# HOMOTOPY THEORY

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## 1. CW Approximation

The CW approximation theorem says that every space is weakly equivalent to a CW complex.

**Theorem 1.1** (CW Approximation)**.** *There exists a functor $\Gamma$ from the homotopy category of spaces to itself such that for every space $X$, $\Gamma X$ is a CW complex and there is a natural transformation $\gamma : \Gamma \to id$ such that $\gamma : \Gamma X \to X$ is a weak equivalence. If $X$ is $n$-connected then $\Gamma X$ has one vertex and no $q$-cells for $1 \leq q \leq n$.*

*Proof.* Without loss of generality we may assume $X$ is connected. The inclusion of a point in $X$ then gives a surjection on $\pi_0(X)$. We inductively define $X_q$ and $\gamma_q : X_q \to X$. Assume we have a CW complex $X_q$ and $\gamma_q$ that induces an surjection on $\pi_q$ and an isomorphism on the lower homotopy groups. We construct an new space $X'_{q+1}$ by attaching a $q+1$ dimensional disk $D^{q+1}_f$ along a map $f$ for each $f \in \pi_q(X)$ in the kernel of $\gamma_q$. Since $\gamma_{q*}(f)$ is nullhomotopic we can extend $\gamma_q$ to $X'_{q+1}$ by defining it on $D^{q+1}_f$ as the nullhomotopy of $f$. Now $\pi_q(X'_{q+1})$ is isomorphic to $\pi_q(X)$.

We then construct $X_{q+1}$ as

$$X_{q+1} = X'_{q+1} \vee \bigvee_{j \in A} S^{q+1}_j$$

where $A$ is a generating subset of $\pi_{q+1}(X)$. We then define $\gamma_{q+1} : X_{q+1} \to X$ by defining it as $j$ on $S^{q+1}_j$ and the extension of $\gamma_q$ on $X'_{q+1}$. Thus $\pi_{q+1}(X_{q+1})$ surjects onto $\pi_{q+1}(X)$. Cellular approximation of maps shows that this construction does not change $\pi_i$ for $i < q$ because $\pi_i$ only depends on the $i+1$ skeleton which hasn't been altered. Thus we now have a sequence of inclusions $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots$ and maps $\gamma_q : X_q \to X$ such that for each $q$, $\gamma_q$ induces an isomorphism on $\pi_{q-1}$ and a surjection on $\pi_q$. Let $\Gamma X$ be the colimit of the $X_q$ and $\gamma$ be the map $\Gamma X \to X$ induced by the $\gamma_q$. From the construction we can see that if $X$ is $n$-connected then $X_q$ is just a vertex for each $q \leq n$ and so $\Gamma X$ has no $q$-cells for $q \leq n$.

□

## 2. Homotopy Excision

**Definition 2.1.** Given spaces $A, B \subset X$, $(X; A, B)$ is called an *excisive triad* if $X$ is contained in the union of the interiors of $A$ and $B$.

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The excision theorem for homology says that the inclusion of pairs $(A, A\cap B) \hookrightarrow (X, B)$ induces an isomorphism on homoology groups. We would like to have a similar theorem for homotopy groups, but unfortunately we only have an isomorhpism for the lower homotopy groups.

**Definition 2.2.** $f : X \to Y$ is an $n$-equivalence if it induces an isomorphism on $\pi_q$ for $q < n$ and an surjection on $\pi_n$. Thus a map is a weak equivalence if and only if it is an $n$-equivalence for all $n$.

**Theorem 2.1** (Homotopy Excision)**.** *If $(X; A, B)$ is an excisive triad with $(A, A\cap B)$ $m$-connected and $(B, A\cap B)$ $n$-connected then $(A, A\cap B) \hookrightarrow (X, B)$ is a $m+n$-equivalence.*

The proof can be found in Allen Hatcher's *Algebraic Topology* (page 361) or J. P. May's *A Concise Course in Algebraic Topology* (page 86).

**Corollary 2.2.** *If $f : X \to Y$ is an $n$-equivalence with $X, Y$ $(n-1)$-connected then $p : (Mf, X) \to (Cf, *)$ is a $2n$-equivalence. Here $Mf$ is the mapping cylinder $(X \times [0,1] \cup Y)/\{(x,1) \sim f(x)\}$ and $X$ is considered a subspace of $Mf$ under the inclusion $x \mapsto (x, 0)$. $Cf$ is the mapping cone which is given by identifying the image of $X$ in $Mf$ to point and $p$ is the natural map that sends $(x, 0)$ to the basepoint and is the identity on everything else.*

*Proof.* Consider the subsets of $Cf$ defined by $A = Y \cup \{(x,t) : 0 \leq t \leq 2/3\}$, $B = \{(x,t) : 1/3 \leq t \leq 1\}$. Then $(Cf; A, B)$ form an excisive triad and $A\cap B \simeq X$. We can see that $(A, A\cap B)$ is $n$-connected because $f_*$ maps $\pi_n(X)$ onto $\pi_n(Y)$ and therefore every element of $\pi_n(A)$ is represented by an element of $\pi_n(X)$. Thus every element of $\pi_n(A)$ is homotopic to an element of $\pi_n(A\cap B)$ which implies $\pi_n(A, A\cap B) = 0$ and since $X, Y$ are $(n-1)$-connected and we have just shown $\pi_n(A, A\cap B) = 0$ we may conclude that $(A, A\cap B)$ is $n$-connected. Letting $CX = X \times [0,1]/\{(x,1) \simeq *\}$ denote the cone on $X$ with the natural inclusion $X \hookrightarrow CX$ $x \mapsto (x, 0)$ we get the long exact sequence

$$\cdots \to \pi_q(CX) \to \pi_q(CX, X) \to \pi_{q-1}(X) \to \pi_{q-1}(CX) \to \ldots.$$

Since $CX$ is contractible it has trivial homotopy groups which implies $\pi_q(CX, X) \cong \pi_{q-1}(X)$ and the fact that $X$ is $(n-1)$-connected implies $(CX, X)$ is $n$-connected. Since $(B, A\cap B)$ is homotopic to $(CX, X)$, $(X, A\cap B)$ is $n$-connected. Thus the homotopy excision theorem implies that the natural map $(A, A\cap B) \to (Cf, B)$ is a $2n$-equivalence. We also have homotopy equivalences $(Mf, X) \simeq (A, A\cap B)$ and $(Cf, B) \simeq (Cf, *)$ and $p$ is given by the composite $(Mf, X) \to (A, A\cap B) \to (Cf, B) \to (Cf, *)$. Since each map is a $2n$-equivalence or a homotopy equivalence the composition is a $2n$-equivalence. □

**Corollary 2.3.** *Let $i : A \to X$ be a cofibration and an $n$-equivalence between $(n-1)$-connected spaces. Then the natural map $q : (X, A) \to (X/A, *)$ is a $2n$-equivalence.*

*Proof.* The previous corollary implies that $p : (Mi, A) \to (Ci, *)$ is a $2n$-equivalence. $(Mi, A)$ is homotopy equivalent to $(X, A)$ and $(Ci, *)$ is homotopy equivalence to $(X/A, *)$ because the inclusion of $A$ is a cofibration. $q$ is given by the composition

$$(X, A) \xrightarrow{\simeq} (Mi, A) \xrightarrow{p} (Ci, *) \xrightarrow{\simeq} (X/A, *)$$

so it is a $2n$-equivalence. □

## 3. Freudenthal Suspension

For each $q$ there is a map $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ defined by $f \mapsto f \wedge id : S^q \wedge S^1 \to X \wedge S^1$. This is an element of $\pi_{q+1}(\Sigma X)$ because $S^q \wedge S^1 \cong S^{q+1}$.

**Theorem 3.1** (Freudenthal Suspension)**.** *If $X$ is $(n-1)$-connected and the inclusion of the base point $* \hookrightarrow X$ is a cofibration then $\Sigma$ induces an isomorphism on $\pi_q$ for $q < 2n-1$ and a surjection for $q = 2n-1$.*

*Proof.* Write $\Sigma X$ as $C_-X \cup C_+X$ where the $-$ and $+$ denote two different copies of $X \wedge [0,1]$ glued along $\{(x,0)\}$ (here we are using the reduced cone, as opposed to the unreduced cone used earlier). Then $C_-X \cap C_+X = X$. The same argument as before shows that $\pi_i(X) \cong \pi_{i+1}(C_+X, X)$. Since $C_-X$ is contractible, $\pi_i(\Sigma X, C_-X) \cong \pi_i(\Sigma X)$. Furthermore, $\Sigma$ is given by the composition

$$\pi_i(X) \to \pi_{i+1}(C_+X, X) \to \pi_{i+1}(\Sigma X, C_-X) \to \pi_{i+1}(\Sigma X)$$

because the first map is $f \mapsto f \wedge [0,1]$. We would like to apply homotopy excision to the second map. Note that to do this we actually need to replace $C_+X, C_-X$ with open neighborhoods of $C_+X, C_-X$ so that $(\Sigma X; C_-X, C_+X)$ is an excisive triad. Since $X$ is $(n-1)$-connected, $(C_\pm X, X)$ is $n$-connected (again because $\pi_{i+1}(CX, X) \cong \pi_i(X)$) and this implies that the second map is an isomorphism for $i+1 < 2n$ and a surjection for $i+1 = 2n$. Thus $\Sigma$ induces an isomorphism $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ for $q < 2n-1$ and a surjection for $i = 2n-1$. □

The Freudenthal suspension Theorem has one very important application in that it allows us to calculate some homotopy groups, namely $\pi_n(S^n)$. We have seen that the Hopf fibration $S^1 \to S^3 \to S^2$ gives a long exact sequence

$$\cdots \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \ldots.$$

Since $\pi_i(S^3) = 0$ for $i = 1, 2$ we have $\pi_1(S^1) \cong \pi_2(S^2)$. We have also already calculated $\pi_1(S^1) \cong \mathbb{Z}$. Freudenthal suspension implies that $\Sigma : \pi_q(S^q) \to \pi_{q+1}(S^{q+1})$ is an isomorphism for $q \geq 2$ so we may conclude that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n$.

It also helps to calculate stable homotopy groups. These are defined as $\pi_q^s(X) = \mathrm{colim}\pi_{q+n}(\Sigma^n X)$. Freudenthal suspension implies that the sequence $\pi_q(X) \to \pi_{q+1}(\Sigma X) \to \pi_{q+2}(\Sigma^2 X) \to \ldots$ is eventually constant and this is in fact where the term stable homotopy group comes from.

## 4. Hurewicz

There is an isomorphism $\tilde{H}_i(X) \to \tilde{H}_{i+1}(\Sigma X)$ which we also denote by $\Sigma$. The long exact sequence

$$\cdots \to \tilde{H}_{i+1}(CX) \to H_{i+1}(CX, X) \to \tilde{H}_i(X) \to \tilde{H}_i(CX) \to \ldots$$

gives an isomorphism $\tilde{H}_i \to H_{i+1}(CX, X)$ because $\tilde{H}_i(CX) = 0$ for all $i$. Since $H_i(CX, X) \cong \tilde{H}_i(\Sigma X)$ we can set $\Sigma$ equal to the composition

$$\tilde{H}_i(X) \to H_{i+1}(CX, X) \to \tilde{H}_{i+1}(X).$$

We fix generators $i_n$ for $\tilde{H}_n(S^n)$ such that $\Sigma i_n = i_{n+1}$. We then construct $h : \pi_n(X) \to \tilde{H}_n(X)$ by $h([f]) = f_*(i_n)$ (since $f$ is a map $S^n \to X$ it induces a map $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(X)$). This is a homomorphism because $[f+g]$ is given by the composition

$$S^n \to S^n \vee S^n \overset{f \vee g}{\to} X \vee X \to X$$

where the first map is the map that sends one hemisphere to the first copy of $S^n$ and the other hemisphere to the other copy of $S^n$. Then the induced map on $\tilde{H}_n$ sends $i_n \mapsto i_n \oplus i_n \mapsto f_*(i_n) \oplus g_*(i_n) \mapsto f_*(i_n) + g_*(i_n)$ so $h$ is in fact a homomorphism. In the case where $n = 1$ the fact that $\tilde{H}_n(X)$ is abelian implies that $h$ induces a map from $\pi_1'(X)$ (the abelianization of $\pi_1(X)$) to $\tilde{H}_1(X)$. We can now state the Hurewicz theorem which gives a connection between homology groups and homotopy groups.

**Theorem 4.1** (Hurewicz)**.** *If $X$ is $(n-1)$-connected then $h$ is an isomorphism if $n > 1$ and if $n = 1$ the induced map on $\pi_1'(X) \to \tilde{H}_1(X)$ is an isomorphism.*

*Proof.* If $X$ is a sphere then $h([id]) = i_n$ and our previous calculation for $\pi_n(S^n)$ shows that the theorem holds in this case. In the general case we may assume that $X$ is a CW-complex with 1 vertex and no $q$-cells for $q < n$ by the CW approximation theorem. Cellular approximation of maps implies that the inclusion $X^{n+1} \hookrightarrow X$ is an $n+1$-equivalence. It is easy to see by the CW construction of homology that this inclusion induces an isomorphism on $\tilde{H}_n$ as well because $\tilde{H}_n$ is determined solely by the terms of the chain complex in degrees $n$ and $n+1$. Thus we may assume without loss of generality that $X = X^{n+1}$.

Let $L = X^n$ and let $K = \vee_{\alpha \in A} S^n_\alpha$ where $A$ is the set of $n+1$ cells in $X$. Define $f : K \to L$ by letting it equal the attaching map of $\alpha$ on $S^n_\alpha$. Then $X$ is the mapping cone of $f$ (unreduced). The maps $K \xrightarrow{f} L \hookrightarrow X$ induces maps on homology and homotopy and our goal is to show that the rows in the commutative diagram

$$\begin{tikzcd}
\pi_n(K) \arrow[r] \arrow[d] & \pi_n(L) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & 0 \\
\tilde{H}_n(K) \arrow[r] & \tilde{H}_n(L) \arrow[r] & \tilde{H}_n(X) \arrow[r] & 0
\end{tikzcd}$$

are exact. Since $K, L$ are wedges of spheres the theorem holds for them and we can then apply the five lemma to get the result for $X$. The diagram is commutative because $h$ is natural. This comes from the fact that if $g : X \to Y$ is a map of spaces and $\sigma : S^n \to X$ is an element of $\pi_n(X)$ then $(f \circ \sigma)_*(i_n) = f_*(\sigma_*(i_n))$. $\tilde{H}_n(L)$ is the free abelian group generated by the $n$-cells and a linear compbination of $n$-cells maps to 0 in $\tilde{H}_n(X)$ if and only if it is the boundary of discs. Since the boundaries of discs are given precisely by the image of $K$ we have exactness of the $\tilde{H}$ sequence at $L$. The cokernel of the map $\tilde{H}_n(L) \to \tilde{H}_n(X)$ should be $H_n(X, L) = \tilde{H}_n(X/L)$, but $X/L$ is a wedge of $n+1$ spheres so it has trivial $\tilde{H}_n$. Thus the bottom row is exact.

For $n > 1$ we may write $f$ as the composition $K \to Mf \xrightarrow{p} L$ where $p$ is the natural homotopy equivalence. This gives us the commutative diagram

$$\begin{tikzcd}
\pi_n(K) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_n(Mf) \arrow[r] \arrow[d, "p_*"] & \pi_n(Mf, K) \arrow[r] \arrow[d] & 0 \\
\pi_n(K) \arrow[r] & \pi_n(L) \arrow[r] & \pi_n(X) \arrow[r] & 0
\end{tikzcd}$$

in which the two left vertical arrows are isomoprphisms. The top row is exact because the cokernel of the last map is $\pi_{n-1}(K)$ and this is trivial because $K$ is a wedge of $n$ spheres. The right vertical arrow is an isomorphism by the first corollary to the homotopy excision theorem. Thus the bottom row is exact and we are done with the case $n > 1$. In the case with $n = 1$ the top row is not exact because the kernel of the map $\pi_1(Mf) \to \pi_1(Mf, K)$ is the *normal subgroup generated* by $\pi_1(K)$. But then it becomes exact when we take the abelianization of the whole diagram and this gives us the $n = 1$ case.

□

## References

[1] May, J. P. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[2] Hatcher, Allen. Algebraic Topology. Cambridge University Press. 2002.