Chapter 3

SEPARATION PROPERTIES, PRINCIPAL PIVOT TRANSFORMS, CLASSES OF MATRICES

In this chapter we present the basic mathematical results on the LCP. Many of these results are used in later chapters to develop algorithms to solve LCPs, and to study the computational complexity of these algorithms. Here, unless stated otherwise, I denotes the unit matrix of order n. M is a given square matrix of order n. In tabular form the LCP (q, M) is

w	z	q
I	$-M$	q

$$w \geqq 0, \qquad z \geqq 0, \qquad w^T z = 0 \tag{3.1}$$

Definition: Subcomplementary Sets of Column Vectors

A vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ where $y_r \in \{w_r, z_r\}$ for $r = 1, \ldots, i-1, i+1, \ldots, n$ is known as a **subcomplementary vector of variables** for the LCP (3.1). The complementary pair (w_i, z_i) is known as the **left-out complementary pair of variables** in the subcomplementary vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$. Let $A_{.j}$ be the column vector associated with y_j in (3.1). The ordered set $(A_{.1}, \ldots, A_{.i-1}, A_{.i+1}, \ldots, A_{.n})$ is known as a **subcomplementary set of column vectors** for the LCP (3.1), and $(I_{.i}, -M_{.i})$ is the **left-out complementary pair of column vectors** in this subcomplementary set of column vectors.

Sometimes we have to refer to subcomplementary sets which are complementary sets with several elements missing. For this, we adopt the following notation. Let $\mathbf{J} \subset \{1, \ldots, n\}$, $\mathbf{J} \neq \emptyset$, $\mathbf{J}$ a proper subset. The vector $(y_j : j \in \mathbf{J})$ where $y_j \in \{w_j, z_j\}$

for all $j \in \mathbf{J}$ is said to be a subcomplementary vector of variables for (3.1) associated with the subset $\mathbf{J}$. Let t_j be the complement of y_j and let $A_{.j}$ be the column vector associated with y_j in (3.1), and let $B_{.j}$ be the complement of $A_{.j}$, for $j \in \mathbf{J}$. Then $\{A_{.j} : j \in \mathbf{J}\}$ is said to be a subcomplementary set of column vectors associated with $\mathbf{J}$, and $\{B_{.j} : j \in \mathbf{J}\}$ is its complement. The subcomplementary vector $(t_j : j \in \mathbf{J})$ is the complement of the subcomplementary vector $(y_j : j \in \mathbf{J})$.

3.1 LCPs ASSOCIATED WITH PRINCIPALLY NONDEGENERATE MATRICES

If $y = (y_1, \ldots, y_n)$ is a complementary vector of variables for (3.1), define

$$\begin{aligned} \mathbf{Z}(y) &= \{j : j \text{ such that } y_j = z_j\} \\ \mathbf{W}(y) &= \{j : j \text{ such that } y_j = w_j\}. \end{aligned} \tag{3.2}$$

Theorem 3.1 *If y is a complementary vector of variables for (3.1), it is a complementary basic vector iff the principal subdeterminant of M corresponding to the subset $\mathbf{Z}(y)$ is nonzero.*

Proof. Let the cardinality of $\mathbf{Z}(y)$ be r. Let A be the complementary matrix associated with y. For $j \in \mathbf{W}(y)$, $A_{.j} = I_{.j}$ and for $j \in \mathbf{Z}(y)$, $A_{.j} = -M_{.j}$. If $r = 0$, $A = I$ and its determinant is 1. If $r > 0$, by expanding the determinant of A in terms of its elements in the jth column for each $j \in \mathbf{W}(y)$ in some order, we see that the determinant of A is $(-1)^r$ (principal subdeterminant of M corresponding to the subset $\mathbf{Z}(y)$). Since y is a complementary basic vector iff the determinant of A is nonzero, the result follows. $\square$

As an example, let $n = 4$, and consider the LCP (q, M). Let $y = (w_1, z_2, w_3, z_4)$ be a complementary vector of variables for this problem. The corresponding complementary matrix is

$$\begin{pmatrix} 1 & -m_{12} & 0 & -m_{14} \\ 0 & -m_{22} & 0 & -m_{24} \\ 0 & -m_{32} & 1 & -m_{34} \\ 0 & -m_{42} & 0 & -m_{44} \end{pmatrix}$$

and its determinant is determinant $\begin{pmatrix} -m_{22} & -m_{24} \\ -m_{42} & -m_{44} \end{pmatrix}$, which is non-zero iff the principal subdeterminant of M corresponding to the subset $\mathbf{Z}(y) = \{2, 4\}$ is non-zero. Thus, in this problem, y is a complementary basic vector iff the principal subdeterminant of M corresponding to the subset $\mathbf{Z}(y)$ is non-zero.

Corollary 3.1 *Every complementary vector of variables is a basic vector for (3.1) iff M is a nondegenerate matrix. This follows from Theorem 3.1 and the definition of nondegeneracy of a matrix.*

Corollary 3.2 *The complementary cone associated with the complementary vector of variables y for (3.1) has a nonempty interior iff the principal subdeterminant of M corresponding to the subset $\mathbf{Z}(y)$ is nonzero.*

Proof. If A is the corresponding complementary matrix, the complementary cone is $\text{Pos}(A)$, and it has nonempty interior iff the determinant of A is nonzero. So the result follows from Theorem 3.1.

$\square$

Corollary 3.3 *Every complementary cone in the class $\mathcal{C}(M)$ has a nonempty interior iff M is a nondegenerate matrix. This follows from Corollary 3.2.*

Theorem 3.2 *The LCP (q, M) has a finite number of solutions for each $q \in \mathbf{R}^n$ iff M is a nondegenerate matrix.*

Proof. Let $(\hat{w}, \hat{z})$ be a solution of the LCP (q, M). Let $A_{.j} = -M_{.j}$ if $\hat{z}_j > 0$, $I_{.j}$ otherwise; and $\alpha_j = \hat{z}_j$ if $\hat{z}_j > 0$, $\hat{w}_j$ otherwise. Then $(A_{.1}, \dots, A_{.n})$ is a complementary set of column vectors and $q = \sum_{j=1}^n \alpha_j A_{.j}$. In this manner each solution of the LCP (q, M) provides an expression of q as a nonnegative linear combination of a complementary set of column vectors. There are only $2^{\mathbf{n}}$ complementary sets of column vectors. If $q \in \mathbf{R}^n$ is such that the LCP (q, M) has an infinite number of distinct solutions, there must exist a complementary set of column vectors, say $(A_{.1}, \dots, A_{.n})$, such that q can be expressed as a nonnegative linear combination of it in an infinite number of ways. So there exist at least two vectors $\alpha^t = (\alpha_1^t, \dots, \alpha_n^t)^T \geqq 0$, $t = 1, 2$ such that $\alpha^1 \neq \alpha^2$ and $q = A\alpha^1 = A\alpha^2$. So $A(\alpha^1 - \alpha^2) = 0$, and since $\alpha^1 \neq \alpha^2$, $\{A_{.1}, \dots, A_{.n}\}$ is linearly dependent. By Theorem 3.1, this implies that M is degenerate.

Conversely suppose M is degenerate. So, by Theorem 3.1, there exists a complementary set of column vectors, say $\{A_{.1}, \dots, A_{.n}\}$ which is linearly dependent. So there exists a $\beta = (\beta_1, \dots, \beta_n) \neq 0$ such that $\sum_{j=1}^n \beta_j A_{.j} = 0$. Let $\delta = \text{Maximum}$ $\{|\beta_j| : j = 1 \text{ to } n\}$. Since $\beta \neq 0$, $\delta > 0$. Define $\overline{q} = \delta \sum_{j=1}^n A_{.j}$. Let $(y_1, \dots, y_n)$ be the complementary vector associated with $(A_{.1}, \dots, A_{.n})$. Define a solution $(w(\lambda), z(\lambda))$ by

$$\begin{aligned} \text{Complement of} \quad & y_j = 0,\ j = 1 \text{ to } n \\ & y_j = \delta + \lambda\beta_j,\ j = 1 \text{ to } n\,. \end{aligned} \tag{3.3}$$

Then $(w(\lambda), z(\lambda))$ is a solution of the LCP $(\overline{q}, M)$ for each $0 \leqq \lambda \leqq 1$, and since $\beta \neq 0$, each of these solutions is distinct. So if M is degenerate, there exist a $q \in \mathbf{R}^n$ such that the LCP (q, M) has an infinite number of distinct solutions.

$\square$

Example 3.1

Consider the following LCP

w_1	w_2	z_1	z_2	
1	0	−1	−1	−2
0	1	−1	−1	−2

$w_1, w_2, z_1, z_2 \geqq 0,\; w_1 z_1 = w_2 z_2 = 0$

We have

$$\begin{aligned} q = (-2,-2)^T &= (-M_{.1}) + (-M_{.2}) \\ 0 &= (-M_{.1}) - (-M_{.2})\,. \end{aligned}$$

These facts imply that $(w_1, w_2; z_1, z_2) = (0, 0; 1+\theta, 1-\theta)^T$ is a complementary solution to this LCP for all $0 \leqq \theta \leqq 1$.

The set of q for which the number of complementary solutions for the LCP (q, M) is infinite, is always a subset of the union of all degenerate complementary cones. Also if the LCP (q, M) has an infinite number of complementary solutions, q must be degenerate in it (that is, q can be expressed as a linear combination of $(m-1)$ or less column vectors of $(I : -M)$).

Result 3.1 If q is nondegenerate in the LCP (q, M) of order n (that is, if in every solution to the system of equations $w - Mz = q$, at least n of the variables in the system are non-zero), every complementary solution of the LCP (q, M) must be a complementary BFS, and so the number of complementary solutions to the LCP (q, M) is finite and $\leqq 2^{\mathbf{n}}$.

Proof. In every complementary solution of the LCP (q, M) at most n variables can be positive by the complementarity constraint, and hence exactly n variables have to be positive by the nondegeneracy of q, that is one variable from every complementary pair of variables must be strictly positive. Consider a complementary solution $(\overline{w}, \overline{z})$ in which the positive variable from the complementary pair $\{w_j, z_j\}$ is y_j say, for $j = 1$ to n and suppose y_j has value $\overline{y}_j > 0$ in the solution. Let $A_{.j} = I_{.j}$ if $y_j = w_j$, or $-M_{.j}$ otherwise. So

$$q = \sum_{j=1}^{n} \overline{y}_j A_{.j}\,.$$

If $\{A_{.1}, \ldots, A_{.n}\}$ is linearly dependent, let the linear dependence relation be

$$0 = \sum_{j=1}^{n} \alpha_j A_{.j}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)^T \neq 0$. Suppose $\alpha_1 \neq 0$. Let $\lambda = -(\overline{y}_1/\alpha_1)$, then $\overline{y}_1 + \lambda\alpha_1 = 0$. From the above two equations, we have

$$q = \sum_{j=1}^{n} (\overline{y}_j + \lambda\alpha_j) A_{\cdot j} = \sum_{j=2}^{n} (\overline{y}_j + \lambda\alpha_j) A_{\cdot j}$$

that is, q is expressed as a linear combination of $\{A_{\cdot 2}, \ldots, A_{\cdot n}\}$ which is a subset of $n-1$ columns of $(I \vdots -M)$, contradicting the nondegeneracy of q. So $\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$ must be linearly independent, that is $A = (A_{\cdot 1} \vdots \cdots \vdots A_{\cdot n})$ is a complementary basis, and hence the representation of q as a linear combination of the columns of A is unique, and $(\overline{w}, \overline{z})$ is a complementary BFS. Thus under the nondegeneracy assumption of q, every complementary solution for the LCP (q, M) must be a complementary BFS. Since the total number of complementary bases is $\leqq 2^{\mathbf{n}}$, this implies that there are at most 2^n complementary solutions in this case.

□

3.2 PRINCIPAL PIVOT TRANSFORMS

Let $y = (y_j)$ be a complementary basic vector associated with the complementary basis A for (3.1). Let t_j be the complement of y_j for $j = 1$ to n (i. e., $t_j = w_j$ if $y_j = z_j$, $t_j = z_j$ if $y_j = w_j$). Let $B_{\cdot j}$ be the complement of $A_{\cdot j}$ for $j = 1$ to n, and $B = (B_{\cdot 1}, \ldots, B_{\cdot n})$. Obtain the canonical tableau of (3.1) with respect to the basic vector y, and after rearranging the variables suppose it is

basic vector	$y_1 \ldots y_n$	$t_1 \ldots t_n$	
y	I	$-D$	$\overline{q}$

(3.4)

Then the matrix D is known as the **principal pivot transform** (PPT in abbreviation) of M associated with the complementary basic vector y or the corresponding complementary basis A of (3.1). Clearly $D = -A^{-1}B$. Also (3.4) can be viewed as the system of equations of an LCP in which the complementary pairs are (y_j, t_j), $j = 1$ to n. Remembering that the variables in (3.4) are just the variables in (3.1) arranged in a different order, we can verify that the canonical tableau of (3.4) with respect to its basic vector $(w_1, \ldots, w_n)$ is (3.1). This clearly implies that M is a PPT of D. Hence the property of being a PPT is a mutual symmetric relationship among square matrices of the same order.

Example 3.2

Consider the LCP (q, M) where

$$M = \begin{pmatrix} -1 & -2 & 0 & -1 \\ -1 & 1 & -1 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 0 & 2 \end{pmatrix}.$$

The LCP (q, M) is

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	
1	0	0	0	1	2	0	1	q_1
0	1	0	0	1	−1	1	2	q_2
0	0	1	0	0	1	−1	1	q_3
0	0	0	1	0	2	0	−2	q_4

$w_j, z_j \geqq 0$, $w_j z_j = 0$ for all j.

(z_1, w_2, z_3, w_4) is a complementary basic vector for this problem. The canonical tableau with respect to it is

z_1	w_2	z_3	w_4	w_1	z_2	w_3	z_4	
1	0	0	0	1	2	0	1	q'_1
0	1	0	0	−1	−2	1	2	q'_2
0	0	1	0	0	−1	−1	−1	q'_3
0	0	0	1	0	2	0	−2	q'_4

Thus the matrix

$$D = \begin{pmatrix} -1 & -2 & 0 & -1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \end{pmatrix}$$

is a PPT of M and vice versa.

Each complementary basic vector for (3.1) leads to a PPT of M. We thus get a class of matrices containing M, such that each matrix in the class is a PPT of each other matrix in the class. Some of the matrices in the class may be equal to the others as matrices (for example, it can be verified that every PPT of I is equal to I). This class of matrices is known as the **principal pivot transform class of** M.

Single and Double Principal Pivot Steps

If $y = (y_1, \ldots, y_n)$ is a complementary basic vector for (3.1), then y_r can be replaced in this basic vector by its complement, to yield another complementary basic vector for (3.1), iff the rth diagonal element in the PPT of M corresponding to y is nonzero. If this condition is satisfied, the pivot operation of replacing y_r by its complement, is known as a **single principal pivot step in the rth position in the complementary basic vector** y.

Suppose for $r \neq s$, the rth and sth diagonal elements in $M' = (m'_{ij})$, the PPT of M corresponding to the complementary basic vector y, are both zero. Then it is not possible to make a single principal pivot step either in the rth position, or in the sth position, in the complementary basic vector y. However, suppose $m'_{rs} \neq 0$ and $m'_{sr} \neq 0$. In this case we can perform two consecutive pivot steps, in the first one replacing y_r in the basic vector by the complement of y_s, and in the second one replacing y_s in the resulting basic vector by the complement of y_r. In the canonical tableau obtained at the end of these two pivot steps, the column vector associated with the complement of y_s is $I_{.r}$ and the column vector associated with the complement of y_r is $I_{.s}$. So, now interchange rows r and s in the canonical tableau. After this interchange it can be verified that in the new canonical tableau the column vector associated with the basic variable from the jth complementary pair, in the new complementary basic vector, is $I_{.j}$, for all j (including $j = r$ and s). This operation (one pivot step in position (r, s) replacing y_r in the basic vector by the complement of y_s, followed by another pivot step in position (s, r) replacing y_s in the resulting basic vector by the complement of y_r, followed by an interchange of rows r and s in the resulting canonical tableau) is called a **double principal pivot step in positions r and s in the complementary basic vector** y. Clearly, this double principal pivot step in positions r and s can only be carried out if the order two determinant $\begin{pmatrix} m'_{rr} & m'_{rs} \\ m'_{sr} & m'_{ss} \end{pmatrix} \neq 0$. If this order two determinant is nonzero, and one of its diagonal entries, say m'_{rr}, is nonzero; carrying out the double principal pivot in positions r and s in the complementary basic vector y, can be verified to have exactly the same effect as carrying out two single principal pivot steps, first in position r in y, and then in position s in the complementary basic vector resulting from the first. In general, in the algorithms discussed in the following chapters, a double principal pivot in positions r and s will only be performed if the diagonal entry in the PPT of M in at least one of the two positions r and s is zero (i. e., either $m'_{rr} = 0$ or $m'_{ss} = 0$ or both).

Example 3.3

Consider the following LCP

basic variable	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	
w_1	1	0	0	0	−1	1	−1	−1	q_1
w_2	0	1	0	0	−1	0	0	1	q_2
w_3	0	0	1	0	0	0	−1	−1	q_3
w_4	0	0	0	1	1	−1	1	0	q_4
	$w_j, z_j \geqq 0$, and $w_j z_j = 0$ for all j								

In this problem, in the complementary basic vector w, single principal pivot steps are only possible in positions 1 and 3. Carrying out a single principal pivot in the complementary basic vector w in position 1 leads to the following

basic variable	z_1	w_2	w_3	w_4	w_1	z_2	z_3	z_4	
z_1	1	0	0	0	−1	−1	1	1	q'_1
w_2	0	1	0	0	−1	−1	1	2	q'_2
w_3	0	0	1	0	0	0	−1	−1	q'_3
w_4	0	0	0	1	1	0	0	−1	q'_4

In the above canonical tableau, we have also rearranged the column vectors so that the basic variables, and the nonbasic variables, appear together and in their proper order.

We can make a double principal pivot step in the complementary basic vector w, in positions 2, 4 in this problem, because the determinant of the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is non-zero. Carrying out this double principal pivot step requires replacing the basic variable w_2 in the basic vector (w_1, w_2, w_3, w_4) by z_4, then replacing the basic variable w_4 in the resulting basic vector (w_1, z_4, w_3, w_4) by z_2, and finally interchanging rows 2 and 4 in the resulting canonical tableau. This is carried out below.

basic variable	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	
w_1	1	1	0	0	−2	1	−1	0	q'_1
z_4	0	1	0	0	−1	0	0	1	q'_2
w_3	0	1	1	0	−1	0	−1	0	q'_3
w_4	0	0	0	1	1	−1	1	0	q'_4
w_1	1	1	0	1	−1	0	0	0	q''_1
z_4	0	1	0	0	−1	0	0	1	q''_2
w_3	0	1	1	0	−1	0	−1	0	q''_3
z_2	0	0	0	−1	−1	1	−1	0	q''_4
w_1	1	1	0	1	−1	0	0	0	q''_1
z_2	0	0	0	−1	−1	1	−1	0	q''_4
w_3	0	1	1	0	−1	0	−1	0	q''_3
z_4	0	1	0	0	−1	0	0	1	q''_2

Block Principal Pivoting

Consider the LCP (q, M), (3.1). Let $\mathbf{J} \subset \{1, \ldots, n\}$ be such that $M_{\mathbf{JJ}}$, the principal submatrix of M corresponding to the subset $\mathbf{J}$, is nonsingular. Define the complementary vector $y = (y_j)$ by

$$y_j = \begin{cases} w_j, & \text{for } j \notin \mathbf{J} \\ z_j, & \text{for } j \in \mathbf{J} \end{cases}$$

and let A be the complementary matrix corresponding to y. Since $M_{\mathbf{JJ}}$ is nonsingular, A is a basis. Let t_j be the complement of y_j for each $j = 1$ to n, and let $t = (t_j)$. Multiplying (3.1) on the left by A^{-1} and rearranging the variables leads to the LCP

y	t	
I	$-D$	q'
$y, t \geq 0,$	$y^T t = 0$	

where

$$D_{\mathbf{JJ}} = (M_{\mathbf{JJ}})^{-1}, D_{\mathbf{J\bar{J}}} = -(M_{\mathbf{JJ}})^{-1} M_{\mathbf{J\bar{J}}}$$
$$D_{\mathbf{\bar{J}J}} = M_{\mathbf{\bar{J}J}}(M_{\mathbf{JJ}})^{-1}, D_{\mathbf{\bar{J}\bar{J}}} = M_{\mathbf{\bar{J}\bar{J}}} - M_{\mathbf{\bar{J}J}}(M_{\mathbf{JJ}})^{-1} M_{\mathbf{J\bar{J}}}$$
$$q'_{\mathbf{J}} = -(M_{\mathbf{JJ}})^{-1} q_{\mathbf{J}}, q'_{\mathbf{\bar{J}}} = q_{\mathbf{\bar{J}}} - M_{\mathbf{\bar{J}J}}(M_{\mathbf{JJ}})^{-1} q_{\mathbf{J}}\,.$$

Here $\overline{\mathbf{J}} = \{1, \ldots, n\} \setminus \mathbf{J}$, and $M_{\mathbf{J}\overline{\mathbf{J}}}$ is the submatrix $(m_{ij} : i \in \mathbf{J}, j \in \overline{\mathbf{J}})$, etc.; and $q_{\mathbf{J}} = (q_j : j \in \mathbf{J})$, etc. D is of course the PPT of M corresponding to the complementary basic vector y. The above LCP (q', D) is said to have been obtained from the LCP (q, M) in (3.1) by a **block principal pivot step in positions J** (or by **block principal pivoting** on $-M_{\mathbf{JJ}}$) in (3.1).

Corollary 3.4 *If M is a nondegenerate matrix, a single principal pivot step in any position is always possible in every complementary basic vector.*

Proof. Follows from Corollary 3.1 and the argument used in Theorem 3.1.

□

Corollary 3.5 *A square matrix M of order n is nondegenerate (that is, principally nondegenerate to be specific) iff every diagonal entry in every PPT of M is non-zero.*

Proof. Follows from Corollary 3.1.

□

Theorem 3.3 *If M is a* PD *or a P-matrix, or a nondegenerate matrix in general; starting with a complementary basic vector $y^1 = (y_1^1, \ldots, y_n^1)$, any other complementary basic vector $y^2 = (y_1^2, y_2^2, \ldots, y_n^2)$ for (3.1), can be obtained by performing a sequence of single principal pivot steps.*

Proof. In these cases, by Corollary 3.1 every complementary vector of variables is a complementary basic vector. Hence if y^1 and y^2 have $n - r$ common variables; each of the variables in y^1 which is not in y^2, can be replaced by its complement, to lead to y^2 after r single principal pivot steps.

□

Theorem 3.4 *All PPTs of a nondegenerate matrix are nondegenerate.*

Proof. Let M be nondegenerate. Let y, $\hat{y}$ be distinct complementary vectors of variables associated with the complementary matrices A, $\hat{A}$ respectively in (3.1). Since M is nondegenerate, A is a complementary basis. Let (3.4) be the canonical tableau of (3.1) with respect to y. So D is the PPT of M corresponding to y. We will now prove that D is nondegenerate. Look at (3.4). The complementary matrix corresponding to the complementary vector of variables $\hat{y}$ in (3.4) is $A^{-1}\hat{A}$, and this matrix is nonsingular since both A and $\hat{A}$ are. Hence $\hat{y}$ is a complementary basic vector for (3.4). Since $\hat{y}$ is an arbitrary complementary vector of variables, this implies that all complementary vectors of variables in (3.4) are basic vectors.
Hence by Corollary 3.1, D is nondegenerate.

□

Theorem 3.5 *All PPTs of a P-matrix are P-matrices.*

Proof. Let $M = (m_{ij})$ be a P-matrix of order n. Consider a single principal pivot step on (3.1) in any position, say position 1. The pivot matrix corresponding to this pivot step is P, which is the same as the unit matrix of order n, with the exception that its first column vector is $(-1/m_{11}, -m_{12}/m_{11}, \ldots, -m_{1n}/m_{11})^T$. Let M' be the

PPT of M obtained after this pivot step. Let $\mathbf{J} = \{j_1, \dots, j_r\} \subset \{1, \dots, n\}$, $\mathbf{J} \neq \emptyset$, and let Δ be the principal subdeterminant of M' corresponding to the subset $\mathbf{J}$. We will now prove that $\Delta > 0$. We consider two cases separately.

Case 1: $1 \notin \mathbf{J}$. Let $y = (y_1, \dots, y_n)$ where $y_j = w_j$ if $j \notin \mathbf{J} \cup \{1\}$, or z_j otherwise. Let A, $\overline{A}$ be the complementary bases corresponding to y, in the original LCP (3.1) and in the canonical tableau for (3.1) obtained after the single principal pivot step in position 1, respectively. So $\overline{A} = PA$. Let Δ_1 be the principal subdeterminant of M corresponding to the subset $\{1\} \cup \mathbf{J}$. We have $\Delta = (-1)^r$ (determinant of $\overline{A}$) $= (-1)^r$ (determinant of PA) $= (-1)^r$ (determinant of P) (determinant of A) $= (-1)^r(-1/m_{11})(-1)^{r+1}\Delta_1 = (\Delta_1/m_{11}) > 0$, because $m_{11} > 0$ and $\Delta_1 > 0$ since M is a P-matrix.

Case 2: $1 \in \mathbf{J}$. In this case let $y = (y_1, \dots, y_n)$ where $y_j = z_j$ if $j \in \mathbf{J} \setminus \{1\}$, or w_j otherwise. Let A, $\overline{A}$ be the complementary bases corresponding to y, in the original LCP (3.1), and in the canonical tableau for (3.1) obtained after the single principal pivot step in position 1, respectively. Then $\overline{A} = PA$. Let Δ_2 be the principal subdeterminant of M determined by the subset $\mathbf{J} \setminus \{1\}$. As in Case 1, we have $\Delta = (-1)^r$ (determinant of $\overline{A}$) $= (-1)^r$ (determinant of P) (determinant of A) $= (-1)^r(-1/m_{11})(-1)^{r-1}\Delta_2 = (\Delta_2/m_{11}) > 0$, since both Δ_2, m_{11} are strictly positive because M is a P-matrix.

Hence the principal subdeterminant of M' corresponding to the subset $\mathbf{J}$ is strictly positive. This holds for all subsets $\mathbf{J} \subset \{1, \dots, n\}$. So M' is itself a P-matrix.

Thus the property of being a P-matrix is preserved in the PPTs of M obtained after a single principal pivot step on (3.1). By Theorem 3.3 any PPT of M can be obtained by making a sequence of single principal pivot steps on (3.1). So, applying the above result repeatedly after each single principal pivot step, we conclude that every PPT of M is also a P-matrix.

□

Theorem 3.6 *If all the diagonal entries in every PPT of M are strictly positive, M is a P-matrix.*

Proof. By the hypothesis of the theorem all principal subdeterminants of M of order 1 are strictly positive.

Induction Hypothesis: Under the hypothesis of the theorem, all principal subdeterminants of M of order less than or equal to r are strictly positive.

We will now prove that under the hypothesis of the theorem, the induction hypothesis implies that any principal subdeterminant of M of order $r + 1$ is also strictly positive. Let Δ_1 be the principal subdeterminant of M corresponding to the subset $\{j_1, \dots, j_r, j_{r+1}\} \subset \{1, 2, \dots, n\}$. Carry out a single principal pivot step in position j_{r+1} in (3.1) and let M' be the PPT of M obtained after this step. Since M' is a PPT of M it also satisfies the hypothesis of the theorem. So by the induction hypothesis, all principal subdeterminants of M' of order r or less are strictly positive, and so Δ, the principal subdeterminant of M' corresponding to the subset $\{j_1, \dots, j_r\}$, is > 0. As in the proof of Theorem 3.5 we have $\Delta = \Delta_1/m_{j_{r+1},j_{r+1}}$, that is $\Delta_1 = m_{j_{r+1},j_{r+1}}\Delta$, and since $m_{j_{r+1},j_{r+1}} > 0$, $\Delta > 0$, we have $\Delta_1 > 0$. So under the hypothesis of the theorem, the induction hypothesis implies also that all principal subdeterminants of M of order $r+1$ are strictly positive. Hence by induction, all principal subdeterminants of M are strictly positive, and hence M is a P-matrix.

□

Corollary 3.6 *The following conditions (i) and (ii) are equivalent*

(i) all principal subdeterminants of M are strictly positive

(ii) the diagonal entries in every PPT of M are strictly positive.

Proof. Follows from Theorem 3.5, 3.6.

□

Corollary 3.7 *If M is a P-matrix, in making any sequence of single principal pivot steps on (3.1), the pivot element will always be strictly negative.*

□

Theorem 3.7 *Let M' be a PPT of M obtained after carrying out exactly one single principal pivot step. Then M' is* PD *if M is* PD. *And M' is* PSD *if M is* PSD.

Proof. Let $M = (m_{ij})$. Let $u = (u_1, \ldots, u_n)^T \in \mathbf{R}^n$. Define $v = (v_1, \ldots, v_n)^T$ by

$$v - Mu = 0\,. \tag{3.5}$$

Suppose $M' = (m'_{ij})$ is the PPT of M obtained after making a single principal pivot step in (3.5) in position r. So $m_{rr} \neq 0$. After this single principal pivot step in position r, (3.5) becomes

$$(v_1, \ldots, v_{r-1}, u_r, v_{r+1}, \ldots, v_n)^T - M'(u_1, \ldots, u_{r-1}, v_r, u_{r+1}, \ldots, u_n)^T = 0\,. \tag{3.6}$$

For any $u \in \mathbf{R}^n$ and v defined by (3.5), let $\xi = (u_1, \ldots, u_{r-1}, v_r, u_{r+1}, \ldots, u_n)$, $\eta = (v_1, \ldots, v_{r-1}, u_r, v_{r+1}, \ldots, v_n)$. Since $v_r = M_{r.}u$ and $m_{rr} \neq 0$, as u varies over all of $\mathbf{R}^n$, ξ also varies over all of $\mathbf{R}^n$. Also, as u varies over all the nonzero points in $\mathbf{R}^n$, ξ does the same. Since (3.6) is obtained from (3.5) by a pivot step, they are equivalent. So for any $u \in \mathbf{R}^n$ and v defined by (3.5), (3.6) also holds. Now $u^T M u = u^T v = \xi^T \eta = \xi^T M' \xi$. These facts imply that $\xi^T M' \xi \geqq 0$ for all $\xi \in \mathbf{R}^n$ iff $u^T M u \geqq 0$ for all $u \in \mathbf{R}^n$ and $\xi^T M' \xi > 0$ for all $\xi \neq 0$ iff $u^T M u > 0$ for all $u \neq 0$. Hence M is PD iff M' is PD. And M' is PSD iff M is PSD.

□

Theorem 3.8 *Let M'' be a PPT of M obtained after carrying out exactly one double principal pivot step. Then M'' is* PD *if M is* PD. *And M'' is* PSD *if M is* PSD.

Proof. Let $M = (m_{ij})$. Let $u = (u_1, \ldots, u_n)^T \in \mathbf{R}^n$. Define $v = (v_1, \ldots, v_n)^T$ by (3.5). Suppose $M'' = (m''_{ij})$ is the PPT of M obtained after making a double principal pivot step in positions r and s. This implies that

$$\Delta = \text{ determinant } \begin{pmatrix} -m_{ss} & -m_{sr} \\ -m_{rs} & -m_{rr} \end{pmatrix} \neq 0\,,$$

as otherwise the double principal pivot step in positions r and s cannot be carried out on (3.5). For any $u \in \mathbf{R}^n$ and v defined by (3.5) define $\xi = (u_1, \ldots, u_{s-1}, v_s, u_{s+1}, \ldots, u_{r-1}, v_r, u_{r+1}, \ldots, u_n)^T$, $\eta = (v_1, \ldots, v_{s-1}, u_s, v_{s+1}, \ldots, v_{r-1}, u_r, v_{r+1}, \ldots, v_n)^T$. Then after this double principal pivot step in positions r and s, (3.5) gets transformed into

$$\eta - M''\xi = 0\,. \tag{3.7}$$

Since (3.7) is obtained by performing two pivots on (3.5), they are equivalent. So for any $u \in \mathbf{R}^n$ and v defined by (3.5), (3.7) holds and we have $u^T M u = u^T v = \xi^T \eta = \xi^T M'' \xi$. Also, since $\Delta \neq 0$, as u varies over all of $\mathbf{R}^n$, so does ξ; and as u varies over all nonzero points in $\mathbf{R}^n$ so does ξ. These facts imply that $\xi^T M'' \xi \geqq 0$ for all $\xi \in \mathbf{R}^n$ iff $u^T M u \geqq 0$ for all $u \in \mathbf{R}^n$ and $\xi^T M'' \xi > 0$ for all $\xi \neq 0$ iff $u^T M u > 0$ for all $u \neq 0$. Hence M'' is PD iff M is PD, and M'' is PSD iff M is PSD.

□

Theorem 3.9 *If M is a* PD *matrix, all its PPTs are also* PD.

Proof. By Theorem 3.3 when M is PD, every PPT of M can be obtained by carrying out a sequence of single principal pivot steps on (3.1). By applying the argument in Theorem 3.7 repeatedly after each single principal pivot step in the sequence, we conclude that all PPTs of M are also PD, if M is.

□

Theorem 3.10 *If M is* PSD, *any PPT of M can be obtained by making a sequence of single or double principal pivot steps on (3.1). Also, all these PPTs of M are also* PSD.

Proof. Let $y = (y_1, \ldots, y_n)$ be a complementary basic vector of (3.1). Starting with the complementary basic vector w, perform single principal pivot steps in position j for as many $j \in \mathbf{Z}(y)$ as possible in any possible order. If this leads to the complementary basic vector y, we are done by repeated use of the result in Theorem 3.7 after each single principal pivot step. Suppose y has not yet been obtained and no more single principal pivot steps can be carried out in the remaining positions $j \in \mathbf{Z}(y)$. Let $u = (u_1, \ldots, u_n)$ be the complementary basic vector at this stage. Let $\mathbf{U} = \{j : j \text{ such that } u_j \neq y_j\}$. So $\mathbf{U} \neq \emptyset$, $\mathbf{U} \subset \mathbf{Z}(y)$. And for each $j \in \mathbf{U}$, we have $u_j = w_j$, $y_j = z_j$. Let t_j denote the complement of u_j, $j = 1$ to n. Let the canonical tableau of (3.1) at this stage be

basic vector	$u_1, \ldots, u_n$	$t_1, \ldots, t_n$	q
u	I	$-M'$	q'

(3.8)

M' is the PPT of M corresponding to $\mathbf{U}$, it is PSD by repeated use of Theorem 3.7. We have $-m'_{jj} = 0$ for each $j \in \mathbf{U}$ (as single principal pivot steps cannot be carried out in these positions). If $\mathbf{U}$ is a singleton set, this would imply that the set of column vectors corresponding to y in (3.8) is linearly dependent, a contradiction, since y is a complementary basic vector. So cardinality of $\mathbf{U}$ is $\geqq 2$. Let $r \in \mathbf{U}$. Since $m'_{rr} = 0$ and

M' is PSD, by Result 1.6 we have $m'_{ri} + m'_{ir} = 0$ for all $i = 1$ to n. Search for an $s \in \mathbf{U}$ such that $m'_{sr} \neq 0$. If an s like this does not exist, again the set of column vectors corresponding to y in (3.8) is linearly dependent, and y is not a complementary basic vector, a contradiction. So there always exists an $s \in \mathbf{U}$ such that $m'_{sr} \neq 0$. Since $m'_{rs} + m'_{sr} = 0$, $m'_{rs} \neq 0$ too. So the determinant

$$\begin{pmatrix} m'_{rr} & m'_{rs} \\ m'_{sr} & m'_{ss} \end{pmatrix}$$

is nonzero, and a double principal pivot step can be carried out in (3.8) in positions r, s. The complementary basic vector obtained after this double principal pivot step contains two more variables in common with y than u does, and the PPT of M corresponding to it is also PSD by Theorem 3.8. Delete r, s from $\mathbf{U}$. In the resulting canonical tableau, make as many single principal pivot steps in positions $j \in \mathbf{U}$ as possible, deleting such j from $\mathbf{U}$ after each step. Or make another double principal pivot step in positions selected from $\mathbf{U}$ as above, and continue the same way until $\mathbf{U}$ becomes empty. At that stage we reach the canonical tableau with respect to y. By repeated use of Theorems 3.7, 3.8, the PPT of M with respect to y is also PSD.

□

3.2.1 Principal Rearrangements of a Square Matrix

Let M be a given square matrix of order n. Let $p = (i_1, \dots, i_n)$ be a permutation of $(1, \dots, n)$. The square matrix P of order n whose rows are $I_{i_1 .}, I_{i_2 .}, \dots, I_{i_n .}$ in that order, is the permutation matrix corresponding to the permutation p. P is obtained essentially by permuting the rows of the unit matrix I of order n using the permutation p. The matrix $M' = PMP^T$ is known as the principal rearrangement of M according to the permutation p. Clearly M' is obtained by first rearranging the rows of M according to the permutation p, and in the resulting matrix, rearranging the columns again according to the same permutation p.

As an example let $n = 3$, and

$$p = (3, 1, 2)\,, \quad M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\,, \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

then

$$PM = \begin{pmatrix} m_{31} & m_{32} & m_{33} \\ m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}\,, \quad M' = PMP^T = \begin{pmatrix} m_{33} & m_{31} & m_{32} \\ m_{13} & m_{11} & m_{12} \\ m_{23} & m_{21} & m_{22} \end{pmatrix}$$

and M' here is the principal rearrangement of M according to the permutation p.

The following results can be obtained directly using the definition. Let M' be the principal rearrangement of M according to the permutation p associated with the permutation matrix P. Then M' is a P-matrix, iff M is. For all $y \in \mathbf{R}^n$, $y^T M y = (Py)^T M' (Py)$. So M' is a PSD, or PD, or NSD, or ND matrix iff M has the same property. Also, M' is principally degenerate (or nondegenerate) iff M has the same property.

3.3 LCPs ASSOCIATED WITH P-MATRICES

Properties of P-Matrices

The following Theorems 3.11, 3.12 are important properties of P-matrices due to D. Gale and H. Nikaido (see reference [3.24]).

Theorem 3.11 *Let $F = (f_{ij})$ be a P-matrix of order n. Then the system of linear inequalities*

$$\begin{aligned} Fx &\leqq 0 \\ x &\geqq 0 \end{aligned} \tag{3.9}$$

has "$x = 0$" as its unique solution.

Proof. The theorem is easily verified to be true for $n = 1$. We will prove the theorem for all n by induction.

Induction Hypothesis: If T is a P-matrix of order $r \leqq n-1$, then the system of inequalities $T\xi \leqq 0$, $\xi \geqq 0$, $\xi \in \mathbf{R}^r$ has "$\xi = 0$" as its unique solution.

Under the induction hypothesis we will now prove that the statement of the theorem holds for the matrix F which is a P-matrix of order n. Since F is a P-matrix, it is nonsingular, and hence F^{-1} exists. Let $B = F^{-1} = (b_{ij})$. From standard results in the theory of determinants (for example, see Chapter 3 in F. E. Hohn, *Elementary Matrix Algebra*, Macmillan, 2nd edition, 1964) it is known that b_{ii} = (principal subdeterminant of F corresponding to the subset $\{1, \ldots, i-1, i+1, \ldots, n\}$)/determinant of F. So $b_{ii} > 0$ for all i, since F is a P-matrix. Thus each column of B has at least one positive entry. Let $\overline{x} \in \mathbf{R}^n$ satisfy (3.9). Select a column of B, say $B_{.1}$. Let $\theta =$ minimum$\{\overline{x}_i/b_{i1} : i$ such that $b_{i1} > 0\}$, and suppose this minimum is attained by $i = s$. So $\theta = \overline{x}_s/\overline{b}_{s1} \geqq 0$, and $(\overline{x}_j/b_{j1}) \geqq \theta$, for all j such that $b_{j1} > 0$. From this and the fact that $\overline{x} \geqq 0$, we have $\overline{\eta} = (\overline{\eta}_1, \ldots, \overline{\eta}_n)^T = \overline{x} - \theta B_{.1} \geqq 0$ and $\overline{\eta}_s = 0$. Also $F\overline{\eta} = F\overline{x} - \theta F B_{.1} = F\overline{x} - \theta I_{.1} \leqq 0$. Let T be the matrix of order $n-1$ obtained by striking off the sth row and the sth column from F. Since F is a P-matrix, its principal submatrix T is also a P-matrix. Let $\overline{\xi} = (\overline{\eta}_1, \ldots, \overline{\eta}_{s-1}, \overline{\eta}_{s+1}, \ldots, \overline{\eta}_n)^T$. Since $\overline{\eta}_s = 0$ and $F\overline{\eta} \leqq 0$, we have $T\overline{\xi} \leqq 0$. Also since $\overline{\eta} \geqq 0$, $\overline{\xi} \geqq 0$ too. So $T\overline{\xi} \leqq 0$, $\overline{\xi} \geqq 0$. Since T is a P-matrix of order $n-1$, by the induction hypothesis, $\overline{\xi} = 0$. $\overline{\xi} = 0$, $\overline{\eta}_s = 0$ together imply that $\overline{\eta} = 0$. So $F\overline{\eta} = 0$, that is $F(\overline{x} - \theta I_{.1}) = 0$. Then $F\overline{x} = \theta I_{.1} \geqq 0$. However from (3.9), $F\overline{x} \leqq 0$. So $F\overline{x} = 0$, and since F is nonsingular, $\overline{x} = 0$.

Thus under the induction hypothesis the statement of the theorem also holds for F which is a P-matrix of order n. The statement of the theorem is easily verified for $n = 1$. Hence, by induction, the statement of the theorem is true for all n.

∎

Theorem 3.12 The Sign Nonreversal Property: *Let F be a square matrix of order n. For $x \in \mathbf{R}^n$ let $y = Fx$. Then F is said to reverse the sign of x if $x_i y_i \leqq 0$ for all i. If F is a P-matrix it reverses the sign of no vector except zero.*

Proof. For this proof we need only to consider the case $x \geq 0$. For if F reverses the sign of an $\overline{x} \not\geqq 0$, let $\mathbf{J} = \{j : \overline{x}_j < 0\}$, let D be the diagonal matrix obtained from the unit matrix by multiplying its jth column by -1 for each $j \in \mathbf{J}$. The matrix $F^* = DFD$ is again a P-matrix, since F^* is obtained by simply changing the signs of rows and columns in F for each $j \in \mathbf{J}$. And F^* reverses the sign of $\hat{x} = D\overline{x}$, where $\hat{x} \geqq 0$.

Now suppose that $x \geqq 0$ and that F reverses the sign of x. Let $\mathbf{P} = \{j : x_j > 0\}$. Assume that $\mathbf{P} \neq \emptyset$. Let A be the principal submatrix of F corresponding to $\mathbf{P}$. Let χ be the vector of x_j for $j \in \mathbf{P}$. The fact that F reverses the sign of x implies that A reverses the sign of χ. Since $\chi > 0$, this implies that $A\chi \leqq 0$. Since A is a P-matrix $A\chi \leqq 0$, $\chi \geqq 0$ implies $\chi = 0$ by Theorem 3.11, a contradiction. So x must be zero.

□

Unique Solution Property of LCPs Associated with P-Matrices

Theorem 3.13 *Let M be a P-matrix. The LCP (q, M) has a unique solution for each $q \in \mathbf{R}^n$. Also, when the complementary pivot algorithm of Section 2.2 is applied on the LCP (q, M), it finds the solution.*

Proof. Suppose when the complementary pivot algorithm is applied on the LCP (q, M) it ends in ray termination. As in the proof of Theorem 2.1 this implies that there exists a $z^h \geq 0$, $w^h \geqq 0$, $z_0^h \geqq 0$ satisfying $w^h = Mz^h + e_n z_0^h$; $w_i^h z_i^h = 0$ for all i. So $z_i^h(M_{i.}z^h) + z_i^h z_0^h = 0$. This implies that $z_i^h(M_{i.}z^h) = -z_i^h z_h^0 \leqq 0$ for all i. So M reverses the sign of $z^h \geq 0$, which is a contradiction to Theorem 3.12. So, when the complementary pivot method is applied on the LCP (q, M) associated with a P-matrix, it cannot end in ray termination, it has to terminate with a solution of the LCP. This also proves that every P-matrix is a Q-matrix.

Now we will prove that if M is a P-matrix, for any $q \in \mathbf{R}^n$, the LCP (q, M) has exactly one solution, by induction on n, the order of the problem.

Suppose $n = 1$. $M = (m_{11})$ is a P-matrix, iff $m_{11} > 0$. In this case $q = (q_1)$. If $q_1 \geqq 0$, $(w = (w_1) = (q_1);\ z = (z_1) = (0))$ is the only solution to the LCP (q, M). If $q_1 < 0$, $(w = (w_1) = (0);\ z = (z_1) = (-q_1/m_{11}))$ is the only solution to the LCP (q, M). Hence the theorem is true for $n = 1$.

Induction Hypothesis: Suppose any LCP of order $(n-1)$ or less, associated with a P-matrix, has a unique solution for each of its right hand side constant vectors.

Now we will prove that under the induction hypothesis, the LCP (q, M) where M is a P-matrix of order n, has a unique solution for any $q \in \mathbf{R}^n$. We have shown above that it has at least one solution, say $(\tilde{w}; \tilde{z})$. For each $j = 1$ to n let $u_j = z_j$, if

$\tilde{z}_j > 0$; or w_j otherwise; and let v_j be the complement of u_j. Then $u = (u_1, \ldots, u_n)$ is a complementary feasible basic vector of variables associated with the BFS $(\tilde{w}; \tilde{z})$ for (3.1). Obtain the canonical tableau for (3.1) with respect to the complementary feasible basic vector u, and suppose it is

$u_1, \ldots, u_n$	$v_1, \ldots, v_n$	q
I	$-\widetilde{M}$	$\tilde{q}$

(3.10)

$\tilde{q} \geqq 0$ by our assumptions here. (3.10) can itself be viewed as the LCP $(\tilde{q}, \widetilde{M})$, one solution of this LCP is $(u = \tilde{u} = \tilde{q};\ v = \tilde{v} = 0)$. $\widetilde{M}$ is a PPT of M, by Theorem 3.5, $\widetilde{M}$ is also a P-matrix. So all the principal submatrices of $\widetilde{M}$ are also P-matrices. So the principal subproblem of the LCP $(\tilde{q}, \widetilde{M})$ in the variables $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$; $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ is an LCP of order $(n-1)$ associated with a P-matrix, and by the induction hypothesis this principal subproblem has a unique solution. One solution of this principal subproblem is $(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, \tilde{u}_{i+1}, \ldots, \tilde{u}_n;\ \tilde{v}_1, \ldots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \ldots, \tilde{v}_n) = (\tilde{q}_1, \ldots, \tilde{q}_{i-1}, \tilde{q}_{i+1}, \ldots, \tilde{q}_n;\ 0, \ldots, 0, 0, \ldots, 0)$. If the LCP $(\tilde{q}, \widetilde{M})$, (3.10), has an alternate solution $(\hat{u}; \hat{v}) \neq (\tilde{u}; \tilde{v})$ in which $\hat{v}_i = 0$, its principal subproblem in the variables $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$; $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ will have an alternate solution $(\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots, \hat{u}_n;\ \hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$, a contradiction. So, if the LCP $(\tilde{q}, \widetilde{M})$ has an alternate solution $(\hat{u}; \hat{v}) \neq (\tilde{u}; \tilde{v})$, then $\hat{v}_i$ must be strictly positive in it, and by complementarity $\hat{u}_i$ must be zero. Since this holds for each $i = 1$ to n, $\hat{v} > 0$, $\hat{u} = 0$. So $\hat{u} - \widetilde{M}\hat{v} = \tilde{q}$, $\hat{u} = 0$, $\hat{v} > 0$. Since $\tilde{q} \geqq 0$, this implies that $\widetilde{M}\hat{v} = -\tilde{q} \leqq 0$, $\hat{v} > 0$, a contradiction to Theorem 3.11, since $\widetilde{M}$ is a P-matrix. Hence under the induction hypothesis the LCP $(\tilde{q}, \widetilde{M})$ has a unique solution, which implies that the equivalent LCP (q, M) has a unique solution also. Since this holds for any $q \in \mathbf{R}^n$, under the induction hypothesis, the LCP (q, M) of order n has a unique solution for each $q \in \mathbf{R}^n$ when M is a P-matrix. Hence, by induction the theorem is true.

□

Theorem 3.14 *Let M be a given square matrix of order n. Suppose the LCP (q, M) has at most one solution for each $q \in \mathbf{R}^n$. Then M is a P-matrix.*

Proof. So, the number of solutions of the LCP (q, M) is either 1 or 0 and hence is finite for all q, which implies that M is nondegenerate by Theorem 3.2. So the determinant of M is nonzero, and hence M^{-1} exists.

Proof is by induction on n, the order of the matrix M. We first verify that the theorem is true if $n = 1$. In this case $q = (q_1)$, $M = (m_{11})$. Since M is shown to be nondegenerate under the hypothesis of the theorem, $m_{11} \neq 0$. If $m_{11} < 0$; when $q_1 > 0$, $(w = (q_1), z = 0)$, $(w = 0, z = q_1/(|m_{11}|))$ are two distinct solutions of the LCP (q, M). Hence under the hypothesis of the theorem $m_{11} \nless 0$. So, $m_{11} > 0$, which implies that the theorem is true when $n = 1$.

Induction Hypothesis: If F is a square matrix of order $r \leqq n - 1$, such that the LCP (γ, F) has at most one solution for each $\gamma \in \mathbf{R}^r$, then F is a P-matrix.

Under the hypothesis of the theorem, and the induction hypothesis, we will now prove that M has to be a P-matrix too.

Consider the principal subproblem of the LCP (q, M) in the variables $\omega = (w_2, \dots, w_n)$, $\xi = (z_2, \dots, z_n)$. This is an LCP of order $n-1$ associated with the principal submatrix of M determined by the subset $\{2, \dots, n\}$. If there exists a $\tilde{q} = (\tilde{q}_2, \dots, \tilde{q}_n)^T$ for which this principal subproblem has two distinct solutions, namely, $(\overline{\omega}, \overline{\xi})$ and $(\hat{\omega}, \hat{\xi})$, choose $\tilde{q}_1$ to satisfy $\tilde{q}_1 > \text{Maximum}\{|\sum_{j=2}^n \overline{z}_j m_{1j}|, |\sum_{j=2}^n \hat{z}_j m_{1j}|\}$, and let $\overline{w}_1 = \tilde{q}_1 + \sum_{j=2}^n \overline{z}_j m_{1j}$, $\overline{z}_1 = 0$, $\hat{w}_1 = \tilde{q}_1 + \sum_{j=2}^n \hat{z}_j m_{1j}$, $\hat{z}_1 = 0$, $\overline{w} = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_n)$, $\overline{z} = (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_n)$, $\hat{w} = (\hat{w}_1, \hat{w}_2, \dots, \hat{w}_n)$, $\hat{z} = (\hat{z}_1, \hat{z}_2, \dots, \hat{z}_n)$, $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n)^T$. Then $(\overline{w}; \overline{z})$, $(\hat{w}; \hat{z})$ are two distinct solutions of the LCP $(\tilde{q}, M)$, contradicting the hypothesis of the theorem. So the principal subproblem of the LCP (q, M) in the variables ω, ξ has at most one solution for each of its right hand side constant vectors. By the induction hypothesis this implies that the principal submatrix of M determined by the subset $\{2, \dots, n\}$ is a P-matrix.

A similar argument can be made for each principal subproblem of the LCP (q, M) of order $(n-1)$, and this implies that all principal submatrices of M of order $(n-1)$ are P-matrices, by the induction hypothesis. Hence all the principal subdeterminants of M of order $\leqq (n-1)$ are strictly positive. In particular, the diagonal entries of M are strictly positive. It only remains to be proved that the determinant of M itself is strictly positive. We have already seen that M^{-1} exists. The canonical tableau of (3.1) with respect to the complementary basic vector $(z_1, \dots, z_n)$ is

z	w	
I	$-\overline{M}$	$\overline{q}$

(3.11)

where $\overline{M} = M^{-1}$ and $\overline{q} = -M^{-1}q$. The LCP in (3.11) has at most one solution for each $\overline{q} \in \mathbf{R}^n$. So by the previous arguments all diagonal entries in the matrix $\overline{M}$ have to be strictly positive. However since $\overline{M} = (\overline{m}_{ij}) = M^{-1}$, $\overline{m}_{11} =$ (principal subdeterminant of M corresponding to the subset $\{2, \dots, n\}$)/(determinant of M). Since the principal subdeterminant of M corresponding by the subset $\{2, \dots, n\}$ has been shown to be strictly positive, $\overline{m}_{11} > 0$ implies that the determinant of M is strictly positive. Hence under the hypothesis of the theorem, and the induction hypothesis, the matrix M of order n has to be a P-matrix. So, by induction the theorem is true in general.

□

Corollary 3.8 *Let M be a given square matrix of order n. If the LCP (q, M) has at most one solution for each $q \in \mathbf{R}^n$, then it has exactly one solution for each $q \in \mathbf{R}^n$. This follows from Theorems 3.13, 3.14.*

□

Theorem 3.15 *Let M be a given square matrix of order n. The LCP (q, M) has a unique solution for each $q \in \mathbf{R}^n$ iff M is a P-matrix.*

□

Proof. Follows from Theorems 3.13, 3.14.

Strict Separation Property

The strict separation property is a property of the matrix M, and does not depend on the right hand side constants vector q. An LCP associated with the matrix M (or the class of complementary cones $\mathcal{C}(M)$) is said to satisfy the **strict separation property** if the following conditions are satisfied.

(i) Every subcomplementary set of column vectors is linearly independent.

(ii) If $(A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n})$ is any subcomplementary set of column vectors, the hyperplane which is its linear hull strictly separates the points represented by the left out complementary pair of column vectors $(I_{.i}, -M_{.i})$.

From (i) and (ii), it is clear that every complementary set of column vectors has to be linearly independent for the strict separation property to be satisfied.

Example 3.4

Let $M = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. Here $n = 2$. The points representing the column vectors of I, $-M$ are plotted in Figure 3.1.

Since $n = 2$ here, in this case each subcomplementary set consists of exactly one of the column vectors from $\{I_{.1}, I_{.2}, -M_{.1}, -M_{.2}\}$. The linear hull of any subcomplementary set of vectors in this example is the straight line through the vector in that subcomplementary set and the origin.

Consider the subcomplementary set of column vectors $\{I_{.1}\}$. The left out complementary pair of column vectors in this set is $(I_{.2}, -M_{.2})$. The linear hull of $\{I_{.1}\}$, which is the horizontal axis in Figure 3.1, strictly separates the points $I_{.2}, -M_{.2}$, since neither of these points is on this straight line and they are on opposite sides of it. In a similar manner it can be verified that both properties (i) and (ii) discussed above are satisfied in this example. Hence any LCP associated with the matrix M in this example satisfies the strict separation property.

Example 3.5

Let $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Here again, $n = 2$. The points representing the column vectors of I, $-M$ in this case are plotted in Figure 1.3. Consider the subcomplementary set of column vectors $\{I_{.2}\}$ in this example. Its linear hull is the vertical axis in Figure 1.3, and it strictly separates the left-out complementary pair of column vectors $(I_{.1}, -M_{.1})$. In a similar manner, it can be verified that the strict separation property holds in this case.

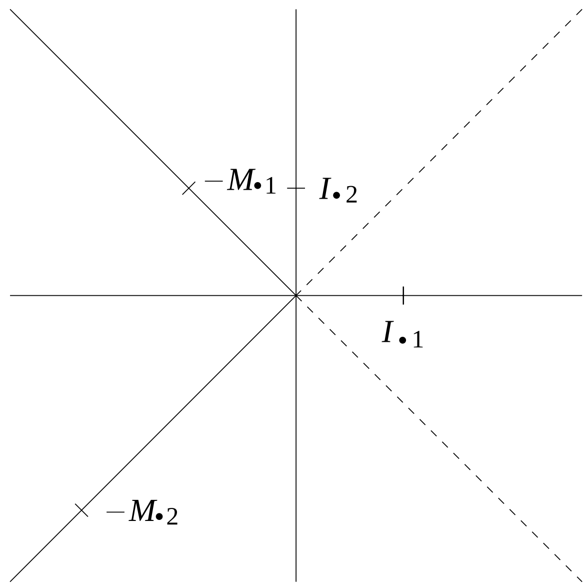

Figure 3.1 Illustration of Strict Separation

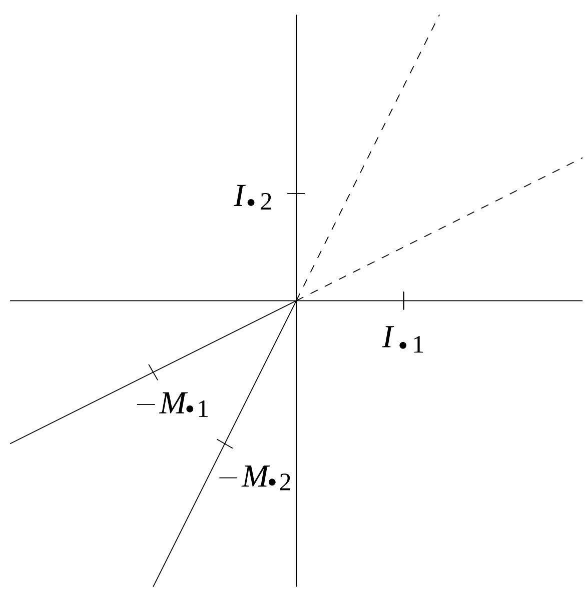

Figure 3.2 Violation of the Strict Separation Property

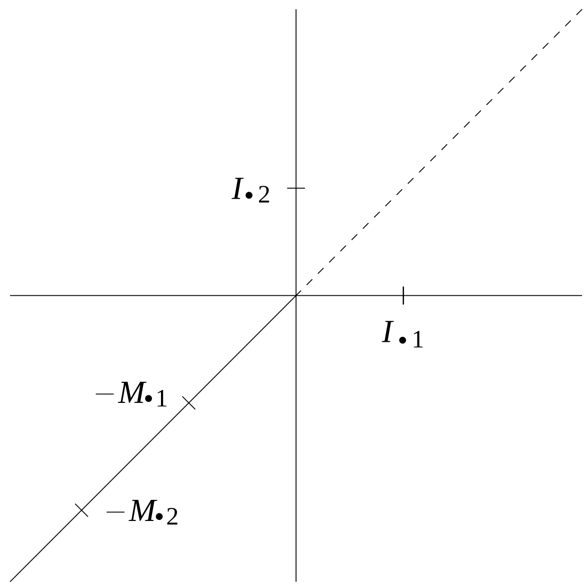

Figure 3.3 Another Example of Violation of the Strict Separation Property.

Example 3.6

Let $M = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Here $n = 2$, and the column vectors of I, $-M$ are plotted in Figure 3.2. Consider the subcomplementary set of column vectors $\{-M_{.1}\}$ here. Both the points in the left-out complementary pair $(I_{.2}, -M_{.2})$ are on the same side of the linear hull of $\{-M_{.1}\}$ here, and hence the strict separation property is not satisfied by the LCPs associated with the matrix M here.

Example 3.7

Let $M = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. See Figure 3.3. Consider the subcomplementary set of column vectors $\{-M_{.1}\}$ here. The point $-M_{.2}$ from the left-out complementary pair $(I_{.2}, -M_{.2})$ lies on the straight line which is the linear hull of the subcomplementary set of column vectors $\{-M_{.1}\}$. So the strict separation property is not satisfied in this example.

Corollary 3.9 *If an LCP associated with the matrix M satisfies the strict separation property, M is nondegenerate. This follows from the definitions.*

□

Theorem 3.16 *The LCP associated with a matrix M satisfies the strict separation property iff M is a P-matrix.*

Proof. Suppose M is a P-matrix. Property (i) required for strict separation property is obviously satisfied because M is nondegenerate (Corollary 3.1).

Let $(A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n})$ be any subcomplementary set of column vectors where $A_{.j} \in \{I_{.j}, -M_{.j}\}$ for each $j \neq i$. Let $\mathbf{H}$ be the hyperplane which is the linear hull of $\{A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n}\}$. By Corollary 3.1, the complementary sets of column vectors $(A_{.1}, \dots, A_{.i-1}, I_{.i}, A_{.i+1}, \dots, A_{.n})$ and $(A_{.1}, \dots, A_{.i-1}, -M_{.i}, A_{.i+1}, \dots, A_{.n})$ are both linearly independent, and hence neither $I_{.i}$ nor $-M_{.i}$ lie on the hyperplane $\mathbf{H}$. Suppose both $I_{.i}$ and $-M_{.i}$ are on the same side of the hyperplane $\mathbf{H}$ in $\mathbf{R}^n$. See Figure 3.4. In this case the interiors of the complementary cones $\text{Pos}(A_{.1}, \dots, A_{.i-1}, I_{.i}, A_{.i+1}, \dots, A_{.n})$ and $\text{Pos}(A_{.1}, \dots, A_{.i-1}, -M_{.i}, A_{.i+1}, \dots, A_{.n})$ have a nonempty intersection, and if $\overline{q}$ is a point in the intersection, then $\overline{q}$ is in the interior of two complementary cones, and the LCP $(\overline{q}, M)$ has two distinct solutions; a contradiction to Theorem 3.13, since M is a P-matrix. So $I_{.i}$ and $-M_{.i}$ cannot be on the same side of the hyperplane $\mathbf{H}$. Since neither of these points is on $\mathbf{H}$, and they are not on the same side of $\mathbf{H}$, these points are on either side of $\mathbf{H}$, that is $\mathbf{H}$ separates them strictly. Since this holds for any subcomplementary set of column vectors and the corresponding left-out complementary pair of column vectors, the strict separation property holds when M is a P-matrix.

Suppose the strict separation property is satisfied. By Corollary 3.9 M is nondegenerate. So all the principal subdeterminants of M are nonzero. It remains to be proved that they are all positive. Let $y = (y_1, \dots, y_n)$ be any complementary vector of variables for the LCP (q, M). Let t_j be the complement of y_j for $j = 1$ to n. Since M is nondegenerate, $(y_1, \dots, y_n)$ is a complementary basic vector of variables by Corollary 3.1. Obtain the canonical tableau of (3.1), with respect to the complementary basic vector y. Suppose it is

$y_1 \dots y_n$	$t_1 \dots t_n$	q
I	$-M'$	q'

(3.12)

where $M' = (m'_{ij})$ is the PPT of M corresponding to the complementary basic vector y. Now look at the subcomplementary vector of variables $(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$.

The column corresponding to y_j in (3.12) is $I_{.j}$, for $j = 1$ to n. For convenience, call the coordinates along the axis of coordinates, as $x_1, \dots, x_n$. Since the column of y_j in (3.12) is $I_{.j}$, the hyperplane in $\mathbf{R}^n$ which contains the columns of y_j in (3.12) for all $j = 1, \dots, i-1, i+1, \dots, n$, is the coordinate hyperplane $\mathbf{H} = \{x : x_i = 0\}$.

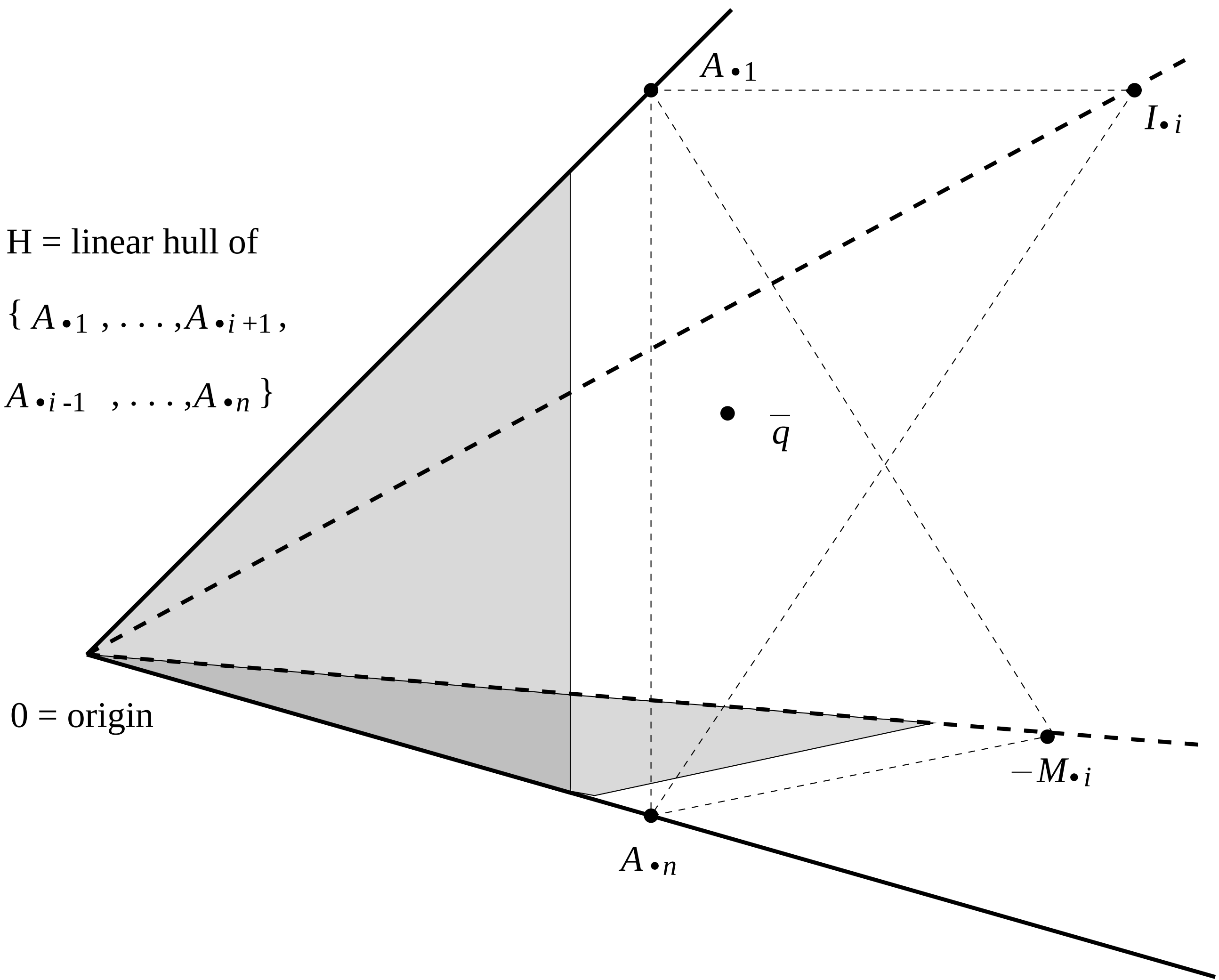

Figure 3.4 $I_{.i}$ and $-M_{.i}$ are both on the same side of $\mathbf{H}$. Interiors of the complementary cones $\text{Pos}(A_{.1}, \ldots, A_{.i-1}, I_{.i} A_{.i+1}, \ldots, A_{.n})$ and $\text{Pos}(A_{.1}, \ldots, A_{.i-1}, -M_{.i} A_{.i+1}, \ldots, A_{.n})$ have a nonempty intersection.

Among the left-out complementary pair of column vectors $(I_{.i}, -M'_{.i})$, since the ith component in the column vector $I_{.i}$ is $+1$, it is on the side on $\mathbf{H}$ corresponding to the inequality $x_i > 0$. So by the strict separation property, the point $-M'_{.i}$ is on the side of $\mathbf{H}$ corresponding to the inequality $x_i < 0$, which implies that $-m'_{ii} < 0$, or $M'_{ii} > 0$. Thus the ith diagonal element in M' is strictly positive. In a similar manner we see that if the strict separation property holds, then all the diagonal elements in all PPTs of M are strictly positive. By Theorem 3.6 this implies that M is a P-matrix.

∎

A class of convex polyhedral cones in $\mathbf{R}^n$ is said to **partition** $\mathbf{R}^n$ if

a) Every cone in the class has a nonempty interior.

b) The union of the cones in the class is $\mathbf{R}^n$.
c) The interiors of any pair of cones in the class are disjoint.

Theorem 3.17 *Let M be a given square matrix of order n. The class of complementary cones $\mathcal{C}(M)$ partitions $\mathbf{R}^n$ iff M is a P-matrix.*

Proof. If M is a P-matrix, the result that the class of complementary cones $\mathcal{C}(M)$ partitions $\mathbf{R}^n$ follows from Corollary 3.1 and Theorem 3.13.

To prove the converse, suppose that $\mathcal{C}(M)$ partitions $\mathbf{R}^n$. Since every complementary cone in $\mathcal{C}(M)$ has a nonempty interior, by Corollary 3.2, M must be nondegenerate. Hence all complementary sets of column vectors are linearly independent. If the strict separation property is not satisfied, there exists a subcomplementary set of column vectors, say $(A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n})$ such that the hyperplane H which is its linear hull contains both the points in the left out complementary pair $(I_{.i}, -M_{.i})$ on the same side of it. As in the proof of Theorem 3.16, this implies that the interiors of the complementary cones $\text{Pos}(A_{.1}, \dots, A_{.i-1}, I_{.i}, A_{.i+1}, \dots, A_{.n})$ and $\text{Pos}(A_{.1}, \dots, A_{.i-1}, -M_{.i}, A_{.i+1}, \dots, A_{.n})$ have a nonempty intersection; a contradiction, since $\mathcal{C}(M)$ partitions $\mathbf{R}^n$. Hence, if $\mathcal{C}(M)$ partitions $\mathbf{R}^n$, the strict separation property is satisfied, and by Theorem 3.16 this implies that M is a P-matrix.

Hence the class of complementary cones $\mathcal{C}(M)$ partitions $\mathbf{R}^n$ iff M is a P-matrix.

□

Example 3.8

Let $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The complementary cones are the quadrants in $\mathbf{R}^2$, drawn in Figure 1.3, and obviously this class of complementary cones partitions $\mathbf{R}^n$. For any n in general $\mathcal{C}(I)$ is the class of orthants of $\mathbf{R}^n$, and these obviously partition $\mathbf{R}^n$. As mentioned earlier the class of complementary cones is a generalization of the class of orthants of $\mathbf{R}^n$ (orthants of $\mathbf{R}^n$ are the special class of complementary cones obtained by taking $M = I$), and $\mathcal{C}(M)$ possesses the property of partitioning $\mathbf{R}^n$ as long as M is a P-matrix. This was first proved by Samelson, Thrall and Wesler in [3.69].

Corollary 3.10 *Let M be a given square matrix of order n. The following conditions are mutually equivalent.*

i) All principal subdeterminants of M are strictly positive.
ii) The LCP (q, M) has a unique solution for each $q \in \mathbf{R}^n$.
iii) The LCP (q, M) has at most one solution for each $q \in \mathbf{R}^n$.
iv) The diagonal entries in all PPTs of M are strictly positive.
v) LCPs associated with M satisfy the strict separation property.
vi) The class of complementary cones $\mathcal{C}(M)$ forms a partition of $\mathbf{R}^n$.

Proof. Follows from Theorems 3.15, 3.16, 3.17, 3.6 and Corollaries 3.6, 3.8.

□

Theorem 3.18 *Consider the LCP (3.1) in which M is a P-matrix. Suppose $(\overline{w}, \overline{z})$ is the unique solution of the LCP with $\overline{z}_1 = 0$. Let $\omega = (w_2, \ldots, w_n)$, $\xi = (z_2, \ldots, z_n)$. If $(y_2, \ldots, y_n)$, with $y_j \in \{w_j, z_j\}$ for $j = 2$ to n, is a complementary feasible basic vector for the principal subproblem of (3.1) in ω, ξ; $(w_1, y_2, \ldots, y_n)$ is a complementary feasible basic vector for (3.1).*

Proof. By Result 2.2 and Theorem 3.13, $\overline{\omega} = (\overline{w}_2, \ldots, \overline{w}_n)$, $\overline{\xi} = (\overline{z}_2, \ldots, \overline{z}_n)$ is the unique solution of the principal subproblem in ω, ξ. Since $\overline{w}$, $\overline{z}$ is the unique solution of (3.1), and $\overline{z}_1 = 0$, we have $\sum_{j=2}^{n} m_{2j}\overline{z}_j + q_1 = \overline{w}_1 \geqq 0$. Under degeneracy, there may be several complementary feasible basic vectors (all differing in the zero valued basic variables) for the principal subproblem in ω, ξ, but the BFS corresponding to each of them must be $\overline{\omega}$, $\overline{\xi}$ by the uniqueness of the solution. Also, the column vector of w_1 in (3.1) is $I_{.1}$. So, when we compute the basic solution of (3.1) corresponding to the basic vector $(w_1, y_2, \ldots, y_n)$, we get $w_j = \overline{w}_j$, $z_j = \overline{z}_j$ for $j = 2$ to n, $\overline{z}_1 = 0$ and $w_1 = \sum_{j=2}^{n} m_{2j}\overline{z}_j + q_1 = \overline{w}_1 \geqq 0$, which is the solution $(\overline{w}, \overline{z})$ of (3.1). So, $(w_1, y_2, \ldots, y_n)$ is a complementary feasible basic vector for (3.1).

$\square$

Higher Order Separation Theorems

Theorem 3.19 *Let M be a P-matrix of order n and let $\mathbf{J}$, $\overline{\mathbf{J}}$ be a partition of $\{1, \ldots, n\}$ with $\mathbf{J}$, $\overline{\mathbf{J}}$ both being nonempty. Let $\{A_{.j} : j \in \mathbf{J}\}$, $\{A_{.j} : j \in \overline{\mathbf{J}}\}$ be the corresponding partition of a complementary set of vectors. Let $\{B_{.j} : j \in \overline{\mathbf{J}}\}$ be the complement of the subcomplementary set $\{A_{.j} : j \in \overline{\mathbf{J}}\}$. If $\mathbf{H}$ is a hyperplane in $\mathbf{R}^n$ satisfying*

i) $\mathbf{H}$ contains the origin 0 and all the vectors in the subcomplementary sets $\{A_{.j} : j \in \mathbf{J}\}$.

ii) All the vectors in the subcomplementary set $\{A_{.j} : j \in \overline{\mathbf{J}}\}$ lie in one of the closed half-spaces, $\mathbf{H}^{\geqq}$, defined by $\mathbf{H}$, then at least one of the vectors in $\{B_{.j} : j \in \overline{\mathbf{J}}\}$ lies strictly on the other side of $\mathbf{H}$ in the other open half-space $\mathbf{H}^{<}$ defined by $\mathbf{H}$.

Proof. Consider the system (3.13)

$$w - Mz = 0\,. \tag{3.13}$$

Perform principal pivot steps in (3.13) to transform the complementary set of vectors $\{A_{.j} : j \in \mathbf{J} \cup \overline{\mathbf{J}}\}$ into the set of unit vectors. This is a nonsingular linear transformation that preserves separation properties. If u_j denotes the variable in (3.13) associated with $A_{.j}$, and v_j denotes its complement, this transforms (3.13) into

$$u - \overline{M}v = 0 \tag{3.14}$$

where $\overline{M}$ is also a P-matrix because it is a principal pivot transform of the P-matrix M. Let $\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$ denote the principal submatrix of $\overline{M}$ corresponding to the subset $\overline{\mathbf{J}}$. Let

$\overline{\mathbf{H}} = \{x : \sum_{j=1}^{n} a_j x_j = 0\}$ be the transform of $\mathbf{H}$. Since $A_{.j}$ is transformed into $I_{.j}$, by (i) we have $a_j = 0$ for each $j \in \mathbf{J}$, and by (ii) we have $a_{\overline{\mathbf{J}}} = (a_j : j \in \overline{\mathbf{J}}) \geqq 0$. So the row vector $a = (a_j) \geqq 0$ and since $\overline{\mathbf{H}}$ is a hyperplane $a \geq 0$, that is $a_{\overline{\mathbf{J}}} \geq 0$. (A vector $y = (y_j) \geq 0$ means that each y_j is nonnegative and at least one y_j is strictly positive). For $j \in \overline{\mathbf{J}}$, $B_{.j}$ is now transformed into $-\overline{M}_{.j}$. The vector $(a(-\overline{M}_{.j}) : j \in \overline{\mathbf{J}}) = -a_{\overline{\mathbf{J}}}\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$. Since $\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$ is itself a P-matrix and $a_{\overline{\mathbf{J}}} \geq 0$, by Theorem 3.11 at least one of the components of $a_{\overline{\mathbf{J}}}\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$ is strictly positive, that is $a(-\overline{M}_{.j}) < 0$ for at least one $j \in \overline{\mathbf{J}}$. That is, at least one of the $-\overline{M}_{.j}$ for $j \in \overline{\mathbf{J}}$ lies in the open half-space $\overline{\mathbf{H}}^{<} = \{x : \sum_{j=1}^{m} a_j x_j < 0\}$ not containing the unit vectors. In terms of the original space this implies that at least one of the $B_{.j}$, $j \in \overline{\mathbf{J}}$ is contained in the open half-space $\mathbf{H}^{<}$ defined by $\mathbf{H}$ not containing the complementary set of vectors $\{A_{.j} : j \in \mathbf{J} \cup \overline{\mathbf{J}}\}$.

∎

Theorem 3.20 *Let M be a P-matrix of order n, $\mathbf{J}$ a nonempty proper subset of $\{1, \ldots, n\}$ and let $\{A_{.j} : j \in \mathbf{J}\}$ be a subcomplementary set of vectors. Let $\mathbf{H}$ be a hyperplane in $\mathbf{R}^n$ that contains the origin 0 and all the vectors in the set $\{A_{.j} : j \in \mathbf{J}\}$. Then $\mathbf{H}$ strictly separates at least one pair of the left out complementary pairs of vectors $\{I_{.j}, -M_{.j}\}$ for $j \in \overline{\mathbf{J}} = \{1, \ldots, n\} \setminus \mathbf{J}$.*

Proof. Choose the subcomplementary set $\{A_{.j} : j \in \overline{\mathbf{J}}\}$ arbitrarily and transform the system (3.13) into (3.14) as in the proof of Theorem 3.19. Using the notation in the proof of Theorem 3.19, suppose this transforms $\mathbf{H}$ into $\overline{\mathbf{H}} = \{x : \sum_{j=1}^{n} a_j x_j = 0\}$. Since $A_{.j}$ is transformed into $I_{.j}$ and $\mathbf{H}$ contains $A_{.j}$ for $j \in \mathbf{J}$, $\overline{\mathbf{H}}$ must contain $I_{.j}$ for $j \in \mathbf{J}$, that is $a_j = 0$ for all $j \in \mathbf{J}$. Since $\overline{\mathbf{H}}$ is a hyperplane, we must have $a \neq 0$, that is $a_{\overline{\mathbf{J}}} = (a_j : j \in \overline{\mathbf{J}}) \neq 0$. Define $\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$ as in the proof of Theorem 3.19, it is a P-matrix as noted there. By the sign nonreversal theorem for P-matrices of D. Gale and H. Nikaido, Theorem 3.12, if $(y_j : j \in \overline{\mathbf{J}}) = a_{\overline{\mathbf{J}}}\overline{M}_{\overline{\mathbf{J}}\overline{\mathbf{J}}}$, $a_j y_j > 0$ for at least one $j \in \overline{\mathbf{J}}$. Since $a_j = 0$ for $j \in \mathbf{J}$, these facts imply that there exists at least one $j \in \overline{\mathbf{J}}$ satisfying the property that $aI_{.j}$ and $a(-\overline{M}_{.j})$ have strictly opposite signs, that is $\overline{\mathbf{H}}$ separates the complementary pair of vectors $\{I_{.j}, -\overline{M}_{.j}\}$ strictly. In terms of the original space, this implies that $\mathbf{H}$ strictly separates the complementary pair of vectors $\{I_{.j}, -M_{.j}\}$ for that $j \in \overline{\mathbf{J}}$.

∎

Comment 3.1 Theorem 3.2 is from K. G. Murty [3.47, 3.48]. Theorem 3.5 is due to A. W. Tucker [3.78]. The proofs of Theorems 3.7, 3.8 given here are attributed to P. Wolfe. The fact that the LCP (q, M) of order n has a unique solution for all $q \in \mathbf{R}^n$ is originally established [3.69]. The inductive proof of Theorem 3.13 given here, and Theorems 3.14, Corollary 3.6 are from K. G. Murty [3.47, 3.49].

A Variant of the LCP

We now discuss some results from K. G. Murty [3.51]. Let M be a given square matrix of order n and q a given column vector of order n. Let $\mathbf{J}$ be a given subset of $\{1, \ldots, n\}$.

The **generalized** LCP with data q, M, $\mathbf{J}$ is the problem of finding column vectors $w \in \mathbf{R}^n$, $z \in \mathbf{R}^n$ satisfying:

$$\begin{array}{llll} w & -Mz & = q & \\ w_j z_j & & = 0 & \text{for all } j = 1 \text{ to } n \\ w_j, & & z_j \geqq 0 & \text{for all } j \notin \mathbf{J} \\ w_j, & & z_j \leqq 0 & \text{for all } j \in \mathbf{J}\,. \end{array} \tag{3.15}$$

We will use the notation $(q, M, \mathbf{J})$ to denote this generalized LCP. Notice that if $\mathbf{J} = \emptyset$, the generalized LCP $(q, M, \emptyset)$ is the same as the usual LCP (q, M) that we have been discussing so far. We will now prove some results about the uniqueness of the solution to this generalized LCP.

Theorem 3.21 *Let M be a given square matrix of order n, and $\mathbf{J}$ a given subset of $\{1, \ldots, n\}$. With M, $\mathbf{J}$ fixed, the generalized LCP $(q, M, \mathbf{J})$ has a unique solution for each $q \in \mathbf{R}^n$ iff M is a P-matrix.*

Proof. In (3.15), make the following transformation of variables: $w_i = u_i$ for $i \notin \mathbf{J}$, $-u_i$ for $i \in \mathbf{J}$; $z_i = v_i$ for $i \notin \mathbf{J}$, $-v_i$ for $i \in \mathbf{J}$. After making these substitutions, multiply both sides of the ith equation in it by -1 for each $i \in \mathbf{J}$. Let $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T$. After these transformation the problem becomes:

$$\begin{aligned} u - \overline{M} v &= \overline{q} \\ u \geqq 0,\ v &\geqq 0 \\ u^T v &= 0 \end{aligned} \tag{3.16}$$

where $\overline{M}$ is the matrix obtained from M by multiplying each entry in the ith row of M by -1 for each $i \in \mathbf{J}$, and then multiplying each entry in the ith column of the resulting matrix by -1 for each $i \in \mathbf{J}$. So the value of a principal subdeterminant of $\overline{M}$ is exactly equal to the corresponding principal subdeterminant of M. Thus $\overline{M}$ is a P-matrix, iff M is. The column vector $\overline{q}$ is obtained by multiplying the ith entry in q by -1 for each $i \in \mathbf{J}$. (3.16) is equivalent to (3.15). If $(\hat{w}, \hat{z})$ is a solution of (3.15), then the corresponding (u, v) obtained as above is a solution of (3.16) and vice versa. But (3.16) is the usual LCP $(\overline{q}, \overline{M})$, and hence by Theorem 3.13 it has a unique solution for each $\overline{q} \in \mathbf{R}^n$ iff $\overline{M}$ is a P-matrix. Consequently (3.15) has a unique solution for each $q \in \mathbf{R}^n$ iff M is a P-matrix.

∎

Now let M be a given square matrix of order n, and consider the usual LCP (q, M), (3.1), again. The column vector q is **nondegenerate** in (3.1), if q is not in the linear hull of any set of $(n-1)$ columns of $(I, -M)$. There are $2^{\mathbf{n}}$ complementary sets of column vectors in the LCP (q, M), and number these sets in some order, from $l = 1$ to $2^{\mathbf{n}}$. Let A_l denote the matrix whose columns are the columns in the lth complementary set of column vectors (in that order), for $l = 1$ to $2^{\mathbf{n}}$. If M is a P-matrix, by Corollary 3.1, A_l is nonsingular and hence is a complementary basis for

(3.1), for each $l = 1$ to $2^{\mathbf{n}}$. Let $\mathcal{A}$ denote the set of all these complementary bases, that is $\mathcal{A} = \{A_l : l = 1, 2, \dots, 2^{\mathbf{n}}\}$.

It is clear from the definitions, that if q is nondegenerate in the LCP (q, M) and A is a complementary basis for the LCP (q, M) and $\hat{q} = (\hat{q}_j) = A^{-1}q$, then $\hat{q}_j \neq 0$ for each $j = 1$ to n. (Since $\hat{q} = A^{-1}q$, we have $q = A\hat{q} = \sum_{j=1}^{n} \hat{q}_j\ A_{.j}$. If $\hat{q}_j = 0$ for some j, then q can be expressed as a linear combination of $(n-1)$ column vectors of $(I\ \vdots -M)$, contradicting the hypothesis that q is nondegenerate in (3.1)).

We will now discuss some important results on the LCP (q, M) when M is a P-matrix and q is nondegenerate, from [3.51].

Theorem 3.22 *Let M be a given P-matrix of order n, and let q be nondegenerate in the LCP (q, M). Then for each subset $\mathbf{J} \subset \{1, \dots, n\}$, there exists a unique complementary basis $A \in \mathcal{A}$ satisfying the property that if $\hat{q} = (\hat{q}_j) = A^{-1}q$, then $\hat{q}_j < 0$ for all $j \in \mathbf{J}$ and $\hat{q}_j > 0$ for all $j \notin \mathbf{J}$.*

Proof. Since q is nondegenerate, for any $A \in \mathcal{A}$ all the components in $A^{-1}q$ are nonzero. Suppose $\hat{q} = A^{-1}q$ is such that $\hat{q}_j < 0$ for all $j \in \mathbf{J}$ and $\hat{q}_j > 0$ for all $j \notin \mathbf{J}$. Let $(y_1, \dots, y_n)$ be the complementary vector of variables corresponding to the complementary basis A. Let $(\hat{w}; \hat{z})$ be the solution defined by:

$$\begin{aligned} y_j &= \hat{q}_j, \quad \text{for } j = 1 \text{ to } n \\ \text{Complement of } y_j &= 0, \quad \text{for } j = 1 \text{ to } n\,. \end{aligned}$$

Then $(\hat{w}, \hat{z})$ is a solution of the generalized LCP $(q, M, \mathbf{J})$. However, by Theorem 3.21, the generalized LCP $(q, M, \mathbf{J})$ has a unique solution, since M is a P-matrix. This implies that there exists a unique complementary basis $A \in \mathcal{A}$ such that if $\hat{q} = A^{-1}q$, then $\hat{q}_j < 0$ for all $j \in \mathbf{J}$ and $\hat{q}_j > 0$ for all $j \notin \mathbf{J}$.

□

Example 3.9

Let

$$\widetilde{M}(3) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Here $n = 3$, and there are eight complementary bases. Verify that $\widetilde{M}(3)$ is a P-matrix. The LCP $(q, \widetilde{M}(3))$ corresponding to this data will be discussed in Example 4.1 of Chapter 4. From there, we see that for $A \in \mathcal{A}$, $\overline{q} = A^{-1}q$, the updated right hand side constants vector is as tabulated below.

Complementary Basic Vector Corresponding to the Complementary Basis	$\overline{q}^T =$ Transpose of the Updated Right Hand side Constants Vector
(w_1, w_2, w_3)	$(-1, -1, -1)$
(w_1, w_2, z_3)	$(-1, -1, \ \ 1)$
(w_1, z_2, z_3)	$(-1, \ \ 1, -1)$
(w_1, z_2, w_3)	$(-1, \ \ 1, \ \ 1)$
(z_1, z_2, w_3)	$(\ \ 1, -1, -1)$
(z_1, z_2, z_3)	$(\ \ 1, -1, \ \ 1)$
(z_1, w_2, z_3)	$(\ \ 1, \ \ 1, -1)$
(z_1, w_2, w_3)	$(\ \ 1, \ \ 1, \ \ 1)$

As an example let $\mathbf{J} = \{2\}$. We verify that the complementary basis corresponding to the complementary basic vector (z_1, z_2, z_3) is the unique complementary basis in this problem satisfying the property that the jth updated right hand side constant is negative for $j \in \mathbf{J}$ and positive for $j \notin \mathbf{J}$. In a similar manner, the statement of Theorem 3.22 can be verified to be true in this example for all subsets $\mathbf{J} \subset \{1, 2, 3\}$.

3.3.1 One-to-One Correspondence Between Complementary Bases and Sign Vectors

Given any vector of "+" and "−" sign symbols in $\mathbf{R}^n$, Theorem 3.22 states that if M is a P-matrix of order n and q is nondegenerate in the LCP (q, M), then there exists a unique complementary basis for the LCP (q, M) satisfying the property that the signs of the components in the updated right hand sides constants vector with respect to that complementary basis, are exactly the given vector of signs.

Corollary 3.11 *Let M be a given P-matrix of order n, and let q be a given column vector which is nondegenerate for the LCP (q, M). The number of complementary basis $A \in \mathcal{A}$ such that if $\hat{q} = (\hat{q}_i) = A^{-1}q$, then exactly r of the q_i are strictly negative, is $\binom{n}{r}$. This follows from Theorem 3.22.*

Corollary 3.12 *Let M be a given P-matrix of order n, and let q be a given column vector which is nondegenerate for the LCP (q, M). There is a* **one-to-one correspondence** *between the $2^{\mathbf{n}}$ complementary basic vectors for this problem, and the $2^{\mathbf{n}}$ sign vectors for the components in the updated q. This follows from Theorem 3.22.*

The result in Theorem 3.22 and Corollary 3.12 displays the nice combinatorial structure of the LCP (q, M) when M is a P-matrix and q is nondegenerate. As we move from one complementary basic vector to another, the sign pattern of the components in the updated q vector changes distinctly. The problem of solving the LCP (q, M) in this case, is the same as that of finding the complementary basic vector that corresponds to the sign vector consisting of all $+$ signs under this one-to-one correspondence.

3.4 OTHER CLASSES OF MATRICES IN THE STUDY OF THE LCP

In this section we provide a brief summary of some of the other classes of matrices used by many researchers in the study of the LCP.

The Weak Separation Property

This is a property of the matrix M, and does not depend on the right hand side constants vector q. An LCP associated with the matrix M (or the class of complementary cones $\mathcal{C}(M)$) is said to satisfy the weak separation property if, given any subcomplementary set of column vectors $(A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n})$, there exists a hyperplane $\mathbf{H}$ in $\mathbf{R}^n$ which contains the points 0, and $A_{.t}$, $t = 1, \dots, i-1, i+1, \dots, n$, and separates (not necessarily strictly) the points represented by the left out complementary pair of column vectors $I_{.i}$, $-M_{.i}$. See reference [3.48]. As an example let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding complementary cones are drawn in Figure 3.5, verify that the weak separation property holds, but not the strict separation property. Also see Figure 3.6.

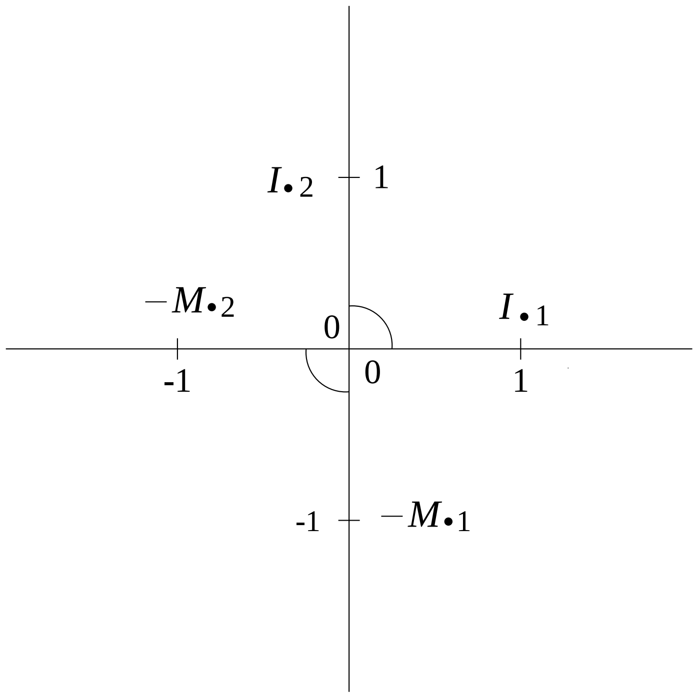

Figure 3.5 The Complementary Cones when $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The Complementary Cones $\text{Pos}\{I_{.1}, -M_{.2}\}$, $\text{Pos}\{-M_{.1}, I_{.2}\}$ are both degenerate, they are the coordinate lines. The Weak Separation Property Holds.

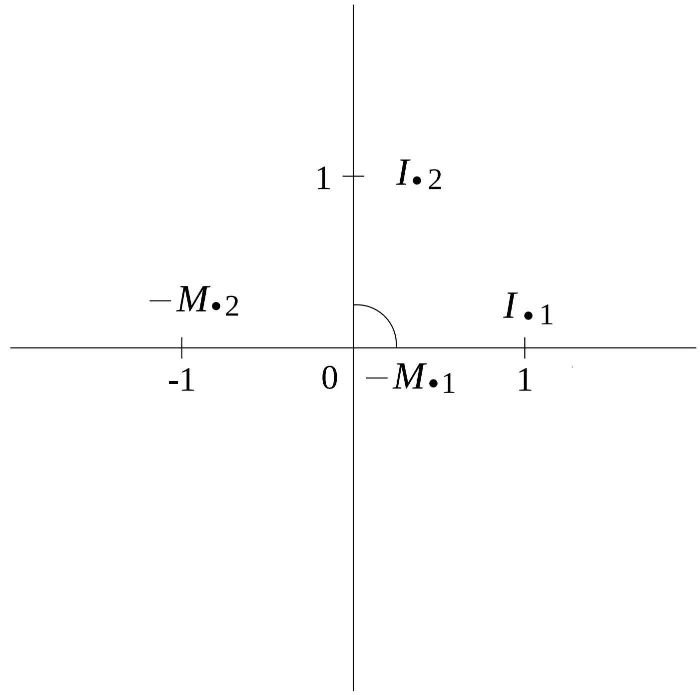

Figure 3.6 The Complementary Cones when $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The Cones $\text{Pos}(I_{.1}, -M_{.2})$, $\text{Pos}(-M_{.1}, -M_{.2})$, $\text{Pos}(-M_{.1}, I_{.2})$ are all degenerate and their Union is the Horizontal Coordinate Line, and the Nonnegative Half of the Vertical Coordinate Line.

The square matrix M of order n is said to be a **weak separation matrix** if it satisfies the weak separation property. Using arguments similar to those in the proof of Theorem 3.16, it can be verified that M is a weak separation matrix iff the diagonal entries in M and all the PPTs of M are nonnegative. See reference [3.48], and also Exercise 3.1.

P_0-Matrices: A square matrix M of order n belongs to this class iff all its principal subdeterminants are $\geqq 0$.

The union of all the complementary cones in $\mathcal{C}(M)$ may not even be convex when M is a P_0-matrix. For example, consider $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The complementary cones for this case are plotted in Figure 3.6. The complementary pivot algorithm may not be able to process the LCP (q, M) when M is a P_0-matrix. For example, on the LCP in which M is the matrix given above, and $q = (-1, 0)^T$, the complementary pivot algorithm ends up in ray termination, even though the LCP has a solution.

Z-Matrices: A square matrix $M = (m_{ij})$ of order n is said to be a Z-matrix iff $m_{ij} \leqq 0$ for every $i \neq j$. A very efficient special algorithm for solving the LCP (q, M) when M is a Z-matrix has been developed by R. Chandrasekaran, and this is discussed in Section 8.1.

Matrices with Dominant Principal Diagonal: A square matrix $M = (m_{ij})$ of order n belongs to this class if $|m_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |m_{ij}|$ for each $i = 1$ to n.

Generalized Diagonally Dominant: A square matrix M is said to be a generalized diagonally dominant if there exists a positive diagonal matrix T such that AT is strictly diagonally dominant.

M-Matrices: A square matrix M of order n is said to be an M-matrix if it is a Z-matrix which is also a P_0-matrix. In the literature these matrices are also called K_0-matrices in some references (see S. R. Mohan [3.46]). Nonsingular M-matrices are precisely Z-matrices which are also P-matrices (in the literature these are also known as Minkowski-matrices or K-matrices and some authors refer to these as M-matrices. See the paper [3.22] by M. Fiedler and V. Ptak for the properties of these matrices. If M is a nonsingular M-matrix, then its inverse $M^{-1} \geqq 0$.)

Comparison Matrix: Given a square matrix $M = (m_{ij})$, its comparison matrix is $A = (a_{ij})$ where $a_{ii} = |m_{ii}|$ for $i = 1$ to n and $a_{ij} = -|m_{ij}|$ for all $i \neq j$, $i, j = 1$ to n.

H-Matrix: A square matrix M is said to be a H-matrix if its comparison matrix (which is a Z-matrix) is a P-matrix.

Semi-Monotone Matrices (E_0-Matrices): The square matrix M of order n is said to be semi-monotone iff for all $x \in \mathbf{R}^n$, $x \geq 0$, there exists an index i such that $x_i > 0$ and $M_{i.}x \geqq 0$. This class of matrices has also been called the class of L_1-matrices. The matrix M belongs to this class iff the LCP (q, M) has a unique solution whenever $q > 0$. If M is symmetric, then it is semi-monotone iff it is copositive.

Strictly Semi-Monotone Matrices: The square matrix M of order n belongs to this class if for every $x \in \mathbf{R}^n$, $x \geq 0$, there exists an index i such that $x_i > 0$ and $M_{i.}x > 0$. Equivalently, let $\widetilde{M}$ refer to any nonempty principal submatrix of M, or M itself. Then M is strictly semi-monotone, iff the system

$$\begin{aligned} \widetilde{M}\tilde{z} &< 0 \\ \tilde{z} &\geq 0 \end{aligned}$$

has no solution $\tilde{z}$, for all such $\widetilde{M}$. B. C. Eaves [3.21] calls this class of matrices L_*. See also the papers [1.3] of R. W. Cottle and G. B. Dantzig, [1.16] by S. Karamardian, and [3.40] of C. E. Lemke (Lemke calls this class of matrices E).

If M is symmetric, M is strictly semi-monotone iff it is strictly copositive. A matrix M is strictly semi-monotone if the LCP (q, M) has a unique solution whenever $q \geqq 0$. This class is the same as the class of $\overline{Q}$ or completely Q-matrices.

Fully Semi-Monotone: A square matrix of M of order n belongs to this class if M and all its PPTs are semi-monotone. See R. W. Cottle and R. E. Stone [3.13]. The square matrix M is fully semi-monotone iff the LCP (q, M) has a unique solution whenever q is in the interior of any nondegenerate complementary cone.

S-Matrix: A matrix M, not necessarily square, belongs to this class if the system

$$Mx > 0, \ x \geq 0$$

has a solution x. See [3.40] by C. E. Lemke.

$\overline{Q}$, or Completely Q-Matrices: A square matrix of order n belongs to this class if the matrix, and all its principal submatrices are Q-matrices. In [3.9] R. W. Cottle has proved that this class is exactly the same as the class of strictly semi-monotone matrices. See Exercises 3.10, 3.11.

V-Matrices: The square matrix M of order n belongs to this class if every principal submatrix $\widetilde{M}$ of M has the property that there is no positive column vector z such that the last coordinate of $\widetilde{M}\tilde{z}$ is nonpositive and the remaining ones are zero. In [2.38] L. Van der Heyden constructed a new algorithm for the LCP and showed that it will always obtain a solution to the LCP (q, M), provided M is a V-matrix. In [3.9] R. W. Cottle has proved that this class of matrices is the same as the class of strictly semi-monotone matrices, or the class of $\overline{Q}$-matrices. See Exercises 3.10, 3.11.

Q_0-Matrices: A square matrix M of order n belongs to this class if the union of all the complementary cones in $\mathcal{C}(M)$ is convex. In some early papers on the LCP this class was denoted by K. We have the following theorem on this class of matrices.

Theorem 3.23 *If M is a Q_0-matrix, the union of all the complementary cones in $\mathcal{C}(M)$ is* $\text{Pos}(I, -M)$.

Proof. Let $\mathbf{K}(M)$ denote the union of all the complementary cones in $\mathcal{C}(M)$. Every solution of the LCP (q, M) is a $(\overline{w}, \overline{z})$ satisfying $\overline{w} = M\overline{z} + q$, $\overline{w}, \overline{z} \geqq 0$ and $\overline{w}^T\overline{z} = 0$, and hence $(\overline{w}, \overline{z})$ give the coefficients in an expression for q as a nonnegative linear combination of the columns of $(I \ \vdots -M)$. So if $q \in \mathbf{K}(M)$, then $q \in \text{Pos}(I \ \vdots -M)$, that is, $\mathbf{K}(M) \subset \text{Pos}(I \ \vdots -M)$. Now, let $\mathbf{\Gamma} \subset \{I_{.j}, -M_{.j}, j = 1 \text{ to } n\}$. For any $j = 1$ to n, if $q = I_{.j}$, $(w = I_{.j}, z = 0)$ is a solution of the LCP (q, M); and if $q = -M_{.j}$, $(w = 0, z = I_{.j})$ is a solution of the LCP (q, M). So $\mathbf{\Gamma} \subset \mathbf{K}(M)$. Since M is a Q_0-matrix by hypothesis $\mathbf{\Gamma} \subset \mathbf{K}(M)$ implies that $\text{Pos}(\mathbf{\Gamma}) \subset \mathbf{K}(M)$, that is, $\text{Pos}(I \ \vdots -M) \subset \mathbf{K}(M)$. All these facts together imply that $\mathbf{K}(M) = \text{Pos}(I \ \vdots -M)$.

□

$\overline{Q}_0$-Matrices: The square matrix M of order n belongs to this class if it, and all its principal subdeterminants are Q_0-matrices.

Adequate Matrices: A square matrix of order n belongs to this class if it is a P_0-matrix, and whenever a principal submatrix of M corresponding to a subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ is singular, the sets of vectors $\{M_{i.} : i \in \{i_1, \ldots, i_r\}\}$, $\{M_{.i} : i \in \{i_1, \ldots, i_r\}\}$ are both linearly dependent. This class of matrices has been defined by A. W. Ingleton [3.31]. He proved that if M is adequate, for any $q \in \mathbf{R}^n$, there exists at most one w such that (w, z) is a solution of the LCP (q, M). Also, if M is invertible and adequate, it is a P-matrix.

L_2-Matrices: A square matrix of order n is said to be an L_2-matrix if for each $z \geq 0$ satisfying $w = Mz \geqq 0$ and $w^T z = 0$; there exists a $\hat{z} \neq 0$ satisfying $\hat{w} = -(\hat{z}^T M)^T$, $w \geqq \hat{w} \geqq 0$, $z \geqq \hat{z} \geqq 0$.

$E^*(d)$-Matrices: Let $d \in \mathbf{R}^n$ be given. The square matrix M of order n belongs to this class if $z = 0$ in every solution of the LCP (d, M). Thus if M is an $E^*(d)$ matrix, the LCP (d, M) has the unique solution $(w = d,\ z = 0)$ if $d \geqq 0$, and no solutions if $d \ngeqq 0$.

$E(d)$-Matrices: Let $d \in \mathbf{R}^n$ be given. The square matrix M of order n belongs to this class, if whenever $(\overline{w}, \overline{z})$ is a solution of the LCP (d, M) with $\overline{z} \neq 0$, there exists an $x \geqq 0$ such that $y = -M^T x \geqq 0$, and $\overline{z} \geqq x$, $\overline{w} \geqq y$.

$L(d)$-Matrices: Let $d \in \mathbf{R}^n$ be given. The square matrix M of order n belongs to this class if it is both an $E(d)$-matrix and also an $E(0)$-matrix.

$L^*(d)$-Matrices: Let $d \in \mathbf{R}^n$ be given. The square matrix M of order n belongs to this class if it is both an $E^*(d)$-matrix and also an $E^*(0)$-matrix.

The classes of matrices $E(d)$, $E^*(d)$, $L(d)$, $L^*(d)$ have been defined by C. B. Garcia [3.25]. He has shown that if $d > 0$, and M is an $L(d)$ matrix, then the LCP (q, M) can be processed by the variant of the complementary pivot algorithm in which the original column of the artificial variable z_0 is taken to be $-d$.

Regular Matrices: The square matrix M of order n is said to be a regular matrix (denoted by R-matrix) if there exists no $z \in \mathbf{R}^n$, $t \in \mathbf{R}^1$ satisfying

$$\begin{aligned} z \geqq 0, t &\geqq 0 \\ M_{i.} z + t &= 0 \quad \text{if } i \text{ is such that } z_i > 0 \\ M_{i.} z + t &\geqq 0 \quad \text{if } i \text{ is such that } z_i = 0. \end{aligned}$$

So the matrix M is a regular matrix iff for all $\lambda \geqq 0$, the only solution to the LCP $(\lambda e, M)$ is $(w = \lambda e,\ z = 0)$. S. Karmardian [1.16] introduced this class of matrices and proved that all regular matrices are Q-matrices.

R_0-Matrices: These are matrices M for which the LCP $(0, M)$ has a unique solution. This is exactly the class $E^*(0)$ defined earlier. These matrices have also been called **superregular matrices**. If M belongs to this class there exists no $z \in \mathbf{R}^n$ satisfying

$$\begin{aligned} z &\geq 0 \\ M_{i.} z &= 0 \quad \text{for } i \text{ is such that } z_i > 0 \\ M_{i.} z &\geqq 0 \quad \text{for } i \text{ is such that } z_i = 0. \end{aligned}$$

This class includes all regular matrices. In particular the matrix $M = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ is an R_0-matrix, but not regular.

A degenerate complementary cone $\text{Pos}(A_{.1}, \ldots, A_{.n})$ is said to be **strongly degenerate** if there exists $\alpha = (\alpha_1, \ldots, \alpha_{.n}) \geq 0$ satisfying $\sum_{j=1}^{n} \alpha_j A_{.j} = 0$, **weakly**

degenerate if no such α exists. As an example, let

$$M = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} .$$

For this matrix M, the degenerate complementary cone $\mathrm{Pos}(-M_{.1}, -M_{.2}, I_{.3})$ is strongly degenerate because $0 = (-M_{.1}) + (-M_{.2})$. The degenerate complementary cone $\mathrm{Pos}(I_{.1}, I_{.2}, -M_{.3})$, is weakly degenerate since it is impossible to express 0 as $\alpha_1 I_{.1} + \alpha_2 I_{.2} + \alpha_3(-M_{.3})$ with $\alpha_1, \alpha_2, \alpha_3 \geqq 0$ and at least one of $\alpha_1, \alpha_2, \alpha_3$ strictly > 0.

Clearly, a square matrix M is an R_0-matrix iff there exists no strongly degenerate complementary cone in$\mathcal{C}(M)$.

N-Matrix: A square matrix of order n belongs to this class if all its nonempty principal subdeterminants are strictly negative. See M. Kojima and R. Saigal [3.39] in which they prove that if M is an N-matrix, then the LCP (q, M) has either 0, 1, 2 or 3 solutions for any q.

U-Matrix: A square matrix of order n belongs to this class iff the LCP (q, M) has a unique solution whenever q is in the interior of $\mathbf{K}(M)$ = the union of all complementary cones in $\mathcal{C}(M)$. See R. W. Cottle and R. E. Stone [3.13].

INS-Matrices: A square matrix M of order n is said to be an INS-Matrix (Invariant Number of Solutions) iff the number of solutions of the LCP (q, M) is the same for all q contained in the interior of $\mathbf{K}(M)$. See R. W. Cottle and R. E. Stone [3.13], R. E. Stone [3.70, 3.71].

INS$_k$-Matrices: A square matrix M of order n is called an INS$_k$-Matrix if for every q in the interior of $\mathbf{K}(M)$, the LCP (q, M) has exactly k distinct solutions.

W-Matrices: Let M be a given real square matrix of order n. For any $\mathbf{J} \subset \{1, \ldots, n\}$ define the complementary matrix $A(\mathbf{J})$ associated with the subset $\mathbf{J}$ to be the square matrix of order n in which

$$(A(\mathbf{J}))_{.j} = \begin{cases} -M_{.j}, & \text{if } j \in \mathbf{J} \\ I_{.j}, & \text{if } j \notin \mathbf{J} . \end{cases}$$

The matrix M is said to be a W-matrix iff

$$\mathrm{Pos}(A(\mathbf{J})) \cap \mathrm{Pos}(A(\overline{\mathbf{J}})) = \{0\}$$

for every $\mathbf{J} \subset \{1, \ldots, n\}$ and $\overline{\mathbf{J}} = \{1, \ldots, n\} \setminus \mathbf{J}$. This definition is due to M. W. Jeter and W. C. Pye, they have shown that every W-matrix is a U-matrix.

3.5 Exercises

3.1 Let M be a given square matrix of order n. Let $\mathbf{\Gamma} = \{1, \dots, n\}$. If $\mathbf{S} \subset \mathbf{\Gamma}$ define

$$\begin{aligned} f(\mathbf{S}) &= 1, \quad \text{if } \mathbf{S} = \emptyset \\ &= \text{principal subdeterminant of } M \text{ corresponding to } \mathbf{S}, \text{ if } \mathbf{S} \neq \emptyset. \end{aligned}$$

Prove that M is a weak separation matrix iff there exists no nonempty subset $\mathbf{S} \subset \mathbf{\Gamma}$ satisfying the property that for some $j \in \mathbf{S}$, $f(\mathbf{S})$ and $f(\mathbf{S} \setminus \{j\})$ are both non-zero and have strictly opposite signs. Using it, prove that a square matrix is a weak separation matrix iff the diagonal entries of all its PPTs are $\geqq 0$.

Prove that every nondegenerate weak separation matrix is a P-matrix and that every square matrix which is not a weak separation matrix must have a negative principal subdeterminant. Show that all P_0-matrices are weak separation matrices.

Prove that if the LCP (q, M) has more than one solution, and M is a weak separation matrix, then $q \geqq 0$ (K. G. Murty [3.48, 1.26]).

3.2 Prove that the two definitions given for strictly semi-monotone matrices are equivalent.

3.3 Prove that every copositive plus matrix which contains a strictly positive column vector, is a Q-matrix.

3.4 Prove that all PPTs of a P_0-matrix are P_0-matrices.

3.5 Prove that the square matrix M of order n is a P_0-matrix iff for all $y \in \mathbf{R}^n$, $y \neq 0$, there exists an i such that $y_i \neq 0$ and $y_i(M_{i.}y) \geqq 0$ (Fiedler and Ptak [3.23]).

3.6 If M is a P_0-matrix, prove that there exists an $x \geq 0$ such that $Mx \geqq 0$ (B. C. Eaves [3.21]).

3.7 If M is a P_0-matrix and $x > 0$ satisfies $Mx = 0$, prove that there exists a $y \geq 0$ such that $y^T M = 0$.

3.8 If M is a P_0-matrix and (q, M) has a nondegenerate complementary BFS, then prove that it is the unique complementary feasible solution. Construct a numerical example to show that the converse could be false (B. C. Eaves [3.21]).

3.9 Prove that every Q-matrix is an S-matrix (C. E. Lemke [3.40]).

3.10 Prove that if M is a square matrix of order n which is an S-matrix, and every $(n-1) \times (n-1)$ principal submatrix of M is strictly semi-monotone then M itself is strictly semi-monotone; using this prove that the class of strictly semi-monotone matrices is the same as the class of completely Q-matrices (R. W. Cottle [3.9]).

3.11 Prove that the classes of matrices, strictly semi-monotone, $\overline{Q}$, V, are the same (R. W. Cottle [3.9]).

3.12 If M is a square symmetric matrix of order n, prove that the following conditions are equivalent.

(i) M is strictly copositive,

(ii) M is strictly semi-monotone,

(iii) for all $q \geqq 0$, the LCP (q, M) has a unique solution (F. Pereira [3.59]).

3.13 If M is a square matrix of order n which is principally nondegenerate, prove that the number of complementary feasible solutions for the LCP (q, M) has the same parity (odd or even) for all $q \in \mathbf{R}^n$ which are nondegenerate. As an example, when

$$M = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

show that the number of complementary feasible solutions for the LCP (q, M) is always an even number $\geqq 2$ whenever q is nondegenerate (K. G. Murty [1.26, 3.47]).

3.14 Prove that if the number of complementary feasible solutions for the LCP (q, M) is a constant for all q which are nondegenerate, then that constant must be equal to 1, and M must be a P-matrix. (K. G. Murty [1.26, 3.47]).

3.15 If $y^T q + y^T M y$ is bounded below on the set $y \geqq 0$, then prove that the LCP (q, M) has a solution and it can be computed by using the complementary pivot algorithm (B. C. Eaves [3.21]).

3.16 Let q, M be matrices of orders $n \times 1$, $n \times n$ respectively. If there exists an $x \in \mathbf{R}^n$, $x \geq 0$ such that $q^T x < 0$, $M^T x \leqq 0$, prove that the LCP (q, M) has no solution (C. B. Garcia [3.25]).

3.17 Prove that the classes of matrices $E(d)$ and $E^*(d)$ are the same whenever either $d > 0$, or $d < 0$ (C. B. Garcia [3.25]).

3.18 Prove that the semi-monotone class of matrices is $\bigcap_{d>0} E(d)$. Also, prove that the class L of matrices is $\bigcap_{d>0} L(d)$. Verify that the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

is an $L(d)$ matrix for $d = (2,3,1)^T$, but not an L-matrix (C. B. Garcia [3.25]).

3.19 Let $d > 0$ and suppose M is an $L^*(d)$ matrix. For any $q \in \mathbf{R}^n$, prove that when the variant of the complementary pivot algorithm in which the original column of the artificial variable z_0 is taken to be $-d$, is applied on the LCP (q, M), it terminates with a solution of the LCP (S. Karamardian, [1.16], C. B. Garcia [3.25]).

3.20 Let M be a copositive plus matrix. Prove that the set of solutions of the LCP (q, M) is nonempty and bounded iff the optimum objective value in the following LP is zero

$$\begin{array}{lrl} \text{Maximize} & e^T u & \\ \text{Subject to} & M^T u & \leqq 0 \\ & q^T u & \leqq 0 \\ & u & \geqq 0\,. \end{array}$$

In particular, prove that if M is copositive plus and the LCP (q, M) has a nondegenerate complementary BFS, then the set of solutions of the LCP (q, M) is bounded (O. L. Mangasarian [3.42]).

3.21 Let M be a copositive plus matrix. If the system: $Mx > 0$, $x \geqq 0$ has a solution $x \in \mathbf{R}^n$, prove that the set of solutions of the LCP (q, M) is nonempty and bounded, for every $q \in \mathbf{R}^n$ (O. L. Mangasarian [3.42], J. Parida and K. L. Roy [3.56]).

3.22 Prove that every regular matrix is a Q-matrix (S. Karamardian [1.16]).

3.23 Prove that if M is a P_0-matrix then the following are equivalent

(i) M is an R_0-matrix,
(ii) M is a regular matrix,
(iii) M is a Q-matrix.

(M. Aganagic and R. W. Cottle [3.2]).

3.24 If M is a P-matrix, prove that the system $Mx > 0$, $x > 0$ has a feasible solution.

3.25 Let M be a P-matrix of order n and let $q \in \mathbf{R}^n$. Consider the quadratic program:

$$\begin{array}{ll} \text{minimize} & z^T(Mz+q) \\ \text{subject to} & Mz+q \geqq 0 \\ & z \geqq 0\,. \end{array} \tag{3.17}$$

Prove the following

(i) (3.17) has a unique local minimum $\overline{z}$ which is the global minimum with objective value 0. In this case $(\overline{w} = M\overline{z} + q, \overline{z})$ is the unique solution of the LCP (q, M).

(ii) If $\overline{z}$ is the unique local minimum for (3.17), let $\pi = \overline{z}^T$, $\mu = (M\overline{z} + q)^T$. Then $(\overline{z}, \overline{\pi}, \overline{\mu})$ is the unique KKT point for (3.17) (Y. C. Chang [3.7]).

3.26 The square matrix M of order n is a nonsingular M-matrix iff the following property holds. Let $(\overline{w}, \overline{z})$ be the solution of the LCP (q, M). Then $\overline{z}$ is the unique vector in the region $\mathbf{X} = \{z : Mz + q \geqq 0, z \geqq 0\}$ satisfying $\overline{z} \in \mathbf{X}$ and $z \geqq \overline{z}$ for any $z \in \mathbf{X}$ (R. W. Cottle and A. F. Veinott, Jr. [3.14]).

3.27 Let M be a Z-matrix which is also a P-matrix of order n, and $q^1, q^2 \in \mathbf{R}^n$ satisfying $q^2 \geqq q^1$. If (w^i, z^i) is a solution of the LCP (q^i, M) for $i = 1, 2$, prove that $z^1 \geqq z^2$ (R. W. Cottle, G. H. Golub, and R. S. Sacher [3.11]).

3.28 Let M be an N-matrix. Then prove that either $M < 0$ or there exists a $d > 0$ such that $Md > 0$. Also prove that a square matrix M is an N-matrix iff all proper principal subdeterminants of M^{-1} are positive and the determinant of M^{-1} is < 0 (M. Kojima and R. Saigal [3.39]).

3.29 Let M be an N-matrix. Prove the following. If $M < 0$, (q, M) has no solutions for $q \ngeqq 0$ and exactly two solutions for $q > 0$. If $M \not< 0$, and $q \not> 0$, the LCP (q, M) has a unique solution. If $M \not< 0$, and $q > 0$, the LCP (q, M) has 2 or 3 solutions. If $M \not< 0$, $q \geq 0$ and $q_i = 0$ for at least one i, the LCP (q, M) has exactly two solutions (M. Kojima and R. Saigal [3.39]).

3.30 If M is an M-matrix prove that the union of all the degenerate complementary cones is the set of all $q \in \mathbf{R}^n$ for which the LCP (q, M) has an infinite number of solutions. Also, in this case, prove that the LCP (q, M) has infinitely many solutions iff q is in the boundary of $\mathbf{K}(M)$, which is the union of all complementary cones in $\mathcal{C}(M)$ (S. R. Mohan [3.46]).

3.31 Prove that every U-matrix is a fully semi-monotone matrix (R. W. Cottle and R. E. Stone [3.13]).

3.32 Prove that the LCP (q, M) has an even number of solutions for each $q \in \mathbf{R}^n$ which is nondegenerate, if there exists a $z > 0$ such that $zM < 0$, or equivalently if $(x = 0,\ y = 0)$ is the only solution to the system

$$\begin{aligned} Ix - My &= 0 \\ x, y &\geqq 0 \end{aligned}$$

(R. Saigal [3.63]).

3.33 Consider the LCP (q, M) where M is an adequate matrix. If $(\overline{w}, \overline{z})$, $(\hat{w}, \hat{z})$ are any two solutions of this LCP, prove that $\overline{w} = \hat{w}$ (A. W. Ingleton [3.31]).

3.34 Let M be a square nondegenerate matrix of order n. For some $q^* \in \mathbf{R}^n$, if the LCP (q^*, M) has a unique solution (w^*, z^*) and $w^* + z^* > 0$, then prove that M is a Q-matrix (A. W. Ingleton [3.31]).

3.35 If M is an L-matrix and an R_0-matrix prove that it must also be an R-matrix and a Q-matrix.

Prove that if M is R_0-matrix which is copositive, then it must be an R-matrix and a Q-matrix.

If M is an L_2-matrix and a Q-matrix, prove that it must be an R_0-matrix.

If M is an L-matrix, prove that the following are equivalent:

(i) M is a Q-matrix,
(ii) M is an R-matrix,
(iii) M is an R_0-matrix, and
(iv) M is an S-matrix.

Is every Q-matrix which is an L_1-matrix, also an R_0-matrix? (J. S. Pang [3.53]).

3.36 Prove that copositive plus and strictly copositive matrices are L-matrices.

3.37 Prove that every P_0-matrix is semi-monotone, and that every Q-matrix is an S-matrix.

3.38 If M is an L-matrix, prove that it is a Q-matrix iff it is an S-matrix (B. C. Eaves [3.21]).

3.39 Prove that the system: $Mx = 0$, $x > 0$, is inconsistent if either M is an L_1-matrix and a Q-matrix, or M is a Q-matrix which is copositive.

If M is an L_1-matrix and a Q-matrix, prove that every nonzero $\overline{z}$ that leads to solution of the LCP $(0, M)$ must have at least two nonzero components.

If M is a Q-matrix which is copositive, prove that any vector $\overline{z}$ satisfying $\overline{z}^T M \overline{z} = 0$ and $(M + M^T)\overline{z} = 0$, and leads to a solution of the LCP $(0, M)$ must be the zero vector.

If M is a Q-matrix which is symmetric and copositive, prove $x = 0$ is the only feasible solution to the system: $Mx = 0$, $x \geqq 0$.

If M is a Q-matrix which is symmetric and copositive plus, prove that it must be strictly copositive.

If M is a copositive plus matrix prove that the following are equivalent:

(i) M is a Q-matrix,
(ii) M is a R-matrix,
(iii) M is a R_0-matrix,
(iv) M is an S-matrix.

In addition, if M is also symmetric, then prove that each of the above is equivalent to

(v) M is strictly copositive,
(vi) $x = 0$ is the only feasible solution of the system: $Mx = 0$, $x \geqq 0$

(J. S. Pang [3.53]).

3.40 Let M be a nondegenerate Q-matrix of order n. Prove that the number of distinct solutions of the LCP (q, M) is $\leqq 2^n - 1$ for any $q \in \mathbf{R}^n$ (A. Tamir [3.75]).

3.41 If M is a square matrix all of whose principle subdeterminants are negative and there exists an $x > 0$ such that $Mx > 0$, then M is a Q-matrix (R. Saigal [3.65]).

3.42 Prove that any square matrix of order 2 with all diagonal entries zero cannot be a Q-matrix. Show that this result is not true for higher order matrices by considering

$$M = \begin{pmatrix} 0 & 3 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

which is a Q-matrix since $M^{-1} > 0$ (M. Jetter and W. Pye [3.33]).

3.43 If M is a square matrix of order n such that there exists a $z > 0$ satisfying $z^T M < 0$ then the LCP (q, M) has an even number of solutions for all nondegenerate q (R. Saigal [3.63]).

3.44 If M is copositive plus and the LCP (q, M) has a solution $(\overline{w}, \overline{z})$ which is a nondegenerate BFS of "$w - MZ = q$, $w \geqq 0$, $z \geqq 0$", prove that the set of solutions of the LCP (q, M) is a bounded set. However, show that the existence of a nondegenerate

BFS solution is not necessary for the set of solutions of the LCP (q, M) to be bounded. (Hint: try $q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (O. L. Mangasarian [3.42]).

3.45 If M is a copositive plus matrix of order n, for any $q \in \mathbf{R}^n$, the set of solutions of the LCP (q, M) is nonempty and bounded if the following system has a solution $x \in \mathbf{R}^n$.

$$Mx + q > 0, \ x \geqq 0 \tag{3.18}$$

(O. L. Mangasarian [3.42]).

3.46 If M is a copositive Q-matrix, prove that the system

$$\begin{aligned} Mx &= 0 \\ x &> 0 \end{aligned}$$

is inconsistent.

3.47 If M is a symmetric, copositive plus Q-matrix, prove that M must be strictly copositive (J. S. Pang [3.53]).

3.48 If M is a copositive plus matrix of order n, the solution set of the LCP (q, M) is nonempty and bounded for each $q \in \mathbf{R}^n$ iff M is a Q-matrix. This happens iff the system "$Mx > 0, \ x \geqq 0$" has a solution $x \in \mathbf{R}^n$ (O. L. Mangasarian [3.42]).

3.49 If the nondegenerate matrix M is the limit of a convergent sequence of nondegenerate Q-matrices, prove that M is a Q-matrix (M. Aganagic and R. W. Cottle [3.2]).

3.50 Suppose M is a Q-matrix of order n. Let $\mathbf{J} \subset \{1, \ldots, n\}$ be such that $M_{j.} \geqq 0$ for a $j \in \mathbf{J}$. Then the principal submatrix of M determined by the subset $\{1, \ldots, n\} \setminus \mathbf{J}$ must be a Q-matrix.

3.51 Let M be a Q-matrix of order n. If $\{A_{.1}, \ldots, A_{.j-1}, A_{.j+1}, \ldots, A_{.n}\}$ is a subcomplementary set, there exists a hyperplane $\mathbf{H}$ in $\mathbf{R}^n$ containing 0 and all the vectors in this subcomplementary set such that $I_{.j}$ and $-M_{.j}$ do not lie in the same open half-space corresponding to this hyperplane $\mathbf{H}$. Also, if M is a nondegenerate Q-matrix, there exists a hyperplane $\mathbf{H}$ of the type described above, which strictly separates $I_{.j}$ and $-M_{.j}$ (M. Aganagic and R. W. Cottle [3.2]).

3.52 If M is a Q_0-matrix satisfying the property that the LCP (q, M) has a unique solution for each q in the interior of $\mathbf{K}(M)$, prove that M must be a P_0-matrix. Also, if M is a P_0-matrix with only one zero principal subdeterminant and has the property that $\mathbf{K}(M) \neq \mathbf{R}^n$, then prove that $\mathbf{K}(M)$ is a closed half-space and that the LCP (q, M) has a unique solution whenever q is in the interior of $\mathbf{K}(M)$ (R. W. Cottle and R. E. Stone [3.13]).

3.53 If M is a symmetric matrix of order n satisfying

$$m_{ii} > 0 \quad \text{for all } i$$
$$m_{ij} \leqq 0 \quad \text{for all } j \neq i$$

prove that M is copositive iff it is PSD.

3.54 Prove that the LCP (q, M) has a unique solution for all $q > 0$ iff for all $x \geq 0$ there exists an i such that $x_i > 0$, $y = (y_1, \ldots, y_n)^T = Mx$ and $y_i \geqq 0$.

3.55 If M is a symmetric matrix of order n, the following are equivalent

(i) M is copositive;

(ii) for all $x \geq 0$ there exists an i such that $x_i > 0$ and $y = (y_1, \ldots, y_n)^T = Mx$, $y_i \geqq 0$;

(iii) (q, M) has a unique solution for all $q > 0$.

3.56 If M is a symmetric matrix of order n, the following are equivalent

(i) M is strictly copositive;

(ii) M is a Q-matrix and the LCP (q, M) has a unique solution for all $q \in \{I_{.1}, \ldots, I_{.n}\}$ (F. J. Pereira [3.59]).

3.57 Prove that a H-matrix with positive diagonals is a P-matrix (J. S. Pang [3.55]).

3.58 Prove that M-matrices and generalized diagonally dominant matrices are H-matrices.

3.59 Prove that if M is a strictly semi-monotone matrix and q is nondegenerate in the LCP (q, M), then the LCP (q, M) has an odd number of solutions (B. C. Eaves [3.21]).

3.60 Prove that a square matrix M of order n is a Z-matrix iff for each $q \in \mathbf{R}^n$ for which the set $\mathbf{X}(q, M) = \{x : Mx + q \geqq 0,\ x \geqq 0\} \neq \emptyset$, there exists a least element $\tilde{x} \in \mathbf{X}(q, M)$ (given $\mathbf{K} \subset \mathbf{R}^n$, an element $\overline{x} \in \mathbf{K}$ is said to be a least element in $\mathbf{K}$ if $\overline{x} \leqq x$ for all $x \in \mathbf{K}$. If a least element exists, it is clearly unique) satisfying $\tilde{x}^T(M\tilde{x} + q) = 0$ (A. Tamir [3.73]).

3.61 Prove that a square matrix M of order n is a nonsingular M-matrix (i. e., a Z-matrix which is also a P-matrix) iff for each $q \in \mathbf{R}^n$, the set $\mathbf{X}(q, M) = \{x : Mx + q \geqq 0, x \geqq 0\}$ has a least element $\tilde{x}$ which is the only vector in $\mathbf{X}(q, M)$ satisfying $x^T(Mx + q) = 0$ (R. W. Cottle and A. F. Veinott, Jr. [3.14]).

3.62 Prove that a square matrix which has either a zero row or a zero column cannot be a Q-matrix.

3.63 If M is a Q-matrix and PSD, is M^T also a Q-matrix? (Hint: Check $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$).

3.64 Let M be a PSD matrix and A a PD matrix of order n. Let $(w(\varepsilon), z(\varepsilon))$ denote the solution of the LCP $(q, M + \varepsilon A)$ for some $q \in \mathbf{R}^n$ and $\varepsilon > 0$. If the LCP (q, M) has a solution, prove that the $\text{limit}_{\varepsilon \to 0^+} z(\varepsilon)$ exists, and if this limit is $\overline{z}$, it is the point that minimizes the norm $\|Az\|$ in the set $\{z : (w = Mz + q, z)$ is a solution of the LCP $(q, M)\}$. If the LCP (q, M) has no solution, prove that $\text{limit}_{\varepsilon \to 0^+} \|z(\varepsilon)\| = +\infty$ (A. Gana [5.6]).

3.65 Let $-M$ be a Z matrix. A well-known theorem states that if there exists an $x \geqq 0$ such that $x^T M < 0$ in this case, then M^{-1} exists and $-M^{-1} \geqq 0$. Using this theorem, prove the following:

(a) If M satisfies all the above properties, there exist $y_{ij} \geqq 0$ for all i, j such that

$$I_{.j} = \sum_{i=1}^{n} (-y_{ij}) M_{.i}, \qquad \text{for all } j\,.$$

(Hint: Use the fact that $M^{-1} \leqq 0$.)

(b) Under the same conditions on M, $\text{Pos}(I \,\vdots\, -M) = \text{Pos}(-M)$.

(c) Under the same conditions on M the LCP (q, M) has a solution iff $-M^{-1}q \geqq 0$. Also, if $-M^{-1}q \geqq 0$, then a solution to the LCP is $(w, z) = (0, -M^{-1}q)$ (R. Saigal).

3.66 Let M be a square matrix of order n satisfying the property "if $Mx \leqq 0$, then x must be nonnegative". Prove the following.

(a) M^{-1} must exist.

(b) $-M^{-1} \geqq 0$. (Hint: Use the fact that $(M(M^{-1}))_{.j} = I_{.j} \geqq 0$.)

(c) In this case $\text{Pos}(-M) \supset \text{Pos}(I)$.

3.67 Let M be an arbitrary square matrix of order n. Consider the LCP (q, M). Prove that the following property "the LCP has a solution whenever q is such that the system

$w - Mz = q, w \geqq 0,\ z \geqq 0$ has a feasible solution and for all such q the LCP has a solution in which $w = 0$" holds iff $\text{Pos}(-M) \supset \text{Pos}(I)$ [i. e., $\text{Pos}(I \vdots -M) = \text{Pos}(-M)$].

Also prove that this property holds iff for all x such that $Mx \leqq 0$, x must be nonnegative (A. K. Rao).

3.68 Let M be a square matrix of order n with non-positive off-diagonal elements. If M is a P-matrix, prove that it has a nonnegative inverse (M. Fiedler and V. Ptak [3.22]).

3.69 Let M be a square matrix of order n. Let $q \in \mathbf{R}^n$. The matrix M is said to be a Q_0-matrix if the LCP (q, M) has a complementary feasible solution whenever the system

$$\begin{aligned} w - Mz &= q \\ w \geqq 0, z &\geqq 0 \end{aligned}$$

has a feasible solution.

i) Prove that M is a Q_0-matrix iff the union of all the complementary cones in $\mathcal{C}(M)$ is a convex set.
ii) Prove that the matrix M is a Q_0-matrix iff the LCP (q, M) satisfies: "if $q^1, q^2 \in \mathbf{R}^n$ are such that (q^1, M) has a complementary feasible solution, and $q^2 \geqq q^1$, then (q^2, M) also has a complementary feasible solution" (A. K. Rao).

3.70 If M is a square matrix which is positive semidefinite, and q is nondegenerate in the LCP (q, M), prove that the number of solutions of the LCP (q, M) is either 0 or 1.

3.71 If M is a square matrix of order n which is positive semidefinite, prove that the intersection of the interiors of any pair of complementary cones in $\mathcal{C}(M)$ is empty.

3.72 If M is a square matrix of order n which is positive semidefinite, and q lies in the interior of a complementary cone in $\mathcal{C}(M)$, prove that the LCP (q, M) has a unique solution.

3.73 Let M be a M-matrix (i. e., a Z-matrix which is also a P_0-matrix). Let $w(\varepsilon)$, $z(\varepsilon)$ be the solution of the LCP $(q, M + \varepsilon I)$. If the LCP (q, M) has a solution, prove that $\text{limit}_{\varepsilon \to 0^+} z(\varepsilon)$ exists, and if this limit is $\overline{z}$, it is the least element of $\{z : z \geqq 0, Mz + q \geqq 0\}$ (i. e., $\overline{z} \leqq z$ for all z in this set). If the LCP (q, M) does not have a solution, then $\text{limit}_{\varepsilon \to 0^+} \|z(\varepsilon)\|$ is $+\infty$ (A. Gana [5.6]).

3.74 Consider the LCP (q, M) of order n. Suppose the matrix M is not a P-matrix, but its principal submatrix of order $n - 1$ obtained by deleting row i and column i

from it is a P-matrix for a given i. Discuss an efficient algorithm for computing all the solutions of this LCP (V. C. Prasad and P. K. Sinha [3.60]).

3.75 Let M be a square nondegenerate matrix. Prove that the number of complementary feasible solutions for the LCP (q, M), is either even for all q that are nondegenerate, or odd for all q that are nondegenerate (K. G. Murty [3.50]).

3.76 Given $q \in \mathbf{R}^n$ and a square matrix M of order n, q is said to be **nondegenerate with respect to** M, if q does not lie in the linear hull of any set of $n-1$ or less column vectors of $(I \;\vdots\; -M)$.

Let M be a nondegenerate Q-matrix of order n satisfying the property for some $q \in \mathbf{R}^n$ which is nondegenerate with respect to M, the LCP (q, M) has an odd number of solutions. Prove that small perturbations in the entries of M still leave it as a nondegenerate Q-matrix (A. Tamir).

3.77 Let M be a square matrix of order 2 and let I be the identity matrix of order 2. Prove that M is a Q-matrix iff the LCPs $(-I_{.1}, M)$ and $(-I_{.2}, M)$ both have complementary feasible solutions (L. M. Kelly and L. T. Watson [3.38]).

3.78 Let

$$M = \begin{pmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 0.4 & -0.1 \end{pmatrix}, \quad \hat{q} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and let I be the identity matrix of order 3. Show that the LCPs $(-I_{.1}, M)$, $(-I_{.2}, M)$, $(-I_{.3}, M)$ all have complementary feasible solutions, but the LCP $(\hat{q}, M)$ does not have a complementary feasible solution. This clearly shows that the result in Exercise 3.77 cannot be generalized for $n > 2$ (L. M. Kelly and L. T. Watson [3.38]).

3.79 Consider the following matrix

$$M(\varepsilon) = \begin{pmatrix} 21 & 25 & -27 & -36-\varepsilon \\ 7 & 3 & -9 & 36+\varepsilon \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{pmatrix}$$

and let I be the identity matrix of order 4.

(a) Show that $M(\varepsilon)$ is a nondegenerate matrix for all $0 \leqq \varepsilon < 1$.
(b) Show that $M(0)$ is a Q-matrix.
(c) Show that $M(\varepsilon)$ is not a Q-matrix for $0 < \varepsilon < 1$. In particular, let $q(\varepsilon) = (1-\varepsilon)32I_{.3} + \varepsilon(0.26, -0.02, 30.8, -0.08)^T$. Show that the LCP $(q(\varepsilon), M(\varepsilon))$ has no complementary feasible solution when $0 < \varepsilon < 1$.

These results clearly establish that small perturbations in its elements might change a nondegenerate Q-matrix into a nondegenerate non Q-matrix (L. M. Kelly and L. T. Watson [3.38]).

3.80 Let M be a given square matrix of order n. Prove that the set of complementary feasible solutions for the LCP (q, M) is a bounded set for every $q \in \mathbf{R}^n$, iff $(w, z) = (0, 0)$ is the unique solution of the LCP $(0, M)$.

3.81 The set of nondegenerate Q-matrices is closed in the relative topology of the set of nondegenerate matrices.

Let M be a given nondegenerate Q-matrix of order n. Let $\beta > 0$, and let ∂M be a square matrix of order n satisfying the properties that

a) $M + \lambda \partial M$ is a nondegenerate Q-matrix for all $0 \leqq \lambda < \beta$,
b) $M + \beta \partial M$ is nondegenerate.

Then prove that $M + \beta \partial M$ is also a Q-matrix.

Using the same arguments, prove the following: Suppose $M^1, M^2, \dots$ is a given infinite sequence of nondegenerate Q-matrices satisfying the property that it converges to a limit, $\overline{M}$. If $\overline{M}$ is also nondegenerate, prove that $\overline{M}$ is a Q-matrix (L. T. Watson [3.79], and M. Aganagic and R. W. Cottle [3.2]).

3.82 Let M be a square matrix of order n satisfying the following properties:

a) $m_{ij} \geqq 0$ for all $i \neq j$, and $m_{ii} \leqq 0$.
b) There exists a row vector $\pi \in \mathbf{R}^n$ satisfying $\pi > 0$ and $\pi M < 0$.

Property b) is easily satisfied by $\pi = e$, if a) holds and $|m_{ii}| > \sum_{j \neq i} m_{ij}$ for each i. Prove the following:

i) If M satisfies properties a), b) above, then $\text{Pos}(I) \subset \text{Pos}(-M)$.
ii) If M satisfies properties a), b) above, then either the LCP (q, M) has a solution in which $w = 0$, or it has no solution at all.
iii) If M satisfies properties a), b) above, the LCP (q, M) has a solution iff

$$\begin{aligned} -Mz &= q \\ z &\geqq 0 \end{aligned}$$

has a solution. And if $\overline{z}$ is a feasible solution of the above system then $(\overline{w} = 0, \overline{z})$ is a solution of the LCP (q, M).
iv) If M satisfies conditions a), b) above, and if $q \geq 0$, the the LCP (q, M) has $2^{\mathbf{n}}$ distinct solutions (R. Saigal [3.64]).

3.83 Consider the LCP (q, M) where M is a square matrix of order n all of whose nonempty principal subdeterminants are strictly negative. Prove the following:

i) The matrix

$$\begin{pmatrix} -1 & 2 & -2 & 2 & \cdots \\ 2 & -1 & 2 & -2 & \cdots \\ -2 & 2 & -1 & 2 & \cdots \\ 2 & -2 & 2 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & -1 \end{pmatrix}$$

satisfies the property that all its nonempty principal subdeterminants are strictly negative.

ii) If all the nonempty principal subdeterminants of M are strictly negative, either $M < 0$ or there exists an $x > 0$ satisfying $Mx > 0$.

(iii) All the nonempty principal subdeterminants of M are strictly negative iff all the proper principal subdeterminants of M^{-1} are strictly positive and the determinant of M^{-1} is strictly negative.

(iv) If all the nonempty principal subdeterminants of M are strictly negative and $M < 0$, then the LCP (q, M) has a solution whenever $q \geqq 0$, and no solution whenever $q \ngeqq 0$. Also when $q > 0$, it has exactly two solutions.

v) If all the nonemtpy principal subdeterminants of M are strictly negative and $M \not< 0$, then the LCP (q, M)

a) has a unique solution whenever $q \ngeqq 0$,

b) has exactly three solutions whenever $q > 0$,

c) has exactly two solutions, with one solution degenerate, whenever $q \geqq 0$ with at least one $q_i = 0$.

Hence establish that any matrix $M \not< 0$ whose nonempty principal subdeterminants are strictly negative, is a Q-matrix.

Also prove that in this case, if $q \geq 0$, and $w_i = 0$ in some solution of the LCP (q, M), then that $w_i > 0$ in all other solutions of the LCP (q, M).

vi) Whenever M is such that all the nonempty principal subdeterminants of M are strictly negative, the LCP (q, M) has either 0, 1, 2 or 3 solutions for any $q \in \mathbf{R}^n$ (M. Kojima and R. Saigal [3.39]).

3.84 If M is a Q-matrix, prove that the system

$$\begin{aligned} Mz &> 0 \\ z &\geqq 0 \end{aligned}$$

has a solution z.

3.85 Let M be a given square matrix of order n. For $j = 1$ to n, let $A_{.j} \in \{I_{.j}, -M_{.j}\}$. Then $(A_{.1}, \ldots, A_{.n})$ is a complementary set of column vectors for the LCP (q, M) and we call the matrix with $A_{.1}, \ldots, A_{.n}$ as its columns in this order, a **complementary submatrix** of $(I \;\vdots\; -M)$. Obviously there are $2^{\mathbf{n}}$ such matrices, and let these be $A^1, \ldots, A^{2^{\mathbf{n}}}$. On these, some may be nonsingular and some singular. Let there be

l nonsingular complementary submatrices, and let all the $2^{\mathbf{n}} - l$ remaining complementary submatrices be singular. Rearrange the complementary submatrices in the sequence $A^1, \ldots, A^{2^{\mathbf{n}}}$, so that the first l of these are nonsingular, and all the remaining are singular. So the complemenatry cone $\text{Pos}(A^t)$ has a nonempty interior iff $1 \leqq t \leqq l$, and has an empty interior if $l + 1 \leqq t \leqq 2^{\mathbf{n}}$.

Prove that M is a Q-matrix iff

$$\bigcup_{t=1}^{l} \text{Pos}(A^t) = \mathbf{R}^n$$

that is, iff the union of all the complementary cones with a nonempty interior is $\mathbf{R}^n$.

3.86 Using the same notation as in Exercise 3.85 for any fixed i between 1 to n, the subcomplementary set of column vectors $(A^t_{.1}, \ldots, A^t_{.i-1}, A^t_{.i+1}, \ldots, A^t_{.n})$ is linearly independent for $1 \leqq t \leqq l$, and let $\mathbf{H}^t_i$ denote the hyperplane in $\mathbf{R}^n$ which is the subspace of $\mathbf{R}^n$ containing all the column vectors in this subcomplementary set.

If there exists an i between 1 to n such that $I_{.i}$ and $-M_{.i}$ are both in one of the open half-spaces determined by $\mathbf{H}^t_i$, for each $t = 1$ to l, then prove that M is not a Q-matrix.

3.87 A Finite Procedure for Checking Whether a Given Square Matrix M of Order n is a Q-Matrix

Using the same notation as in Exercise 3.85, let D^t be $(A^t)^{-1}$ for $t = 1$ to l. For each $t = 1$ to l, select one of the rows of D^t, for example the i_tth for $t = 1$ to l, leading to the set of row vectors $\{D^t_{i_t.} : t = 1 \text{ to } l\}$. For each t, i_t can be chosen in n different ways, and hence there are n^l different sets of row vectors $\{D^t_{i_t.} : t = 1 \text{ to } l\}$ obtained in this manner. For each such sets define the following system of linear inequalities in the variables $q = (q_1, \ldots, q_n)^{\mathcal{T}}$

$$D^t_{i_t.} q < 0, \ t = 1 \text{ to } l \,. \tag{3.19}$$

So there are n^l different systems of inequalities of the form (3.19) depending on the choice of the rows from the matrices D^t.

(i) (3.19) is a system of l strict linear inequalities in n variables $q_1, \ldots, q_n$. Prove that the system (3.19) has a feasible solution q, iff the following system (3.20) is infeasible:

$$\begin{aligned} \sum_{t=1}^{l} \pi_t D^t_{i_t.} &= 0 \\ \sum_{t=1}^{l} \pi_t &= 1 \\ \pi_t &\geqq 0 \text{ for all } t = 1 \text{ to } l \end{aligned} \tag{3.20}$$

that is, it has no feasible solution $\pi = (\pi_t)$.

(ii) Prove that M is a Q-matrix iff each of the n^l systems of the form (3.19) is infeasible, that is, none of them has a feasible solution q.

(iii) Remembering that $l \leqq 2^{\mathbf{n}}$, construct a finite procedure for checking whether a given square matrix M of order n is a Q-matrix, using the above results. Comment on the practical usefulness of such a procedure (D. Gale, see [3.2]).

3.88 A square matrix M is called a Q_0-matrix if the union of all complementary cones in $\mathcal{C}(M)$ is a convex set.

(i) Prove that M is a Q_0-matrix iff $w - Mz = q$, $w \geqq 0$, $z \geqq 0$ has a feasible solution implies that the LCP (q, M) has a complementary feasible solution.

(ii) Prove that M is a Q_0-matrix iff

$$\begin{array}{c} (q^0, M) \text{ has a complemenatry feasible solution} \\ \text{implies} \\ (q, M) \text{ has a complemenatry feasible solution for all } q \geqq q^0 . \end{array}$$

(iii) Prove that every 1×1-matrix is a Q_0 matrix. Also develop necessary and sufficient condition for a 2×2 matrix to be a Q_0-matrix.

(iv) Consider the matrices

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} -10 \\ 2 \\ 5 \end{pmatrix}.$$

Show that $w - Mz = q$, $w \geqq 0$, $z \geqq 0$ has a feasible solution, but the LCP (q, M) has no complementary feasible solution. Also, in this case verify that all the proper principal submatrices of M are Q_0-matrices (by (i), this implies that there are matrices which are not Q_0-matrices, but all of whose proper submatrices are Q_0-matrices).

3.89 A Finite Characterization for Q_0-Matrices

Given a square matrix M of order n, using the notation and results in Exercises 3.85, 3.87, prove that M is a Q_0-matrix iff

$$\text{Pos}(I \ \vdots -M) = \bigcup_{t=1}^{l} \text{Pos}(A^t) .$$

Using this, show that M is a Q_0-matrix, iff each of the following n^l systems

$$\begin{aligned} \sum_{t=1}^{l} \pi_t D^t_{i_t .} - \mu \quad &= 0 \\ -\mu M &\geqq 0 \\ \sum_{t=1}^{l} \pi_t \quad &= 1 \\ \pi \geqq 0, \ \mu \geqq 0 & \end{aligned}$$

are infeasible (i. e., none of them have a feasible solution (π, μ)). This provides a method for checking whether a given square matrix of order n, is a Q_0-matrix or not, using at most a finite amount of computation.

3.90 Prove that every PPT of a Q-matrix is a Q-matrix.

3.91 Let M be a square matrix of order n. Prove that all nonempty principal submatrices of M are Q-matrices iff any of the following three equivalent conditions hold.

i) For all nonempty principal submatrices $\overline{M}$ of M (including M itself), the system

$$\begin{aligned} \overline{M}y &\leqq 0 \\ y &\geq 0 \end{aligned}$$

has no solution.

ii) For every vector $x \geq 0$, there exists an index j such that $x_j > 0$ and $(Mx)_j > 0$.

iii) For every $q \geqq 0$ the LCP (q, M) admits the unique solution $(w; z) = (q; 0)$ (R. W. Cottle [3.9]).

3.92 Row and Column Scalings of Matrices

Given a square matrix of order n, multiply its rows by positive numbers $\alpha_1, \ldots, \alpha_n$ respectively. Multiply the columns of the resulting matrix by positive numbers $\beta_1, \ldots, \beta_n$ respectively. The final matrix M', is said to have been obtained from M by row scaling using the positive vector of scales $\alpha = (\alpha_1, \ldots, \alpha_n)$, and column scaling using the positive vector of scales $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

(i) Prove that, to every LCP associated with the matrix M; there is a corresponding LCP associated with the matrix M', that can be obtained by dividing each constraint by a suitable positive number and appropriate scaling of the variables (i. e., choose appropriate units for measuring it); and vice versa.

(ii) Prove that M is a P-matrix iff M' is.

(iii) Assume that M is an asymmetric P-matrix which is not a PD matrix. It is possible that M' is PD (e. g., let $M = \begin{pmatrix} 1 & 0 \\ -10 & 1 \end{pmatrix}$. Obtain M' using $\alpha = (100, 1)$, $\beta = (1, 1)$ and verify that the resulting matrix is PD). If M is either a lower triangular or an upper triangular P-matrix, show that positive scale vectors α, β exists, such that the resulting matrix is PD.

(iv) Let

$$M = \begin{pmatrix} 1 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & -3 & \varepsilon \end{pmatrix}$$

where ε is a positive number. Verify that M is a P-matrix. When ε is sufficiently small, prove that there exist no positive scale vectors α, β which will transform this matrix into a PD matrix by scaling.

If M is a P-matrix which is not PSD, the LCP (q, M) is equivalent to the nonconvex quadratic program

$$\begin{array}{ll} \text{Minimize} & z^T(Mz+q) \\ \text{Subject to} & z \geqq 0 \\ & Mz+q \geqq 0\,. \end{array}$$

And yet, if we can find positive row and column scale vectors α, β that will convert M into a PD matrix M' by scaling, this problem can be transformed into an equivalent convex quadratic programming problem. For this reason, the study of scalings of P-matrices that transform them into PD matrices is of interest. Prove that every P-matrix of order 2 can be scaled into a PD-matrix. Characterize the class of P-matrices which can be transformed into PD matrices by scaling (R. Chandrasekaran and K. G. Murty).

3.93 Let D be a given square matrix of order n and let I be the unit matrix of order n. Let c, b be given column vectors in $\mathbf{R}^n$. Let

$$q = \begin{pmatrix} c \\ b \end{pmatrix}, \quad M = \begin{pmatrix} D & I \\ -I & 0 \end{pmatrix}.$$

With this data, prove that LCP (q, M) always has a solution, and that the solution is unique if D is a P-matrix (B. H. Ahn [9.4]).

3.94 Let M be a Z-matrix of order n. Prove that M is a P-matrix if the LCPs $(0, M)$ and (e_n, M) have unique solutions.

3.95 Let M be a given square matrix of order n, and let D be an arbitrary diagonal matrix with positive diagonal elements. Prove that the following are equivalent.

i) M is a P-matrix.
ii) $(I-E)D+EM$ is a P-matrix for all diagonal matrices $E=(E_{ij})$ of order n satisfying $0 \leqq E_{ii} \leqq 1$ for all i.
iii) $(I-E)D+EM$ is nonsingular for all diagonal matrices $E=(E_{ij})$ of order n satisfying $0 \leqq E_{ii} \leqq 1$ for all i (M. Aganagic [3.1]).

3.96 Develop an efficient method based on the complementary pivot algorithm to check whether a given square matrix is an M-matrix (K. G. Ramamurthy [3.61]).

3.97 Prove that a Z-matrix which is also a Q-matrix must be a P-matrix. Also prove that every M-matrix is a U-matrix.

3.98 Prove that a symmetric matrix is semi-monotone iff it is copositive. Prove that a symmetric matrix M is strictly semi-monotone iff it is strictly copositive.

3.99 If M is a fully semi-monotone matrix and $(\overline{w}, \overline{z})$ is a solution of the LCP (q, M) and $\overline{w} + \overline{z} > 0$, prove that $(\overline{w}, \overline{z})$ is the unique solution of this LCP.

3.100 (Research Problem) Given a square matrix M of order n, develop finite sets of points $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ in $\mathbf{R}^n$, constructed using the data in M, satisfying the properties

(i) M is a Q-matrix if the LCP (q, M) has a solution for each $q \in \mathbf{\Gamma}_1$,

(ii) M is a Q_0-matrix if the LCP (q, M) has a solution for each $q \in \mathbf{\Gamma}_2$.

3.101 Let M be a P-matrix of order n. Let $\mathbf{J} \subset \{1, 2, \ldots, n\}$, $\overline{\mathbf{J}} = \{1, 2, \ldots, n\} \setminus \mathbf{J}$. Let $(A_{.j} : j \in \mathbf{J})$ be a subcomplementary vector corresponding to $\mathbf{J}$. For each $j \in \overline{\mathbf{J}}$, let $\{A_{.j}, B_{.j}\} = \{I_{.j}, -M_{.j}\}$. Is the following conjecture — "there exists a hyperplane containing the linear hull of $(A_{.j} : j \in \mathbf{J})$ which separates the convex hull of $\{A_{.j} : j \in \overline{\mathbf{J}}\}$ from the convex hull of $\{B_{.j} : j \in \overline{\mathbf{J}}\}$", — true?

3.102 Let M be a square matrix of order n. M is said to be totally principally degenerate iff all its principal subdeterminants are zero. Prove that M is totally principally degenerate iff it is a principal rearrangement of an upper triangular matrix with zero diagonal elements. Use this to develop an efficient algorithm to check whether a matrix is totally principally degenerate (T. D. Parsons [4.15]).

3.103 Let M be a square matrix of order n which is not an R_0-matrix (i. e., the LCP $(0, M)$ has $(w = 0,\ z = 0)$ as the unique solution). Show that there exists a square matrix $\hat{M} = (\hat{m}_{ij})$ of order n, satisfying

$$\begin{aligned} \hat{m}_{nn} &= 0 \text{ and} \\ \hat{m}_{in} &= 0 \text{ or } 1 \text{ for all } i = 1 \text{ to } n-1 \end{aligned}$$

such that for any $q \in \mathbf{R}^n$, the LCP (q, M) can be transformed into an equivalent LCP $(\hat{q}, \hat{M})$, by performing a block principal pivot step, some principal rearrangements, and row scalings.

Use this to show the following

a) Every Q-matrix of order 2 must be an R_0-matrix.

b) Every Q-matrix which is also a PSD matrix, must be an R_0-matrix.

Verify that the result in (a) does not generalize to $n > 2$, using the matrix

$$M = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 10 & 10 & 0 \end{pmatrix}.$$

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