root system. We will be able to classify root systems by classifying their bases.

Definition 4.2. A subset $B \subset \Phi$ is a *base* for Φ if B is a basis for E and every $\alpha \in \Phi$ can be written as

$$\alpha = \sum_{\beta\in B} c_\beta \beta$$

where the c_β are integers and all have the same sign.

Lemma 4.3. *If α,β are elements of a base B for Φ then $(\alpha,\beta) \leq 0$*

Theorem 4.4. *Every root system has a base.*

Proof. If $\dim E = 1$ the result is clear. Otherwise we may choose some $v \in E$ such that $(v,\alpha) \neq 0$ for all $\alpha \in \Phi$. We then let $\Phi_+ = \{\alpha \in \Phi : (v,\alpha) > 0\}$ and define

$$B = \{\beta \in \Phi_+ : \beta \neq \alpha + \gamma \,\forall \alpha,\gamma \in \Phi_+\}.$$

Suppose there is some $\alpha \in \Phi_+ \setminus B$ such that α cannot be written as $\alpha = \sum_{\beta\in B} c_\beta\beta$ where all the c_β are integers ≥ 0. Then we may choose such an α with (v,α) minimal. Since $\alpha \notin B$, there exist $\beta_1,\beta_2 \in \Phi_+$ such that $\alpha = \beta_1+\beta_2$. One of the β_i cannot be written as $\sum_{\beta\in B} c_\beta\beta$ with all the c_β are nonnegative integers, but $(v,\alpha) = (v,\beta_1)+(v,\beta_2)$ contradicting the minimality of (v,α). Thus every $\alpha \in \Phi_+$ can be written as $\sum_{\beta\in B} c_\beta\beta$ and therefore every element of Φ can be written as an integral linear combination of elements of B with all coefficients of the same sign since $\alpha \in \Phi_+$ or $-\alpha \in \Phi_+$ for all $\alpha \in \Phi$.

Since Φ is in the span of B, B spans all of E. Suppose that $0 = \sum_{\beta\in B} c_\beta\beta$. Then we have some element w such that

$$w = \sum_{\beta\in B: c_\beta>0} c_\beta\beta = \sum_{\beta\in B: c_\beta<0} -c_\beta\beta.$$

But then

$$(w,w) = \sum_{\alpha,\beta: c_\alpha>0, c_\beta<0} c_\alpha(-c_\beta)(\alpha,\beta) \leq 0$$

so $w = 0$. Thus for each $\beta \in B$, $c_\beta = (\beta,w) = 0$. Therefore B is a basis for E. $\square$

Definition 4.3. Given a root system Φ the *Weyl group* of Φ, denoted by $W(\Phi)$, is the group generated by the reflections

$$s_\alpha : x \mapsto x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$$

with $\alpha \in \Phi$.

Lemma 4.5. *If B is a base for Φ then every element of Φ can be written as $g(\beta)$ with $g \in W(\Phi), \beta \in B$. Furthermore, $W(\Phi)$ is generated by $W_0 = \{s_\beta : \beta \in B\}$.*

Thus we can always reconstruct the root system from a base.

Theorem 4.6. *Given two bases for Φ, B and B', there exists some $g \in W(\Phi)$ such that $g(B) = B'$.*

Definition 4.4. Given a base B, the *Dynkin diagram* is the graph obtained by taking the vertices to be the set B and connecting two vertices α, β with n edges where

$$n = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\frac{2(\alpha,\beta)}{(\beta,\beta)}.$$

Furthermore, we direct the edge from α to β if $(\beta,\beta) > (\alpha,\alpha)$.

The two previous theorems tell us that every root system has a unique Dynkin diagram corresponding to it. In fact, a root system is determined by it's Dynkin diagram:

Proposition 4.7. *If Φ and Φ' are root systems (spanning E and E' respectively) with the same Dynkin diagram, then there is a linear transformation $\phi : E \to E'$ such that $\Phi' = \phi(\Phi)$ and $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\phi(\alpha),\phi(\beta))}{(\phi(\alpha),\phi(\alpha))}$ for all $\alpha, \beta \in \Phi$.*

Proof. Choose bases B, B' for Φ and Φ' respectively. Define ϕ by sending each $\beta \in B$ to the corresponding element of B' and extend by linearity to all of E. Given $\alpha \in \Phi$, $\alpha = g(\beta)$ for some $g \in W(\Phi), \beta \in B$. Then $\alpha = s_{\beta_1} \circ \cdots \circ s_{\beta_n}(\beta)$ where each $\beta_i \in B$. But then

$$\phi(\alpha) = s_{\phi(\beta_1)} \circ \cdots \circ s_{\phi(\beta_n)}(\phi(\beta))$$

which is in Φ'. Since Φ' is generated by $W(\Phi')$ acting on B' and $W(\Phi')$ is generated by $\{s_{\phi(\beta)} : \beta \in B\}$, ϕ maps Φ surjectively onto Φ'. $\square$

Lemma 4.8. *Φ is irreducible if and only if it's Dynkin diagram is connected.*

Thus we need to classify connected Dynkin diagrams.

5. Classifying Dynkin Diagrams

We first introduce a slight variation on bases:

Definition 5.1. Given a finite dimensional real inner product space E an *admissible set* $A \subset E$ is an set $\{v_1, \ldots, v_n\}$ of normalized linearly independent vectors such that $(v_i, v_j) \leq 0$ if $i \neq j$ and $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$.

If we take a base and normalize each of the vectors then we get an admissible set and any subset of an admissible set is admissible. We can take the Dynkin diagram for an admissible set just as we can for a base. We now describe all connected Dynkin diagrams for admissible sets.

Lemma 5.1. *Let A be an admissible set of order n. Then in the Dynkin diagram there are at most $n-1$ pairs of vertices with an edge between them.*

Proof. Let $A = \{v_1, \ldots, v_n\}$ and $v = \sum_{i=1}^{n} v_i$. Then

$$(v, v) = n + \sum_{i \neq j} (v_i, v_j)$$

and $(v, v) > 0$ because $v \neq 0$. If there is an edge between v_i and v_j then $-2(v_i, v_j) \geq 1$, so if N is the number of pairs v_i, v_j with an edge between them we have

$$n > \sum_{i > j} -2(v_i, v_j) \geq N.$$

□

Corollary 5.2. *The Dynkin diagram of an admissible set has no cycles.*

Lemma 5.3. *No vertex in the Dynkin diagram of an admissible set has ≥ 4 edges.*

Proof. Let $\{v_1, \ldots, v_n\}$ be the set of vertices such that there is an edge between v and v_i. Since there are no cycles, $(v_i, v_j) = 0$ for all $i \neq j$. Thus there exists a $v_0 \in E$ such that $Span\{v, v_1, \ldots, v_n\} = Span\{v_0, v_1, \ldots, v_n\}$ and $\{v_0, v_1, \ldots, v_n\}$ form an orthonormal basis for this subspace. Then $v = \sum_{i=0}^{n} (v, v_i) v_i$, so

$$1 = (v, v) = \sum_{i=0}^{n} (v, v_i)^2 > \sum_{i=1}^{n} \frac{1}{4} = \frac{n}{4}.$$

Thus v has at most 3 neighbors and it is easy to check that it can't have more than 3 edges. □

Lemma 5.4. *Let $S = \{v_1, \ldots, v_n\}$ be a subset of an admissible set A such that there is a single edge between v_i and v_{i+1} for $1 \leq i \leq n-1$, so the Dynkin diagram of S is a line. Then if $v = \sum_{i=1}^{n} v_i$, $A' = (A \setminus S) \cup \{v\}$ is an admissible set and its Dynkin diagram is the graph obtained from shrinking S to a point in the diagram of A.*

Proof.

$$(v, v) = n + \sum_{i > j} 2(v_i, v_j) = n - (n-1) = 1.$$

If $w \in A \setminus S$ then w shares an edge with at most one v_i (otherwise there would be a cycle) so

$$4(v, w)^2 = 4(v_i, w)^2 \in \{0, 1, 2, 3\}.$$

□

Corollary 5.5. *If the Dynkin diagram of an admissible set has a vertex incident to three edges, then either one of these is a double edge and the vertex at the other end of the double edge may be incident to three edges or there are no other vertices incident to three edges.*

Proof. Simply apply the previous lemma to the path connecting two vertices that are incident to three edges. □

Lemma 5.6. *Let $S = \{v_1, \ldots, v_n\}$ be a subset of an admissible set A that forms a line in the Dynkin diagram (as in the previous lemma), then if $v = \sum_{i=1}^{n} i v_i$ we have $(v, v) = \frac{n(n+1)}{2}$.*

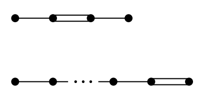
Proof.

$$(v,v) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n-1} 2(v_i, v_{i+1}) i(i+1) = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n-1} i^2 + i$$

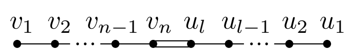
$$= n^2 - \frac{n(n-1)}{2} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

□

Lemma 5.7. *Let A be an admissible set with a double edge in its Dynkin diagram. Then the Dynkin diagram is one of the following:*



Proof. We already know the diagram must be of the form



where $l \geq n$. Let $v = \sum_{i=1}^{n} iv_i$ and $u = \sum_{i=1}^{l} iu_i$. Then by the previous lemma $(v,v) = \frac{n(n+1)}{2}$ and $(w,w) = \frac{l(l+1)}{2}$. But $(v_n, u_l)^2 = 1/2$ and $(v_i, u_j) = 0$ otherwise, so we can calculate

$$(v,u)^2 = (nv_n, lu_l)^2 = \frac{n^2 l^2}{2}.$$

The Cauchy-Schwarz inequality gives us

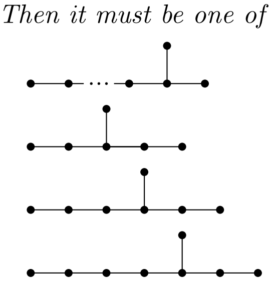
$$\frac{n^2 l^2}{2} = (v,u)^2 < (v,v)(u,u) = \frac{n(n+1)}{2} \frac{l(l+1)}{2}.$$

Thus $(n+1)(l+1) > 2nl$, which implies $nl < n + l + 1$. This gives us

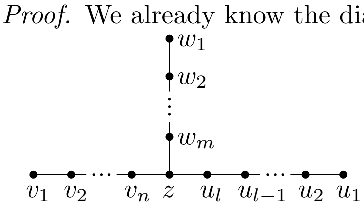
$$(n-1)(l-1) = nl - (n+l+1) + 2 < 2.$$

Therefore $l = 1$ or $n = 2, l = 2$. □

Lemma 5.8. *Let A be an admissible set with a branch point in its Dynkin diagram. Then it must be one of the following:*



Proof. We already know the diagram must be of the form



where $n \geq m \geq l$. Let $v = \sum_{i=1}^{n} iv_i$, $u = \sum_{i=1}^{l} iu_i$, $w = \sum_{i=1}^{m} iw_i$, $\hat{v} = v/\|v\|$, $\hat{u} = u/\|u\|$, and $\hat{w} = w/\|w\|$. There exists some z_0 such that $\{\hat{v}, \hat{u}, \hat{w}, z_0\}$ is an orthonormal basis for the subspace spanned by $\{v, u, w, z\}$. Then $z = (z, \hat{v})\hat{v} + (z, \hat{u})\hat{u} + (z, \hat{w})\hat{w} + (z, z_0)z_0$. Since $(z, z) = 1$ and $(z, z_0) \neq 0$ we must have

$$1 > (z, \hat{v})^2 + (z, \hat{u})^2 + (z, \hat{w})^2.$$

But

$$(z, \hat{v})^2 = \frac{(z, nv_n)^2}{(v, v)} = \left(\frac{n^2}{4}\right)\left(\frac{2}{n(n+1)}\right) = \frac{n}{2(n+1)}$$

and similarly for u and w.

Thus

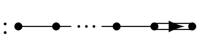
$$\frac{1}{n+1} + \frac{1}{m+1} + \frac{1}{l+1} > 1.$$

Since $\frac{1}{l+1}$ is the largest of the three terms, we must have $\frac{1}{l+1} > 1/3$ and thus $l < 2$ which implies $l = 1$. Then $\frac{1}{n+1} + \frac{1}{m+1} > 1/2$ and since $\frac{1}{m+1}$ is the larger of the two we must have $\frac{1}{m+1} > 1/4$ which implies $m < 3$, so $m = 1$ or $m = 2$. If $m = 1$, n can take on any value, but if $m = 2$ then $\frac{1}{n+1} > 1/6$, so $n < 5$ and thus we have $n = 2, 3$ or 4. □

Now considering the possible Dynkin diagrams for a root space, we simply need to consider all the ways that arrows may be put in for the double and triple edges. This gives us the following complete list of possible Dynkin diagrams (where the subscript denotes the number of vertices):

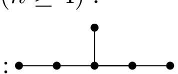
Theorem 5.9. *The following is a complete list of all Dynkin diagrams of irreducible root systems:*

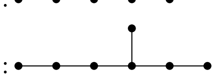
$A_n (n \geq 1)$:

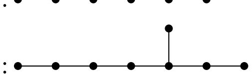
$B_n (n \geq 2)$: 

$C_n (n \geq 3)$:

$D_n (n \geq 4)$: 

E_6 : 

E_7 : 

E_8 : 

F_4 :

G_2 :

Theorem 5.10. *Let Φ be a root system for a semisimple Lie algebra L. If L is simple then Φ is irreducible.*

Thus our classification of connected Dynkin diagrams gives us a classification of semisimple Lie algebras. It turns out that every Dynkin diagram in Theorem 5.9 is realized as the Dynkin diagram of some Lie algebra, so we have a complete list of the simple complex Lie algebras. Since every semisimple complex Lie algebra is a direct sum of simple Lie algebras, and every Lie algebra is an extension of a solvable Lie algebra by a semisimple Lie algebra we have a classification of complex Lie algebras.

Example 5.2. The Dynkin diagram corresponding to the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ is A_n.

Example 5.3. Let S be the $n+1$ by $n+1$ matrix:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$$

Then we define

$$\mathfrak{so}(2n+1, C) := \{x \in \mathfrak{gl}(2n+1,\mathbb{C}) : x^t S = -Sx\}$$

where x^t denotes the transpose of x. The Dynkin diagram of $\mathfrak{so}(2n+1, C)$ is B_n.

Example 5.4. Let

$$T = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

and define

$$\mathfrak{so}(2n,\mathbb{C}) := \{x \in \mathfrak{gl}(2n,\mathbb{C}) : x^t T = -Tx\}.$$

The Dynkin diagram of $\mathfrak{so}(2n,\mathbb{C})$ is D_n.

Example 5.5. Let

$$U = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and define

$$\mathfrak{sp}(2n,\mathbb{C}) := \{x \in \mathfrak{gl}(2n,\mathbb{C}) : x^t U = -Ux\}.$$

The Dynkin diagram of $\mathfrak{sp}(2n,\mathbb{C})$ is C_n.

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References

[1] Karin Erdmann and Mark J. Wildon. Introduction to Lie Algebras. Springer-Verlang. 2006.