The Branch and Bound Approach: A personal account

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1 My introduction to Operations Research and the Traveling Salesman Problem(TSP)

When I was a graduate student trying to do research in statistics at the Indian Statistical Institute (ISI), Calcutta (now called Kolkata) in the late 1950s, the subject "Operations Research" was unknown in India. If any one mentions "I am studying Operations Research" in India at that time, everyone around him/her would have said "what is that subject, we never heard of it before?"

ISI used to attract many foreign visiting faculty in those days. Once a young American professor of mathematics visited, and the anouncement mentioned that he will give a series of lectures on the newly evolving subject "Operations Research". He arrived with his young wife who was extraordinarily beautiful, but also very sociable and talkative. All male students like me at ISI used to gather around her whenever she appeared on campus.

Curiosity about the new subject drew many students at ISI to his lectures. He discussed the beginnings of Operations Reaearch in the effort to optimize the costs of operations in World War II. He mentioned that it is the branch of applied mathematics that has a large number of challenging problems that are easy to state but hard to solve. As an example he mentioned the TSP involving n cities and square cost matrix c of order n. He said that no one yet knows how to solve versions of the TSP involving more than a very small number of cities. That intrigued me a lot, and to me this TSP appeared a lot more interesting and challenging than the research problems in statistics I was trying to solve at that time.

This visiting professor's beautiful wife used to attend his lectures. Most of the time she comes to the lectures along with her husband and sits on the back row busily doing her knitting. Male students like me used to occasionally turn our head backwards to peek at her.

Here I can mention a humorous incident that occured at that time. I was staying in the dormitory of ISI located in a separate building next to the campus building. One morning while I was getting ready in my dorm room to go to campus, a messenger came and told me that the Director of ISI wants to see me right away. So, I got ready quickly and hurried to the Director's Office. As I was approaching it, through the half-open door I saw the American professor's wife sitting inside in a chair talking. My heart skipped a beat at the thought that perhaps she was complaining to the Director about my staring at her, and may be he called to reprimand me.

Trembling with fear I knocked and opened the door slowly. It was a big relief when the Director smiled on seeing me, and said "Murty come in. We are just waiting for you. I am sure you met our visiting professor's wife before, She wants to go sightseeing today. The Institute will provide transport. Do you mind accompanying her to the Kali temple, Victoria Memorial, etc., and show her around?". I agreed, and we started right away.

On the way she asked me "Mr. Murty, what are you doing?". I replied "Mam, I am thinking of doing Ph.D. research in Operations Research". Like my Indian friends, she did not ask what it is, I thought being the American professor's wife she knows about the new subject.

We spent a very long and busy day sightseeing. On our way back, on the left side of College Street, we saw a well-lighted set of huge buildings. At that time the following conversation ensued:

She: Murty, what are those huge buildings?
Me: Mam, those are the Calcutta Medical College Hospital buildings. It is the largest hospital in Asia with 1100 beds.
She: Murty, then you probably spend a lot of time there!
Me: Mam, why do you think so, I am quite healthy.
She: Sorry, that is not what I meant. Didn't you say that you are doing research on operations?!!

2 Fulbright travel grant + ISI scholarship for 1-year study at the Case Institute of Technology(CIT)

After that incident I made up my mind to do a Ph.D. in Operations Research (OR) with perhaps the TSP as the thesis topic, and started thinking about a solution approach to the TSP. However, there was no one in India who could guide me in this work at that time, so it was clear that for my Ph.D. in OR I have to go to some American university. But being born into a poor family, I could not afford it. So, I started looking aound for a scholarship opportunity. Fortunately, that same year the Fulbright Foundation in India with their main office at the American Embassy in Calcutta; started offering travel grants for a 1-year study in the USA.

I showed the Fulbright grant anouncement to a senior administrator of ISI, Mr. S C Sen, who was very friendly and treated me like his son. He was very generous and told me that if I suceed in getting this Fulbright Travel grant, he would complement it with a small grant from ISI for my living expenses using funds provided by a United Nations Agency, under the condition that I agree to return to ISI and work for at least 3 years after that year.

Mr. S C Sen also offered his full support for me to apply to the Fulbright grant. He asked me to get him a textbook on OR. ISI had one of the best libraries among places of higher learning in India at that time. The head librarian helped me search. All we could find was just one newly published text book on OR authored by some faculty members from the Case Institute of Technology (CIT) in Cleveland, Ohio. On seeing that book Mr Sen encouraged me to apply for a Fulbright Travel Grant for a 1-year study at the CIT.

Then I met the Head of the Fulbright Grants Division at the American Embassy in Calcutta. Surprisingly he was from Ohio. When I told him about my desire for a 1-year study at CIT in Cleveland, he told me what a nice place Cleveland was, and encouraged me to apply. With his help and that of Mr S C Sen, I arrived at CIT in September 1961.

3 My introduction to the Assignment Problem

In Fall 1961 I enrolled in 5 courses at CIT, one of which is a survey course on deterministic OR in which the instructor introduced us to the assignment problem and the efficient Hungarian Method (HM) for solving it. In that class I realized that the set of tours T for the TSP with the cost matrix c of order n is a subset of $A =$ the set of assignments of order n. The HM for the assignment problem with c as the cost matrix outputs a minimum cost assignment a_0 in A. If a_0 is a tour, it is a minimum cost tour, an optimum solution of the TSP with the same cost matrix and we are done. But what to do if a_0 is not a tour? I realized that in this case the objective value of a_0 is a **lower bound** for the minimum objective value in the TSP.

4 The assignment ranking approach to the TSP

My first thought was that if an algorithm could be developed to rank the assignments in A in increasing order of cost, starting from the minimum cost assignment a_0 given by the HM, then the first assignment in this ranked sequence which is a tour, is a minimum cost tour.

So I started thinking about an algorithm for ranking the assignments in A in increasing order of cost. Within a short time I had this ranking algorithm.

This ranking algorithm was developed as a means for solving the TSP, but I did not think of publishing it by itself at that time. Much later when I joined the graduate program in University of California, Berkeley (UCB) and explained it to George Dantzig, he advised me to submit it to "Operations Research" for publication, which I did, and it appeared in 1968 [2].

The variables in both the assignment problem and the TSP of order n are: $x_{ij}, \;\; i, j = 1, ..., n$, all binary variables. The ranking algorithm is based on partitioning the set of assignments A into various subsets. Each subset in this partition is characterized by a pair of subsets of variables in the problem S_1, S_2. Denoting it by $A(S_1, S_2)$ = set of all assignments $x = (x_{i,j}) \in A$ satsfying: $x_{pq} = 1, \; 0$, if $x_{pq} \in S_1, S_2$, respectively. The ranking algorithm generates these subsets S_1, S_2 corresponding to each subset in the partition; and they of course satisfy consistency conditions like: the set of variables in S_1 made equal to 1 in all assignments in $A(S_1, S_2)$ do not violate the constraints in the assignment problem, etc..

In each step of this ranking algorithm, the next element in the ranked sequence is a minimum cost assignment, a_k, in one of the subsets in the partition, say $A(S_1, S_2)$. Then in this step the ranking algorithm partitions the set $A(S_1, S_2) \backslash \{a_k\}$ into smaller subsets, the partitioning of this set is like a **branching step**, even though I did not call it by that name at the time. Each of the newly generated subsets is of the form $A(S'_1, S'_2)$ where S'_2 contains one new variable not in $S_1 \bigcup S_2$, and S'_1 contains one or more variables not in $S_1 \bigcup S_2$.

This assignment ranking algorithm offers an approach to solve this TSP, so I started writing it down in a paper under the tentative title "An assignment ranking approach to solve the TSP". In that process, the cost of each assignment in the ranked sequence gives an improving lower bound for the cost of an optimum tour until a tour appears in the sequence for the first time.

But soon I realized that in order to solve the TSP, it is not necessary to rank all assignments in the ranked sequence strictly. For example, if the next element in the ranked sequence, a_k, is the minimum cost assignment in the subset $A(S_1, S_2)$ at that stage, and S_1 contains all the variables corresponding to arcs in a subtour, then obtaining a_k is of no use for solving the TSP, since the subset of assignments $A(S_1, S_2)$ at this stage of ranking contains no tour at all. So for the specific goal of solving the TSP the ranking algorithm can be improved significantly.

5 Improved version of the assignment ranking process to solve the TSP

Soon this process lead me to an improved version of the algorithm for the TSP. In the new version each step involved the partitioning of a single subset of assignments of the form $A(S_1, S_2)$ into exactly two subsets of the same form, obtained by setting a new variable not in $S_1 \bigcup S_2$ to 0, 1 respectively in the two subsets. This requires solving two assignment problems in each step to get lower bounds for a minimum cost tour in each of these newly generated subsets of assignments.

I tested this algorithm on a few problems involving a small number of cities, but solving by hand the many assignment problems involved in applying it, was becoming difficult. I really wanted to solve the 20-city problem of Croes in the literature, but it turned out to be too difficult by hand. Then one of my classmates in that OR course, Caroline Karel, came to my rescue. She told me that she just wrote a computer code for the HM already, and would be happy to let me use it. With her code I solved the

20-city problem of Croes easily. With this I prepared the paper "The TSP: Solution by a method of ranking assignments" listing Caroline Karel as a co-author for letting me use her code.

I had no experience in publishing papers at that time, so I started looking around for a faculty member to discuss my paper and tell me what the next step should be to send it for publication. But I did not find any of the faculty members at CIT interested in the TSP, much less willing to read a new approach by a student for it. So, finally I approached John Little from whom I took a Q-ing theory course in Fall 1961. He said that his area of research is Q-ing, but agreed to read my paper and give comments. I felt so happy that I addded his name also as a 3rd author on the paper.

6 Time for me to return to ISI

After some time John Little gave me his comments on my paper. He said that instead of using the cost of an optimum assignment as the lower bound, which requires applying the whole HM in each step; the "total reduction" obtained in the initial step of the HM can itself be used as the lower bound at that stage, and this strategy may simplify the algorithm.

But by then it was getting to be time for me to return to ISI. Actually John Little was also leaving CIT to accept a position at MIT. He agreeed to get the algorithm tested at MIT, revise it, and then submit it to the journal "Operations Research" for publication, and I agreed.

7 The final name for the method

Later in correspondence John Little told me that one of his students at MIT, D. Sweeney, suggested the name "Branch and Bound" for the method, and that he was adding his name to the list of authors.

8 References

1. Katta G. Murty, C. Karel, John D C Little , 1962, "The TSP: Solution by a method of ranking assignments", can be seen on my webpage http://www-personal.umich.edu/ ˜ murty/ , near the top under "Selected Publications"

2. Katta G. Murty, 1968, "An algorithm for ranking all the assignments of the assignment problem in increasing order of cost", *Operations Research*, 16, 3, May-June 1968, 682-687.

3. J D C Little, K G Murty, C Karel, D. Sweeney, 1963, "An algorithm for the TSP", *Operations Research*, 11, Dec 1963, 972-989.