1. CW Approximation

The CW approximation theorem says that every space is weakly equivalent to a CW complex.

**Theorem 1.1 (CW Approximation).** There exists a functor $\Gamma$ from the homotopy category of spaces to itself such that for every space $X$, $\Gamma X$ is a CW complex and there is a natural transformation $\gamma : \Gamma \rightarrow \text{id}$ such that $\gamma : \Gamma X \rightarrow X$ is a weak equivalence. If $X$ is $n$-connected then $\Gamma X$ has one vertex and no $q$-cells for $1 \leq q \leq n$.

**Proof.** Without loss of generality we may assume $X$ is connected. The inclusion of a point in $X$ then gives a surjection on $\pi_0(X)$. We inductively define $X_q$ and $\gamma_q : X_q \rightarrow X$. Assume we have a CW complex $X_q$ and $\gamma_q$ that induces an surjection on $\pi_q$ and an isomorphism on the lower homotopy groups. We construct an new space $X'_q$ by attaching a $q+1$ dimensional disk $D^{q+1}$ along a map $f$ for each $f \in \pi_q(X)$ in the kernel of $\gamma_q$. Since $\gamma_q*(f)$ is nullhomotopic we can extend $\gamma_q$ to $X'_q$ by defining it on $D^{q+1}$ as the nullhomotopy of $f$. Now $\pi_q(X'_q)$ surjests onto $\pi_q(X)$. We then construct $X_{q+1}$ as

$$X_{q+1} = X_{q+1}' \cup \bigcup_{j \in A} S^{q+1}_j$$

where $A$ is a generating subset of $\pi_{q+1}(X)$. We then define $\gamma_{q+1} : X_{q+1} \rightarrow X$ by defining it as $j$ on $S^{q+1}_j$ and the extension of $\gamma_q$ on $X'_{q+1}$. Thus $\pi_{q+1}(X_{q+1})$ surjects onto $\pi_{q+1}(X)$. Cellular approximation of maps shows that this construction does not change $\pi_i$ for $i < q$ because $\pi_i$ only depends on the $i+1$ skeleton which hasn’t been altered. Thus we now have a sequence of inclusions $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ and maps $\gamma_q : X_q \rightarrow X$ such that for each $q$, $\gamma_q$ induces an isomorphism on $\pi_{q-1}$ and a surjection on $\pi_q$. Let $\Gamma X$ be the colimit of the $X_q$ and $\gamma$ be the map $\Gamma X \rightarrow X$ induced by the $\gamma_q$. From the construction we can see that if $X$ is $n$-connected then $X_q$ is just a vertex for each $q \leq n$ and so $\Gamma X$ has no $q$-cells for $q \leq n$. □

2. Homotopy Excision

**Definition 2.1.** Given spaces $A, B \subset X$, $(X; A, B)$ is called an excisive triad if $X$ is contained in the union of the interiors of $A$ and $B$. 

*Date: 7/30/10.*
The excision theorem for homology says that the inclusion of pairs \((A, A \cap B) \hookrightarrow (X, B)\) induces an isomorphism on homology groups. We would like to have a similar theorem for homotopy groups, but unfortunately we only have an isomorphism for the lower homotopy groups.

**Definition 2.2.** \(f : X \to Y\) is an \(n\)-equivalence if it induces an isomorphism on \(\pi_q\) for \(q < n\) and an surjection on \(\pi_n\). Thus a map is a weak equivalence if and only if it is an \(n\)-equivalence for all \(n\).

**Theorem 2.1** (Homotopy Excision). If \((X; A, B)\) is an excisive triad with \((A, A \cap B)\) \(m\)-connected and \((B, A \cap B)\) \(n\)-connected then \((A, A \cap B) \hookrightarrow (X, B)\) is a \(m + n\)-equivalence.

The proof can be found in Allen Hatcher’s *Algebraic Topology* (page 361) or J. P. May’s *A Concise Course in Algebraic Topology* (page 86).

**Corollary 2.2.** If \(f : X \to Y\) is an \(n\)-equivalence with \(X, Y\) \((n - 1)\)-connected then \(p : (Mf, X) \to (Cf, *)\) is a 2\(n\)-equivalence. Here \(Mf\) is the mapping cylinder \((X \times [0, 1] \cup Y)/\{(x, 1) \sim f(x)\}\) and \(X\) is considered a subspace of \(Mf\) under the inclusion \(x \mapsto (x, 0)\). \(Cf\) is the mapping cone which is given by identifying the image of \(X\) in \(Mf\) to point and \(p\) is the natural map that sends \((x, 0)\) to the basepoint and is the identity on everything else.

**Proof.** Consider the subsets of \(Cf\) defined by \(A = Y \cup \{(x, t) : 0 \leq t \leq 2/3\}, B = \{(x, t) : 1/3 \leq t \leq 1\}\). Then \((Cf; A, B)\) form an excisive triad and \(A \cap B \simeq X\). We can see that \((A, A \cap B)\) is \(n\)-connected because \(f_\ast\) maps \(\pi_n(X)\) onto \(\pi_n(Y)\) and therefore every element of \(\pi_n(A)\) is represented by an element of \(\pi_n(X)\). Thus every element of \(\pi_n(A)\) is homotopic to an element of \(\pi_n(A \cap B)\) which implies \(\pi_n(A, A \cap B) = 0\) and since \(X, Y\) are \((n - 1)\)-connected and we have just shown \(\pi_n(A, A \cap B) = 0\) we may conclude that \((A, A \cap B)\) is \(n\)-connected. Letting \(CX = X \times [0, 1]/\{(x, 1) \simeq *\}\) denote the cone on \(X\) with the natural inclusion \(X \hookrightarrow CX\) \(x \mapsto (x, 0)\) we get the long exact sequence

\[ \cdots \to \pi_q(CX) \to \pi_q(CX, X) \to \pi_{q-1}(X) \to \pi_{q-1}(CX) \to \cdots \]

Since \(CX\) is contractible it has trivial homotopy groups which implies \(\pi_q(CX, X) \cong \pi_{q-1}(X)\) and the fact that \(X\) is \((n - 1)\)-connected implies \((CX, X)\) is \(n\)-connected. Since \((B, A \cap B)\) is homotopic to \((CX, X)\), \((X, A \cap B)\) is \(n\)-connected. Thus the homotopy excision theorem implies that the natural map \((A, A \cap B) \to (Cf, B)\) is a 2\(n\)-equivalence. We also have homotopy equivalences \((Mf, X) \simeq (A, A \cap B)\) and \((Cf, B) \simeq (Cf, *)\) and \(p\) is given by the composite \((Mf, X) \to (A, A \cap B) \to (Cf, B) \to (Cf, *)\). Since each map is a 2\(n\)-equivalence or a homotopy equivalence the composition is a 2\(n\)-equivalence.

**Corollary 2.3.** Let \(i : A \to X\) be a cofibration and an \(n\)-equivalence between \((n - 1)\)-connected spaces. Then the natural map \(q : (X, A) \to (X/A, *)\) is a 2\(n\)-equivalence.

**Proof.** The previous corollary implies that \(p : (Mi, A) \to (Ci, *)\) is a 2\(n\)-equivalence. \((Mi, A)\) is homotopy equivalent to \((X, A)\) and \((Ci, *)\) is homotopy equivalence to \((X/A, *)\) because the inclusion of \(A\) is a cofibration. \(q\) is given by the composition

\[ (X, A) \cong (Mi, A) \xrightarrow{p} (Ci, *) \cong (X/A, *) \]

so it is a 2\(n\)-equivalence.
3. FREUDENTHAL SUSPENSION

For each $q$ there is a map $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ defined by $f \mapsto f \land \text{id} : S^q \land S^1 \to X \land S^1$. This is an element of $\pi_{q+1}(\Sigma X)$ because $S^q \land S^1 \cong S^{q+1}$.

**Theorem 3.1** (Freudenthal Suspension). If $X$ is $(n-1)$-connected and the inclusion of the base point $* \to X$ is a cofibration then $\Sigma$ induces an isomorphism on $\pi_q$ for $q < 2n - 1$ and a surjection for $q = 2n - 1$.

**Proof.** Write $\Sigma X$ as $C_-X \cup C_+X$ where the $-$ and $+$ denote two different copies of $X \land [0, 1]$ glued along $\{(x, 0)\}$ (here we are using the reduced cone, as opposed to the unreduced cone used earlier). Then $C_-X \cap C_+X = X$. The same argument as before shows that $\pi_i(X) \cong \pi_{i+1}(C_+X, X)$. Since $C_+X$ is contractible, $\pi_i(\Sigma X, C_-X) \cong \pi_i(\Sigma X)$. Furthermore, $\Sigma$ is given by the composition

$$
\pi_i(X) \to \pi_{i+1}(C_+X, X) \to \pi_{i+1}(\Sigma X, C_-X) \to \pi_{i+1}(\Sigma X)
$$

because the first map is $f \mapsto f \land [0, 1]$. We would like to apply homotopy excision to the second map. Note that to do this we actually need to replace $C_+X, C_-X$ with open neighborhoods of $C_+X, C_-X$ so that $(\Sigma X; C_-X, C_+X)$ is an excisive triad. Since $X$ is $(n-1)$-connected, $(C_+X, X)$ is $n$-connected (again because $\pi_{i+1}(C_+X, X) = \pi_i(X)$) and this implies that the second map is an isomorphism for $i + 1 < 2n$ and a surjection for $i + 1 = 2n$. Thus $\Sigma$ induces an isomorphism $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ for $q < 2n - 1$ and a surjection for $i = 2n - 1$. □

The Freudenthal suspension Theorem has one very important application in that it allows us to calculate some homotopy groups, namely $\pi_n(S^n)$. We have seen that the Hopf fibration $S^1 \to S^3 \to S^2$ gives a long exact sequence

$$
\ldots \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \ldots
$$

Since $\pi_1(S^1) = 0$ for $i = 1, 2$ we have $\pi_1(S^1) \cong \pi_2(S^2)$. We have also already calculated $\pi_1(S^1) \cong \mathbb{Z}$. Freudenthal suspension implies that $\Sigma : \pi_q(S^1) \to \pi_{q+1}(S^{q+1})$ is an isomorphism for $q \geq 2$ so we may conclude that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n$.

It also helps to calculate stable homotopy groups. These are defined as $\pi^S_q(X) = \text{colim}_n\pi_{q+n}(\Sigma^n X)$. Freudenthal suspension implies that the sequence $\pi_q(X) \to \pi_{q+1}(\Sigma X) \to \pi_{q+2}(\Sigma^2 X) \to \ldots$ is eventually constant and this is in fact where the term stable homotopy group comes from.

4. HUREWICZ

There is an isomorphism $\tilde{H}_i(X) \to \tilde{H}_{i+1}(\Sigma X)$ which we also denote by $\Sigma$. The long exact sequence

$$
\ldots \to \tilde{H}_{i+1}(CX) \to \tilde{H}_{i+1}(CX, X) \to \tilde{H}_i(X) \to \tilde{H}_i(CX) \to \ldots
$$

gives an isomorphism $\tilde{H}_i \to \tilde{H}_{i+1}(CX, X)$ because $\tilde{H}_i(CX) = 0$ for all $i$. Since $\tilde{H}_i(CX, X) \cong \tilde{H}_i(\Sigma X)$ we can set $\Sigma$ equal to the composition

$$
\tilde{H}_i(X) \to \tilde{H}_{i+1}(CX, X) \to \tilde{H}_{i+1}(X).
$$

We fix generators $i_n$ for $\tilde{H}_n(S^n)$ such that $\Sigma i_n = i_{n+1}$. We then construct $h : \pi_n(X) \to \tilde{H}_n(X)$ by $h(f) = f \circ i_n$ (since $f$ is a map $S^n \to X$ it induces a map $f_\ast : \tilde{H}_n(S^n) \to \tilde{H}_n(X)$). This is a homomorphism because $[f + g]$ is given by the composition

$$
S^n \to S^n \lor S^n \overset{f \lor g}{\longrightarrow} X \lor X \to X
$$
where the first map is the map that sends one hemisphere to the first copy of $S^n$ and the other hemisphere to the other copy of $S^n$. Then the induced map on $\tilde{H}_n$ sends $i_n \mapsto i_n \oplus i_n \mapsto f_*(i_n) \oplus g_*(i_n) \mapsto f_*(i_n) + g_*(i_n)$ so $h$ is in fact a homomorphism. In the case where $n = 1$ the fact that $\tilde{H}_n(X)$ is abelian implies that $h$ induces a map from $\pi_1(X)$ (the abelianization of $\pi_1(X)$) to $\tilde{H}_1(X)$. We can now state the Hurewicz theorem which gives a connection between homology groups and homotopy groups.

**Theorem 4.1 (Hurewicz).** If $X$ is $(n-1)$-connected then $h$ is an isomorphism if $n > 1$ and if $n = 1$ the induced map on $\pi_1(X) \to \tilde{H}_1(X)$ is an isomorphism.

**Proof.** If $X$ is a sphere then $h([id]) = i_n$ and our previous calculation for $\pi_n(S^n)$ shows that the theorem holds in this case. In the general case we may assume that $X$ is a CW-complex with 1 vertex and no $q$-cells for $q < n$ by the CW approximation theorem. Cellular approximation of maps implies that the inclusion $X^{n+1} \hookrightarrow X$ is an $n+1$-equivalence. It is easy to see by the CW construction of homology that this inclusion induces an isomorphism on $H_n$ as well because $H_n$ is determined solely by the terms of the chain complex in degrees $n$ and $n + 1$. Thus we may assume without loss of generality that $X = X^{n+1}$.

Let $L = X^n$ and let $K = \vee_{\alpha \in A} S^\alpha_\alpha$ where $A$ is the set of $n + 1$ cells in $X$. Define $f : K \to L$ by letting it equal the attaching map of $\alpha$ on $S^n_\alpha$. Then $X$ is the mapping cone of $f$ (unreduced). The maps $K \xrightarrow{\alpha} L \hookrightarrow X$ induces maps on homology and homotopy and our goal is to show that the rows in the commutative diagram

$$
\begin{array}{cccc}
\pi_n(K) & \xrightarrow{\alpha} & \pi_n(L) & \xrightarrow{} & \pi_n(X) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{H}_n(K) & \xrightarrow{} & \tilde{H}_n(L) & \xrightarrow{} & \tilde{H}_n(X) & \xrightarrow{} & 0
\end{array}
$$

are exact. Since $K, L$ are wedges of spheres the theorem holds for them and we can then apply the five lemma to get the result for $X$. The diagram is commutative because $h$ is natural. This comes from the fact that if $g : X \to Y$ is a map of spaces and $\sigma : S^n \to X$ is an element of $\pi_n(X)$ then $(f \circ \sigma)_*(i_n) = f_*(\sigma_*(i_n))$. $\tilde{H}_n(L)$ is the free abelian group generated by the $n$-cells and a linear combination of $n$-cells maps to 0 in $\tilde{H}_n(X)$ if and only if it is the boundary of discs. Since the boundaries of discs are given precisely by the image of $K$ we have exactness of the $\tilde{H}$ sequence at $L$. The cokernel of the map $\tilde{H}_n(L) \to \tilde{H}_n(X)$ should be $H_n(X, L) = \tilde{H}_n(X/L)$, but $X/L$ is a wedge of $n + 1$ spheres so it has trivial $H_n$. Thus the bottom row is exact.

For $n > 1$ we may write $f$ as the composition $K \to Mf \xrightarrow{p} L$ where $p$ is the natural homotopy equivalence. This gives us the commutative diagram

$$
\begin{array}{cccc}
\pi_n(K) & \xrightarrow{\alpha} & \pi_n(Mf) & \xrightarrow{} & \pi_n(Mf, K) & \xrightarrow{} & 0 \\
\downarrow{id} & & \downarrow{p_*} & & \downarrow{} & & \downarrow{}
\end{array}
$$

in which the two left vertical arrows are isomorphisms. The top row is exact because the cokernel of the last map is $\pi_{n-1}(K)$ and this is trivial because $K$ is a wedge of $n$ spheres. The right vertical arrow is an isomorphism by the first corollary.
to the homotopy excision theorem. Thus the bottom row is exact and we are done with the case $n > 1$. In the case with $n = 1$ the top row is not exact because the kernel of the map $\pi_1(Mf) \to \pi_1(Mf, K)$ is the normal subgroup generated by $\pi_1(K)$. But then it becomes exact when we take the abelianization of the whole diagram and this gives us the $n = 1$ case.

□

References