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Chapter 3

LP Formulations

This is Chapter 3 of "Junior Level Web-Book for Optimization Models for decision Making" by Katta G. Murty.

3.1 Category 2 Decision Making Problems

As defined in Chapter 1, the main feature of these decision problems is that they have **decision variables** whose values the decision maker(s) can control, subject to constraints and bound restrictions on them imposed by the manner in which the relevent system must operate. A solution to the decision problem specifies numerical values for all the decision variables. A **feasible solution** is one which satisfies all the constraints and bound and other restrictions. Even to identify a feasible solution, or to represent the set of all feasible solutions, we need to construct a mathematical model of all the constraints and restrictions. Usually an objective function to be minimized is also specified, then the goal is to find an **optimum solution** which minimizes the objective function value among all the feasible solutions; this leads to an **optimization model**.

Among all optimization models, the **linear programming (LP)** model is the most highly developed. LP theory is very rich and beautiful and extensive, it has efficient computational procedures worked out for any kind of problem related to LP models that practitioners may need in applications. So we begin our study of this category of problems with the study of those that can be modeled with an LP model. This strategy offers the following advantages to the readers.

1) Among optimization models, LP models are the simplest to costruct and understand, so, beginning with their study is a good first step to learn the skill in modeling.

2) The simplex method for solving LPs is an extension of the classical GJ method for solving linear equations. It is easy to understand, and study, and it helps you to appreciate algorithms and how they work.

3) LP has numerous applications in a wide variety of areas. So, knowing when LP may be an appropriate model for a decision problem, how to construct an LP model for such a problem, the algorithms to solve that model, and how to implement the output of that algorithm; are essential skills that anyone aspiring to a decision making career must have.

Ideally a decision problem involving the decision variables $x = (x_1, \ldots, x_n)^T$ can be modeled as an LP if the following conditions are satisfied:

- All the decision variables are continuous variables with possibly only lower and/or upper bound restrictions.
- There is only one objective function to be optimized in the problem. The objective function, and the constraint functions for the constraints are all linear functions; i.e., functions of the mathematical form $a_1x_1 + \ldots + a_nx_n$ where $a_1, \ldots, a_n$ are constants known as the **coefficients** for the function.

The second condition above is known as the **linearity assumption** or **property** and is often stated in words broken into two separate assumptions as given below: A function $f(x)$ of decision variables $x = (x_1, \ldots, x_n)^T$ is a linear function of x if it satisfies the following two assumptions:

Additivity (or separability) assumption: $f(x)$ is the sum of n different functions each one involving only one of the decision variables; i.e., there exist functions of one variable $f_j(x_j)$, $j = 1$ to n such that $f(x) = f_1(x_1) + \ldots + f_n(x_n)$.

Caution: In defining $f(x) = f_1(x_1) + \ldots + f_n(x_n)$ satisfying the additivity assumption, one should make sure that all the component functions $f_1(x_1), \ldots, f_n(x_n)$ are in the same units so that their addition makes sense. For example, if $f_1(x_1)$ is expressed in \$, and $f_2(x_2)$ is expressed in tons, their direct addition does not make sense.

Proportionality assumption: The contribution of each variable x_j to $f(x)$ is proportional to the value of x_j. So, when $f(x)$ satisfies the additivity assumption stated above, this proportionality assumption implies that $f_j(x_j)$ must be of the form $a_j x_j$ for some constant a_j for $j = 1$ to n.

Note on applying LP to real world problems: It has been mentioned earlier that constructing a mathematical model for real world problems often involves simplification, approximation, human judgement, and relaxing features that are difficult to represent mathematically. In many applications the above assumptions for the validity of using an LP model may not hold exactly. Even when a linear approximation to the objective function or a constraint function is acceptable, the coefficients of the variables in it may only be known for the time being, their values may change over time in a random and unpredictable fashion. If the violations in the assumptions are significant or fundamental, then LP is not the appropriate technique to model the problem. An example of this is given in Section 3.8. But when the disparities are minor, an LP model constructed with estimated coefficients as an approximation, may lead to conclusions that are reasonable for the real problem, at least for the time being.

Actually the linearity assumptions provide reasonably good approximations in many applications, this and the relative ease with which LPs can be solved have made LP useful in a vast number of applications.

3.2 The Scope of LP Modeling Techniques Discussed in this Chapter

There are many applications in which the reasonableness of the linearity assumptions can be verified, and an LP model for the problem constructed by direct arguments. We restrict the scope of this chapter to such applications. Even in this restricted set, there are many different classes of applications of LP models. We present some of them in the following sections. In all these applications you can judge intutively that the assumptions needed to handle them using an LP model are satisfied to a reasonable degree of approximation, so we will not highlight this issue again.

Of course LP can be applied on a much larger class of problems. Many important applications involve optimization models with linear constraints, and a nonlinear objective function to be minimized which is piecewise linear and convex. These problems can be transformed into LPs by introducing additional variables. These techniques are discussed in graduate level books and are beyond the scope of this book.

Also, in this chapter we focus only on how to construct a mathematical model for the problem, not on algorithms for solving the models. In Section 3.12 we discuss a special geometric procedure for solving simple LP models involving only two variables. This is to provide geometric intuition, and to discuss all the useful planning information that can be derived from the solution of the LP model.

LPs involving 3 or more variables are solved by numerical procedures which are discussed in Chapter 4.

3.3 Each Inequality Constraint Contains a Hidden New Variable Called its Slack Variable

Linear Equations and Hyperplanes, Linear Inequalities and Half-Spaces, Convex Polyhedra, Convex Polytopes

In an LP model, the conditions on the decision variables form a system of linear constraints consisting of linear equations and/or inequalities (even a bound on a variable is in fact a linear inequality). The set of feasible solutions of such a system of linear constraints is called a **convex polyhedron**. If a convex polyhedron is a bounded set, it is called a **convex polytope**.

In the space R^n of decision variables $x = (x_1, \ldots, x_n)^T$, the set of feasible solutions of a single linear equation

$$a_1x_1 + \ldots + a_nx_n = a_0$$

where the vector of coefficients $(a_1, \ldots, a_n) \neq 0$, is called a **hyperplane**. Each hyperplane in R^n divides it into two **half-spaces** which are on either side of this hyperplane. For example, the two half-spaces into which the hyperplane defined by the above equation divides R^n are the set of feasible solutions of one of these two inequalities: $a_1x_1 + \ldots + a_nx_n \geq a_0$ or $a_1x_1 + \ldots + a_nx_n \leq a_0$.

So, each half-space is the set of feasible solutions of a single linear ineuality; and a hyperplane is the intersection of the two half-spaces into which it divides the space.

From this we can see that every convex polyhedron is the intersection of a finite number of half-spaces. We will provide illustrative examples in R^2 shortly, also you can look up Chapter 3 in reference [1.1].

Notation to Denote Points

We use superscripts to enumerate points in a set. For example the symbol $x^r = (x^r_1, \ldots, x^r_n)^T$ denotes the rth point written as a column vector, and $x^r_1, \ldots, x^r_n$ are its components, i.e., the values of the various decision variables $x_1, \ldots, x_n$ in it.

Slack Variables in Inequality Constraints

Each inequality constraint contains in itself a new nonnegative variable called its **slack variable**. As an example, consider the inequality constraint in original variables x_1, x_2, x_3:

$$2x_1 - 7x_2 - 4x_3 \leq 6$$

This constraint allows its constraint function $2x_1 - 7x_2 - 4x_3$ to take any value less than or equal to 6. The constraint can be written in an equivalent manner as $6 - (2x_1 - 7x_2 - 4x_3) \geq 0$. If we define $s_1 = 6 - (2x_1 - 7x_2 - 4x_3)$, then s_1, a new variable required to be nonegative by the original constraint, is known as the *slack variable* corresponding to this constraint; it represents the amount by which the value of the constraint function $(2x_1 - 7x_2 - 4x_3)$ lies below its upper bound of 6. The original inequality constraint can be written in an equivalent manner in the form of an equation involving its nonnegative slack variable as:

$$\begin{aligned} 2x_1 - 7x_2 - 4x_3 + s_1 &= 6 \\ s_1 &\geq 0 \end{aligned}$$

One should not think that the original inequality has become an equation when its slack variable is introduced, actually the inequality has been transferred from the constraint into the nonnegativity restriction on its slack variable.

In the same manner, if there is a constraint of the form $-8x_1 + 16x_2 - 19x_3 \geq -4$, it can be written in an equivalent manner as $s_2 = -8x_1 + 16x_2 - 19x_3 - (-4) \geq 0$, and the new variable s_2 is the slack variable corresponding to this inequality constraint. This constraint can be written in the form of an equation including its nonnegative slack variable as:

$$\begin{aligned} -8x_1 + 16x_2 - 19x_3 - s_2 &= -4 \\ s_2 &\geq 0 \end{aligned}$$

Notice that the coefficient of the slack variable in the equation into which the original inequality is converted is $+1$ or -1 depending on whether the original inequality is a "$\leq$" or a "$\geq$" inequality.

In some books the name slack variable is only used for those new variables corresponding to "$\leq$" constraints; the new variables corrresponding to "$\geq$" constraints are called *surplus variables* in those books. We will not make this distinction, we will include all of them under the phrase *slack variables.* Remember that each inequality constraint in the problem leads to a different slack variable, and that all slack variables are nonnegative variables.

When all the inequality constraints whose constraint functions involve two or more variables in an LP model are transformed into equations by introducing the appropriate slack variables; the remaining system consists of linear equations and lower and/or upper bound restrictions on individual variables.

Slack variables are just as important as the other variables in the original model. Their values in an optimum solution provide very useful planning information. This will be illustrated later.

Infeasible, Active, Inactive Inequality Constraints WRT a Given Point

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in R^n$ be a given point. Consider an inequality constraint in the "$\geq$" form

$$a_1x_1 + \ldots + a_nx_n \geq a_0.$$

Definitions similar to those given below for this inequality, also hold for inequalities in the "$\leq$" form with appropriate modifications.

$\bar{x}$ is said to be **infeasible** for this constraint if $\bar{x}$ violates it (i.e., if $a_1\bar{x}_1 + \ldots + a_n\bar{x}_n < a_0$), in this case this inequality constraint is infeasible at $\bar{x}$. $\bar{x}$ is **feasible** for this constraint if it is satisfied at this point (i.e., if $a_1\bar{x}_1 + \ldots + a_n\bar{x}_n \geq a_0$).

Suppose $\bar{x}$ is feasible to the above inequality constraint. Then this constraint is said to be an **active** or **tight inequality constraint** at $\bar{x}$ if it holds as an equation there (i.e., if $a_1\bar{x}_1 + \ldots + a_n\bar{x}_n = a_0$), or an **inactive** or **slack inequality constraint** at $\bar{x}$ otherwise (i.e., if $a_1\bar{x}_1 + \ldots + a_n\bar{x}_n > a_0$).

As we will see later, this classification of inequality constraints into active, inactive types at a desired point is the key to solving LPs, and systems of linear inequalities.

Example 3.3.1: Consider the following system of constraints in two variables x_1, x_2:

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The convex polyhedron K, which is the set of feasible solutions of this system is the shaded region in Figure 3.1. In R^2 every hyperplane is a straight line (not so in spaces of dimension 3 or more). To draw the half-space corresponding to $x_1 + x_2 \leq 5$, we draw the straight line L corresponding to $x_1 + x_2 = 5$, and check which side of L contains points satisfying $x_1 + x_2 \leq 5$, and mark that side with an arrow on L as the desired half-space. Other half-spaces corresponding to the remaining inequalities in the above system are drawn in the same way. K is the region common to all the four half-spaces. Since K is bounded it is a convex polytope.

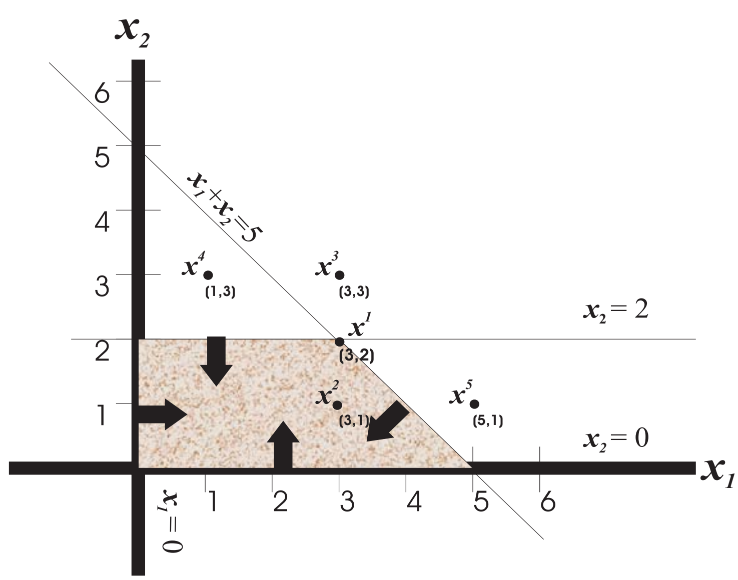

Figure 3.1: A convex polytope

Introducing the slack variables s_1, s_2 corresponding to the 1st, and 2nd constraints respectively, the system becomes

$$\begin{array}{cccccc} x_1 & +x_2 & +s_1 & & = & 5 \\ & x_2 & & +s_2 & = & 2 \\ x_1, & x_2, & s_1, & s_2 & \geq & 0 \end{array}$$

which is an equivalent system of 2 equations in 4 nonnegative variables. Here $s_1 = 5 - x_1 - x_2$, $s_2 = 2 - x_2$. So, the values of the slack variables at the point $x^1 = (x_1^1, x_2^1) = (3, 2)^T$ are $s_1^1 = 5 - x_1^1 - x_2^1 = 0$, $s_2^1 = 2 - x_2^1 = 0$. Since $x_1^1, x_2^1, s_1^1, s_2^1$ are all ≥ 0, we conclude that x^1 is a feasible solution of the original system (we verify that it is in K in Figure 3.1), and that it corresponds to $(x_1^1, x_2^1, s_1^1, s_2^1)^T = (3, 2, 0, 0)^T$ of the transformed system. Also, since the values of both the slack variables s_1, s_2 are 0 at x^1, it indicates that this point satisfies the corresponding constraints in the original system, the 1st and the 2nd, as equations. So, the first two constraints in the original system are active at x^1, while the last two constraints (nonnegativity restrictions) are inactive at x^1.

In the same way we find the following about other points marked in Figure 3.1:

$x^2 = (x_1^2, x_2^2) = (3, 1)^T$ corresponds to $(x_1^2, x_2^2, s_1^2, s_2^2)^T = (3, 1, 1, 1)^T$ of the transformed system, this point is feasible to the original system since all 4 variables are ≥ 0 at it. Also, since s_1, s_2 are both $= 1$ at x^2, it indicates that this point satisfies both the 1st and 2nd constraints in the original syatem as strict inequalities; and the actual numerical values of s_1^2, s_2^2 give measures of how far away x^2 is from satisfying these constraints as equations. All the constraints in the original system are inactive at x^2.

$x^3 = (x_1^3, x_2^3) = (3, 3)^T$ corresponds to $(x_1^3, x_2^3, s_1^3, s_2^3)^T = (3, 3, -1, -1)^T$ of the transformed system, this point is infeasible to the original system since the slack variables are both negative at it. It violates both the 1st and 2nd constraints in the original system. The first two constraints in the original system are infeasible at x^3.

$x^4 = (x_1^4, x_2^4) = (1, 3)^T$ corresponds to $(x_1^4, x_2^4, s_1^4, s_2^4)^T = (1, 3, 1, -1)^T$ of the transformed system, this point is infeasible to original system since the slack variable s_2 is negative at it. It satisfies the 1st constraint in the original system, but violates the 2nd. x^4 is feasible to all the constraints in the original system except the 2nd.

$x^5 = (x_1^5, x_2^5) = (1,3)^T$ corresponds to $(x_1^5, x_2^5, s_1^5, s_2^5)^T = (5,1,-1,1)^T$ of the transformed system, this point is infeasible to original system since the slack variable s_1 is negative at it. It satisfies the 2nd constraint in the original system, but violates the 1st. x^5 is feasible to all the constraints in the original system except the 1st.

Exercises

3.3.1: Transform the following systems of linear constraints into systems in which all the conditions on the variables are either linear equations with nonnegative RHS constants, or bounds on individual variables. In each exercise give the expression for each new variable introduced, in terms of the original variables.

$$\begin{array}{rcl}
\textbf{(i):}\quad 2x_2 - 3x_1 - 17x_3 & \geq & -6 \\
-18x_2 + 7x_4 + 2x_3 & \leq & -7 \\
2x_4 + 8x_3 - 4x_1 - 5x_2 & \geq & 2 \\
-3x_3 + 2x_4 + x_1 & \geq & 0 \\
x_1 - x_2 + x_3 - 4x_4 & = & -2 \\
-2 \leq x_1 \leq 6, x_2 \geq 0, x_3 & \leq & 0.
\end{array}$$

$$\begin{array}{rcl}
\textbf{(ii):}\quad x_1 + x_2 - x_3 - x_4 & \geq & 8 \\
x_1 - x_2 - x_3 + x_4 & \leq & 16 \\
10 \leq x_1 + x_2 + x_3 + x_4 & \leq & 20 \\
-3 \leq -x_1 + x_2 - x_3 + x_4 & \leq & 15 \\
x_1 \geq 6, x_2 \leq 7, x_3 \geq 0, x_4 & \leq & 0.
\end{array}$$

$$\begin{array}{rcl}
\textbf{(iii):}\quad -20 \leq x_1 - x_2 - x_3 - x_4 & \leq & -10 \\
-x_1 + x_2 + x_3 - x_4 & \geq & -6 \\
x_1 + x_2 + x_3 + x_4 & \leq & 100 \\
2x_1 - 3x_2 + 9x_3 & = & 30 \\
4 \leq x_2 \leq 10; x_3, x_4 & \geq & 0.
\end{array}$$

3.3.2: In the following system transform the variables so that all lower bound conditions on individual variables become nonnegativity restrictions on the new variables. Then in the resulting system, introduce appropriate slack variables thereby transforming it into a system of linear equations in nonnegative variables.

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &\geq 6 \\ x_1 + x_2 - x_3 + x_4 &\leq 24 \\ x_1 - x_2 + x_3 + x_4 &= 33 \\ -4 \leq x_1 \leq 10, x_2 \geq -7, x_3 \geq 2, x_4 &\geq 4. \end{aligned}$$

3.3.3: In the following systems there are some variables which only have an upper bound restriction but no lower bound restriction. Transform these into lower bound restrictions on new variables. Then on the resulting systems, follow the instructions in Exercise 3.3.2 to transform them into systems of linear equations in nonnegative variables.

$$\begin{aligned} \textbf{(i):}\quad x_1 + 2x_2 + 2x_3 + 2x_4 &\geq 10 \\ 2x_1 + x_2 + 2x_3 + 2x_4 &\leq 100 \\ 20 \leq 2x_1 + 2x_2 + x_3 + 2x_4 &\leq 110 \\ 2 \leq x_1 \leq 40, x_2 \leq 50, x_3 \leq 60, x_4 &\geq 5 \end{aligned}$$

$$\begin{aligned} \textbf{(ii):}\quad x_1 + x_2 - 2x_3 &\geq 5 \\ x_1 - 2x_2 + x_3 &\leq 25 \\ -2x_1 + x_2 + x_3 &= 13 \\ x_1 \leq 0, x_2 \geq 2, 4 \leq x_3 &\leq 30. \end{aligned}$$

3.3.4: Set up the Cartesian coordinate system and draw the set of feasible solutions of each of the following systems of constraints in variables x_1, x_2.

If K is the set of feasible solutions, is it a convex polytope? Why?

Number the constraints in the system with 1 to 6, and identify each boundary edge of K with the constraint to which it corresponds. Are the points $\bar{x}, \hat{x}, \tilde{x}$ feasible to the system? Which of these points are in K? At each of these points which are feasible, classify the constraints into active, inactive types.

Introduce slack variables corresponding to all the constraints in the system other than the nonnegativity restrictions on x_1, x_2, thereby transforming the system into a system of 4 linear equations in 6 nonnegative variables. Compute the values of all the slack variables at the points $\bar{x}, \hat{x}, \tilde{x}$ respectively. Explain how you can decide whether the point is in K or not from the signs of the values of the slack variables at it.

Explain how the value of a slack variable at the points $\bar{x}, \hat{x}, \tilde{x}$ can be interpreted as a measure of how far away that point is from satisfying the corresponding constraint as an equation.

$$\textbf{(i):} \quad \begin{array}{rcr} 3x_1 + 2x_2 & \leq & 12 \\ x_1 + 2x_2 & \leq & 6 \\ -x_1 + x_2 & \leq & 1 \\ x_2 & \leq & 2 \\ x_1, x_2 & \geq & 0 \end{array}, \qquad \begin{array}{l} \bar{x} = (3,1)^T \\ \hat{x} = (1,0)^T \\ \tilde{x} = (4,1)^T \end{array}$$

$$\textbf{(ii):} \quad \begin{array}{rcr} x_2 - x_1 & \leq & 2 \\ x_2 - x_1 & \geq & -2 \\ x_1 - 2x_2 & \leq & 1 \\ x_2 - 2x_1 & \leq & 1 \\ x_1, x_2 & \geq & 0 \end{array}, \qquad \begin{array}{l} \bar{x} = (2,1)^T \\ \hat{x} = (1,3)^T \\ \tilde{x} = (1,4)^T. \end{array}$$

3.4 Product Mix Problems

Product mix problems are an extremely important class of problems that manufacturing companies face. Normally the company can make a variety of products using the raw materials, machinery, labor force, and other resources available to them. The problem is to decide how much of each product to manufacture in a period, to maximize the total profit subject to the availability of needed resources.

To model this, we need data on the units of each resource necessary to manufacture one unit of each product, any bounds (lower, upper, or both) on the amount of each product manufactured per period, any bounds on the amount of each resource available per period, and the cost or net profit per unit of each product manufactured.

Assembling this type of reliable data is one of the most difficult jobs in constructing a product mix model for a company, but it is very worthwhile. A product mix model can be used to derive extremely useful planning information for the company. The process of assembling all the needed data is sometimes called **input-output analysis** of the company. The coefficients, which are the resources necessary to make a unit of each product, are called **input-output (I/O) coefficients**, or **technology coefficients**.

Example 3.4.1

As an example, consider a fertilizer company that makes two kinds of fertilizers called Hi-phosphate (Hi-ph) and Lo-phosphate (Lo-ph). The manufacture of these fertilizers requires three raw materials called RM 1, 2, 3. At present their supply of these raw materials comes from the company's own quarry which is only able to supply maximum amounts of 1500, 1200, 500 tons/day respectively of RM 1, RM 2, RM 3. Even though there are other vendors who can supply these raw materials if necessary, at the moment they are not using these outside suppliers.

They sell their output of Hi-ph, Lo-ph fertilizers to a wholesaler who is willing to buy any amount that they can produce, so there are no upper bounds on the amounts of Hi-ph, Lo-ph manufactured daily.

At the present rates of operation their Cost Accounting Department estimates that it is costing the quarry \$50, 40, 60/ton respectively to produce and deliver RM 1, 2, 3 at the fertilizer plant. Also, at the present rates of operation, all other production costs (for labor, power, water, maintenance, depreciation of plant and equipment, floorspace, insurance, shipping to the wholesaler, etc.) come to \$7/ton to manufacture Hi-ph, or Lo-ph and deliver to wholesaler.

The sale price of the manufactured fertilizers to the wholesaler fluctuates daily, but their averages over the last one month have been \$222, 107/ton respectively for Hi-Ph, Lo-ph fertilizers. We will use these prices for constructing the mathematical model.

The Hi-ph manufacturing process needs as inputs 2 tons RM 1, and 1 ton each of RM 2, 3 for each ton of Hi-ph manufactured. Similarly the Lo-ph manufacturing process needs as inputs 1 tons RM 1, and 1 ton of RM 2 for each ton of Lo-ph manufactured. So, the net profit/ton of fertilizer manufactured is $\$(222 - 2 \times 50 - 1 \times 40 - 1 \times 60 - 7) = 15$, $(107 - 1 \times 50 - 1 \times 40 - 7) = 10$/respectively of Hi-ph, Lo-ph.

We will model the problem with the aim of determining how much of Hi-ph, Lo-ph to make daily to maximize the total daily net profit from these fertilizer operations. All the relevant data is summarized in Table 3.4.1.

Table 3.4.1

Item	Tons required to make one ton of		Maximum amount of item available daily (tons)
	Hi-ph	Lo-ph	
RM 1	2	1	1500
RM 2	1	1	1200
RM 3	1	0	500
Net profit ($) per ton made	15	10	

So, in this example, the Hi-ph manufacturing process can be imagined as a black box which takes as input a packet consisting of 2 tons RM 1, 1 ton RM 2, and 1 ton RM 3; and outputs 1 ton of Hi-ph. See the following figure. A similar interpretation can be given for the Lo-ph making process.

$$\left.\begin{array}{r}\text{2 tons RM 1}\\ \text{1 ton RM 2}\\ \text{1 ton RM 3}\end{array}\right\} \longrightarrow \boxed{\begin{array}{l}\text{Hi-ph}\\ \text{making}\\ \text{process}\end{array}} \longrightarrow \text{1 ton Hi-ph}$$

Constructing a mathematical model for the problem involves the following steps.

Step 1: Make a list of all the decision variables

The list must be complete in the sense that if an optimum solution providing the values of each of the variables is obtained, the decision maker should be able to translate it into an optimum policy that can be implemented. In product mix models, there is one decision variable for each possible product the company can produce, it measures the amount of that product made per period.

In our example, there are clearly two decision variables; these are:

$$\begin{aligned} x_1 &= \text{the tons of Hi-ph made per day} \\ x_2 &= \text{the tons of Lo-ph made per day} \end{aligned}$$

Associated with each variable in the problem is an **activity** that the decision maker can perform. The activities in this example are:

Activity 1 : to make 1 ton of Hi-ph
Activity 2 : to make 1 ton of Lo-ph

The variables in the problem just define the **levels** at which these activities are carried out. So, one way of carrying out this step is to make a list of all the possible activities that the company can perform, and associate a variable that measures the level at which it is carried out, for each activity.

Even though it is mathematically convenient to denote the decision variables by symbols x_1, x_2, etc., practitioners find it very cumbersome to look up what each of these variables represents in the practical problem. For this reason they give the decision variables suggestive names, for example x_1, x_2 here would be called Hi-ph, Lo-ph instead.

Step 2: Verify that the linearity assumptions and the assumption on the continuity of the decision variables hold

Since all the data is given on a per ton basis, it provides an indication that the linearity assumptions are quite reasonable in this problem. Also, the amount of each fertilizer manufactured can vary continuously within its present range. So, LP is an appropriate model for this problem.

In some applications, variables may be restricted to take only integer values (e.g., if the variable represents the number of empty buses transported from one location to another). Such restrictions make the problem an **integer program**. However, sometimes people ignore the integer restrictions on integer variables and treat them as continuous variables. If the linearity assumptions hold, this leads to the **LP relaxation** of the integer program.

Step 3: Construct the objective function

By the linearity assumptions the objective function is a linear function, it is obtained by multiplying each decision variable by its profit (or cost) coefficient and summing up.

In our example problem, the objective function is the total net daily profit, $z(x) = 15x_1 + 10x_2$, and it has to be maximized.

Step 4: Identify the Constraints on the Decision Variables

Nonnegativity constraints

In product mix models the decision variables are the amounts of various products made per period; these have to be nonnegative to make any practical sense. In linear programming models in general, the nonnegativity restriction on the variables is a natural restriction that occurs because certain activities (manufacturing a product, etc.) can only be carried out at nonnegative levels.

The nonnegativity restriction is a lower bound constraint. Sometimes it may be necessary to impose a positive lower bound on a variable. This occurs if we have a commitment to make a minimum quantity, ℓ_j units say, where $\ell_j > 0$, of product j. Then the lower bound constraint on the decision variable x_j = amount of product j manufactured, is $x_j \geq \ell_j$.

There may be an upper bound constraint on a variable too. This occurs if we know that only a limited quantity, say u_j units, of product j can be either sold in a period or stored for use later on, then $x_j \leq u_j$ is the upper bound constraint on x_j = the amount of product j made in that period.

On some decision variables there may be both a lower and an upper bound constraint.

In our example problem the bound restrictions are: $x_1, x_2 \geq 0$.

Items and the associated constraints

There may be other constraints on the variables, imposed by lower or upper bounds on certain goods that are either inputs to the production process or outputs from it. Such goods that lead to constraints in the model are called **items**. Each item leads to a constraint on the decision variables, and conversely every constraint in the model is associated with an item. Make a list of all the items that lead to constraints.

In the fertilizer problem each raw material leads to a constraint. The amount of RM 1 used is $2x_1 + x_2$ tons, and it cannot exceed 1500, leading to the constraint $2x_1 + x_2 \leq 1500$. Since this inequality compares the amount of RM 1 used to the amount available, it is called a **material balance inequality**. The material balance equations or inequalities corresponding to the various items are the constraints in the problem.

When all the constraints are obtained, the formulation of the problem as an LP is complete. The LP formulation of the fertilizer product mix problem is given below.

$$
\begin{array}{llcccccl}
\text{Maximize} & z(x) = 15x_1 & + & 10x_2 & & & & \text{Item} \\
\text{Subject to} & 2x_1 & + & x_2 & \leq & 1500 & & \text{RM 1} \\
 & x_1 & + & x_2 & \leq & 1200 & & \text{RM 2} \\
 & x_1 & & & \leq & 500 & & \text{RM 3} \\
 & x_1 & \geq 0, & x_2 & \geq & 0 & &
\end{array}
$$

Slack Variables and Their Interpretation

After introducing x_3, x_4, x_5, the slack variables for RM 1, 2, 3 constraints, the fertilizer product mix problem in detached coefficient form is the following.

Tableau 2.2

x_1	x_2	x_3	x_4	x_5	$=$	
2	1	1	0	0	1500	
1	1	0	1	0	1200	
1	0	0	0	1	500	
15	10	0	0	0	$z(x)$	maximize

x_1 to $x_5 \geq 0$

Here, the slack variable x_3 corresponding to the RM 1 constraint is $= 1500 - 2x_1 - x_2$, it represents the amount of RM 1 remaining unutilized in the daily supply, after implementing the solution vector $x = (x_1, x_2)^T$. In the same manner, the slack variables x_4, x_5 corresponding to the RM2, RM3 constraints are respectively $= 1200 - x_1 - x_2$, and $500 - x_1$; and they represent the quantities of RM 2, RM 3 left unused if solution vector x is implemented. So, these slack variable values at the optimum solution contain valuable planning information.

To model any problem as an LP we need to go through the same Steps 1 to 4 given above. Examples from other classes of applications are discussed in the following sections.

Limitations of the Model Constructed Above

In real world applications, typically after each period there may be changes in the profit or cost coefficients, the RHS constants (availabilities of items), and technology coefficients. Also, new products may come on stream and some old products may fade out. So, most companies find it necessary to revise their product mix model and solve it afresh at the beginning of each period.

Other limitations of the LP model constructed above for the fertilizer problem can be noticed. It is based on the assumption that only the raw material supplies coming from the company's own quarry can be used. Businesses are always looking for ways to improve their profits. If our fertilizer company decides to explore whether getting additional supplies from outside vendors will improve their profit, the model changes completely. In the same way, companies have to revise their product mix models to reflect changes in company policies, or market conditions.

3.5 Blending Problems

This is another large class of problems in which LP is applied heavily. Blending is concerned with mixing different materials called the **constituents** of the mixture (these may be chemicals, gasolines, fuels, solids, colors, foods, etc.) so that the mixture conforms to specifications on several properties or characteristics.

To model a blending problem as an LP, the **linear blending assumption** must hold for each property or characteristic. This implies that the value for a characteristic of a mixture is the weighted average of the values of that characteristic for the constituents in the mixture; the weights being the proportions of the constituents. As an example, consider a mixture consisting of 4 barrels of fuel 1 and 6 barrels of fuel 2, and suppose the characteristic of interest is the octane rating (Oc.R). If linear blending assumption holds, the Oc.R of the mixture will be equal to (4 times the Oc.R of fuel 1 + 6 times the Oc.R of fuel 2)/(4 + 6).

The linear blending assumption holds to a reasonable degree of precision for many important characteristics of blends of gasolines, of crude oils, of paints, of foods, etc. This makes it possible for LP to be used extensively in optimizing gasoline blending, in the manufacture of paints, cattle feed, beverages, etc.

The decision variables in a blending problem are usually either the quantities or the proportions of the constituents in the blend.

If a specified quantity of the blend needs to be made, then it is convenient to take the decision variables to be the quantities of the various constituents blended; in this case one must include the constraint that the sum of the quantities of the constituents = the quantity of the blend desired.

If there is no restriction on the amount of blend made, but the aim is to find an optimum composition for the mixture, it is convenient to take the decision variables to be the proportions of the various constituents in the blend; in this case one must include the constraint that the sum of all these proportions is 1.

Example 3.5.1: **A gasoline blending problem:**

There are more than 300 refineries in the USA processing a total of more than 20 million barrels of crude oil daily. Crude oil is a complex mixture of chemical components. The refining process separates crude oil into its components that are blended into gasoline, fuel oil, asphalt, jet fuel, lubricating oil, and many other petroleum products. Refineries and blenders strive to operate at peak economic efficiencies taking into account the demand for various products.

As an example we consider a gasoline blending problem. To keep it simple, we consider only one characteristic of the mixture, the Oc.R. of the blended fuels, in this example. In actual application there are many other characteristics to be considered also.

A refinery takes four raw gasolines, blends them to produce three types of fuel. The company sells raw gasoline not used in making fuels at \$38.95/barrel if its Oc.R is > 90, and at \$36.85/barrel if its Oc.R is ≤ 90.

The cost of handling raw gasolines purchased, and blending them into fuels or selling them as is, is estimated to be \$2/barrel by the Cost Accounting Department. Other data is given below.

Raw gas type	Octane rating (Oc.R)	Available daily (barrels)	Price per barrel
1	68	4000	\$31.02
2	86	5050	33.15
3	91	7100	36.35
4	99	4300	38.75

Fuel type	Minimum Oc.R	Selling price (barrel)	Demand
1	95	$47.15	At most 10,000 barrels/day
2	90	44.95	No limit
3	85	42.99	At least 15,000 barrels/day

The problem is to determine how much raw gasoline of each type to purchase, the blend to use for the three fuels, and the quantities of these fuels to make to maximize total daily net profit.

To model this problem, we will use the quantities of the various raw gasolines in the blend for each fuel as the decision variables, and we assume that the linear blending assumption stated above holds for the Oc.R. Since three different fuels are under consideration, it is convenient to use a double subscript notation to denote the blending decision variables as given below.

$$
\begin{aligned}
RG_i &= \text{barrels of raw gasoline type } i \text{ to purchase/day, } i = 1 \text{ to } 4\\
x_{ij} &= \begin{cases} \text{barrels of raw gasoline type } i \text{ used in making fuel} \\ \text{type } j \text{ per day, } i = 1 \text{ to } 4,\ j = 1,2,3 \end{cases}\\
y_i &= \text{barrels of raw gasoline type } i \text{ sold as is/day}\\
F_j &= \text{barrels of fuel type } j \text{ made/day, } j = 1,\ 2,\ 3.
\end{aligned}
$$

So, the total amount of fuel type 1 made daily is $F_1 = x_{11} + x_{21} + x_{31} + x_{41}$. If this is > 0, by the linear blending assumption its Oc.R will be $(68x_{11} + 86x_{21} + 91x_{31} + 99x_{41})/F_1$. This is required to be ≥ 95. Thus, if $F_1 > 0$, we must have

$$\frac{68x_{11} + 86x_{21} + 91x_{31} + 99x_{41}}{F_1} \geq 95$$

In this form the constraint is not a linear constraint since the constraint function on the left is a ratio of two linear functions of the decision variables, and not a linear function. So, if we write the constraint in this form the model will not be an LP. However we see that this constraint is equivalent to

$$68x_{11} + 86x_{21} + 91x_{31} + 99x_{41} - 95F_1 \geq 0$$

and this is a linear constraint. Also, if F_1 = the amount of fuel type 1 made, is zero, all of $x_{11}, x_{21}, x_{31}, x_{41}$ are zero, and above linear inequality holds automatically. Thus, the Oc.R constraint on fuel type 1 can be represented by the above linear constraint for all $F_1 \geq 0$. Proceeding in a similar manner, we obtain the following LP formulation for this problem.

$$\begin{array}{ll}
\text{Maximize} & 47.15F_1 + 44.95F_2 + 42.99F_3 + y_1(36.85 - 31.02) \\
& +y_2(36.85 - 33.15) + y_3(38.95 - 36.35) + y_4(38.95 \\
& -38.75) - (31.02 + 2)RG_1 - (33.15 + 2)RG_2 \\
& -(36.35 + 2)RG_3 - (38.75 + 2)RG_4 \\
\text{Subject to} & RG_i = x_{i1} + x_{i2} + x_{i3} + y_i, \quad i \;=\; 1, \ldots, 4 \\
& 0 \leq (RG_1, RG_2, RG_3, RG_4) \leq (4000, 5050, 7100, 4300) \\
& F_j = x_{1j} + x_{2j} + x_{3j} + x_{4j}, \quad j \;=\; 1, 2, 3 \\
& 0 \leq F_1 \leq 10,000 \\
& F_3 \geq 15,000 \\
& 68x_{11} + 86x_{21} + 91x_{31} + 99x_{41} - 95F_1 \geq 0 \\
& 68x_{12} + 86x_{22} + 91x_{32} + 99x_{42} - 90F_2 \geq 0 \\
& 68x_{13} + 86x_{23} + 91x_{33} + 99x_{43} - 85F_3 \geq 0 \\
& F_2 \geq 0, \quad x_{ij}, y_i \geq 0, \text{ for all } i, j
\end{array}$$

Blending models are economically significant in the petroleum industry. The blending of gasoline is a very popular application. A single grade of gasoline is normally blended from about 3 to 10 individual components, no one of which meets the quality specifications by itself. A typical refinery might have 20 different components to be blended into 4 or more grades of gasoline, and other petroleum products such as aviation gasoline, jet fuel, and middle distillates; differing in Oc.R and properties such as pour point, freezing point, cloud point, viscosity, boiling characteristics, vapor pressure, etc., by marketing region.

Example 3.5.2: **A powdered grains mixing problem:**

There are four grains G_1 to G_4 that contain nutrients, starch, fiber, protein, and gluten. The composition of these grains, and their prices are given below.

	% Nutrient in grain			
	1	2	3	4
Starch	30	20	40	25
Fiber	40	65	35	40
Protein	20	15	5	30
Gluten	10	0	20	5
Cost (cents/kg.)	70	40	60	80

It is required to develop a minimum cost mixture of these grains for making a new type of multigrain flour subject to the following constraints:

1. For taste considerations, the percent of grain 2 in the mix cannot exceed 20, and the percent of grain 3 in the mix has to be at least 30, and the percent of grain 1 in the mix has to be between 10 to 25.

2. The percent protein content in the flour must be at least 18, the percent gluten content has to be between 8 to 13, and the percent fiber content at most 50.

We will now formulate the problem of finding the composition of the least costly blend of the grains to make the flour, using the proportions of the various grains in the blend as decision variables. Let:

$$p_i \quad = \quad \text{proportion of grain } i \text{ in the blend, } i = 1 \text{ to } 4.$$

$p = (p_1, p_2, p_3, p_4)^T$ is the vector of decision variables in this problem. The linear blending assumptions are quite reasonable in this problem. From them, we derive the percent protein content in the blend corresponding to p to be $20p_1 + 15p_2 + 5p_3 + 30p_4$, hence the constraint on protein content in the flour is $20p_1 + 15p_2 + 5p_3 + 30p_4 \geq 18$.

Arguing the same way, we get the following LP model for this problem.

$$
\begin{array}{rl}
\text{Minimize} & 70p_1 + 40p_2 + 60p_3 + 80p_4 \\
\text{Subject to} & 0 \leq p_2 \leq 0.2 \\
& p_3 \geq 0.3 \\
& 0.10 \leq p_1 \leq 0.25 \\
& 20p_1 + 15p_2 + 5p_3 + 30p_4 \geq 18 \\
& 8 \leq 10p_1 + 20p_3 + 5p_4 \leq 13 \\
& 40p_1 + 65p_2 + 35p_3 + 40p_4 \leq 50 \\
& p_1 + p_2 + p_3 + p_4 = 1 \\
& p_4 \geq 0.
\end{array}
$$

The last equality constraint expresses the condition that the sum of the proportions of the various constituents in a mixture must be equal to 1.

3.6 The Diet Problem

A **diet** is a selection of food for consumption in a day.

It has to satisfy many constraints. Perhaps the most important constraint is that it should be palatable to the individual for whom it is intended. This is a very difficult constraint to model mathematically, particularly so if you are restricted to only linear constraints; and the diet is for a human individual. So, most of the early publications on the diet problem have ignored this constraint. Also, these days most of the applications of the diet problem are in the farming sector.

The other important constraint on the diet is that it should meet the MDR (minimum daily requirement) of each nutrient identified as being important for the individual's well-being. This is the only constraint considered in early publications on the diet problem, we will also restrict our scope to only this constraint for the sake of simplicity.

The diet problem is a classic problem, one among the earliest problems formulated as an LP. The first paper on it was published by G. J. Stigler under the title "The Cost of Subsistence" in the *Journal of Farm Economics*, vol. 27, 1945. Those were the war years, food was expensive, and the problem of finding a minimum cost diet was of more than academic interest. Nutrition science was in its infancy in those days, and after extensive discussions with nutrition scientists Stigler identified nine essential nutrient groups for his model. His search of the grocery shelves yielded a list of 77 different available foods. With these, he formulated a diet problem which was an LP involving 77 nonnegative decision variables subject to nine inequality constraints.

Stigler did not know of any method for solving his LP model at that time, but he obtained an approximate solution using a trial and error search procedure that led to a diet meeting the MDR of the nine nutrients considered in the model at an annual cost of $39.93 in 1939 prices! After Dantzig developed the simplex algorithm for solving LPs in 1947, Stigler's diet problem was one of the first nontrivial LPs to be solved by the simplex method on a computer, and it gave the true optimum diet with an annual cost of $39.67 in 1939 prices. So, the trial and error solution of Stigler was very close to the optimum.

The Nobel prize committee awarded the 1982 Nobel prize in economics to Stigler for his work on the diet problem and later work on the functioning of markets and the causes and effects of public regulation.

The units for measuring the various nutrients and foods may be very different, for example carrots may be measured in pounds, chestnuts in kilograms, milk in gallons, orange juice in liters, vitamins in IU, minerals in mg., etc. The data in the diet problem that we are considering consists of a list of nutrients with the MDR for each, a list of available foods with the price and composition (i.e., information on the number of units of each nutrient in each unit of food) of every one of them; and the data defining any other constraints the user wants to place on the diet.

As an example we consider a very simple diet problem in which the nutrients are starch, protein, and vitamins as a group; and the foods are two types of grains with data given below.

Nutrient	Nutrient units/kg. of grain type		MDR of nutrient in units
	1	2	
Starch	5	7	8
Protein	4	2	15
Vitamins	2	1	3
Cost ($/kg.) of food	0.60	0.35	

The activities and their levels in this model are: for $j = 1, 2$

Activity j: to include one kg. of grain type j in the diet, associated level $= x_j$

So, x_j is the amount in kg. of grain j included in the daily diet, $j = 1, 2$, and the vector $x = (x_1, x_2)^T$ is the diet. The items in this model are the various nutrients, each of which leads to a constraint. For example, the amount of starch contained in the diet x is $5x_1 + 7x_2$, which must be ≥ 8 for feasibility. This leads to the formulation given below.

$$
\begin{array}{llcrcll}
\text{Minimize} & z(x) = 0.60x_1 & + & 0.35x_2 & & & \text{Item} \\
\text{Subject to} & 5x_1 & + & 7x_2 & \geq & 8 & \text{Starch} \\
 & 4x_1 & + & 2x_2 & \geq & 15 & \text{Protein} \\
 & 2x_1 & + & x_2 & \geq & 3 & \text{Vitamins} \\
 & x_1 & \geq 0, & x_2 & \geq & 0 &
\end{array}
$$

This simple model contains no constraints to guarantee that the diet is palatable, and does not allow any room for day-to-day variations that contributes to eating pleasure, and hence the solution obtained from it may be very hard to implement for human diet. The basic model can be modified by including additional constraints to make sure that the solution obtained leads to a tasteful diet with ample scope for variety. This sort of modification of the model after looking at the optimum solution to determine its reasonableness and implementability, solving the modified model, and even repeating this whole process several times, is typical in practical applications of optimization.

We human beings insist on living to eat rather than eating to live. And if we can afford it, we do not bother about the cost of food. It is also impossible to make a human being eat a diet that has been determined as being optimal. For all these reasons, it is not practical to determine human diet using an optimization model.

However, it is much easier to make cattle and fowl consume the diet that is determined as being optimal for them. Almost all the companies in the business of making feed for cattle, other farm animals, birds, etc. use LP extensively to minimize their production costs. The prices and supplies of various grains, hay, etc. are constantly changing, and feed makers solve the diet model frequently with new data values, to make their buy-decisions and to formulate the optimum mix for manufacturing the feed.

3.7 The Transportation Problem

An essential component of our modern life is the shipping of goods from where they are produced to markets worldwide. Nationally, within the USA alone transportation of goods is estimated to cost over $trillion/year.

The aim of this problem is to find a way of carrying out this transfer of goods at minimum cost. Historically it is among the first LPs to be modeled and studied. The Russian economist L. V. Kantorovitch studied this problem in the 1930's and published a book on it, *Mathematical Methods in the Organization and Planning of Production*, in Russian in 1939. In the USA, F. L. Hitchcock published a paper "The Distribution of a Product From Several Sources to Numerous Localities" in the *Journal of Mathematics and Physics*, vol. 20, 1941, where he developed an algorithm similar to the primal simplex algorithm for finding an optimum solution to the problem. And T. C. Koopmans published a paper "Optimum Utilization of the Transportation System" in *Econometrica*, vol. 17, 1949, in which he developed an optimality criterion for a basic solution to the transportation problem in terms of the dual basic solution (discussed later on). The early work of L. V. Kantorovitch and T. C. Koopmans in these publications is part of their effort for which they received the 1975 Nobel prize in economics.

The classical single commodity transportation problem is concerned with a set of nodes or places called **sources** which have a commodity available for shipment, and another set of places called **sinks** or **demand centers** or **markets** which require this commodity. The data consists of the **availability** at each source (the amount available there to be shipped out), the **requirement** at each market, and the cost of transporting the commodity per unit from each source to each market. The problem is to determine the quantity to be transported from each source to each market so as to meet the requirements at minimum total shipping cost.

Example 3.7.1: **Iron ore shipping problem:**

As an example, we consider a small problem where the commodity is iron ore, the sources are mines 1 and 2 that produce the ore, and the markets are three steel plants that require the ore. Let c_{ij} = cost (cents per ton) to ship ore from mine i to steel plant j, $i = 1, 2$, $j = 1, 2, 3$. The data is given below. To distinguish between different data elements, we show the cost data in normal size letters, and the supply and requirement data in larger size letters.

	c_{ij} (cents/ton)			Availability at
	$j = 1$	2	3	mine (tons) daily
Mine $i = 1$	11	8	2	800
2	7	5	4	300
Requirement at plant (tons) daily	400	500	200	

The activities in this problem are: to ship one ton of the commodity from source i to market j. It is convenient to represent the level at which this activity is carried out by the double subscripted symbol x_{ij}. In this example x_{ij} represents the amount of ore (in tons) shipped from mine i to plant j.

The items in this model are the ore at various locations. Consider ore at mine 1. There are 800 tons of it available, and the amount of ore shipped out of this mine, $x_{11}+x_{12}+x_{13}$, cannot exceed the amount available, leading to the constraint $x_{11} + x_{12} + x_{13} \leq 800$. Likewise, considering ore at steel plant 1, at least 400 tons of it is required there, so the total amount of ore shipped to this plant has to be ≥ 400, leading to the constraint $x_{11} + x_{21} \geq 400$. The total amount of ore available at both mines 1, 2 together is $800 + 300 = 1100$ tons daily; and the total requirement at plants 1, 2, 3 is also $400 + 500 + 200 = 1100$ tons daily. Clearly, this implies that all the ore at each mine will be shipped out, and the requirement at each plant will be met exactly; i.e., all the constraints will hold as equations. Therefore we have the following LP formulation for this problem.

$$
\begin{array}{llllllllll}
\text{Min. } z(x) = & 11x_{11} & +\,8x_{12} & +\,2x_{13} & +\,7x_{21} & +\,5x_{22} & +\,4x_{23} & & & \text{Item} \\
 & & & & & & & & & \text{Ore at} \\
\text{S. to} & x_{11} & +\;x_{12} & +\;x_{13} & & & & & = 800 & \text{mine 1} \\
 & & & & x_{21} & +\;x_{22} & +\;x_{23} & & = 300 & \text{mine 2} \\
 & x_{11} & & & +\;x_{21} & & & & = 400 & \text{plant 1} \\
 & & x_{12} & & & +\;x_{22} & & & = 500 & \text{plant 2} \\
 & & & x_{13} & & & +\;x_{23} & & = 200 & \text{plant 3}
\end{array}
$$

$$x_{ij} \geq 0 \quad \text{for all } i = 1, 2,\ j = 1, 2, 3$$

The Special Structure of the Transportation Problem

As an LP, the simple transportation problem discussed above has a very special structure. It can be represented very compactly in a two dimensional array in which row i corresponds to source i; column j corresponds to demand center j; and (i, j), the cell in row i and column j, corresponds to the shipping route from source i to demand center j. Inside the cell (i, j), record the decision variable x_{ij} which represents the amount of commodity shipped along the corresponding route, and enter the unit shipping cost on this route in the lower right-hand corner of the cell. The objective function in this model is the sum of the variables in the array multiplied by the cost coefficient in the corresponding cell. Record the availabilities at the sources in a column on the right-hand side of the array; and similarly the requirements at the demand centers in a row at the bottom of the array. Then each constraint other than any bound constraints on individual variables is a constraint on the sum of all the variables either in a row or a column of the array, and it can be read off from the array as shown below for the iron ore example.

Array Representation of the Transportation Problem

	Steel Plant			
	1	2	3	
Mine 1	x_{11} 11	x_{12} 8	x_{13} 2	= 800
Mine 2	x_{21} 7	x_{22} 5	x_{23} 4	= 300
	= 400	= 500	= 200	

$x_{ij} \geq 0$ for all i, j. Minimize cost.

Supplies, requirements in large size numbers

Any LP, whether it comes from a transportation or a different context, that can be represented in this special form of a two dimensional array is called a **transportation problem**. The constraints in the example problem are equations, but in general they may be equations or inequalities.

Integer Property in the Transportation Model

In a general LP, even when all the data are integer valued, there is no guarantee that there will be an optimum integer solution. However, the special structure of the transportation problem makes the following theorem possible.

Theorem 3.7.1 *In a transportation model, if all the availabilities and requirements are positive integers, and if the problem has a feasible solution, then it has an optimum solution in which all the decision variables* x_{ij} *assume only integer values.*

This theorem follows from the results discussed in Chapter 6. In fact in that chapter we discuss the primal simplex algorithm for the transportation problem, which terminates with an integer optimum solution for it when the conditions mentioned in the theorem hold.

A word of caution. The statement in Theorem 3.7.1 does not claim that an optimum solution to the problem must be an integer vector when the conditions stated there hold. There may be many alternate optimum solutions to the problem and the theorem only guarantees that at least one of these optimum solutions will be an integer vector.

The practical importance of the integer property will become clear from the next section.

The Balanced Transportation Problem

As mentioned above, the constraints in a transportation problem may be equations or inequalities. However, when the following condition holds

$$\left.\begin{array}{l}\text{total material avail-}\\ \text{able = sum of avail-}\\ \text{abilities at all sources}\end{array}\right\} = \left\{\begin{array}{l}\text{total material required =}\\ \text{sum of the requirements at}\\ \text{all the markets}\end{array}\right.$$

to meet the requirements at the markets, all the material available at every source will be shipped out and every market will get exactly as much as it requires, i.e., all constraints hold as equations. That's why this condition is a **balance condition**, and when it holds, and all the constraints are equations, the problem is called a **balanced transportation problem**. As formulated above, the iron ore problem is a balanced transportation problem.

The Limitations of this Transportation Model

We discussed this model mainly to introduce you to the art of optimizing commodity distribution costs using a mathematical model. For real world goods distribution problems this model is perhaps too simplistic. One limitation comes from its assumption that the shipping activity from each source to each sink takes place separately from the shipping between other source-sink pairs. Actually if sinks 1, 2 are along the route from source 1 to sink 3, then in reality the shippings from source 1 to sinks 1, 2 will probably be combined into the truck going from source 1 to sink 3. Also, this model ignores the timing and scheduling of various shipments, and the importance of packing all the shipments into the least number of vehicles. Advanced network models discussed in graduate level OR books remove some of these limitations, but even they cannot capture all the complicated features in most real world distribution problems. So, ultimately some heuristic adjustments and human judgement are essential to handle them.

However, even this simple model finds many important applications in a variety of problems, some of them not dealing with distribution of goods at all. For example, in Section 6.2 we discuss an application of this simple transportation model for deciding from which of several depots in the city to allocate a bus to each customer trip at a bus rental company.

3.8 The Assignment Problem

This is a very important optimization model that finds many applications in a variety of problems. This problem appears when there are two sets of objects with each set containing the same number of elements. For the sake of illustration, let us call one set the set of *machines*, and the other the set of *jobs*. Suppose

n = the number of machines = the number of jobs.

The problem deals with forming a set of couples, each couple consisting of a job and a machine. Forming the ordered couple (job i, machine j) can be interpreted as **assigning** or **allocating** job i to machine j for being carried out. For each possible coupling, an effectiveness coefficient is given, for instance this coefficient for assigning job i to machine j, denoted by c_{ij} may be the *cost* (or *reward*) of forming that couple.

Each job can be assigned to any one of the n machines, and each machine can be assigned only one of the n jobs. The aim is to find an assignment of each job to a machine, which minimizes the sum of the cost coefficients (or maximizing the sum of the reward coefficients) of the n couples formed.

As an example consider a company that has divided their marketing area into n zones based on the characteristics of the shoppers, their economic status, etc. They want to appoint a director for each zone to run the marketing effort there. They have already selected n candidates to fill the positions. The total annual sales in a zone would depend on which candidate is appointed as director there. Based on the candidates skills, demeanor, and background, it is estimated that \$$c_{ij}$ million in annual sales will be generated in zone j if candidate i is appointed as director there, and this (c_{ij}) data is given. The problem is to decide which zone each candidate should be assigned to, to maximize the total annual sales in all the zones (each zone gets one candidate and each candidate goes to one zone). We provide the data for a sample problem with $n = 6$ in Table 3.8.1 given below.

In this problem candidate 1 can go to any one of the n zones (so, n possibilities for candidate 1). Then candidate 2 can go to any one of the other zones (so, $n - 1$ possibilities for candidate 2 after candidate 1's zone is fixed). And so on. So, the total number of possible ways of assigning the candidates to the zones is $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$. For the example problem with $n = 6$ there are $6! = 720$ ways of assigning candidates to jobs. As n grows, $n!$ grows very rapidly. Real

Table 3.8.1

	c_{ij} = annual sales volume in \$million if candidate i is assigned to zone j					
Zone j =	1	2	3	4	5	6
Candidate i = 1	1	2	6	10	17	29
2	3	4	8	11	20	30
3	5	7	9	12	22	33
4	13	14	15	16	23	34
5	18	19	21	24	25	35
6	26	27	28	31	32	36

world applications of the assignment problem typically lead to problems with $n = 100$ to 10,000, and the number of possible assignments in these models is finite but very very large. So, it is not practical to evaluate each alternative separately to select the best as we did in Category 1. We have to construct a mathematical model for it and solve it by an efficient algorithm.

To model this problem, define the decision variables x_{ij} for $i, j = 1$ to $n = 6$ taking only two values 0 and 1 with the following interpretation:

$$x_{ij} = \begin{cases} 0 & \text{if Candidate } i \text{ not assigned to Zone } j \\ 1 & \text{if Candidate } i \text{ assigned to Zone } j \end{cases}$$

$$\begin{aligned} \text{Maximize } z_c(x) &= \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}x_{ij} \\ \text{Subject to } \sum_{j=1}^{6} x_{ij} &= 1 \quad \text{for } i = 1 \text{ to } 6 \qquad (3.8.1) \\ \sum_{i=1}^{6} x_{ij} &= 1 \quad \text{for } j = 1 \text{ to } 6 \\ x_{ij} &\geq 0 \quad \text{for all } i, j \end{aligned}$$

$$\text{and } x_{ij} = 0 \text{ or } 1 \quad \text{for all } i, j \qquad (3.8.2)$$

Since each candidate has to be assigned to exactly one zone, and each zone gets exactly one candidate, with the above definition of the variables, the model for this problem is (3.8.1), (3.8.2) given above.

(3.8.1), (3.8.2) is actually an integer programming problem since the decision variables can only take integer vales (fractional values for x_{ij} have no meaning in the problem). But we see that (3.8.1) is itself a special transportation problem with all the availabilities and requirements equal to 1. So, by the integer property (Theorem 3.7.1 of Section 3.7), it has an optimum solution in which all the variables have integer values only. Actually when (3.8.1) is solved by the Simplex method discussed in Chapter 4 or 6, the optimum solution obtained will be an integer solution. So, the constraint (3.8.2) can be ignored without any loss of generality for solving this problem.

That's why the assignment problem is always considered to be an LP, even though we are looking for a 0-1 optimum solution for it.

Application of the Assignment Model to a Marriage Problem, An Example Where Linearity Assumptions Are Inappropriate

This problem was proposed as an application of LP to sociology in a paper "The Marriage Problem" in *American Journal of Mathematics*, vol. 72, 1950, by P. R. Halmos and H. E. Vaughan. It is concerned with a club consisting of an equal number of men and women (say n each), who know each other well. The data consists of a rating (or happiness coefficient) c_{ij} which represents the amount of happiness that man i and woman j acquire when they spend a unit of time together, $i, j =$ 1 to n. The coefficients c_{ij} could be positive, 0, or negative. If $c_{ij} > 0$, man i and woman j are happy when together. If $c_{ij} < 0$, they are unhappy when together, and so acquire unhappiness only. Hence, in this setup unhappiness is a negative value for happiness and vice versa. To keep the model simple, it is assumed that the remaining life of all club members is equal, and time is measured in units of this lifetime. The problem is to determine the fraction of this lifetime that man i, woman j should spend together for $i, j = 1$ to n, to maximize the overall club's happiness.

As an example, we consider a club with $n = 5$, consisting of 5 men and 5 women and the happiness ratings (c_{ij}) given below. These happiness ratings are on a scale of -100 to $+\ 100$ where $-\ 100$ represents "very unhappy" and $+\ 100$ represents "very happy".

	c_{ij} for woman $j =$				
	1	2	3	4	5
man $i = 1$	78	−16	19	25	83
2	99	98	87	16	92
3	86	19	39	88	17
4	−20	99	88	79	65
5	67	98	90	48	60

There are 25 activities in this problem. These are, for $i, j = 1$ to 5

Activity: Man i and woman j to spend one unit of time together. Associated level $= x_{ij}$.

Thus x_{ij} is the fraction of their lifetime that man i and woman j spend together. The items in this model are the lifetimes of the various members of the club. Halmos and Vaughan made the *monogamous assumption*, i.e., that at any instant of time, a man can be with only one woman and vice versa. Under this assumption, man 1's lifetime leads to the constraint that the sum of the fractions of his lifetime that he spends with each woman should be equal to 1, i.e., $x_{11}+x_{12}+x_{13}+x_{14}+x_{15} = 1$. Similar constraints result from other members of the club.

Under the linearity assumptions, the club's happiness is $(\sum c_{ij}x_{ij}$: over $i, j = 1$ to 5). These things lead to the conclusion that the marriage problem in this example is the assignment problem given below. It is a special transportation problem in which the number of sources is

	Woman 1	2	3	4	5	
Man 1	x_{11} 78	x_{12} -16	x_{13} 19	x_{14} 25	x_{15} 83	$=1$
2	x_{21} 99	x_{22} 98	x_{23} 87	x_{24} 16	x_{25} 92	$=1$
3	x_{31} 86	x_{32} 19	x_{33} 39	x_{34} 88	x_{35} 17	$=1$
4	x_{41} -20	x_{42} 99	x_{43} 88	x_{44} 79	x_{45} 65	$=1$
5	x_{51} 67	x_{52} 98	x_{53} 90	x_{54} 48	x_{55} 60	$=1$
	$=1$	$=1$	$=1$	$=1$	$=1$	

$x_{ij} \geq 0$ for all i, j. Maximize objective.

equal to the number of demand centers, all the availabilities and requirements are 1, and the constraints are equality constraints. Since all variables are ≥ 0, and the sum of all the variables in each row and column of the array is required to be 1, all variables in the problem have to lie between 0 and 1. By Theorem 3.7.1 this problem has an optimum solution in which all x_{ij} take integer values, and in such a solution all x_{ij} should be 0 or 1. One such solution, for example, is given below.

$$x = (x_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In this solution man 1 (corresponding to row 1) spends all his lifetime with woman 5 (corresponding to column 5). So in this solution we can think of man 1 **being assigned** to woman 5, etc. Hence an integer solution to this problem is known as an **assignment**, and the problem of finding an optimum solution to this problem that is integral is called the **assignment problem**. In the optimum assignment, each man lives ever after with the woman he is assigned to and vice versa, and there is never any divorce!

For the marriage problem the conclusion that there exists an optimum marriage policy that maximizes the overall club's happiness without any divorce is extremely interesting. Extending this logic to the whole society itself, one can argue that there exists a solution pairing each man with a woman in society that maximizes the society's overall happiness without any divorce. Natural systems have a tendency to move towards an optimum solution, and if such a divorceless optimum solution exists, one would expect it to manifest itself in nature. Why, then, is there so much divorce going on, and why is the frequency of divorce increasing rather than declining? This seems to imply that the conclusion obtained from the model - that there exists an optimum marriage policy that maximizes society's happiness without any divorce - is false. If it is false, some of the assumptions on which the model is based must be invalid. The major assumptions made in constructing the model are the linearity assumptions needed to express the club's overall happiness as the linear function $\sum(c_{ij}x_{ij}$: over $i, j = 1$ to $n)$. Let us examine the proportionality and additivity assumptions that lead to the choice of this objective function carefully.

The proportionality assumption states that the happiness acquired by a couple is proportional to the time they spend together. In practice though, a couple may begin their life together in utter bliss, but develop a mutual dislike for each other as they get to know each other over time. After all, the proverb says: "Familiarity breeds contempt". For most married couples, the rate of happiness they acquire per unit time spent together increases for some time after their wedding, then peaks and either remains flat, or begins to decline. So, the actual total happiness acquired by the couple as a function of the time spent together behaves as a highly nonlinear function. Thus the proportionality assumption is not reasonable for the marriage problem.

The additivity assumption states that the society's happiness is the sum of the happiness acquired by the various members in it. In particular, this states that a person's unhappiness cancels with another person's happiness. In reality these things are quite invalid. History has many instances of major social upheavals just because there was one single unhappy person. The additivity assumption is quite inappropriate for determining the society's happiness as a function of the happiness of its members.

Finally the choice of the objective of maximizing society's happiness is itself quite inappropriate. In determining their marriage partners, most people are guided by the happiness they expect to acquire, and do not care what impact it will have on society. It is extremely hard to force people to do something just because it is good for the society as a whole.

In summary, for studying the marriage problem, and that of rampant divorce, the linearity assumptions and the choice of the objective of maximizing society's happiness seem very inappropriate. Divorce is not a problem that can be solved by mathematics however elegant and sophisticated its level may be. To tackle it needs a fundamental change in the behavior and attitudes of people.

If divorce has become a serious social problem, it is perhaps an indication that our education system with its emphasis on science, technology, and individual success in business, and the role of TV in our daily routines, is training our kids to be self-centered. Unfortunately, many of our religious instituitions with their emphasis on conversions and growth are operating more and more like businesses these days. It is also an indication that our kids are not noticing that we share this earth with other human beings and many other creatures, and that they are not learning the importance of compromising and accommodating other's points of view.

We discussed this problem here mainly to provide an example where the linearity assumptions are totally inappropriate.

This points out the importance of checking the validity of the mathematical model very carefully after it is constructed. Things to review are: Is the objective function appropriate? Are all the constraints relevant or essential, or can some of them be eliminated or modified? Are any decision variables missing? Is the data fairly reliable? Etc.

3.9 A Multi-Period Production Planning Problem

So far we have discussed a variety of static one period problems. Now we will discuss a **multi-period problem**. As an example we will consider the problem of planning the production, storage, and marketing of a product whose demand and selling price vary seasonally. An important feature in this situation is the profit that can be realized by manufacturing the product in seasons during which the production costs are low, storing it, and putting it in the market when the selling price is high. Many products exhibit such seasonal behavior, and companies and businesses take advantage of this feature to augment their profits. An LP formulation of this problem has the aim of finding the best production-storage-marketing plan over the planning horizon, to maximize the overall profit. For constructing a model for this problem we need reasonably good estimates of the demand and the expected selling price of the product in each period of the planning horizon. We also need data on the availability and cost of raw materials, labor, machine times etc. necessary to manufacture the product in each period; and the availability and cost of storage space.

Period	Production cost($/ton)	Prod. capacity(tons)	Demand* (tons)	Sell price ($/ton)
1	20	1500	1100	180
2	25	2000	1500	180
3	30	2200	1800	250
4	40	3000	1600	270
5	50	2700	2300	300
6	60	2500	2500	320

* Demand is the maximum amount that can be sold in period

As an example, we consider the simple problem of a company making a product subject to such seasonal behavior. The company needs to make a production plan for the coming year divided into 6 periods of 2 months each, to maximize net profit (= sales revenue − production and storage costs). Relevant data is in the table given above. The production cost there includes the cost of raw material, labor, machine time etc., all of which fluctuate from period to period. And the production capacity arises due to limits on the availability of raw material and hourly labor.

Product manufactured during a period can be sold in the same period, or stored and sold later on. Storage costs are $2/ton of product from one period to the next. Operations begin in period 1 with an initial stock of 500 tons of the product in storage, and the company would like to end up with the same amount of the product in storage at the end of period 6.

The decision variables in this period are, for period $j = 1$ to 6

$$
\begin{aligned}
x_j &= \text{product made (tons) during period } j \\
y_j &= \text{product left in storage (tons) at the end of period } j \\
z_j &= \text{product sold (tons) during period } j
\end{aligned}
$$

In modeling this problem the important thing to remember is that inventory equations (or material balance equations) must hold for the product for each period. For period j this equation expresses the following fact.

$$
\left.\begin{array}{l}\text{Amount of product in storage} \\ \text{at the beginning of period } j + \\ \text{the amount manufactured dur-} \\ \text{ing period } j\end{array}\right\} = \left\{\begin{array}{l}\text{Amount of product sold during} \\ \text{period } j + \text{the amount left in} \\ \text{storage at the end of period } j\end{array}\right.
$$

The LP model for this problem is given below.

$$\begin{array}{ll} \text{Maximize} & 180(z_1+z_2)+250z_3+270z_4+300z_5+320z_6 \\ & -20x_1-25x_2-30x_3-40x_4-50x_5-60x_6 \\ & -2(y_1+y_2+y_3+y_4+y_5+y_6) \\ \\ \text{Subject to} & x_j, y_j, z_j \geq 0 \ \text{ for all } j = 1 \text{ to } 6, \\ & x_1 \leq 1500, x_2 \leq 2000, x_3 \leq 2200, \\ & x_4 \leq 3000, x_5 \leq 2700, x_6 \leq 2500 \\ & z_1 \leq 1100, z_2 \leq 1500, z_3 \leq 1800, \\ & z_4 \leq 1600, z_5 \leq 2300, z_6 \leq 2500 \end{array}$$

$$\begin{array}{rcl} y_1 &=& 500+x_1-z_1 \\ y_2 &=& y_1+x_2-z_2 \\ y_3 &=& y_2+x_3-z_3 \\ y_4 &=& y_3+x_4-z_4 \\ y_5 &=& y_4+x_5-z_5 \\ y_6 &=& y_5+x_6-z_6 \\ y_6 &=& 500 \end{array}$$

3.10 Examples Illustrating Some of the Approximations Used in Formulating Real World Problems

The following examples, based on, or similar to the case "R. Wilson, "Red Brand Canners", Stanford Business Cases, Graduate School of Business, Stanford University" illustrate some of the approximations used in formulating real world problems using LP models, and in estimating the coefficients in the model.

Example 3.10.1: Orange Juice Concentrate Production

A company makes frozen concentrate for orange juice in grades G_1, G_2, G_3, and sells them wholesale to juice makers, using oranges that they buy from local farmers. They usually buy oranges sorted in two sizes S_1 at \$28/box, and S_2 at \$20/box. This year because of frost in the growing area the farmers harvested the crop early and delivered 400,000 boxes of oranges without sorting to the company at \$19/box.

From a sample of delivered boxes, it has been estimated that 30% of the delivered crop is of size S_1, 60% of size S_2, and 10% is below standard which should be discarded. The sorting has been estimated to cost the company \$2/box. Data on the processing of oranges into concentrates is given below.

Grade	+Inputs/Case	*Demand (cases)
G_1	2 boxes S_1	55,000
G_2	1.5 boxes S_1 or S_2	150,000
G_3	1 box S_2	220,000

+The process for each grade is different

*Maximum that can be sold

The cost accounting department has estimated the net profit (\$/case) of G_1, G_2, G_3 using the following procedure.

Effective average cost/box of S_1 or S_2 (since 0.1 fraction to be discarded)	$\frac{19+2}{0.9} = \$23.33$
Avg. cost/box based on old prices (since S_1, S_2 in ratio 30% to 60%, or 1 to 2)	$\frac{28+20\times 2}{1+2} = \$22.67$
Excess avg. cost/box this year	$\$\ 23.33 - 22.67 = \$0.66$

Allocating the excess average cost of \$0.66/box this year in the ratio of 1:2 to S_1, S_2, the cost/box of S_1 comes to $\$(28 + 0.66(1/3)) = \$28.22$; and that of S_2 comes to $\$20 + 0.66(2/3) = \$20.44$. Using these figures, the net profit/case of G_1, G_2, G_3 is computed in the following table.

Quantity (\$/case)	G_1	G_2	G_3
Input cost	2(28.22)= 56.44	*(1×28.22+ 2×20.44)/3= 34.55	20.44
‡Other variable costs(labor, etc.)	8.33	7.50	6.25
‡Overhead	2.25	1.75	1.50
Total cost	67.02	43.80	28.19
Sale price	86.25	60.35	40.25
Net profit	19.23	16.55	12.06

*Assuming S_1, S_2 are used in ratio 1:2 to make G_2

‡This is other data given in the problem

It is required to construct a mathematical model to determine an optimum production plan for the company that maximizes the total net profit next year.

Formulation: The decision variables in the model are the following:

$$\begin{aligned} g_i &= \text{cases of } G_i \text{ made, } i = 1, 2, 3 \\ s_{11}, s_{12} &= \text{boxes of } S_1 \text{ used for making } G_1, G_2 \text{ respectively} \\ s_{22}, s_{23} &= \text{boxes of } S_2 \text{ used for making } G_2, G_3 \text{ respectively.} \end{aligned}$$

Here is the model:

$$\begin{aligned} \text{Maximize } z = \; 19.23g_1 &+ 16.55g_2 + 12.06g_3 \\ \text{s. to} \quad g_1 &= s_{11}/2 \\ g_2 &= (s_{12} + s_{22})/1.5 \\ g_3 &= s_{23} \\ s_{11} + s_{12} &\leq 0.3 \times 400,000 \\ s_{22} + s_{23} &\leq 0.6 \times 400,000 \\ 0 \leq g_1 &\leq 55,000 \\ 0 \leq g_2 &\leq 150,000 \end{aligned}$$

$$
\begin{aligned}
0 \le g_3 &\le 220,000 \\
s_{11}, s_{12}, s_{22} s_{23} &\ge 0.
\end{aligned}
$$

In writing the constraints on oranges used, we assumed that it is alright to leave some of the available oranges unused, that's why these constraints are listed as "$\le$" inequalities. If the optimum solution of this model leaves some oranges unused, may be the company can sell them to fruit retailers to be sold in the open markets.

Example 3.10.2: Red Brand Canners

RBC cans and distributes a variety of fruit and vegetable products. For the coming season they have the opportunity of buying upto 3,000,000 lbs of current crop tomatoes at an average delivered price of 18 cents/lb. It has been estimated that 20% of the crop is grade "A", and the remaining grade "B". They make three canned tomato products:

P_1 = whole tomatoes
P_2 = tomato juice
P_3 = tomato paste.

Selling price for these products has been set in light of the long-term marketing strategy of the company, and potential sales has been forecast at those prices. These and other production requirements are given below.

Product	Selling price ($/case)	Demand forecast cases	Input/case lbs tomatoes
P_1	12.00	800,000	18
P_2	13.75	50,000	20
P_3	11.36	80,000	25

RBC uses a numerical scale that ranges from 0 to 10 to record the quality of raw produce and products, the higher number representing better quality. "A" tomatoes averaged 9 pts./lb, and "B" tomatoes averaged 5 pts./lb. The minimum average input quality for

P_1 is 8 pts./lb

P_2 is 6

P_3 can be made entirely from “B” grade tomatoes. Cost accounting at RBC used the following procedure to compute the net profit/case of each product.

Cost($/case)	Product		
	P_1	P_2	P_3
Total input	3.24	3.60	4.50
Other Costs			
Direct labor	3.54	3.96	1.62
Variable OHD	0.72	1.08	0.78
Variable selling	1.20	2.55	1.14
Packaging	2.10	1.95	2.31
Total of other costs	7.56	9.54	5.85
Total cost	10.80	13.14	10.35
Selling Price	12.00	13.50	11.36
Net Profit	1.20	0.36	1.01

It is required to formulate the problem of determining the optimal canning policy for this season’s crop as an LP.

Scenario 1: The decision variables for this model are:

θ = total lbs. of tomatoes from current crop purchased

x_j = lbs. of “A” tomatoes used to produce P_j, $j = 1, 2, 3$

y_j = lbs. of “B” tomatoes used to produce P_j, $j = 1, 2, 3$

p_j = cases of P_j produced, $j = 1, 2, 3$.

The constraint on input quality for P_1, P_2 produced are:

$\frac{9x_1+5y_1}{x_1+y_1} \geq 8$ or equivalently $x_1 \geq 3y_1$ for P_1

$\frac{9x_2+5y_2}{x_2+y_2} \geq 6$ or equivalently $3x_2 \geq y_2$ for P_2

Using these the model for this problem under this scenario is

$$\begin{aligned}
\text{maximize } 1.20p_1 &+ 0.36p_2 + 1.01p_3 \\
\text{s. to } p_1 &= (x_1 + y_1)/18 \\
p_2 &= (x_2 + y_2)/20 \\
p_3 &= (x_3 + y_3)/25 \\
0 \le \theta &\le 3,000,000 \\
x_1 + x_2 + x_3 &= 0.2\theta \\
y_1 + y_2 + y_3 &= 0.8\theta \\
x_1 &\ge 3y_1 \\
3x_2 &\ge y_2 \\
0 \le p_1 &\le 800,000 \\
0 \le p_2 &\le 50,000 \\
0 \le p_3 &\le 80,000 \\
x_j, y_j &\ge 0, \quad j = 1, 2, 3.
\end{aligned}$$

Scenario 2: In Scenario 1 we allocated tomato input cost purely proportional to the total quantity used, at 18 cents/lb in computing the net profit coefficients for the three produdts P_1, P_2, P_3. In this scenario instead of using the average cost of 18 cents/lb for all tomatoes equally, we will use imputed costs of "A", "B" tomatoes calculated in proportion to their average quality. So, if c_1, c_2 are the imputed costs cents/lb of "A", "B" tomatoes, then $(c_1/9) = (c_2/5)$.

Another equation that c_1, c_2 satisfy is obtained by equating the price of 18×3,000,000 cents of the whole lot to $600,000c_1 + 2,400,000c_2$ since the lot consists of 600,000 lbs. of "A" and 2,400,000 lbs. of "B"; leading to $600,000c_1 + 2,400,000c_2 = 18 \times 3,000,000$.

From these two equations we see that $c_1 = 27.93$, $c_2 = 15.52$ in cents/lb.

So, the cost per point-lb is $(c_1/9) = (c_2/5) = 3.104$ cents.

Since P_1 needs 18 lbs. of average quality of 8 or more, the tomato input cost for P_1 should be $18 \times 8 \times 3.104$ cents/case = \$4.47/case. Similarly the tomato input cost for P_2 should be $20 \times 6 \times 3.104$ cents/case

= \$3.74/case. And since P_3 can be made purely from "B" tomatoes, its tomato input cost is $25 \times 5 \times 3.104$ cents/case = \$2.79/case. Using these tomato input costs, and the rest of the data from the table in scenario 1, we compute the net profit coefficients for the three products as below.

Cost(\$/case)	Product		
	P_1	P_2	P_3
Tomato input	4.47	3.72	3.88
All others	7.56	9.54	5.85
Total cost	12.03	13.26	9.73
Selling price	12.00	13.50	11.36
Net Profit	−0.03	0.24	1.63

The constraints in the model under this scenario are the same as in the model under Scenario 1, but the objective function changes to that of maximizing $z = -0.03p_1 + 0.24p_2 + 1.63p_3$.

3.11 Material for Further Study

In this chapter we discussed examples of decision problems that can be modeled as linear programs using a direct approach, i.e., without requiring any transformations of variables. these may be termed as *simple applications* of linear programming.

There are many other decision problems that can be modeled as linear programs using transformations of variables, for example those involving minimization of piecewise linear convex objective functions subject to linear constraints. These advanced formulation techniques will be discussed in a Master's level sequel to this book: *Optimization Models for Decision Making*, Volume 2. To understand the full potential of linear programming as a vehicle for modeling decision making problems, one has to study these advanced modeling techniques.

3.12 Geometric Method for Solving LPs in Two Variables

LPs involving only two variables can be solved geometrically by drawing a diagram of the feasible region (i.e., the set of feasible solutions) in R^2 = the two dimensional Cartesian plane. The optimum solution is identified by tracing the line corresponding to the set of feasible solutions that give a specific value to the objective function and then moving this line parallel to itself in the optimal direction as far as possible.

In R^2 a linear equation in the variables represents a straight line, hence the set of all points giving a specific value to the objective function is a straight line. Each straight line divides R^2 into two half-spaces, and every linear inequality represents a half-space.

As an example, consider the fertilizer product mix problem in Example 3.4.1. The constraint $2x_1 + x_2 \leq 1500$ requires that any feasible solution (x_1, x_2) to the problem should be on one side of the line represented by $2x_1 + x_2 = 1500$, the side that contains the origin (because the origin makes $2x_1 + x_2 = 0 < 1500$). This side is indicated by an arrow on the line in Figure 3.2. Likewise, all the constraints can be represented by the corresponding half-spaces in the diagram. The feasible region is the set of points in the plane satisfying all the constraints; i.e., the shaded region in Figure 3.2.

Let $z(x)$ be the linear objective function that we are trying to optimize. Select any point, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ say, in the feasible region, and compute the objective value at it, $z(\bar{x}) = \bar{z}$, and draw the straight line represented by $z(x) = \bar{z}$. This straight line has a nonempty intersection with the feasible region since the feasible point $\bar{x}$ is contained on it. For any value $z_0 \neq \bar{z}$, $z(x) = z_0$ represents a straight line which is parallel to the line represented by $z(x) = \bar{z}$.

If $z(x)$ is to be maximized, move the line $z(x) = z_0$ in a parallel fashion by increasing the value of z_0 beginning with $\bar{z}$, as far as possible while still maintaining a nonempty intersection with the feasible region. If $\hat{z}$ is the maximum value of z_0 obtained in this process, it is the maximum value of $z(x)$ in the problem, and the set of optimum solutions is the set of feasible solutions lying on the line $z(x) = \hat{z}$.

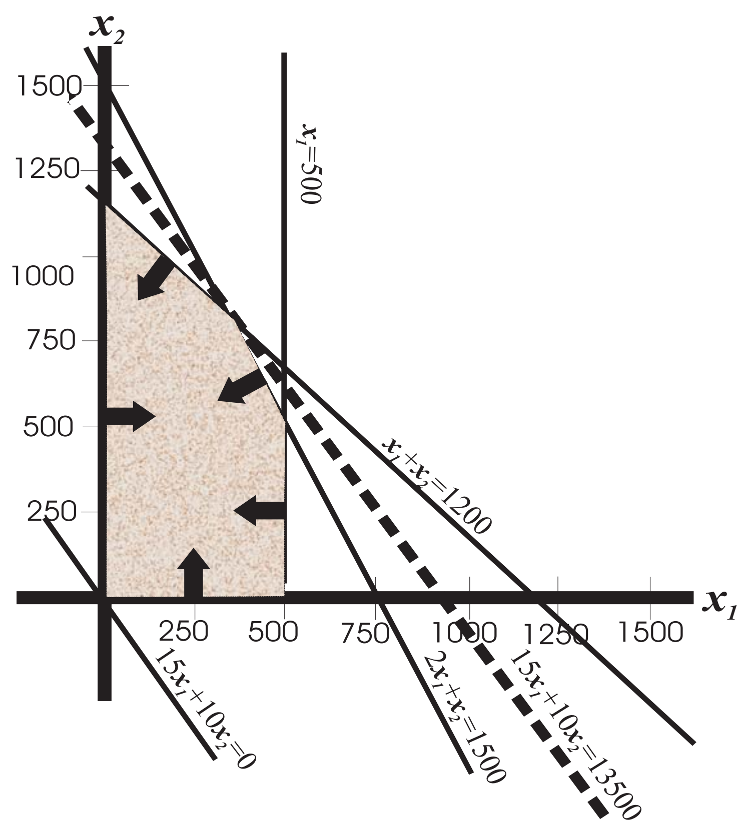

Figure 3.2: Fertilizer product mix problem.

On the other hand, if the line $z(x) = z_0$ has a nonempty intersection with the feasible region for every $z_0 \geq \bar{z}$, then $z(x)$ is **unbounded above** on this set. In this case $z(x)$ can be made to diverge to $+\infty$ on the feasible region, and the problem has no finite optimum solution.

If the aim is to minimize $z(x)$, then decrease the value of z_0 beginning with $\bar{z}$ and apply the same kind of arguments.

In the fertilizer product mix problem we start with the feasible point $\bar{x} = (0, 0)$ with an objective value of 0. As z_0 is increased from 0, the line $15x_1 + 10x_2 = z_0$ moves up keeping a nonempty intersection with the feasible region, until the line coincides with the dashed line $15x_1 + 10x_2 = 13,500$ in Figure 3.2 passing through the point of intersection of the two lines

$$\begin{aligned} 2x_1 + x_2 &= 1500 \\ x_1 + x_2 &= 1200 \end{aligned}$$

which is $\hat{x} = (300, 900)$. For any value of $z_0 > 13,500$ the line $15x_1 + 10x_2 = z_0$ does not intersect with the feasible region. Hence, the optimum objective value in this problem is \$13,500, and the optimum solution of the problem is $\hat{x} = (300, 900)$. Hence the fertilizer maker achieves his maximum daily profit of \$13,500 by manufacturing 300 tons of Hi-ph, and 900 tons of Lo-ph daily.

Optimum Solution of an LP Is Characterized by a System of Linear Equations

So, the optimum solution of the fertilizer product mix problem is characterized by the system of linear equations obtained by treating the inequality constraints corresponding to the items RM 1 and RM 2 as equations.

In general it is a fundamental fact that the optimum solution of every LP is characterized by a system of linear equations containing all the equality constraints in the original problem, and a subset of the inequality constraints treated as equations. All computational algorithms for solving LPs have the goal of determining which inequality constraints in the problem are active at an optimum solution.

Can the Geometric Method be Extended to Higher Dimensions?

The feasible region of LPs involving n variables is a subset of R^n. So, if $n \geq 3$, it is hard to visualize geometrically. Hence this simple geometric method cannot be used to solve LPs involving 3 or more variables directly. Fortunately there are now efficient computational algorithms to solve LPs involving any number of variables. We discuss these in later chapters, and LPs in higher dimensional spaces can be solved efficiently using them.

When the objective function is $z(x)$, the main idea in the geometric method described above is to identify the straight line $z(x) = z_0$ for some z_0, and to move this line parallel to itself in the desired direction, keeping a nonempty intersection with the feasible region. In LPs with $n \geq 3$ variables, $z(x) = z_0$ defines a hyperplane in R^n and not a straight line (a hyperplane in R^2 is a straight line). An approach for solving LPs in these higher dimensional spaces, based on the above idea of parallel sliding of the objective plane, would be very efficient. However, checking whether the hyperplane still intersects the feasible region after a small parallel slide requires full-dimensional visual information which is not available currently for $n \geq 3$. So it has not been possible to adopt this parallel sliding of the objective hyperplane to solve LPs in spaces of dimension ≥ 3. The simplex algorithm for solving LPs discussed in the sequel uses an entirely different approach. It takes a path along line segments called **edges** on the boundary of the feasible region, moving from one corner point to an adjacent one along an edge in each move, using local one-dimensional information collected in each step. As an example, to solve the fertilizer product mix problem starting with the feasible corner point 0, the simplex algorithm takes the edge path from 0 to the corner point $(500, 0)$, then from $(500, 0)$ to $(500, 500)$, and finally from $(500, 500)$ to the optimum solution $(300, 900)$) in Figure 3.2.

3.13 What Planning Information Can Be Derived from an LP Model?

Finding the Optimum Solutions

We can find an optimum solution for the problem, if one exists, by solving the model using the algorithms discussed later on. These algorithms can actually identify the set of all the optimum solutions if there are alternate optimum solutions. This may be helpful in selecting a suitable optimum solution to implement (one that satisfies some conditions that may not have been included in the model, but which may be important).

For the fertilizer product mix problem, we found out that the unique optimum solution is to manufacture 300 tons Hi-ph, and 900 tons Lo-ph, leading to a maximum daily profit of $13,500.

Infeasibility Analysis

We may discover that the model is **infeasible** (i.e., it has no feasible solution). If this happens, there must be a subset of constraints that are mutually contradictory in the model (maybe we promised to deliver goods without realizing that our resources are inadequate to manufacture them on time). In this case the algorithms will indicate how to modify the constraints in order to make the model feasible. After making the necessary modifications, the new model can be solved.

Values of Slack Variables at an Optimum Solution

The values of the slack variables at an optimum solution provide useful information on which supplies and resources will be left unused and in what quantities, if that solution is implemented.

For example, in the fertilizer product mix problem, the optimum solution is $\hat{x} = (300, 900)$. At this solution, RM 1 slack is $\hat{x}_3 = 1500 - 2\hat{x}_1 - \hat{x}_2 = 0$, RM 2 slack is $\hat{x}_4 = 1200 - \hat{x}_1 - \hat{x}_2 = 0$, and RM 3 slack is $\hat{x}_5 = 500 - \hat{x}_1 = 200$ tons. The fact that RM 1, RM 2 slacks are zero at the optimum solution is clear because the optimum solution is obtained by treating the RM 1, RM 2 inequality constraints as equations.

Thus, if this optimum solution is implemented, the daily supply of RM 1 and RM 2 will be completely used up, but 200 tons of RM 3 will be left unused. This shows that the supplies of RM 1, RM 2 are very critical to the company, and that there is currently an oversupply of 200 tons of RM 3 daily that cannot be used in the optimum operation of the Hi-ph and Lo-ph fertilizer processes.

This also suggests that it may be worthwhile to investigate if the maximum daily profit can be increased by lining up additional supplies of RM 1, RM 2 from outside vendors, if additional capacity exists in the Hi-ph, Lo-ph manufacturing processes. A useful planning tool for this investigation is discussed next.

Marginal Values and Their Uses

Each constraint in an LP model is the material balance constraint of some item, the RHS constant in that constraint being the availability or the requirement of that item.

Definition: The **marginal value** of that item (also called the marginal value corresponding to that constraint) is defined to be the rate of change in the optimum objective value of the LP, per unit change in the RHS constant in the associated constraint.

For example, in the fertilizer product mix problem, the marginal value of RM 1 (and of the corresponding first constraint in the model) is the rate of change in the maximum daily profit per unit change in the supply of RM 1 from its present value of 1500. These rates are also called **dual variables**, or the **shadow prices of the items**. These are the variables in another linear programming problem that is in **duality relationship** with the original problem. In this context the original problem is called the **primal problem**, and the other problem is called the **dual problem**. The derivation of the dual problem is discussed in Chapter 5.

If $b = (b_1, \dots, b_m)^T$ is the vector of RHS constants in an LP model, and $f(b)$ denotes the optimum objective value in the LP as a function of the RHS constants vector, then the marginal value corresponding to constraint 1 is therefore the limit of $[f((b_1+\epsilon, b_2, \dots, b_m)^T) - f(b)]/\epsilon$ as $\epsilon \to 0$. So, one crude way of getting this marginal value is to select a small nonzero quantity ϵ, and then take $[f((b_1+\epsilon, b_2, \dots, b_m)^T) - f(b)]/\epsilon$ as an approximation to this marginal value.

As an example, let us consider the fertilizer product mix problem again. The present RHS constants vector is $(1500, 1200, 500)^T$, and we computed the optimum objective value to be $f((1500, 1200, 500)^T) =$ \$13,500. To get the marginal value of RM 1 (item corresponding to the first constraint) we can change the first RHS constant to $1500 + \epsilon$ and solve the new problem by the same geometric method discussed above. For small values of ϵ the straight line representing the constraint $2x_1 + x_2 = 1500 + \epsilon$ is obtained by a slight tilt of the straight line corresponding to $2x_1 + x_2 = 1500$. From this it can be seen that when ϵ is small, the optimum solution of the perturbed problem is the solution of the system of equations

$$\begin{aligned} 2x_1 + x_2 &= 1500 + \epsilon \\ x_1 + x_2 &= 1200 \end{aligned}$$

which is $x^1(\epsilon) = (300 + \epsilon, 900 - \epsilon)$, with an optimum objective value of $f((1500 + \epsilon, 1200, 500^T) = \$13{,}500 + 5\epsilon$. So, the marginal value of RM 1 in the fertilizer problem is

$$\lim_{\epsilon \to 0} \frac{13500 + 5\epsilon - 13500}{\epsilon}$$

= \$5/ton in terms of net profit dollars. Remember that the current price of RM 1 delivered to the company is \$50/ton. This indicates that as long as the price charged by an outside vendor per ton of RM 1 delivered is $\leq \$50 + 5 = 55$/ton, it is worth getting additional supplies of RM 1 from that vendor. \$55/ton delivered is the breakeven price for acquiring additional supplies of RM 1 for profitability.

In the same way if we change the 2nd RHS constant in the fertilizer problem from 1200 to 1200 $+\epsilon$, it can be verified that the optimum solution of the perturbed problem for small values of ϵ is the solution of

$$\begin{aligned} 2x_1 + x_2 &= 1500 \\ x_1 + x_2 &= 1200 + \epsilon \end{aligned}$$

which is $x^2(\epsilon) = (300 - \epsilon, 900 + 2\epsilon)$, with an optimum objective value of $f((1500, 1200 + \epsilon, 500^T)) = \$13{,}500 + 5\epsilon$. So, the marginal value of RM 2 in the fertilizer problem is

$$\lim_{\epsilon \to 0} \frac{13500 + 5\epsilon - 13500}{\epsilon}$$

= \$5/ton in terms of net profit dollars. The current price of RM 2 delivered to the company is \$40/ton. This indicates that the breakeven price for acquiring additional supplies of RM 2 for profitability is \$40 + 5 = \$45/ton delivered.

There is currently 200 tons excess supply of RM 3 daily that is not being used. Clearly, changing the availability of RM 3 from the present 500 tons daily to 500 + ϵ tons daily for small values of ϵ will have no effect on the optimum solution of the problem. So $f((1500, 1200, 500 + \epsilon)^T) = f((1500, 1200, 500)^T) = \$13{,}500$. Therefore the marginal value of RM 3 is zero, and there is no reason to get additional supplies of RM 3, as no benefit will accrue from it.

This type of analysis is called **marginal analysis**. It helps companies to determine what their most critical resources are, and how the requirements or resource availabilities can be modified to arrive at much better objective values than those possible under the existing requirements and resource availabilities.

Summary: Marginal values in an LP are associated with the RHS constants in it. Each of them is defined as a limit when the limit exists.

In some LPs the limits defining marginal values may not exist. In these LPs, marginal values do not exist.

The criterion for deciding whether marginal values exist or not in a given LP is discussed in Chapter 5. When they exist, marginal values can be computed very easily from the final output of the simplex method when the LP is solved by the simplex method. So, there is no need to resort to their fundamental definition to compute the marginal values as done above.

Evaluating the Profitability of New Products

One major use of marginal values is in evaluating the profitability of new products. It helps to determine whether they are worth manufacturing, and if so at what level they should be priced so that they are profitable in comparison with existing product lines.

We will illustrate this again using the fertilizer product mix problem. Suppose the company's research chemist has come up with a new fertilizer that he calls **lushlawn**. Its manufacture requires per ton, as inputs

3 tons of RM 1, 2 tons of RM 2, and 2 tons of RM 3

At what rate/ton should lushlawn be priced in the market, so that it is competitive in profitability with the existing Hi-ph, Lo-ph that the company currently makes?

To answer this question, let π_1, π_2, π_3 be the marginal values of RM 1, RM 2, RM 3 in terms of net profit dollars. We computed their vales to be $\pi_1 = 5, \pi_2 = 5, \pi_3 = 0$.

So, the input packet of $(3, 2, 2)^T$ tons of (RM 1, RM 2, RM $3)^T$ needed to manufacture one ton of lushlawn has value to the company of $3\pi_1 + 2\pi_2 + 2\pi_3 = 3 \times 5 + 2 \times 5 + 2 \times 0 = \$25$ in terms of net profit dollars. On the supply side, the delivery cost of this packet of raw materials is $3 \times 50 + 2 \times 40 + 2 \times 60 = \$350$.

So, clearly, for lushlawn to be competitive with Hi-ph, Lo-ph, its selling price in the market/ton should be $\geq \$\ 25 + 350 +$ (its production cost/ton). The company can conduct a market survey and determine whether the market will accept lushlawn at a price $\geq$ this breakeven level. Once this is known, the decision whether to produce lushlawn is obvious.

By providing this kind of valuable planning information, the linear programming model becomes a highly useful decision making tool.

3.14 The Role of LP in the World of Mathematics

Linear Algebra (LA) is the branch of mathematics dealing with modeling, solving, and analyzing systems of linear equations. It is a classical subject that originated more than 2500 years ago. LA does not have techniques for solving systems of linear constraints that include linear inequalities. Until the development of LP in mid-20th century, there were no techniques that can reliably handle systems of linear constraints including inequalities.

In its original form, LP is actually concerned with optimizing a linear objective function subject to linear constraints including inequalities. As we have seen in Section 3.12, the critical issue in solving an

LP is determining which inequality constraints are active at an optimum solution, so that the optimum solution can itself be computed by ignoring the inactive inequality constraints and solving the resulting system of linear equations by LA techniques.

It has been shown that even if there is no optimization to be performed, just finding a feasible solution to a system of linear inequalities itself leads to exactly the same critical issue of determining which inequalities are active at a feasible solution. Also, it will be shown in Chapter 4 that the problem of finding a feasible solution to a system of linear inequalities can itself be posed as an LP (called the *Phase I problem* in Chapter 4) involving optimizing a Phase I linear objective function, subject to a modified system of linear constraints. Solving the original system of linear inequalities is equivalent to solving this Phase I LP.

Another fundamental result in LP theory establishes that an LP involving optimizing a linear function can itself be posed as an equivalent problem of finding any feasible solution to a system of linear inequalities involving no optimization.

These facts clearly show that LP is the subject dealing with either solving systems of linear inequalities, or optimizing a linear function subject to linear constraints that may include inequalities. It also shows that LP is the extension of LA to handle linear inequality constraints. Its development started when George Dantzig developed his simplex method for LP in 1947. This method is the subject of the next Chapter 4.

After calculus, LA and its extension LP are perhaps the most useful branches of mathematics for applications. Appreciation of beauty is a highly individual reaction, but most people who have developed some knowledge of LA, LP would agree that they are the most beautiful areas of mathematics.

3.15 Exercises

3.1: A farmer has a 100 acre farm on which he has decided to grow zucchini squash or corn in the coming season for selling the produce to the local grocery stores at wholesale prices.

Zucchini needs 4 units of water and corn 2 units/acre/week. He has three wells which together can pump upto 220 units of water/week.

Zucchini needs 10 units of fertilizer/acre/season, whereas corn needs 3 units of the same /acre/season. The farmer wants to use at most 450 units of fertilizer per season.

There is a local competition among vegetable growers in the area. They weigh the produce from each competing farm and among 100 acre farms, they give an award for any farm producing more than 600 units of produce/season. The farmer expects zucchini crop to average 20 units of squash/acre/season, and the corn crop to average 12 units of corn/acre/season. The farmer wants to win an award in this competition.

Labor for watering, harvesting, weeding etc. is expected to cost \$150/acre for the zucchini crop, and \$100/acre/season for corn. The sale of zucchini squash is expected to yield \$400/acre/season, and corn \$300/acre/season. Define the net income from the farm to be = the proceeds from sales to grocery stores − money spent on labor for the farm. Formulate the problem of finding how much land to allocate to zucchini and corn to maximize the net income from the farm. Solve it using the geometric method and find an optimum solution.

3.2: A textile firm has spare capacity in its spinning and weaving sections. They would like to accept outside contracts to use up this spare capacity. Each contract would require some length (in units) of a specified fabric to be woven and delivered to the customer.

They have two offers. Contract 1 needs 2 units of spinning mill time and 1 unit of weaving machine time per unit of fabric delivered. Contract 2 requires 1 unit of spinning mill time and 1 unit of weaving machine time per unit fabric delivered. The company estimates that it has a total of 100 units of spinning mill time, and 75 units of weaving machine time that can be devoted to contract work. The net return from Contract 1, 2 will be \$1000, \$1700/unit fabric respectively. It is required to determine how much work to accept from each contract to maximize the total return from them.

Formulate this as an LP. Find the optimum solution of this model using the geometric method, and determine the marginal values associated with spinning mill and weaving machine times.

3.3: A company makes 2 products P_1, P_2 using 3 resources R_1, R_2, R_3. The data is given below.

Resource	Inputs (units/unit) for		Units available
	P_1	P_2	
R_1	2	1	20
R_2	1	2	20
R_3	1	1	12
Net profit (units/unit)	30	20	

Assuming that there are no bounds on the units of each product made, formulate the problem of determining the optimum number of units of each product to make to maximize total net profit. Solve the model by the geometric method and determine the marginal values of the three resources. Give an interpretation of these marginal values.

3.4: Her doctor has informed a girl that she needs to include special antioxidants (SA), and betacarotene related previtamins (BC) in her diet in order to improve her health. A nice way to obtain these nutrients is to eat two tropical fruit, durian and litchies. There are minimum weekly requirements (WR) that she has to meet. These fruit also contain sugary carbohydrates (SC), she wants to limit the units of these that she consumes. Here is all the data.

Nutrient	Units/unit in		WR (units)
	Durian	Litchies	
SA	5	15	≥ 45
BC	20	5	≥ 40
SC	12	2	≤ 60
Cost($/unit)	5	10	

It is required to determine how many units of each of these fruit she should consume weekly in order to meet the requirements at the least cost. Model and solve the problem by the geometric method, and determine the marginal values of each nutrient, and give the interpretation of these marginal values.

3.5: A woman is trying to get as much vitamin K as possible in her diet to improve her health. She started eating a special breed of avacado (BA), and a tropical fruit from Brazil (BF), which are excellent sources of vitamin K. Both BA, BF also contain potassium and leutein. There is a maximum daily limit (MDL) for potassium, and a minimum daily requirement (MDR) from BA, BF, in her diet. Also, BA contains a type of fat, and BF contains sugars, for this reason, she has to limit the quantities of these foods in her diet. Data on the composition of these foods and all the limits is given below.

Nutrient	Composition (units/unit)		Limit
	BA	BF	
Vit K	65	35	Maximize
Potassium	6	8	$\leq$ 48 (MDL)
Leutein	1	1	$\geq$2 (MDR)
Maximum in diet	6	4	

Formulate the problem of determining how much BA, BF to include in her daily diet as an LP. Solve this LP by the geometric method and determine the marginal values of the MDL on potassium, and the MDR on leutein, and explain their interpretation.

3.6: A company makes two products P_1, P_2 using three resources

Resource	Inputs (units/unit) of		Max (units) available
	P_1	P_2	
R_1	7	10	350
R_2	15	5	300
R_3	10	6	240
Net profit ($/unit)	500	300	

R_1, R_2, R_3, whose supply per period is limited. Production data on these products is given above. Formulate the problem of determining the optimum production plan for this company that maximizes the total net profit as an LP.

Solve the LP by the geometric method and show that it has many optimal solutions. Express the general optimum solution of this problem algebraically.

3.7: A small truck manufacturing company makes two specialized truck models, M_1, M_2 in one plant. Plany operations are grouped into four sections: a metal stamping section (MS), and an engine assembly section (EA), in both of which work is carried out for both M_1, M_2; a M_1 final assembly section (MA1), and a M_2 final assembly section (MA2).

The capacity of EA is measured as 4000 units/week. To make one M_1, M_2 needs 2, 1 units respectively of EA capacity.

Similarly the capacity of MS is measured as 5000 units/week. To make one M_1, M_2 needs 1, 2 units respectively of MS capacity.

MA1 can assemble at most 1500 M_1/week; MA2 can assemble at most 2125 M_2/week.

At present the plant is producing 1500 M_1, and 1000 M_2/week. This present solution is used in deriving cost estimates as explained below.

In the present solution M_1 uses (1500/5000) which is 30% of MS capacity, M_2 uses (1000/2500) which is 40% of MS capacity; the remaining 30% of MS capacity is not being used at present. Hence M_1, M_2 are using MS capacity in the ratio 3:4, hence the fixed overhead cost of MS/week is allocated to M_1, M_2 in this ratio.

M_1 uses (1500/2000) which is 75% of EA capacity, M_2 uses (1000/4000) which is 25% of EA capacity. So, EA is operating at full capacity in the present solution. M_1, M_2 are using EA capacity in the ration 0.75:0.25 or 3:1. Hence the fixed overhead cost of EA/week will be allocated to M_1, M_2 in this ratio.

Here are the details of overhead (OH) costs. The total fixed OH costs/week in the MS, EA, MA1, MA2 are $1,750,000; 700,000; 1,275,000; 850,000 respectively. These in the MS, EA are allocated to M_1, M_2 as explained above. The fixed OH costs in MA1, MA2 are allocated 100% to M_1, M_2 respectively. This results in the following:

Sec.	Fixed OH costs/week				Variable OH		Total OH	
	Alloc. to (unit = $\$10^3$)		\$/unit*		\$/unit/week		\$/unit/week	
	M_1	M_2	M_1	M_2	M_1	M_2	M_1	M_2
MS	750	1,000	500	1000	250	100	750	1100
EA	525	175	350	175	150	100	500	325
MA1	1,275	0	850	0	400	1250	0	
MA2	0	850	0	850	0	400	0	1250
Tot							2500	2675

*According to the present solution

Here is the data on all other costs and prices.

Cost (\$/unit)	M_1	M_2
Direct Materials	12,000	9,000
Direct labor		
MS	1500	1000
EA	2000	1500
Final assembly	2500	2000
OH	2500	2675
Total cost	20,500	16,175
Selling price	21,002	17,135
Net profit	502	960

(i): Formulate the problem of finding the best product mix for this company using these cost estimates.

Solve the model geometrically. If the optimum solution is different from the present one, how much does it improve the net profit earned over the present level?

(ii): Find the marginal values of MS, EA capacities.

The company is considering introducing a new economy truck model M_3. The total MS capacity would be sufficient to handle 3000 M_3s, while the total EA capacity would be enough to handle 2500 M_3s/week.

M_3s can be assembled in MA1, each M_3 would require half as much time in MA1 as an M_1.

Suppose M_3 can be sold at a price to yield a net profit of $225. Is it profitable to introduce M_3? If not what is the breakeven net profit for M_3?

(iii): If the capacity of MS can be increased by introducing overtime there at a cost of $250/unit capacity/week, is it worth going overtime in MS?

(Adopted from the Harvard case: C. J. Christenson, "Sherman Motor Company", 1962.)

3.8: A company makes two products P_1, P_2 using two resources R_1, R_2. Here is the data.

Resource	Input (units/unit) for		Max. available (units/period)
	P_1	P_2	
R_1	6	8	48
R_2	7	6	42
Max Demand (units/period)	≤ 4	Unlimited	
Net profit ($/unit)	4	5	

(i): Formulate the problem of determining an optimum product mix for the company. Solve the model geometrically.

(ii): Determine the marginal values of the resources, and of the upper bound on the demand for P_1.

Is it worthwhile launching an advertizing campaign to increase the demand for P_1? Explain clearly.

How much extra money beyond the cost of the present resource units of R_1, R_2 be spent to acquire additional units of these resources and still breakeven?

(iii): Suppose the company wants to adopt the policy that the demand for P_1 should be met exactly. How much will the net profit of the decrease because of this policy?

3.9: A company makes products P_1, P_2 using raw materials R_1, R_2, R_3. Relevant data is given below.

Item	Units/unit input for		Available
	P_1	P_2	(units/day)
R_1	4	5	3000
R_2	2	0	1200
R_3	1	2	900
Demand (units/day)	Unlimited	375	
Net profit ($/unit)	3	5	

Formulate the problem of finding an optimum production plan as an LP.

Solve the LP using the geometric method and find the optimum solution. Find the marginal values of the three raw materials and the demand for P_2.

If the demand for P_2 can be increased from 375 units/day by advertizing locally, is it worth spending money on this advertizement?

The current prices of the resources R_1, R_2, R_3 are $2, 4, 10/unit respectively. If additional supplies of each of these resources can be acquired, which of them has the potential for helping to increase the total daily net profit of the company? For each resource determine the breakeven price/unit at which additional supplies of it can be acquired.

Suppose the company has the opportunity to make a new product P_3. To make one unit of P_3 needs as inputs 2, 1, 2 units of resources R_1, R_2, R_3 respectively. What is the breakeven selling price of P_3 at which it becomes competitive to manufacture?

3.10: A chemicals company makes various chemicals grouped into two groups, the keytone derived products (KET), and the aldehyde derived products (ALD), using three different hydrides: sodium borohydride (SB), hydralls (HY), and lithium aluminium hydride (LAH) as reducing agents. All three hydrides are obtained from an outside supplier. Here is the process data.

Hydride	Requirement (lbs/lb) of		Cost
	KET	ALD	($/lb)
SB	0.236	0.367	10
HY	0.786	0.396	7
LAH	0.079	0.337	27

Data	KET	ALD
Demand (lbs/month)	10,000	5,000
Sale price ($/lb)	31.4	43.6
Net profit* ($/lb)	21.4	28.1

*After subtracting hydride costs

Formulate the problem of determining the optimum quantities of KET and ALD to produce, and solve it.

3.11: A company has four departments D_1 to D_4, and makes two final products A, B by combining four intermediate products P_1 to P_4. The intermediate products are made using three raw materials R_1, R_2, R_3.

Each unit of P_1 is produced by processing one unit of R_1 through D_1 for 15 minutes first, and then through D_3 for 10 minutes.

Each unit of P_2 is produced by processing one unit of R_2 for 15 minutes through D_2 first, and then for 5 minutes through D_3.

Each unit of P_3 is produced by processing one unit of R_3 for 10 minutes through D_1 first, and then for 15 minutes through D_2.

P_4 is purchased from an outside vendor at \$5/unit.

Each unit of A is made by combining together one unit each of P_1, P_2, P_4 in D_4 which takes 10 minutes of that department's time.

Each unit of B is made by combining together one unit each of

P_2, P_3 in D_4 which takes 5 minutes of that department's time.

R_1, R_2, R_3 are all available in unlimited quantities at \$20/unit.

Each of the departments D_1 to D_4 have capacities of 2400 minutes/week and have fixed operating expenses of \$5000/week to provide these capacities.

The selling prices of A, B are \$900, 1,000/unit respectively. At these prices, the demand for A, B is estimated to be 100, 50 units/week respectively.

Formulate the problem of finding an optimum production plan that maximizes total net profit. (Hint: Read wording very carefully.) (From G. Plenert, "Optimizing Theory of Constraints When Multiple Constrained Resources Exist", *European Journal of Operational Research*, 70(1993)126-133.)

3.12: Men begin to develop BPH (Benign Prostratic Hyperplasia) and prostrate cancer as they approach middle age. Cranberries contain a special nutrient which has been shown to delay the onset of these prostrate problems in men. Unfortunately, cranberries are somewhat bitter to eat directly. Also, many of the cranberry juice products in the market contain corn syrup and other sweeteners that many people would like to avoid. For an alternative, a nutrition scientist has developed three types of cranberry sauces, CS_1, CS_2, CS_3, using cranberries, dates and almonds; and has set up a company to manufacture and sell them. Several retail chains have expressed an interest in putting these sauces on their shelves.

The company has been able to find suppliers for the main ingradients, cranberries, dates, and almonds, who can supply at favorable discount prices, but only in limited quantities. To increase supplies beyond these quantities they need to seek higher priced suppliers which will make their products unprofitable. So, for the moment, they have to operate with the quantities provided by the discount suppliers.

Making the sauces involves three operations, each one performed in its own shop. The first is preparatory shop in which batches are made by preparing the ingradients and mixing them in correct proportions. The next operation is sauce making; followed by packing the sauces into containers and preparing cases for dispatching to the retail chains. The man hour requirements for each sauce in each shop, and the availability of labor in each shop are given below.

On the amount of each sauce made in the coming season, we also provide a lower bound (based on the orders for it received already), and an upper bound (based on forecasted maximum sales volumes).

Input	For sauce (units/unit)			Availability	Price
	CS_1	CS_2	CS_3	(units)	($/unit)
Cranberries	6	5	4	2600	150
Dates	2	4	5	2250	300
Almonds	1	2	2	1100	200
Man-hrs in					
Preparatory shop	1	1.5	2	900	15
Sauce making	2	2	2	1200	20
Pack/dispatch	1	2	2	1100	25
Limits (cases)					
Lower	50	150	250		
Upper	100	250	400		
Pack cost ($/case)	20	40	50		
Saleprice ($/case)	2000	2686	2840		

The cost/hour given in the various shops includes the cost of labor and all other production costs except that in the packing shop it does not include the cost of packing materials; these are given separately in the bottommost row in the above tableau. CS_1 is packed in standard containers so its packing cost is low, CS_2, CS_3 are packed in more expensive fancy containers.

Choose an appropriate objective function to optimize, and formulate the problem to determine an optimum production plan as an LP.

3.13: A department store employs part time people with flexible hours to work as sales people in the store. For the coming week they have 7 people (P_1 to P_7) who can work. Each has informed the store the maximum number of hours they can work each day. P_1, P_2, P_3 are senior people so their hourly payrate is higher than for the others. P_4 to P_7 are new recruits, so their hourly rate is lower. Also, each person is guaranteed a certain minimum number of hours depending on their seniority. Folowing table gives all this data, and also the estimated person-hour requirements at the store each day.

It is required to determine how many hours to assign to each person each day so as to meet the requirements of the store at minimum cost, subject to all the constraints. Formulate this problem as an LP.

Person	Max. hous availability on						Rate	Min hrs
	Mon	Tue	Wed	Thur	Fri	Sat	($/hour)	/week
P_1	6	0	6	0	6	3	12	7
P_2	0	7	0	7	5	0	12	7
P_3	3	4	5	0	0	0	12	7
P_4	3	0	0	0	6	6	10	10
P_5	0	3	7	7	8	0	10	10
P_6	0	0	5	5	5	8	10	10
P_7	4	6	6	0	0	6	10	10
Person. -hrs needed	10	15	20	13	22	17		

It is required to determine how many hours to assign to each person each day so as to meet the requirements at the store at minimum cost, subject to all the constraints. Formulate as an LP.

3.14: Consider the Red Brand Canners problem discussed under Scenarion 1, in Example 3.10.2. Suppose the lot of 3,000,000 lbs of tomatoes offered to the company for buying consists of 3 different grades of tomatoes; "A" grade at an average of 9 points/lb, "B" grade at an average of 6 points/lb, and "C" grade with an average of 3 points/lb. Suppose the lot contains 600,000 lbs of "A", 1600,000 lbs of "B", and 800,000 lbs of "C" tomatoes. Here assume that P_3 has no minimum average quality requirement, but P_1, P_2 have minimum average quality requirements as mentioned under this scenario. Formulate the problem of determining the optimum canning policy for this season's crop, as an LP.

3.15: A farmer is planning to grow a special breed of beet root which deelops a sugar content of 19% in its roots (much higher than the normal beet root), and also gives higher than average yields. This beet root needs a special type of fertilizer with fractional content of nutrients n, P, K, Fe equal to 0.317, 0.130, 0.050, 0.018 respectively, and the rest inert materials. The farmer found a company selling this fertilizer, but their price is far too high. So the farmer has decided to develop his/her own fertilizer mix with this composition by mixing some other reasonably priced fertilizers available in the market. There are 5 fertilizers F_1 to F_5 available with compositions and prices as given in the following table (besides the nutrients mentioned above, each fertilizer consists of inert materials). Formulate the problem of determining how the farmer should mix F_1 to F_5 in order to develop a mixture with the composition needed for the beet crop at minimum cost.

Nutrient	Fractional content in					Requirement
	F_1	F_2	F_3	F_4	F_5	
N	0.10	0.45	0	0.20	0.05	0.317
P	0.10	0	0.25	0.05	0.30	0.130
K	0.10	0	0.05	0.15	0	0.050
Fe	0.01	0	0.03	0.02	0.03	0.018
Price($/ton)	220	180	160	150	175	

3.16: A container terminal in Hong Kong port hires trucks with drivers and uses them as IT (Internal Trucks) for moving inbound containers unloaded from vessels from the dock to the storage yard for temporary storage until the customer picks them up, and outbound containers temporarily stored in the storage yard to the dock for loading into the vessel when it arrives, and a variety of other tasks in the terminal. Terminals operate round the clock, every day. They divide the working day into 24 hourly periods numbered 1 to 24, the 1st beginning at 12 midnight (0th hour) and the 24th ending at 12 midnight the next day; and estimate the number of ITs needed in each period based on workload that depends on vessel arrivals and departures, and the contents of those vessels. Following table gives estimates of their

IT requirements each day during a particular season.

Period	ITs needed	Period	ITs needed	Period	ITs needed
1	40	9	70	17	120
2	38	10	70	18	120
3	50	11	68	19	106
4	50	12	68	20	100
5	48	13	100	21	90
6	48	14	100	22	80
7	46	15	96	23	60
8	46	16	84	24	50

They have a certain number of hired trucks arriving for work each hour of the day on the hour. Each truck works continuously for 4 hours after reporting to the terminal, then takes a one hour meal break, and then works another period of 4 hours before departing for the day.

In any period if the number of trucks on duty is greater than the number of trucks needed during that period, some which are not needed will be idle, but the driver has to be paid even when the truck is idle as long as he/she is on duty.

It is required to determine how many hired trucks should be asked to report for duty at each hour of the day, so as to meet the requirements of the terminal, while minimizing the total number of trucks hired daily during this season. Formulate this problem as an LP by ignoring the integer restrictions on the decision variables. (Adopted from Murty, Liu, Wan, and Linn, "DSS for Operations in a Container Terminal", *Decision Support Systems*, 39(2005)309-332.

3.17: Work force scheduling in a bank: The workload (work of tellers, data entry people, etc.) in busy branches of a bank usually varies with the time of the day, typically in the shape of an inverted-U curve with the peak reaching around 1 PM. For efficient use of the resources, the manpower available should also vary correspondingly. Banks usually achieve this variable capacity by employing part-time personnel in their branches where workload variation WRT time of the day is highly pronounced. Here is all the relevant information for the busy downtown branch of a bank for weekdays (Monday to Friday) in a particular period.

Work begins at 9 AM daily, and the regular bank hours go on until 5 PM. But workers do work behind closed doors from 5 PM to 7 PM.

Regular employees (i.e., full-time employees) work in two shifts: both from 9 AM to 5 PM but with lunch break either from 11 AM to 12 noon, or from 12 noon to 1 PM respectively. The payrate for regualr employees comes to $20/hour at work. Regular employees can work between 5 PM to 7 PM if necessary, this will be counted as overtime for them, and they get extra pay for this work at their normal rate of $20/hour. There are a toal of 25 regular employees on the Bank's payroll for this work; the company has the option of specifying how many of them should use 11 AM to 12 noon, or 12 noon to 1 PM as lunch hour.

When part-timers are hired for a day, they work an integer number of hours between 1 hour to 5 hours without any lunchbreak, between 9 AM to 7 PM, and are paid at the hourly rate of $15/hour. They can be hired to begin their work period anytime between 9 AM to 6 PM always on the hour.

Number the hours between 9 AM to 7 PM from 1 to 10, with the 1st hour being 9 AM to 10 AM, and so on. In the current period, the bank estimates that the man-hours required in the ith hour is 14, 25, 26, 38, 55, 60, 51, 29, 14, 9 respectively for $i = 1$ to 10.

The bank wants to determine their workforce scheduling plan during this period, to meet their workload requirements at minimum cost. Define all the relevant decision variables clearly, and model the problem as an LP by ignoring the integer restrictions on the decision variables. (Adopted from Shyam L. Moondra, "An LP Model for Work Force Scheduling for Banks", *Journal of Bank Research*, Winter 1976.)

3.18: The following table gives the estimated need for part time workers in a department store in the current period in hourly intervals between 9 hours to 21 hours Monday to Friday.

Interval	min. need	Interval	min. need
9 to 10	14	15 to 16	18
10 to 11	18	16 to 17	17
11 to 12	18	17 to 18	13
12 to 13	16	18 to 19	13
13 to 14	14	19 to 20	13
14 to 15	19	20 to 21	12

All part time workers who fill these position work in shifts of 5 hours on the day they report for duty, with a one hour break in the middle; and they can begin their shift anytime between 9 hours and 16 hours always on the hour. Most of these workers like to have their break between 12 to 14 hours; so anybody who had to work continuously between 12 to 14 hours is paid 5% extra pay on that day. It is required to determine the number of workers to be hired to fill this need at minimum cost, and how many of them begin their shift on each hour. Ignoring the integer requirements on the decision variables, formulate this problem as an LP.

3.19: A paper company has a machine that cuts master reels of paper of width 82 in. into reels of smaller widths. All the reels have the same length of paper on them. They have standing orders for reels in the following widths from their regular customers.

Width	No. reels on order/week*
58 in.	2000
26 in. 10,000	
24 in.	12,000
23 in.	12,000

Min. on order. Can supply some more.

A *cutting pattern* specifies how many reels of widths 58, 26, 24, 23 ins. are cut from a master roll. For example if 2 rolls of width 26 in. and 1 roll of width 24 in. are cut from a master roll in a pattern; the remaining 6 in. width is too small to be useful for any of the orders, it will be wasted in this pattern. This is called *trim waste*, and in this pattern it amounts to $(6/82)100 = 7.32\%$. Companies try to minimize this trim waste as much as possible by minimizing the number of master rolls cut to fulfill the customer orders.

This company uses the following 12 cutting patterns for cutting the rolls for these orders.

Pattern no.	No. rolls of width cut				Waste
	58 in	26 in	24 in	23 in	(in. width)
1	1	0	1	0	0
2	1	0	0	1	1
3	0	3	0	0	4
4	0	2	1	0	6
5	0	2	0	1	7
6	0	1	2	0	8
7	0	1	1	1	9
8	0	1	0	2	10
9	0	0	3	0	10
10	0	0	2	1	10
11	0	0	1	2	12
12	0	0	0	3	13

The problem is to determine how many master rolls to cut according to each of the above patterns so as to meet the weekly requirements using the smallest possible number of master rolls. Formulate as an LP, ignoring the integer restrictions on the decision variables.

3.20: The fineness of spun yarn is measured by its count, the higher the count the finer the yarn. A textile spinning mill produces cotton yarn in counts 20s and 40s on their spinning frames. Considering the avilable machinery, quantity of cotton on hand and other resources, it estimates that it can produce at most 40,000 yarn in the 20s, and 32,000 units in 40s in a particular period.

The spun yarn in each count can be reeled on a reeling frame and sold as is (i.e., as reels of 20s or 40s); or processed further on machines called doubling frames, and then reeled on reeling frames and sold as doubled yarn (i.e., as reels of 2/20s or 2/40s). The average production of the four varieties, and also the maximum number of available frame shifts in the doubling and reeling shops are given in the following table together with the margin (net profit)/unit yarn in each variety.

Variety	Production/frame shift (in units)		Margin
	Doubling	Reeling	($/unit)
20s	–	27	2.80
2/20s	160	42	3.15
40s	–	17	4.25
2/40s	52	27	4.75
Max frame-shifts available	150	3100	

The problem is to determine how many units in each variety should be produced for maximizing the total margin. Formulate this problem as an LP. (Adopted from N. Srinivasan, "Product Mix Planning for Spinning Mills", Indian Statistical Institute, 1975.)

3.21: There are 5 different areas of specialization in a department

Weekday	No. who want to, but can't attend seminar on this day in area				
	S_1	S_2	S_3	S_4	S_5
Monday	6	5	7	3	6
Tuesday	5	4	5	4	3
Wednesday	7	3	4	3	5
Thursday	4	4	3	4	3
Friday	5	4	5	5	4

at the University of Michigan. Let us denote them S_1 to S_5. Now-a-days because of the tight job market, many graduate students are taking courses in several areas, even though they may specialize in one area. Each area holds a seminar with a guest speaker once per week. The department wants to hold these 5 seminars at the rate of one per day, Monday to Friday afternoons each week. For each of the 25 area-weekday combinations, a survey of the graduate student body was conducted to find how many of them who want to attend the seminar in that area will find it difficult to attend on that day because of other conflicts, and the data is given above.

The decision to be made is the assignment of weekdays to areas for holding the seminars. Select an appropriate objective function to optimize for making this decision, and formulate the problem of finding an optimal decision.

3.22: A company makes 6 types of herbal mixtures H_1 to H_6 that people use for a variety of health benefits, in three plants.

Mixture	Man hrs./unit in		Cost ($/unit)
	Preparatory	Packing	
Plant 1 can make H_1, H_2, H_3			
H_1	2.5	2.2	98.0
H_2	2.7	5.0	115.5
H_3	2.2	4.1	88.4
Available (man hrs/period)	630	620	
Plant 2 can make H_4, H_5, H_6			
H_4	2.1	2.2	105.0
H_5	2.7	2.6	119.8
H_6	2.4	2.0	97.7
Available (man hrs/period)	675	620	

Each plant consists of two shops. The first is the preparatory shop where the ingradients are assembled, cleaned, processed, and the mixture prepared in batches; and the other is the packing shop where the prepared mixtures are packed in special packages and dispatched to customers. Plant 3 actually has two preparatory shops specializing in different products, but only one large packing shop serving both of them.

At present they have adequate supplies of ingradients for meeting customer demands at present levels, so the critical resource in each shop is man hours of trained and skilled manpower. The equipment available in the various plants is different, so the resource use/unit of the same mixture in different plants may not be the same. Data on what each plant can make, and the input-output data is provided here.

Plant 3 can make H_2, H_3, H_4, H_5

Mixture	Man hrs./unit in			Cost
	Prep 1	Prep 2	Packing	($/unit)
H_2	2.4		2.3	107.2
H_3		2.1	1.6	89.75
H_4	2.1		1.7	103.0
H_5		2.4	1.9	116.8
Available (man hrs/period)	700	700	1240	

For the next period the company has the following orders to fulfill. Formulate the problem of finding an optimum production schedule at the three plants for meeting the orders.

Mixture	Orders (units)	Mixture	Orders (units)
H_1	150	H_4	230
H_2	250	H_5	260
H_3	200	H_6	160

3.23: The gas blending division of a petroleum refinery blends butane (B), Straight run gasoline (SR), catalytically cracked gasoline (CC), and reformate (RE), into two grades of motor fuel, regular and premium. The availability and properties of the components to be blended are given below (Oc.R is octane rating, and VP is the Reid vapor pressure measured in psi (pounds/square inch)).

Component	Oc. R	VP	Availability (m bar./week)		Price
			Summer	Winter	($/barrel)
B	105	65	3	4	42
SR	80	8	7	8	32
CC	95	5	5	6	35
RE	102	4	4	5	36

Data on the demand for gasoline in the two seasons is given below (demand given in m barrels/week is the maximum amount that can be sold in their marketing region; price is given in \$/barrel).

Season	Type	Demand	Specs. on		Price
			Oc.R	VP	
Summer	Premium	8	≥ 93	≤ 13	46
Summer	Regular	12	≥ 87	≤ 8	40
Winter	Premium	6	≥ 93	≤ 13	44
Winter	Regular	11	≥ 87	≤ 11	39

Additional quantities of SR beyond the availability mentioned in the first table can be obtained, if necessary, at the rate of \$34.50/barrel. Each barrel of supply purchased by the division is estimated to cost an average of \$0.75 for handling and processing in the division.

Select the appropriate objective function to optimize, and formulate the problem of determining an optimum production schedule in each season.

3.24: A fertilizer blending company buys the basic fertilizer ingradients, nitrates, phosphates, and potash from suppliers; blends them together with some other inert ingradients available in unlimited supply; to produce three different fertilizer mixes 5-10-5, 10-5-10, 10-10-10 (these are the percentages by weight of nitrates, phosphates, and potash respectively in the mix) during the growing season. Here is the data.

Ingradient	Availability (tons/week)	Price (\$/ton)
Nitrates	1000	200
Phosphates	1800	90
Potash	1200	150
Inert	Unlimited	5

Every thing produced can be sold, but there is a sales committment of 6000 tons of 5-10-5 per week. The costs of handling, mixing, packaging, and dispatching, are the same for all the three mixes, and are estimated at \$25/ton of mixture made. Competitive selling prices for the fertilizer mixes (in \$/ton) are 220 for 5-10-5, 250 for 10-5-10, and 300 for 10-10-10.

Select the appropriate objective function to optimize, and formulate the problem of determining how much fertilizer of each type to produce.

3.25: Cosider a diet problem with minimum daily requirements (MDR) on protein, calories, vitamins A, B, C; and a siet consisting of foods milk, lettuce, peanuts, and fortified soy cake (FSC). Here is the data on the composition and prices of foods.

Food	Units/unit of					Price
	Protein (gms)	Calories	Vit. A (units)	Vit. B (mg)	Vit. C (mg)	(\$/unit)
Milk (quarts)	35	666	1550	0.35	13	0.80
FSC (each)	17	77	550	0.05	0	0.25
Lettuce (head)	3	60	1400	0.20	25	0.90
Peanuts (servings)	100	1650	0	0.40	0	0.45
MDR	70	2000	5000	2	80	

Formulate the problem of finding a diet that meets the requirements at minimum cost. Transform the problem into one in which all the constraints are equations.

3.26: This problem involves determining the amounts of food groups F_1 to F_6 (respectively, milk (pints), meat (lbs), eggs (doz.), bread (ozs.), greens (ozs.), orange juice (pints)) to include in a diet to meet nutritional requirements and quantity restrictions at minimum cost. The nutrients are A (vit. A), Fe (iron), C (calories), P (protein), CH (cholosterol), CAR (carbohydrates). UB is upper bound for inclusion in the diet in the same units in which food is measured. Cost is given in \$/unit. Formulate as an LP.

Nutr.	Units/unit in Food						Requirement
	F_1	F_2	F_3	F_4	F_5	F_6	
A	720	107	7,080	0	134	1,000	$\geq$ 5,000
Fe	0.2	10.1	13.2	0.75	0.15	1.2	$\geq$ 12.5
C	344	1,460	1,040	75	17.4	240	$\geq$ 1,400
							$\leq$ 1800
P	18	151	78	2.5	0.2	4	$\geq$63
CH	10	20	120	0	0	0	$\leq$ 55
CAR	24	27	10	15	1.1	52	$\leq$ 165
Cost	0.65	3.4	1.4	0.2	0.15	0.75	
UB	6	1	0.25	10	10	4	

3.27: A farmer is making feed for livestock using hay (H), corn (C), oil seed cake (OSC), and oats (O). The farmer wants to make sure that each animal gets at least one lb of protein, four lbs of carbohydrates, and eight lbs of roughage to eat. Data on the composition and prices of the constituents is given below.

It is required to determine the optimum proportion of the constituents in the feed mix, and the amount of feedmix to give to each animal in order to meet all the requirements stated above at minimum cost. Formulate as an LP.

Item	Item fraction in			
	H	C	OSC	O
Protein	0.05	0.1	0.4	0.02
Carbohydrates	0.20	0.3	0.1	0.15
Roughage	0.4	0.2	0.1	0.3
Price ($/lb)	0.2	0.4	0.6	0.3

3.28: A machine tool manufacturer is planning to exhibit one of their finest lathes, and a top of the line milling machine at an international machine tool exhibition. They want to keep both the machines in operation during the full 100 hours of the exhibition making simple products that can be given away to potential customers visiting their booth. For these products, three are under consideration; an ash tray (A), a paperweight (P), and a metal ruler (M).

Each unit of A, P, M need 0.1, 0.3, 0.2 hours operation on the lathe, and then 0.4, 0.2, 0.4 hours operation on the milling machine respectively, to make. But each of these products use an expensive form of brass stock. Each unit of A, P. M need 0.2, 0.3, 0.1 lbs of brass stock to make.

It is required to find out how many units of A, P, M, to make during the exhibition in order to keep both the lathe and the milling machine fully occupied during the entire 100 hours of exhibition time while minimizing the amount of brass stock used. Formulate this problem as an LP. (From P. W. Marshall, "LP: A Technique for Analyzing Resource Allocation Problems", Harvard Business School Cases, 1971.)

3.29: A Couple saved a lot of money in one year, and are looking to invest upto $40,000 in 5 different investment opportunities. Data on them is given below.

They want to make sure that the fraction invested in industrial opportunities is no more than 0.6, and that the weighted average (with the amounts invested as the weights) maturity period of their investments is no more than 14 years.

Formulate the problem of determining how much to invest in each opportunity in order to maximize the annual return from all the investments subject to the constraints stated above.

Opportunity	Annual return	Maturity
A. Govt. bonds	6%	10 years
B. Govt. bonds	5%	5 years
C. Preferred shares (industry)	7%	20 years
D. Bonds (industry)	4%	6 years
E. CDs	5.5%	7 years

3.30: A shoe company has to allocate production of 5 styles of shoes to three overseas manufacturers denoted by PRC, BYC, SKC. Here is the data on the quote ($/pair of shoes) of each style from each manufaturer, and their production capacity (units of 1000 pairs/month) for each style and overall, and the demand for each style in the same units.

Style	PRC		BYC		SKC		Demand
	Quote	Capa.	Quote	Capa.	Quote	Capa.	
S_1	12.5	120	11.75	40	11.5	120	140
S_2	10.75	80	12.25	100	11.25	160	100
S_3	14.25	80	13.25	50	13.75	80	70
S_4	13.75	50	13.00	70	14.25	120	40
S_5	14.5	60	13.25	90	14.00	30	50
Overall		220		200		230	

Also, BYC and SKC are in the same country, and there is an import quota of 370 (in same units) from both of them put together. Formulate the problem of determining an optimum allocation as an LP.

3.31: An oil refinery processes crude oil in its *distillation tower* (DT) to produce a product called *distillate* and a variety of other products. Distillate is used to make gasoline.

Each barrel of crude processed in the DT yields 0.2 barrels of distillate, 0.75 barrels of by-products, and the remaining 0.05 barrels is lost in the distillation process. The DT has a capacity to process 250,000 barrels of crude/day at an operating cost of $1/barrel of crude.

The refinery gets crude oil at a price $30/barrel. The refinery sells the by-products produced at the DT at $32/barrel.

The Oc. R. (octane rating) of the distillate is 84, it is too low to be used directly as gasoline. So, the refinery keeps only some of the distillate produced at the DT to be blended into gasoline products, and will process the remaining distillate in a *catalytic cracker* (CC) further.

The CC cracks the heavy hydrocarbon compounds in the distillate into lighter compounds. This process produces a high quality product called *gasoline stock* (GS), and several other petroleum by-products. Each barrel of distillate cracked in the CC yields 0.5 barrels of GS, 0.45 barrels of by-products, and the remaining 0.05 barrels of distillate is lost in the cracking process. Cracking costs $2/barrel of distillate cracked. The company sells the by-products of cracking at $40/barrel. The CC can process upto 20,000 barrels of distillate daily. The GS produced by cracking has an Oc. R. of 95.

The company blends the distillate that is not processed through the CC, and the GS produced by the CC into regular and premium gasolines, which have required Oc. R.s of 87, 90 respectively. The cost of blending, and the losses in blending are negligible and can be ignored. The company sells its regular and premium gasolines at \$40, 45/barrel respectively. They can sell upto 25,000 barrels/day of regular and premium respectively.

Formulate the problem of determining an operating policy that maximizes the company's net daily profit.

3.32: Sugar refinery planning: The first step in the production of cane sugar (sugar made from the juice of sugar cane) is the crushing and rolling of sugar cane at sugarcane mills to separate sugar juice and cane stalk. The product called *raw sugar* (RS) is made by clarifying the sugar juice, concentrating and crystallizing it into large brown crystals containing many impurities.

Plants called *refineries* process and purify the RS into finished sugar. This process yields molasses as a by-product, and also a left over fibrous material called bagasse which can be used in the production of feed, paper, and also as fuel. Rum is made by distilling molasses. Also several important chemicals are made by fermenting molasses.

The company operates two refineries R_1, R_2. They buy RS from 8 sugar cane mills S_1 to S_8. The refineries ship molasses to seven customer molasses processing plants C_1 to C_7; and sell finished sugar and bagasse in the open market. Here is the relevant data.

RS supplier	Available (tons/month)	Price (\$/ton)	Shipping cost (\$/ton) to	
			R_1	R_2
S_1	1100	65	9	22
S_2	1475	63	8	27
S_3	2222	68	13	23
S_4	1280	69	11	25
S_5	1950	64	7.5	24
S_6	2050	63	19	28
S_7	1375	645	24	19
S_8	1800	635	22	21

*Does not include shipping cost

Quantity	Production data for R_1	Production data for R_2
% loss in RS in transit from supplier	2.7	2.7
Production %s as % of RS processed		
Finished sugar	33.6	45.6
Molasses	30.4	30.7
Bagasse	36.0	23.7
Production costs		
$/ton RS processed	75	72
Prod. capacity		
tons RS refining/month	8030	8800
Operating range (as % of capacity)	50 to 100	50 to 100

Product	Selling price ($/ton at refinery)
Finished sugar at R_1	400
Finished sugar at R_2	325
Molasses (at R_1 or R_2)	75
Bagasse at R_1 or R_2	55

Molasses requirement (tons/month)

Customer	Required	Shipping cost ($/ton) from R_1	Shipping cost ($/ton) from R_2
C_1	480	50	60
C_2	950	92	65
C_3	610	35	25
C_4	595	32	43
C_5	950	48	29
C_6	117	52	53
C_7	90	45	80

The company would like to determine the production plan that minimizes the total net cost of meeting the demand at the customers exactly. Define all the relevant decision variables and formulate the problem. (Adopted from Harvard Business School case "J. P. Molasses, Inc., 1988.)

3.33: A company processes tomatoes into various final products. For the coming season, they have arranged supplies of four different kinds of tomatoes in the following quantities.

Tomato supply available

Kind	Available (units/week)	Price ($/unit)
I	200,000	40
II	200,000	34
III	250,000	34
iv	150,000	36

The final products which we denote by P_1 to P_5 are cases of: Choice canned tomatoes (CCT), Standard canned tomatoes (SCT), Juice, Paste, and Puree. If processed into canned tomatoes, here are yields from the various kinds of tomatoes.

Kind	Fraction yield of		
	CCT	SCT	Peeling loss
I	0.40	0.45	0.15
II	0.60	0.25	0.15
III	0.20	0.68	0.12
iv	0.30	0.50	0.20

If processed into juice, all kinds of tomatoes are crushed and yield 100% unfiltered juice measured in the same unit as tomatoes.

For processing into paste or puree, the juice has to be condensed to certain specified contents of solids. For making these products, the important characteristic is the solids content (measured in some units/unit of tomatoes) which for kinds I to iv is: 5, 6, 6, 5.5 respectively.

Here is the input-output, and price/cost data.

Product	Ingradients needed (units/case)				*Processing costs ($/case)	Selling price ($/case)
	CCT	SCT	Juice	Solids		
P_1	4	0	1	0	100	450
P_2	0	3	2	0	90	410
P_3	0	0	3	0	80	190
P_4	0	0	0	8	130	240
P_5	0	0	0	4	120	210

All costs other than tomatoes

There are constraints on the quantities the company can sell at the above prices. Their mail lines are canned tomatoes, and their marketing department estimates that they can sell up to 60,000 cases of each of P_1, P_2/week. The company has to produce P_3, P_4, P_5 to make up a full line of commodities in order to protect the company's competitive marketing position. Based on the marketing department's estimates the company wants to make sure that the production of P_3 is between 1000 to 4000 cases/week, and the production of P_4, P_5 is $\geq$ 1000 cases each/week.

Select the appropriate objective function to optimize, and formulate the problem of determining an optimum production plan for this company.

3.34: A company makes cans in three plants P_1, P_2, P_3. All their output is sold to four wholesalers W_1 to W_4. For the coming year W_1 to W_4 agreed to buy 3, 4, 5, 6 million cans from the company respectively. So, the total demand for the company's products is 18 million cans in the coming year. The production capacity at P_1, P_2, P_3 is 6, 7, 8 million cans/year; for a total production capacity of 21 million cans/year. So the company has an excess production capacity for 3 million cans in the coming year.

The total yearly production cost at each plant consists of a fixed cost + a variable cost which is a linear function of the number of cans produced at the plant that year. The fixed costs at P_1, P_2, P_3 are \$6, 7, 8 million/year respectively. The variable costs at P_1, P_2, P_3 are \$2, 1.8, 1.6/can produced. For example, this means that if x cans are produced at $P-1$ in a year, the total production cost at the company will be $\$(6,000,000+2x)$.

There are two scenarios being considered for handling the excess capacity for the coming year.

Scenario 1: Operate P_2, P_3 at full capacity, and let P_1 produce only 3 million cans for the coming year. Under this scenario, the per can production cost at the three plants will be:

Plant	Production cost ($/can)
P_1	$(6,000,000 + 2 × 3,000,000)/3,000,000 = 4
P_2	$(7,000,000 + 1.8 × 7,000,000)/7,000,000 = 2.8
P_3	$(8,000,000 + 1.6 × 8,000,000)/8,000,000 = 2.6

Scenario 2: This scenario disatributes the excess capacity equally among all the plants. So, under this scenario, P_1, P_2, P_3 will produce 5, 6, 7 milliopn cans respectively in the coming year. Under this scenario, the per can production cost at the three plants will be:

Plant	Production cost ($/can)
P_1	$(6,000,000 + 2 × 5,000,000)/5,000,000 = 3.2
P_2	$(7,000,000 + 1.8 × 6,000,000)/6,000,000 = 2.96
P_3	$(8,000,000 + 1.6 × 7,000,000)/7,000,000 = 2.74

Plant	Shipping cost ($/can) to			
	W_1	W_2	W_3	W_4
P_1	0.18	0.13	0.13	0.16
P_2	0.21	0.16	0.11	0.16
P_3	0.16	0.11	0.21	0.16

So, under Scenario 2, the per can production cost at P_1 decreased by \$0.80; but increased by \$0.16, 0.14 at P_2, P_3; resulting in a net change of \$(-0.80 + 0.16 + 0.14) = \$(-0.50), or a saving of \$0.50/can produced. So, under this logic, Scenario 2 seems to be better than Scenario 1.

The selling price to all the wholesalers is a uniform \$5/can delivered. The transportation costs for shipping the cans to the wholesalers from the plants are given above.

(i): Under each of the above scenarios 1, 2, formulate a mathematical model to determine what the maximum total net profit is.

After you learn the algorithm to solve transportation problems from Chapter 6, determine which scenario results in a higher total net profit to the company. Check whether the logic given above that Scenario 2 saves the company \$0.50/can produced is correct.

(ii): Model the problem of determining an optimum production schedule that distributes the excess capacity optimally among the three plants, in the form of a balanced transportation problem.

Again after you learn the algorithm to solve transportation problems from Chapter 6, determine the optimum production schedule for the company.

3.35: A country can be divided into 8 regions WRT the supply (production in the region itself) and demand (within that region) for milk/week. Here is the data (milk is measured in units of million gallons).

Region	Supply	Demand	Excess (+) or deficit (−)
R_1	350	300	+50
R_2	400	100	+300
R_3	550	200	+350
R_4	150	350	−200
R_5	0	100	−100
R_6	50	200	−150
R_7	100	250	−150
R_8	50	250	−200
Total	1650	1750	−100

So, there is a toal deficit of 100 units of milk/week over the whole country which is made up by importing milk regularly from a neighboring country C that has surplus milk. Following table gives data on the average cost of transporting milk in some money units/unit between the excess and deficit regions within the country. Also, in the row of C we provide data on the average cost (same units as above) for milk delivered to each deficit region within the country from C. Formulate the problem of meeting the demand for milk within the whole country that minimizes the distribution costs of milk within the country, and the import costs from C.

From	Transportation Cost to (money units/unit)				
	R_4	R_5	R_6	R_7	R_8
R_1	10	12	8	6	4
R_2	8	4	7	5	4
R_3	9	8	7	6	5
C	95	85	75	115	120

3.36: A company has three plants P_1, P_2, P_3 spread over the country for making a product. There are two wholesalers who buy this product from them. The demand (maximum amount desired by them, company can supply any amount less than this) by these wholesalers for the product in the next two months, months 1, 2, is given below.

Wholesaler	Demand (in units)		Shipping cost (\$/unit) from		
	Month 1	Month 2	P_1	P_2	P_3
W_1	4000	6000	6	12	4
W_2	8000	9000	10	5	7

Inventories are depleted at the beginning of month 1 in each plant, but the units produced in month 1 can be shipped during that month itself towards the demand in that month, or held in inventory at the plant and shipped in the next month. The inventory capacity at P_1, P_2, P_3 is 500, 500, 300 units respectively and it costs \$3/unit to hold in inventory from month 1 to month 2.

Data on the production capacities and costs at each plant is given below.

Plant	Prod. capacity (units)		Prod. cost
	Month 1	Month 2	($/unit)
P_1	4000	5000	65
P_2	5000	4000	63
P_3	5000	4000	64

THe selling price/unit delivered at the wholesalers is $80. Formulate the problem of determining an optimum production, storage, shipping plan for the company.

3.37: A company has been hired to conduct a survey on the economic conditions of people in a region. According to the plan, they need to select a sample of 2500 people to collect data from, satisfying the following constraints:

a): At least 20% of the sample must be in each of the age groups 18 to 30, 30 to 50, 50 to 65 years.

b): At least 20% of the sample must belong to each of the groups, blacks, latin americans, majority whites.

They know that only a fraction of the people contacted will respond and provide the data. Information on the fraction expected to respond in the various subgroups estimated from their past experience is given in the following table.

Group	Fraction expected to respond in age group		
	18 to 30	30 to 50	50 to 65
Blacks	0.25	0.30	0.40
Latin Americans	0.20	0.25	0.45
Whites	0.30	0.20	0.50

It is required to determine how many people to include in the sample in the various subgroups in order to maximize the expected total number responding, subject to all the constraints mentioned above. Formulate this as an LP ignoring the integer requirements on the decision variables.

3.38: A multi-period blending problem: The following table gives data on the fraction (by weight) of three different nutrients N_1, N_2, N_3 in five different foods F_1 to F_5; and the expected cost (in \$/kg) of each food in three periods. There is a demand for a mixture of these foods meeting the required fractions for the three nutrients exactly in each of the periods. The individual foods can be purchased in any period and stored for use in later periods. Storage costs in \$/kg from one period to the next are given in the table. Mixture which is made in one period can be stored and used to meet the demand in later periods, at a storage cost in \$0.02/kg from one period to the next. Formulate the problem of meeting the requirements at minimum total cost. (From C. A. C. Kuip, "Algebraic Languages for Mathematical Programming", *European Journal of Operational Research*, 67(1993)25-51.)

Item	Fraction in Food					Required fraction
	F_1	F_2	F_3	F_4	F_5	
N_1	0.13	0.24	0.12	0.25	0.15	0.2
N_2	0.24	0.12	0.44	0.44	0.48	0.42
N_3	0.03	0.05	0.01	0.01	0.01	0.15
Cost in						Demand*
Period 1	1.7	2.1	1.4	1.7	1.8	50 kg
Period 2	2.0	3.0	2.3	2.0	1.7	60 kg
Period 3	3.0	3.5	2.3	2.3	1.6	70 kg
Storage cost	0.011	0.008	0.014	0.011	0.012	

*for mixture in period.

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