Lecture notes for a class on perfectoid spaces

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Preface

These are notes for a class on perfectoid spaces taught in Winter 2017. The goal of the class was to develop the theory of perfectoid spaces up through a proof of the almost purity theorem, and then explain the proof of the direct summand conjecture.

I have tried to make these notes self-contained, and hopefully accessible to anyone with a background algebraic geometry. In particular, I have not assumed any familiarity with rigid geometry, so the relevant theory of adic spaces is developed from scratch. Likewise, I have not assumed any familiarity with Hochster's network of "homological conjectures", so the direct summand conjecture is proven directly, and not via (a non-trivial) reduction to some other statement. Two exceptions: (a) I have used the language of derived categories in a couple of spots where I think it brings out the essence of the argument faster, and (b) I have used some results in point set topology (of spectral spaces) without proof.

Disclaimers. There are surely many errors, so please use at your own risk. The notes are unstable, and being constantly revised. Also, essentially all references and attributions are missing, and will be added later.

Chapter 1

Conventions on non-archimedean fields

We establish some standard notation about non-archimedean fields.

Definition 1.0.1. A *(complete) non-archimedean or NA field* is a field K equipped with a multiplicative valuation $|\cdot| : K^* \to \mathbf{R}_{>0}$ such that K is complete for the valuation topology. The group $|K^*| \subset \mathbf{R}_{>0}$ is called the *value group* of K.

Remark 1.0.2. Some comments are in order:

1. It is convenient to extend the valuation to a map $|\cdot| : K \to \mathbf{R}_{\geq 0}$ by setting $|0| = 0$. With this extension, the valuation topology on K is the unique group topology with a basis of open subgroups given by $|\cdot|^{-1}((0, \gamma))$ for $\gamma \in \mathbf{R}_{>0}$. This topology defined by a metric on K: $d(x, y) = |x - y|$.

2. Most authors do not impose a completeness hypothesis. However, our later constructions with adic spaces work best for complete rings, so we impose completeness right away.

3. We shall typically work in a mixed/positive characteristic setting, i.e., $|p| < 1$ for some prime p.

4. We shall always assume that K is nontrivially valued, i.e., there exists a nonzero element $t \in K$ with $0 < |t| < 1$.

A NA field comes naturally with some associated rings and ideals:

Definition 1.0.3. Let K be a NA field. The subset $K^\circ := \{x \in K \mid |x| \leq 1\}$ is called the *valuation ring* of K; this is an open valuation subring of K with maximal ideal $K^{\circ\circ} := \{x \in K \mid |x| < 1\}$. The quotient $k := K^\circ / K^{\circ\circ}$ is called the *residue field* of K. Any nonzero element $t \in K^{\circ\circ}$ is called a *pseudo-uniformizer*.

The next exercise shows that given the field K, specifying the topology on K is equivalent to specifying the valuation. In particular, it is meaningful to ask if a topological field K is NA.

Exercise 1.0.4. Fix a NA field K.

1. A subset $S \subset K$ is *bounded* if there exists a nonzero $t \in K$ such that $t \cdot K \subset K^\circ$; equivalently, S is bounded for the metric topology on K. An element $t \in K$ is *power bounded* if the set $t^{\mathbf{N}} := \{t^n \mid n \geq 0\} \subset K$ is bounded. Check that $K^\circ \subset K$ is exactly the set of power bounded elements.

2. Check that $K^{\circ\circ} \subset K$ is exactly the *topologically nilpotent elements* of K, i.e., those $t \in K$ such that $t^n \to 0$ as $n \to \infty$.

3. Fix a pseudouniformizer $t \in K^{\circ\circ}$. Show that the t-adic topology on K° coincides with the valuation topology.

4. Show that the given NA valuation $|\cdot|$ on K can be reconstructed from the valuation ring K°.

The next table records some examples the concepts introduced above.

table:NA

K°	K	pu	Value group
$\mathbf{Z}_p$	$\mathbf{Q}_p$	p	$\lvert p\rvert^{\mathbf{Z}}$
$\mathcal{O}_K$	$K/\mathbf{Q}_p$ finite	π	$\lvert\pi\rvert^{\mathbf{Z}}$ if π is a uniformizer
$\mathbf{F}_p[\![t]\!]$	$\mathbf{F}_p((t))$	t	$\lvert t\rvert^{\mathbf{Z}}$
$\widehat{\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]}$	$\widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$	p	$\lvert p\rvert^{\mathbf{Z}[\frac{1}{p}]}$
$\widehat{\overline{\mathbf{Z}_p}}$	$\mathbf{C}_p := \widehat{\overline{\mathbf{Q}_p}}$	p	$\lvert p\rvert^{\mathbf{Q}}$
K° perfect	K perfect	t	p-divisible

Table 1.1: NA fields

Chapter 2

Perfections and tilting

Recall that a characteristic p ring R is *perfect* is the Frobenius $\phi : R \to R$ is an isomorphism; if instead ϕ is merely assumed to be surjective, we say that R is *semiperfect*. In this chapter, we introduce and study Fontaine's tilting functor: it attaches a perfect ring of characteristic p any commutative ring (and is typically of interest when the latter has mixed/positive characteristic).

def:Tilt **Definition 2.0.1** (The tilting functor)**.** Let R be a ring.

1. If R has characteristic p, set $R_{perf} := \varinjlim_\phi R$ and $R^{perf} := \varprojlim_\phi R$, where $\phi : R \to R$ denotes the Frobenius.

2. (Fontaine) For any ring R, set $R^\flat := (R/p)^{perf} := \lim_\phi R/p$. Unless otherwise specified, this ring is endowed with the inverse limit topology, with each R/p being given the discrete topology.

Remark 2.0.2 (Universal properties of perfections)**.** When R has characteristic p, both R_{perf} and R^{perf} are perfect. The canonical map $R \to R_{perf}$ (resp. $R^{perf} \to R$) is universal for maps into (resp. from) perfect rings. Moreover, the projection $R^{perf} \to R$ is surjective exactly when R is semiperfect.

tingRings **Example 2.0.3.** We record some examples of these concepts.

1. $\mathbf{F}_p[t]_{perf} = \mathbf{F}_p[t^{\frac{1}{p^\infty}}]$ and $\mathbf{F}_p[t]^{perf} = \mathbf{F}_p$.

2. $\mathbf{F}_p[t]^\flat \simeq \mathbf{F}_p$.

3. Say R is a finite type algebra over an algebraically closed field k of characteristic p. Then $R^\flat \simeq k^{\pi_0(\mathrm{Spec}(R))}$ is the algebra of k-valued continuous functions on $\mathrm{Spec}(R)$. To see this, we may assume $\mathrm{Spec}(R)$ is connected and reduced (see Exercise 2.0.4). We must show $k \simeq R^{perf}$. Assume first that $\mathrm{Spec}(R)$ is irreducible. Picking a closed point $x \in \mathrm{Spec}(R)$ gives a map $R \to \widehat{R_x}$, where $\widehat{R_x}$ is the completion of the local ring at x; this map is injective as R is a domain. It is therefore enough to show that $\widehat{R_x}^{perf} \simeq k$. We have $\widehat{R_x} = \lim_n R_x/\mathfrak{m}_x^n$, and thus $\widehat{R_x}^{perf} = \lim_n (R_x/\mathfrak{m}_x^n)^{perf}$. Using Exercise 2.0.4, it is easy to see that $(R_x/\mathfrak{m}_x^n)^{perf} \simeq$

$(R_x/\mathfrak{m}_x)^{perf} \simeq k^{perf}$ for all n, which gives the claim. The generalization to the case where R is not a domain is left to the reader.

4. $(\mathbf{F}_p[t^{\frac{1}{p^\infty}}]/(t))^{perf} \simeq \widehat{\mathbf{F}_p[t^{\frac{1}{p^\infty}}]}$. More generally, if R is a perfect ring of characteristic p and $f \in R$ is a nonzerodivisor, then $(R/f)^{perf}$ is the f-adic completion of R.

5. $(\mathbf{Z}_p)^\flat \simeq \mathbf{F}_p$.

6. $(\widehat{\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]})^\flat \simeq \widehat{\mathbf{F}_p[t^{\frac{1}{p^\infty}}]} \simeq \widehat{\mathbf{F}_p[t]_{perf}} \simeq \Big(\mathbf{F}_p[t]_{perf}/(t)\Big)^{perf}$.

The perfection functors kill nilextensions.

ctionNilp **Exercise 2.0.4.** Let $f : R \to S$ be a map of characteristic p rings that is surjective with nilpotent kernel. Then $R^{perf} \simeq S^{perf}$ and $R_{perf} \simeq S_{perf}$. More generally, the same holds if f factors a power of Frobenius on either ring.

We repeatedly use the following elementary lemma.

:Binomial **Lemma 2.0.5.** *Let R be a ring, and let $t \in R$ be an element such that $p \in (t)$. Given $a, b \in R$ with $a = b \mod t$, we have $a^{p^n} = b^{p^n} \mod t^{n+1}$ for all n..*

Proof. We prove this by induction on n. If $n = 0$, there is nothing to show. Assume inductively that $a^{p^n} = b^{p^n} + t^{n+1}c$ for some $c \in R$. Raising both sides to the p-th power, and using that $p \mid \binom{p}{i}$ for $1 \leq i \leq p-1$, we get

$$a^{p^{n+1}} = b^{p^{n+1}} + p \cdot t^{n+1} \cdot d + t^{p\cdot(n+1)}c^p$$

for some $d \in R$. As $p \in (t)$ and $p \geq 2$, the claim follows. □

The next lemma is critical in future applications. It gives a "strict" description of the $(-)^\flat$ functor.

tAbstract **Lemma 2.0.6.** *Assume R is p-adically complete. The projection map $R \to R/p$ induces a bijection*

$$\lim_{x\mapsto x^p} R \to \lim_{\phi} R/p =: R^\flat$$

of multiplicative monoids.

Proof. We first check injectivity. Fix $(a_n), (b_n) \in \lim_{x\mapsto x^p} R$ with $a_n = b_n \mod p$ for all n. Then $a_{n+k}^{p^k} = a_n$ for all n, k, and similarly for the b's. Applying Lemma 2.0.5 to both sides of $a_{n+k} = b_{n+k} \mod p$ then shows that $a_n = b_n \mod p^{k+1}$ for all n and k. As R is p-adically separated, it follows that $a_n = b_n$ for all n, as wanted.

For surjectivity, fix $(\overline{a_n}) \in \lim_{x\mapsto x^p} R/p$. Choose arbitrary lifts $a_n \in R$ of $\overline{a_n}$. Then $a_{n+k+1}^p = a_{n+k} \mod p$ for all n, k. Lemma 2.0.5 shows that for each n, the sequence $k \mapsto a_{n+k}^{p^k}$ is Cauchy for the p-adic topology, and thus has a limit b_n. Then one checks that $b_{n+1}^p = b_n$ for all n, and that b_n lifts $\overline{a_n}$, proving surjectivity. □

Remark 2.0.7. Note that the construction of (b_n) from $(\overline{a_n})$ in the second half of the proof above is well-defined (i.e., independent of auxiliary choices), and gives an explicit inverse to the projection $\lim_{x\mapsto x^p} R \to \lim_\phi R/p =: R^\flat$.

tTopology **Remark 2.0.8.** (Exercise) In Lemma 2.0.6, if one topologizes R with the p-adic topology and R/p with the discrete topology, then the bijection of Lemma 2.0.6 is a homeomorphism. Indeed, the map is clearly continuous. For continuity in the other direction, note that we have a homeomorphism

$$\lim_{x\mapsto x^p} R \simeq \lim_{x\mapsto x^p} \lim_n R/p^n \simeq \lim \left(\cdots \xrightarrow{x\mapsto x^p} R/p^3 \xrightarrow{x\mapsto x^p} R/p^2 \xrightarrow{x\mapsto x^p} R/p \right).$$

Thus, for each $k \geq 0$, a basic open subgroup $U_k \subset \lim_{x\mapsto x^p} R$ is given by those (a_n) with $a_i \in (p^i)$ for $i \leq k$. Now $a_{n+1}^p = a_n$ for all n, so U_k is exactly those (a_n) with $a_i \in (p^k)$. It suffices to show that the image of U_k in $\lim_\phi R/p$ contains those $(\overline{b_n})$ with $\overline{b_i} = 0$ for all $i \leq 2k$; this follows from the explicit inverse constructed in the proof above.

:SharpMap **Remark 2.0.9** (Sharp map)**.** In Lemma 2.0.6, via projection to the last term, we get multiplicative map

$$\sharp : R^\flat \to R$$

denoted $f \mapsto f^\sharp$. Its image is exactly those $f \in R$ that admit a compatible system $\{f^{\frac{1}{p^k}}\}$ of p-power roots. We shall sometimes call such elements *perfect*.

Using the $\sharp$ map, we can understand valuation rings under tilting; this will be useful when discussing adic spaces later.

ationRing **Lemma 2.0.10.** *If a p-adically complete ring R is a domain (resp. a valuation ring), the same is true for its tilt $R^\flat$. In fact, if $|\cdot| : R \to \Gamma \cup \{0\}$ is the valuation on R, then the map $R^\flat \xrightarrow{\sharp} R \xrightarrow{|\cdot|} \Gamma \cup \{0\}$ gives the valuation on $R^\flat$. In particular, the rank of $R^\flat$ is bounded above*[1] *by the rank of R.*

Proof. We first check that $R^\flat$ is a domain whenever R is a domain. We have $R^\flat \simeq \lim_{x\mapsto x^p} R$ as a multiplicative monoid. Fix elements $(a_n), (b_n) \in \lim_{x\mapsto x^p} R$ with $a_n \cdot b_n = 0$ for all n. Then either a_0 or b_0 vanishes as R is a domain. By symmetry, assume $a_0 = 0$. Then, as the transition maps involve raising to powers, we get $a_n = 0$ for all n, and thus $(a_n) = 0$, so $R^\flat$ has no zero divisors.

Now assume R is a valuation ring. Fix $a, b \in R^\flat$, corresponding to $(a_n), (b_n) \in \lim_{x\mapsto x^p} R$. As R is a valuation ring, we have $a_0 \mid b_0$, or vice versa. Assume $a_0 \mid b_0$ by symmetry. Then, for valuation reasons, we must have $a_n \mid b_n$ for each $n \geq 1$: the element $\frac{a_n}{b_n}$ in the fraction field of R must lie in R (as its p^n-th power does). Thus, $(a_n) \mid (b_n)$ in $\lim_{x\mapsto x^p} R$, and thus $a \mid b$ in $R^\flat$, proving that $R^\flat$ is a domain.

Explicitly, this construction shows that if $|\cdot| : R \to \Gamma \cup \{0\}$ is the valuation, then $|\cdot|^\flat := |\cdot| \circ \sharp : R^\flat \to \Gamma \cup \{0\}$ gives a valuation on $R^\flat$: indeed, given $a = (a_n), b = (b_n) \in R^\flat$, we checked above that $a \mid b$ exactly when $a_0 \mid b_0$, which happens exactly when $|a|^\flat \geq |b|^\flat$ as $|a|^\flat = |a_0|$ (and similarly for b). The assertion about ranks is automatic. $\square$

[1] In this generality, the rank can indeed go down under tilting. For example, if $R = \mathbf{Z}_p$, then $R^\flat = \mathbf{F}_p$. We shall check later that this does not happen for perfectoids.

Chapter 3

Perfectoid fields

In this chapter, we introduce and study perfectoid fields. These are NA fields that contain "lots of" p-power roots. The main result is that the tilt of (the ring of integers of) a perfectoid field K is a perfectoid field $K^\flat$ of characteristic p that reflects the algebraic properties of K. In particular, we formulate (and prove the key special case of) the almost purity theorem for perfectoid fields, equating the Galois theory of K and $K^\flat$.

3.1 Definition and basic properties

Fix a prime number p.

Definition 3.1.1. A perfectoid field K is a NA field with residue characteristic p such that:

- The value group $|K^*| \subset \mathbf{R}_{>0}$ is not discrete.
- K°/p is semiperfect, i.e., the Frobenius map $K^\circ/p \to K^\circ/p$ is surjective.

toidField **Example 3.1.2.** The first condition rules out fields like $\mathbf{Q}_p$ itself. Interesting examples are:

1. Let $K = \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$. Then the value group is $\mathbf{Z}[\frac{1}{p}]$. To calculate the valuation ring, observe that both completions and filtered colimits of valuation rings are valuation rings. It follows that $K^\circ = \widehat{\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]}$: the natural map from the right to the left is an extension of rank 1 valuation rings with the same field of fractions, and must thus be an isomorphism. It is then easy to see that K is perfectoid. A similar analysis applies to $\widehat{\mathbf{Q}_p(\mu_{p^\infty})}$.

2. Let $K = \mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}$. Then the value group $\mathbf{Q}$. As K is algebraically closed, every $t \in K^\circ$ admits a p-th root, so the perfectoidness is clear. Alternately, one may argue directly without using the algebraic closedness of K by observing that $\overline{\mathbf{Z}_p}/p$ is semiperfect (as $\overline{\mathbf{Q}_p}$ is algebraically closed), and that $\overline{\mathbf{Z}_p} \to K^\circ$ is an isomorphism modulo any power of p.

3. Let K be a NA field of characteristic p. Then K is perfectoid if and only if K is perfect. In this case, semiperfectness of K° implies its perfectness, and hence the nondiscreteness of the value group (as long as the valuation is not trivial).

As these examples illustrate, the perfectoid world is very non-noetherian:

Lemma 3.1.3. *Let K be a perfectoid field.*

1. *The value group $|K^*|$ is p-divisible.*
2. *We have $(K^{\circ\circ})^2 = K^{\circ\circ}$. Moreover, $K^{\circ\circ}$ is flat.*
3. *The ring K° is not noetherian.*

Proof. For (1). We temporarily call $x \in K^\circ$ small if $|p| < |x| \leq 1$. We first check the p-divisibility of $|x| \in |K^*|$ for small x. The perfectoidness of K gives a $y, z \in K^\circ$ such that $y^p = x + p \cdot z$ for some $z \in K^\circ$. Taking absolute values and using the NA property shows that $|y|^p = |y^p| = |x|$, so $|x| \in |K^*|$ is divisible by p.

In general, as $|K^*|$ is not discrete, the containment $|p|^{\mathbf{Z}} \subset |K^*|$ must be strict, so we can choose an $x \in K^*$ with $|x| \notin |p|^{\mathbf{Z}}$. After rescaling by a suitable power of p, we can assume x is small. By the total ordering of ideals in K°, we must have $p = xy$ for small y. But then $|p| = |x| \cdot |y|$, so $|p| \in |K^*|$ is divisible by p. A similar argument shows $|p|^{\mathbf{Z}}$ and $|x|$ for x small generate$|K^*|$, so we are done.

For (2). Pick some $f \in K^{\circ\circ}$. By perfectoidness, we can write $f = g^p + ph$ for $g \in K^{\circ\circ}$ and $h \in K^\circ$. The previous proof shows that $p \in (K^{\circ\circ})^2$, so this formula proves that $f \in (K^{\circ\circ})^2$. For flatness, we simply note that any torsionfree module over a valuation ring is flat.

(2) implies (3) by Nakayama's lemma. Alternately, $K^{\circ\circ}$ is not finitely generated as it has elements of arbitrarily small valuation. □

erfectoid **Remark 3.1.4.** The proof above shows that $|K^*| \subset \mathbf{R}_{>0}$ is generated by $|x|$ for $x \in K^\circ$ with $|p| < |x| < 1$. This observation will be useful later in analyzing the value group under tilting.

We shall see later that differential forms tend to vanish in the perfectoid world. An elementary instance of this is:

Exercise 3.1.5. Let K be a perfectoid field. Show that $\Omega^1_{K^\circ/\mathbf{Z}_p}$ is never 0, and yet its p-adic completion always vanishes.

3.2 Tilting

Fix a perfectoid field K. Our goal is to attach to K a perfectoid field $K^\flat$ of characteristic p. We shall do so by first constructing the ring of integers of $K^\flat$ as $K^{\circ\flat}$, and then constructing $K^\flat$ as a suitable localization. For the rest of the section, we fix[1] a pseudouniformizer $\pi \in K^\circ$ with $|p| \leq |\pi| < 1$, so $p \in (\pi)$. We obtain a commutative diagram

$$\begin{array}{ccc}
\lim_{x\mapsto x^p} K^\circ & \xrightarrow{pr_0} & K^\circ \\
\downarrow \simeq & \overset{\sharp}{\nearrow} & \downarrow \\
K^{\circ\flat} := \lim_{x\mapsto x^p} K^\circ/p & \xrightarrow{pr_0} & K^\circ/p \\
\downarrow \simeq & & \downarrow \\
\lim_{x\mapsto x^p} K^\circ/\pi & \xrightarrow{pr_0} & K^\circ/\pi,
\end{array} \tag{3.1}$$

eq:TiltPe

where all the vertical maps are the canonical reduction maps, the top half comes from Lemma 2.0.6 when K has characteristic 0 (and is trivial in characteristic p), the bottom half comes from Exercise 2.0.4 when K has characteristic 0 (and is trivial in characteristic p), and the bottom two horizontal maps are surjective by the semiperfectness of K°/p. We can then topologize $K^{\circ\flat}$ as follows:

Exercise 3.2.1. Check that the following 3 topologies on $K^{\circ\flat}$ are equivalent:

- The inverse limit topology arising via $K^{\circ\flat} \simeq \lim_\phi K^\circ/\pi$.
- The inverse limit topology arising via $K^{\circ\flat} \simeq \lim_{x\mapsto x^p} K^\circ$.
- The inverse limit topology arising via $K^{\circ\flat} \simeq \lim_\phi K^\circ/p$, where the topology on K°/p is the one induced from K (and is thus discrete when K has characteristic 0, but not so in characteristic p).

Recall that we want to show that $K^{\circ\flat}$ is the valuation ring of a perfectoid field $K^\flat$. To this end, we first find a pseudouniformizer:

em:TiltPU

Lemma 3.2.2. *There exists some element $t \in K^{\circ\flat}$ such that $|t^\sharp| = |\pi|$. Moreover, t maps to 0 in K°/π, and this gives an isomorphism $K^{\circ\flat}/t \simeq K^\circ/\pi$.*

Proof. As $p \in (\pi)$, the canonical projections give surjective maps $K^{\circ\flat} \to K^\circ/p \to K^\circ/\pi$. By p-divisibility of the value group, we may choose some $f \in K^\circ$ such that $|f|^p = |\pi|$, and hence $|f| > |\pi|$. In particular, $f \in K^\circ/\pi$ is nonzero. Choose some $g \in K^{\circ\flat}$ lifting f under the canonical map $K^{\circ\flat} \to K^\circ/\pi$. Then $g^\sharp = f \mod \pi$ by the diagram (3.1). By the NA property, this gives $|g^\sharp| = |f|$ since $|f| > |\pi|$. Setting $t = g^p$ and using the multiplicativity of $\sharp$ shows that $|t^\sharp| = |f|^p = |\pi|$.

For the second part, note that $t \in K^{\circ\flat}$ maps to $f^p = \pi = 0$ in K°/π, thus giving a map $K^{\circ\flat}/t \to K^\circ/\pi$. To show this map is an isomorphism, consider the diagram (3.1) again. Choose some $g \in K^{\circ\flat}$ such that g maps to in K°/π. We must show that $g \in (t)$. The diagram shows that $g^\sharp \in K^\circ$ maps to 0 in K°/π, and hence $g^\sharp \in (\pi)$. But $|\pi| = |t^\sharp|$, so $(\pi) = (t^\sharp)$. Hence, we can write

[1] For practical purposes, we may take $\pi = p$ in characteristic 0, and any pseudouniformizer in characteristic p.

$g^\sharp = at^\sharp$ for suitable $a \in K^\circ$. It is then easy to see that a lifts to an element $\tilde{a} = (a_n) \in \lim_{x \mapsto x^p} K^\circ$ along the projection pr_0: simply set

$$a_n = \frac{(g^{\frac{1}{p^n}})^\sharp}{(t^{\frac{1}{p^n}})^\sharp} \in K,$$

and then observe that $a_n^{p^n} = a \in K^\circ$, so $a_n \in K^\circ$ for all n. By construction, we have $g = \tilde{a} \cdot t$ as elements in $\lim_{x \mapsto x^p} K^\circ$. Going down the left vertical arrow, we learn that $g \in (t)$, as wanted. $\square$

Using elements constructed above, the topology on $K^{\circ\flat}$ is seen to be algebraic:

ITopology **Corollary 3.2.3.** *With t as above, $K^{\circ\flat}$ is t-adically complete, and that the t-adic topology coincides with the given topology.*

Proof. Using Lemma 3.2.2, we have a map of inverse systems of rings

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K^{\circ\flat}/(t^{p^n}) & \xrightarrow{std} & \cdots & \longrightarrow & K^\circ/(t^p) & \xrightarrow{std} & K^{\circ\flat}/(t) \\ & & \Big\downarrow{\scriptstyle std\circ\phi^{-n}} & & & & \Big\downarrow{\scriptstyle std\circ\phi^{-1}} & & \Big\downarrow{\scriptstyle std} \\ \cdots & \longrightarrow & K^\circ/\pi & \xrightarrow{\phi} & \cdots & \longrightarrow & K^\circ/\pi & \xrightarrow{\phi} & K^\circ/\pi \end{array}$$

with all vertical maps being isomorphisms. Comparing inverse limits as topological rings proves the claim. $\square$

toidField **Proposition 3.2.4.** *Fix an element t as in Lemma 3.2.2.*

1. *The ring $K^{\circ\flat}$ is a valuation ring, and the ring[2] $K^\flat := K^{\circ\flat}[\frac{1}{t}]$ is a (necessarily perfect) field.*

2. *The ideal $(t^{\frac{1}{p^\infty}})$ is maximal, and the Krull dimension of $K^{\circ\flat}$ is 1.*

3. *The valuation topology on $K^\flat$ coming from (1) coincides with the one induced by the t-adic topology on $K^{\circ\flat}$. In this topology, $K^\flat$ is a perfectoid field, and $K^{\flat,\circ} = K^{\circ\flat}$.*

4. *The value groups and residue fields of K and $K^\flat$ are canonically identified.*

We shall repeatedly use the following: using the presentation $K^{\circ\flat} \simeq \lim_{x \mapsto x^p} K^\circ$ as multiplicative monoids, we learn that an element shows that $a \in K^{\circ\flat}$ is a unit if and only if $a^\sharp$ is a unit.

Proof. 1. Specializing Lemma 2.0.10 to $R = K^\circ$, we learn that $K^{\circ\flat}$ is a valuation ring of rank ≤ 1. In fact, the rank is exactly 1 since we know that $|t|^\flat := |t^\sharp| = |\pi|$ is non-trivial. In particular, inverting any nonzero nonunit in $K^{\circ\flat}$, such as t, produces the fraction field $K^\flat$.

[2] There is abuse of notation here: when K has characteristic 0, the ring $K^\flat$ is *not* the tilt of K in the sense of Definition 2.0.1.

2. The assertion about Krull dimensions follows from (1) as the rank of a valuation ring equals its Krull dimension. For the rest, we already know that $K^{\circ\flat}/t \simeq K^{\circ}/\pi$ by Lemma 3.2.2. As the maximal ideal of K°/π is its nilradical (since K° is a rank 1 valuation ring), the same must be true for $K^{\circ\flat}/t$. But the nilradical of $K^{\circ\flat}/t$ is just the image of $(t^{\frac{1}{p^\infty}})$ (as the latter clearly lies in the nilradical, and the quotient $K^{\circ\flat}/(t^{\frac{1}{p^\infty}})$ is perfect, whence reduced), so $(t^{\frac{1}{p^\infty}})$ must indeed be maximal.

3. As $K^{\circ\flat}$ is a rank 1 valuation ring, the valuation topology coincides with the f-adic topology for any nonzero nonunit f. Taking $f = t$ gives the first part of the claim. Corollary 3.2.3 shows that $K^{\circ\flat}$ is t-adically complete, and thus $K^{\flat}$ is a NA field. Finally, it is clear that $K^{\flat}$ is perfect, and thus $K^{\flat}$ is perfectoid.

4. The claim about residue fields follows from (2) using the identification $K^{\circ}/\pi \simeq K^{\circ\flat}/t$ from Lemma 3.2.2. For value groups, using the notation above, we trivially have $|K^{\flat}|^{\flat} \subset |K^*|$. To show equality, note that $|K^*|$ is generated by $|x|$ for $x \in K^{\circ}$ with $|p| < |x| < 1$ (see Remark 3.1.4). We must show $|x| \in |K^{\flat}|^{\flat}$ for any such x. But this is immediate from Lemma 3.2.2.

□

ltContVal **Remark 3.2.5** (Tilting Continuous Valuations)**.** Proposition 3.2.4 (1) is a special case of the following:

Proposition 3.2.6. *For any continuous valuation $|\cdot| : K^* \to \Gamma$ (of any rank), the function $|\cdot|^{\flat} = |\cdot| \circ \sharp : K^{\flat,*} \to \Gamma$ is also a continuous valuation. This construction identifies the space of continuous valuations on either field.*

Proof. Fix a continuous valuation $|\cdot|$ on K. It is clear that $|\cdot|^{\flat}$ is multiplicative. Moreover, as $\sharp : K^{\flat} \to K$ has trivial fiber over 0, it is clear that $|f|^{\flat} = 0$ if and only if $f = 0$. To check the NA property, fix $f := (f_n), g := (g_n) \in \lim_{x \mapsto x^p} K \simeq K^{\flat}$, so $f^{\sharp} = f_0$ and $g^{\sharp} = g_0$. We must check that

$$|f+g|^{\flat} \leq \max(|f|^{\flat}, |g|^{\flat}).$$

But this follows from

$$|f+g|^{\flat} := |(f+g)^{\sharp}| = |\lim_n (f_n+g_n)^{p^n}| = \lim_n |f_n+g_n|^{p^n} \leq \lim_n \max(|f_n|, |g_n|)^{p^n} = \lim_n \max(|f_0|, |g_0|) = \max(|f$$

where the second equality is obtained by chasing the behaviour of addition across the isomorphism $\lim_{x \mapsto x^p} R \simeq \lim_{x \mapsto x^p} R/p$ for a p-adically complete ring R, and the third equality uses the continuity of $|\cdot|$ on K.

For the second part, write $|\cdot|_{std}$ for the given NA valuation on K. Observe that a valuation $|\cdot| : K^* \to \Gamma$ is continuous if and only if for one (or, equivalently, any) pseudouniformizer $f \in K^{\circ\circ}$, we have $|f|^n \to 0$ as $n \to \infty$. Using this remark, one checks the following about the valuation ring $R \subset K$ attached to $|\cdot|$:

1. R contains $K^{\circ\circ}$ inside its maximal ideal as $|f| < 1$ for any $f \in K^{\circ\circ}$.

2. We have $R \subset K^\circ$. Indeed, if not, then R has an element from $K[\frac{1}{f}] \setminus K^\circ$, i.e., an element of the form a/f^n with $a \in K^\circ$ and $|a|_{std} > |f^n|_{std}$. But then $|f^n/a|_{std} < 1$, so $f^n/a \in K^{\circ\circ}$, so f^n/a lies in the maximal ideal of R. As $a/f^n \in R$ as well, we find an element of the maximal ideal of R that is invertible, which is absurd.

Conversely, one may check any valuation subring $R \subset K$ that satisfies (1) and (2) defines a continuous valuation on K: the key point is that the map $R \to K^\circ$ is a localization of R (at its unique height 1 prime, by the classification of rings between a valuation ring and its fraction field), so $K^{\circ\circ}$ must lie in all primes of R, and thus $|t|^n \to 0$ for $t \in K^{\circ\circ}$. Passing to the quotient, we learn that continuous valuations on K identify bijectively with valuation rings in $K^\circ/K^{\circ\circ}$. Repeating the same argument for $K^\flat$, we conclude using the identification of $K^\circ/K^{\circ\circ}$ with $K^{\flat,\circ}/K^{\flat,\circ\circ}$. □

We give an explicit example of a rank 2 valuation on a perfectoid field.

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Example 3.2.7. Let k be a perfect field of characteristic p, and let $K^\circ = \widehat{W(k)[p^{\frac{1}{p^\infty}}]}$. Then $K = K^\circ[\frac{1}{p}]$ is a perfectoid field with $K^\circ/K^{\circ\circ} \simeq k$. In particular, given any valuation ring $\overline{R} \subset k$, the preimage $R \subset K^\circ$ of $\overline{R}$ is a valuation ring of K whose attached valuation $|\cdot| : K^* \to \Gamma$ is continuous. For an explicit example, set $k = \mathbf{F}_p((t))_{perf}$. Consider its valuation ring $\overline{R} = \mathbf{F}_p[\![t]\!]_{perf} \subset k$. Then the preimage $R \subset K^\circ$ of $\overline{R}$ defines a rank 2 valuation on R with value group $\Gamma := |K^*| \times |t|^{\mathbf{Z}[\frac{1}{p}]} \simeq \mathbf{Z}[\frac{1}{p}] \times \mathbf{Z}[\frac{1}{p}]$ ordered lexicographically.

The main theorem about perfectoid fields is:

Theorem 3.2.8 (Almost purity in dimension 0)**.** *Let L/K be a finite (necessarily separable) extension. Endow L with its natural topology as a finite dimensional K-vector space. Then*

1. *L is perfectoid.*
2. *The field extension $L^\flat/K^\flat$ is finite of the same degree as L/K.*
3. *The association $L \mapsto L^\flat$ defines an equivalence $K_{fet} \simeq K^\flat_{fet}$.*

Example 3.2.9. Let $K = \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$. We explained in Example 3.1.2 that $K^\circ = \widehat{\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]}$. A similar argument shows that if $L = K(\sqrt{p})$, then $L^\circ = \widehat{\mathbf{Z}_p[p^{\frac{1}{2p^\infty}}]}$. It is then easy to calculate that if we fix the isomorphism $K^\flat \simeq \widehat{\mathbf{F}_p((t))_{perf}}$ by requesting $t^\sharp = p$, then $L^\flat = K^\flat(\sqrt{t})$.

We shall prove the full result later, once the language of almost mathematics has been introduced. In fact, the latter will allow us to explicitly construct an inverse to the operation $L \mapsto L^\flat$. Granting the existence of this construction, the result will follow from the following consequence, which we prove directly:

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Proposition 3.2.10. *Assume that $K^\flat$ is algebraically closed. Then K is algebraically closed.*

The proof below is due to Kedlaya.

Proof. We assume that K has characteristic 0. Set $x_0 = 0$. We shall inductively construct a sequence $\{x_n \in K^\circ\}$ such that the following hold for each n:

1. $|P(x_n)| \leq |p|^n$.

2. $|x_{n+1} - x_n| \leq |p|^{\frac{n}{d}}$.

Then (2) shows that $\{x_n\}$ converges to some $x \in K^\circ$, and (1) shows that $|P(x)| = 0$, and thus $P(x) = 0$.

As $x_0 = 0$ is already defined, assume by induction we have constructed $x_0, x_1, ..., x_n$ satisfying the above two properties. Write

$$P(T + x_n) = \sum_{i=0}^{d} b_i T^i,$$

so $b_d = 1$. If $b_0 = 0$, then $P(x_n) = 0$, so we may simply take $x_i = x_n$ for each $i \geq n$. Assume from now on that $b_0 \neq 0$. Consider the quantity

$$c = \min\{|\frac{b_0}{b_j}|^{\frac{1}{j}} \mid j > 0, b_j \neq 0\}.$$

Considering $j = d$ shows that $c \leq |b_0|^{\frac{1}{d}} \leq 1$. By Proposition 3.2.4 (4) and the algebraic closedness of $K^\flat$, we know that $|K^*|$ is a $\mathbf{Q}$-vector space, so $c = |u|$ for some $u \in K$; in fact, we have $u \in K^\circ$ as $|u| = c \leq 1$. We have $\frac{b_i}{b_0} \cdot u^i \in K^\circ$ by construction. Moreover, as the minimum defining c is achieved, there exists $i > 0$ such that $\frac{b_i}{b_0} \cdot u^i$ is a unit.

Using Lemma 3.2.2, choose $t \in K^{\circ\flat}$ with $|t^\sharp| = |p|$. Consider any polynomial $Q(T) \in K^{\circ\flat}[T]$ lifting $\sum_{i=0}^{d} \frac{b_i}{b_0} u^i T^i \in K^\circ/p[T]$ under the identification $K^{\circ\flat}/t \simeq K^\circ/p$. By construction, this is a polynomial of degree > 0 whose constant coefficient and (at least) one non-constant coefficient is a unit. Lemma 3.2.11 shows that there exists some unit $y \in K^{\circ\flat,*}$ such that $Q(y) = 0$.

We shall check that $x_{n+1} = x_n + u \cdot y^\sharp \in K^\circ$ satisfies the analogs of (1) and (2). First, note that

$$P(x_{n+1}) = P(u \cdot y^\sharp + x_n) = \sum_{i=0}^{d} b_i u^i \cdot (y^i)^\sharp = b_0 \cdot \Big(\sum_{i=0}^{d} \frac{b_i}{b_0} u^i \cdot (y^\sharp)^i \Big).$$

Now the parenthesized term is congruent to $Q(y)$ modulo p, and thus 0 modulo p as y is a root of Q. It follows that

$$|P(x_{n+1})| \leq |b_0| \cdot |p| \leq |p|^n \cdot |p| = |p|^{n+1},$$

where we use induction to get $|b_0| = |P(x_n)| \leq |p|^n$. This gives (1), and for (2) we observe that

$$|x_{n+1} - x_n| = |u||y^\sharp| = |u| = c \leq |b_0|^{\frac{1}{d}} = |P(x_n)|^{\frac{1}{d}} \leq |p|^{\frac{n}{d}},$$

where we use that y is a unit in the second equality, and the inductive hypothesis in the last one. □

The following lemma was used above.

outNewton **Lemma 3.2.11.** *Let V be the valuation ring of a complete and algebraically closed NA field. Let $P(T) \in V[T]$ be a polynomial of degree ≥ 1 such that the constant coefficient and (at least) one non-constant coefficient are units in V. Then P vanishes at a unit of V.*

This lemma can be proven using Newton polygons, but we give a direct algebraic proof.

Proof. Write $\mathfrak{m} \subset V$ for the maximal ideal with residue field $k = V/\mathfrak{m}$. Our hypothesis ensures that reduction of $P(T)$ modulo $\mathfrak{m}$ is a non-constant polynomial with unit constant and leading coefficients. We may then choose a pseudouniformizer $t \in V$ such that the reduction modulo t of $P(T)$ is also a non-constant polynomial with unit constant and leading coefficients. We view P as a map $f_P : V[T] \to V[T]$. This map realizes the target $V[T]$ as a torsionfree module over the source $V[T]$ that is finite free when reduced modulo t by our choice of t. Writing $V\langle T\rangle$ for the t-adic completion of $V[T]$, it follows that the t-adic completion $\widehat{f_P} : V\langle T\rangle \to V\langle T\rangle$ is a finite free morphism as well. In particular, the ring $A := V\langle T\rangle/P(T)$ is a finite free V-algebra. As V is henselian, we can decompose $A \simeq \prod_i A_i$ as a finite product with each A_i being a finite free local V-algebra. Reducing modulo $\mathfrak{m}$, this gives $k[T]/P(T) \simeq \prod_i A_i/\mathfrak{m}$ with each factor being local. Now the assumption on $P(T)$ ensures that one of the roots of $P(T)$ over k is a unit. It follows that T maps to a unit in one of the residue fields of $k[T]/P(T)$. As each A_i is local, it follows that T maps to a unit in some A_i. Fix one such index i. As A_i is finite free over V and K is algebraically closed, the ring $A_{i,red}[\frac{1}{t}]$ decomposes as a non-empty product of copies of K. Picking one such copy gives a map $A_i \to K$. This map has image contained inside $V \subset K$ as A_i is integral over V. Thus, we obtain a map $A_i \to V$. As T mapped to a unit in A_i, the same must be true for its image in V. Putting everything together, we have produced a map $V[T]/P(T) \to V$ that carries T to a unit, as wanted. $\square$

yclotomic **Example 3.2.12.** Let $K = \widehat{\mathbf{Q}_p(\mu_{p^\infty})}$, so $K^\circ = \widehat{\mathbf{Z}_p[\mu_{p^\infty}]}$. We have an explicit presentation of K° as the p-adic completion of

$$\mathbf{Z}_p[\epsilon^{\frac{1}{p^\infty}}]/(\frac{\epsilon - 1}{\epsilon^{\frac{1}{p}} - 1}) = \mathbf{Z}_p[\epsilon^{\frac{1}{p^\infty}}]/(1 + \epsilon^{\frac{1}{p}} + \epsilon^{\frac{2}{p}} + ... + \epsilon^{\frac{p-1}{p}}),$$

given by choosing a compatible system $\epsilon_n \in \mu_{p^n}$ of p-power roots of 1, and sending $\epsilon^{\frac{1}{p^n}}$ to ϵ_n. Reducing modulo p, and using that $\frac{x^p - 1}{x-1} = (x-1)^{p-1}$ in characteristic p, we learn that

$$K^\circ/p \simeq \mathbf{F}_p[\epsilon^{\frac{1}{p^\infty}}]/(\epsilon^{\frac{1}{p}} - 1)^{p-1} \simeq \mathbf{F_p}[t^{\frac{1}{p^\infty}}]/(t^{p-1}),$$

where we use the substitution $t \mapsto \epsilon - 1$ (and similarly for p-power roots). By Exercise 2.0.4, we learn that $K^{\circ\flat} = (K^\circ/p)^{perf}$ identifies with the t-adic completion of $\mathbf{F}_p[t^{\frac{1}{p^\infty}}]$, and hence $K^\flat \simeq \widehat{\mathbf{F}_p((t))_{perf}}$.

Remark 3.2.13. Examples 3.2.12 and 2.0.3 show that $K = \widehat{\mathbf{Q}_p(\mu_{p^\infty})}$ and $L = \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$ have isomorphic tilts, i.e., $K^\flat \simeq L^\flat$. In particular, the tilting functor $K \mapsto K^\flat$ is not fully faithful on perfectoid fields $K/\mathbf{Q}_p$. We shall see later that this is a consequence of working over the non-perfectoid base $\mathbf{Q}_p$: the functor $R \mapsto R^\flat$ will be fully faithful on perfectoid fields (in fact, algebras) over a perfectoid base field K.

Chapter 4

Almost mathematics

In this chapter, we introduce Faltings' theory of almost mathematics. This theory is essentially a softening of commutative algebra that is possible when one works over a R equipped with an ideal I such that $I = I^2$; the basic idea is to redevelop the basic notions of commutative algebra whilst systematically ignoring I-torsion modules. This idea was inspired by work of Tate (who observed that ignoring $K^{\circ\circ}$-torsion modules when working over the perfectoid field $K = \widehat{\mathbf{Q}_p(\mu_{p^\infty})}$ was a sensible and useful idea), and is crucial to a finer study of Fontaine's tilting functor. We follow the treatment of Gabber-Ramero [GR].

4.1 Constructing the category of almost modules

Let R be a ring equipped with an ideal I. In this situation, we have the following standard pair of adjoints:

Construction 4.1.1. Restriction of scalars along $R \to R/I$ gives a fully faithful functor

$$i_* : \mathrm{Mod}_{R/I} \to \mathrm{Mod}_R.$$

This functor has a left adjoint i^* given by

$$i^*(M) = M \otimes_R R/I,$$

and a right adjoint $i^!$ given by

$$i^!(M) = \mathrm{Hom}_R(R/I, M) = M[I].$$

We now specialize to the case of interest:

mostSetup **Assumption 4.1.2** (The setup of almost mathematics)**.** Assume $I \subset R$ is a flat ideal and satisfies $I^2 = I$. This implies $I \otimes_R I \simeq I^2 \simeq I$.

The preceding assumption will be in place for the rest of this section.

mostSetup **Example 4.1.3.** Two classes of examples that will be relevant to us are:

- Let K be a perfectoid field. Set $R = K^\circ$ for a perfectoid field K and set $I = K^{\circ\circ}$ to be the maximal ideal. As torsionfree modules over valuation rings are flat, it is easy to see that I is flat. In fact, we can be more explicit: if $t \in K^{\flat,\circ}$ is a pseudo-uniformizer, then $a = t^\sharp$ has a compatible system of p-power roots, and $I = (a^{\frac{1}{p^\infty}})$. In particular, $I = \mathrm{colim}_n(a^{\frac{1}{p^n}})$ is a countable union of free K°-modules, so it is flat of projective dimension ≤ 1.

- Let R be a perfect ring of characteristic p, and let $I = (f^{\frac{1}{p^\infty}})$ for $f \in R$. It is easy to see that $I^2 = I$ in this case. Moreover, I is clearly flat if $f \in R$ is a nonzerodivisor. To verify flatness of I in general, set $M_i = R$ for $i \geq 0$, and consider the inductive system

$$M_0 \xrightarrow{f^{1-\frac{1}{p}}} M_1 \xrightarrow{f^{\frac{1}{p}-\frac{1}{p^2}}} M_2 \to \dots . \to M_n \xrightarrow{f^{\frac{1}{p^n}-\frac{1}{p^{n+1}}}} M_{n+1} \to \dots$$

  Write $M = \mathrm{colim}\, M_n$, so M is a flat. There is an obvious map $M \to I$ given by sending $1 \in M_n$ to $f^{\frac{1}{p^n}} \in I$. This map is surjective by construction, so it suffices to show it is also injective. If $\alpha \in M_n = R$ goes to 0 in I, then $\alpha f^{\frac{1}{p^n}} = 0$. But then $\alpha^{p^m} f = 0$ for all $m \geq n$. By perfectness of R, we learn that $\alpha f^{\frac{1}{p^m}} = 0$ for all $m \geq 0$. In particular, the transition map $M_n \to M_{n+1}$ kills α, so $\alpha = 0$ in M, proving injectivity of $M \to I$.

Exercise 4.1.4. Using the second example in Example 4.1.3, show the following: if R is a perfect ring of characteristic p and $I \subset R$ is the radical of a finitely generated ideal, then R/I has finite flat dimension over R.

In the situation above, one can construct an interesting localization Mod_R^a of the category Mod_R of R-modules. This will be the category of *almost R-modules*. It can defined directly as an abstract category. However, in order to have a tight relationship between this category and Mod_R, we will need the following construction.

Construction 4.1.5 (The almost category in disguise)**.** Let $\mathcal{A} \subset \mathrm{Mod}_R$ be the full subcategory spanned by all R-modules M such that the action map $I \otimes_R M \to M$ is an isomorphism; equivalently, as $I \otimes_R I \simeq I$ via the multiplication map, we can also describe $\mathcal{A}$ as the essential image of the idempotent functor $M \mapsto I \otimes_R M$ on Mod_R; this functor is exact by the flatness of I. Using flatness of I, one checks that $\mathcal{A}$ is abelian subcategory of Mod_R that is closed under taking kernels, cokernels, and images inside Mod_R. We will construct a series of auxiliary functors relating $\mathcal{A}$ with Mod_R, eventually allowing us to realize $\mathcal{A}$ as a *quotient* of Mod_R.

- Write $j_! : \mathcal{A} \to \mathrm{Mod}_R$ for the resulting exact inclusion.

- The inclusion $j_!$ has an exact right adjoint $j^* : \mathrm{Mod}_R \to \mathcal{A}$ given by the formula

$$j^*(M) = I \otimes_R M.$$

  The unit map $N \to j^* j_! N$ is an isomorphism for any $N \in \mathcal{A}$.

Proof. We first note that $I \otimes_R M \in \mathcal{A}$ as $I \otimes_R I \simeq I$, so we have a well-defined functor. The exactness is clear from the flatness of I. For adjointness, fix some $N \in \mathcal{A}$ and $M \in \mathrm{Mod}_R$. We must show that

$$\mathrm{Hom}_{\mathcal{A}}(N, I \otimes_R M) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(N, M).$$

Using the exact triangle

$$I \otimes_R M \to M \to M \otimes_R^L R/I,$$

it is enough to show that $\mathrm{RHom}_R(N, M \otimes_R^L R/I) \simeq 0$. By adjointness, we have

$$\mathrm{RHom}_R(N, M \otimes_R^L R/I) \simeq \mathrm{RHom}_{R/I}(N \otimes_R^L R/I, M \otimes_R^L R/I).$$

But our hypothesis on N tells us that $N \otimes_R I \simeq N$, and that this tensor product is derived (by flatness of I). By associativity of (derived) tensor products, it is enough to show that

$$I \otimes_R^L R/I \simeq 0.$$

This follows from the flatness of I and the hypothesis $I = I^2$.

Granting adjointness, the assertion about the unit map results from the isomorphism $I \simeq I \otimes_R I$. □

- The right adjoint j^* has a further right adjoint j_* given by the formula

$$j_*(M) = \mathrm{Hom}_R(I, M).$$

The counit map $j^* j_* M \to M$ is an isomorphism for any $M \in \mathcal{A}$.

Proof. Fix some $N \in \mathrm{Mod}_R$. Then

$$\mathrm{Hom}_{\mathcal{A}}(j^* N, M) = \mathrm{Hom}_R(I \otimes N, M) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_R(I, M)) = \mathrm{Hom}_R(N, j_*(M)),$$

which proves the adjointness; here we use that Hom-$\otimes$-adjunction the second isomorphism.

For the rest, fix some $M \in \mathcal{A}$. We must show that $I \otimes_R \mathrm{Hom}_R(I, M) \simeq M$ via the natural "evaluation" map. As tensoring with I is exact, it suffices to show the stronger statement that $I \otimes_R^L \mathrm{RHom}_R(I, M) \simeq M$. As $M \in \mathcal{A}$, we have $I \otimes_R^L M \simeq M$, so it is enough to check that the natural map $M \to \mathrm{RHom}_R(I, M)$ induces an isomorphism after tensoring with I. Using the exact triangle

$$\mathrm{RHom}_R(R/I, M) \to M \to \mathrm{RHom}_R(I, M),$$

this reduces to showing that tensoring with I kills the term on the left. But the term on the left admits the structure of an R/I-complex, so we can write

$$I \otimes_R^L \mathrm{RHom}_R(R/I, M) \simeq I \otimes_R^L R/I \otimes_R^L \mathrm{RHom}_R(R/I, M).$$

We now conclude using $I \otimes_R^L R/I \simeq 0$, as before. □

- The composition $i^* j_!$ is 0: we must show that if $M \simeq I \otimes_R M$, then $M \otimes_R R/I \simeq 0$. This follows by observing that $I \otimes_R R/I \simeq 0$.

- The composition $i^! j_*$ is 0: we must show that if $M \simeq I \otimes_R M$, then $\mathrm{Hom}_R(R/I, \mathrm{Hom}_R(I, M)) \simeq 0$. But the adjointness of $\otimes$ and Hom identifies this with $\mathrm{Hom}_R(R/I \otimes_R I, M)$, so we conclude using $R/I \otimes_R I \simeq 0$.

- The composition $j^* i_*$ is 0: we must show that $M \otimes_R I \simeq 0$ if M is I-torsion, but this follows from $M \otimes_R I \simeq M \otimes_R R/I \otimes_{R/I} I \simeq 0$, where the last equality uses $R/I \otimes_R I \simeq 0$.

- The kernel of j^* is exactly $\mathrm{Mod}_{R/I}$: given $M \in \mathrm{Mod}_R$ with $j^*(M) := I \otimes_R M \simeq 0$, we must check that M is I-torsion. Tensoring M with the standard exact sequence $0 \to I \to R \to R/I \to 0$ shows that $M \simeq M/IM$, so M is I-torsion.

The following remark explains why a more naive definition of $\mathcal{A}$ runs into trouble.

Warning 4.1.6. For general $M \in \mathrm{Mod}_R$, the action map $I \otimes_R M \to M$ has image inside $IM \subset M$, and thus if $M \in \mathcal{A}$, then $IM = M$. However, the converse need not be true. Consider $R = k[t]_{perf}$ for a perfect field k of characteristic p. Let $I = (t^{\frac{1}{p^\infty}})$, so $R/I \simeq k$ is the residue field at the origin. Let $M \subset R/t$ be the maximal ideal inside the local ring R/t; this is the image of I/t inside R/t, but does not coincide with I/t. As $I = I^2$, it is easy to see that $M = IM$. However, the action map $I \otimes_R M \to M$ need not be injective. The kernel of this map is $\mathrm{Tor}_1^R(R/I, M)$. To calculate this, we use the defining exact sequence

$$0 \to M \to R/t \to R/I \to 0.$$

Tensoring with R/I, and using that[1] $\mathrm{Tor}_i^R(R/I, R/I) = 0$ for $i > 0$, we learn that

$$\mathrm{Tor}_1^R(M, R/I) \simeq \mathrm{Tor}_1^R(R/t, R/I).$$

The second group is computed to be nonzero using the standard resolution $\left(R \xrightarrow{t} R\right)$ of R/t, so the claim follows.

We can now construct the promised category of almost R-modules.

Proposition 4.1.7. *In the above situation, we have:*

1. *The image of i_* is closed under extensions. In particular, i realizes $\mathrm{Mod}_{R/I}$ as an abelian Serre subcategory of Mod_R, so the quotient $\mathrm{Mod}_R^a := \mathrm{Mod}_R/\mathrm{Mod}_{R/I}$ exists by general nonsense (see [SP, Tag 02MS]).*

[1] This is a general fact about perfect rings. In our case, we may prove this as follows. Writing $I = \cup_n I_n$ with $I_n := (t^{\frac{1}{p^n}})$, we get $\mathrm{Tor}_i^R(R/I, R/I) \simeq \mathrm{colim}\, \mathrm{Tor}_i^R(R/I_n, R/I_n)$; here we use that Tor commutes with direct limits in either variable. Using the standard resolution for $\left(R \xrightarrow{t^{\frac{1}{p^n}}} R\right)$ of R/I_n, we see that the Tor's vanish for $i > 1$. For $i = 1$, we have a canonical identification $\mathrm{Tor}_1^R(R/I_n, R/I_n) \simeq I_n/I_n^2$. We now observe that the transition maps $R/I_n \to R/I_{n+1}$ induce the 0 map $I_n/I_n^2 \to I_{n+1}/I_{n+1}^2$ as $I_n \subset I_{n+1}^2$. It follows that $\mathrm{colim}_n\, \mathrm{Tor}_1^R(R/I_n, R/I_n) = 0$, as wanted.

2. *The quotient functor* $q : \mathrm{Mod}_R \to \mathrm{Mod}_R^a$ *admits fully faithful left and right adjoints. Thus,* q *commutes with all limits and colimits.*

3. *The image of* i *is a "tensor ideal" of* Mod_R*, so the quotient* Mod_R^a *inherits a symmetric monoidal* $\otimes$*-product from* Mod_R*.*

4. *The* $\otimes$*-structure on* Mod_R^a *is closed, i.e., to* $X, Y \in \mathrm{Mod}_R^a$*, one can functorially an object* $\mathrm{alHom}(X, Y) \in \mathrm{Mod}_R^a$ *equipped with a functorial isomorphism*

$$\mathrm{Hom}(Z \otimes X, Y) \simeq \mathrm{Hom}(Z, \mathrm{alHom}(X, Y)).$$

Proof. 1. If an R-module M can be realized as an extension of two R-modules killed by I, then M is itself killed by I^2. But $I = I^2$, so M is also killed by I. Thus, the image of i is closed under extensions. The rest is by category theory, but we shall construct an explicit candidate for Mod_R^a in the proof of (2).

2. We claim that the functor $j^* : \mathrm{Mod}_R \to \mathcal{A}$ introduced above provides an explicit realization of the quotient functor $q : \mathrm{Mod}_R \to \mathrm{Mod}_R^a$. To check this, we must show that:

   - $j^*(\mathrm{Mod}_{R/I}) = 0$: this amounts to show that $I \otimes_R M = 0$ for an R-module M killed by I. But, for such M, we have $I \otimes_R M = I \otimes_R R/I \otimes_{R/I} M = I/I^2 \otimes_R M = 0$ as $I = I^2$.
   - j^* is exact: this follows from the description $j^*(M) = I \otimes_R M$ and the flatness of I.
   - j^* is universal with the previous two properties: let $q' : \mathrm{Mod}_R \to \mathcal{B}$ be an exact functor of abelian categories such that $q'(\mathrm{Mod}_{R/I}) = 0$. Fix some $M \in \mathrm{Mod}_R$. We then have the canonical action map $I \otimes_R M \to M$. The kernel and cokernel of this map are identified with $\mathrm{Tor}_i^R(R/I, I)$ for $i = 1, 0$. In particular, the kernel and cokernel are killed by I, and thus also by q'. As q' is exact, we learn that $q'(I \otimes_R M) \simeq q'(M)$. But $I \otimes_R M =: j_! j^*(M)$, so we have shown that $q' \simeq q' j_! j^*$, so q factors through j^*, as wanted.

3. If M is killed by I, so $M \otimes_R N$ for any R-module N. By category theory, this implies that the symmetric monoidal structure on Mod_R passes to the quotient $\mathcal{A}$. In particular, j^* is symmetric monoidal.

4. Given $M, N \in \mathrm{Mod}_R$, we simply set $\mathrm{alHom}(j^*(M), j^*(N)) = j^*\mathrm{Hom}_R(M, N)$; this is well-defined by (3) as $\mathrm{Hom}_R(M, N)$ is I-torsion if either M or N is so. Checking the rest is left to the reader.

□

Remark 4.1.8 (The topological analog)**.** The construction above can be summarized in the following diagram:

$$\mathrm{Mod}_{R/I} \underset{i^!}{\overset{i^*}{\leftleftarrows}}\xrightarrow{i_*} \mathrm{Mod}_R \underset{j_*}{\overset{j_!}{\leftleftarrows}}\xrightarrow{j^*} \mathrm{Mod}_R^a = \mathcal{A}.$$

Here we have the adjoint pairs (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*). Moreover, each composition in a straight line in the above diagram is the 0 functor: $i^* j_! \simeq 0$, $i^! j_* \simeq 0$, and $i_* j^* \simeq 0$.

The notation is meant to remind the reader of the analogous situation in topology: if X is a topological space with an open subset $j : U \hookrightarrow X$ with complement $i : Z \hookrightarrow X$, then we have a similar diagram of topoi:

$$\mathrm{Shv}(Z)\ \begin{matrix}\xleftarrow{i^*}\\ \xrightarrow{i_*}\\ \xleftarrow{i^!}\end{matrix}\ \mathrm{Shv}(X)\ \begin{matrix}\xleftarrow{j_!}\\ \xrightarrow{j^*}\\ \xleftarrow{j_*}\end{matrix}\ \mathrm{Shv}(U).$$

Moreover, the pairs of functors giving adjoint pairs and the pairs composing to 0 are the same as before. In fact, the analogy can be stretched a bit further. Recall that we may view Mod_R as quasi-coherent sheaves on $X := \mathrm{Spec}(R)$, and $\mathrm{Mod}_{R/I}$ as quasi-coherent sheaves on $Z := \mathrm{Spec}(R/I)$. However, the category Mod_R^a is *not* the category of quasi-coherent sheaves on $U := X - Z$ (or any other open subset of X). Instead, we think of Mod_R^a as quasi-coherent sheaves on some non-existent open $\overline{U} \subset X$ that contains U (as restriction to U factors through j^*).

Exercise 4.1.9. Let $R = k[t^{\frac{1}{p^\infty}}]$ for a perfect field k of characteristic p, and let $I = (t^{\frac{1}{p^\infty}})$. Show that the extension of scalars functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R[t^{-1}]}$ given by $M \mapsto M \otimes_R R[t^{-1}]$ factors through Mod_R^a, and that the resulting functor $\mathrm{Mod}_R^a \to \mathrm{Mod}_{R[t^{-1}]}$ is not an equivalence.

4.2 Almost commutative algebra

We continue in the setup of Assumption 4.1.2. For notational ease and compatibility with Gabber-Ramero, we rewrite some of the functors introduced in the previous section as:

Definition 4.2.1. Fix an R-module M. We say that an element $f \in M$ is *almost zero* if $I \cdot f = 0$. We say that M is *almost zero* if all its elements are almost zero, i.e., M is I-torsion. In general, write

$$M^a := j^* M \in \mathrm{Mod}^{\mathrm{a}}_{\mathrm{R}}, \quad M_* := j_* M^a := \mathrm{Hom}_R(I, M), \quad M_! := j_! M^a = I \otimes_R M,$$

We refer to elements of M_* as *almost elements* of M. A map $f : M \to N$ of R-modules is *almost surjective* (resp. *almost injective*, *almost isomorphism*) if f^a is surjective (resp. injective, isomorphism). We sometimes refer to an almost R-module as an R^a-module, and likewise for algebras.

With the notation above, we have canonical maps $M_! \to M \to M_*$, and they both become isomorphisms on *almostification*, i.e., after applying $(-)^a$. We record some useful examples of the notion of almost elements.

tElements **Example 4.2.2.** Let $R = K^\circ$ for a perfectoid field K and $I = K^{\circ\circ}$.

- If M is I-torsion, then $M_* = 0$.
- If M is a torsionfree R-module, then $M_* \simeq \{m \in M \otimes_{K^\circ} K \mid \epsilon \cdot m \in M \text{ for all } \epsilon \in I\}$.

- $R_* = R$ and $I_* = R$. More generally, given an ideal $J \subset R$, the ideal J_* is principal exactly when
$$c := \sup(|x| \mid x \in J) \leq 1$$
lies in $|K|$. This is clear if J is principal: we have $c = |x|$ for a generator $x \in J$, and $J_* = J$. If J is not principal, then $J = \{a \in K \mid |a| < c\}$ for $c = \sup(|x| \mid x \in J)$ by valuation yoga[2]. It then follows that $J_* = \{a \in K \mid |a| \leq c\}$, which is principal exactly when $c \in |K|$.

- If $t \in R$ is a pseudouniformizer, then $R/t \to (R/t)_*$ is injective, but need not be surjective: the obstruction lies in $\mathrm{Ext}^1_R(I, R)$, which may be nonzero.

- Let $K = \widehat{\mathbf{Q}_p[p^{\frac{1}{p^\infty}}]}$ and let $L := (W(\mathbf{F}_p(t)_{perf})[p^{\frac{1}{p^\infty}}])\widehat{\ }[\frac{1}{p}]$; these are both perfectoid fields, and $K \subset L$. Let $A \subset L^\circ$ be the rank 2 valuation ring in Example 3.2.7. Then A is a K°-algebra, and $A_* = L^\circ$: the cokernel of $A \to L$ is killed by $K^{\circ\circ}$ by construction.

The functor $(-)_!$ is exact, but the functor $(-)_*$ is only left exact. Its higher derived functors will appear (at least implicitly) in some arguments that follow, so we explain how to calculate them.

tElements **Exercise 4.2.3** (Derived functors of $(-)_*$)**.** The bifunctor $\mathrm{Hom}_{R^a}(M^a, N^a)$ can be derived in any variable to convert short exact sequences to long exact sequences (in the usual fashion). In fact, we get these derivatives by the formula
$$\mathrm{Ext}^i_{R^a}(M^a, N^a) := \mathrm{Ext}^i_R(I \otimes M, N) \simeq \mathrm{Ext}^i_R(M, \mathrm{RHom}_R(I, N)).$$
In particular, if $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R^a-modules, we have a long exact sequence
$$\ldots \to \mathrm{Ext}^i_{R^a}(R^a, M') \to \mathrm{Ext}^i_{R^a}(R^a, M) \to \mathrm{Ext}^i_{R^a}(R^a, M'') \to \mathrm{Ext}^{i+1}_{R^a}(R^a, M') \to \ldots,$$
deriving the functor of almost elements. An explicit nonzero example of a higher Ext-group is given in Remark 4.2.5.

We now extend some basic notions of commutative algebra to the almost world:

Definition 4.2.4. Let $M \in \mathrm{Mod}_R$ with image $M^a \in \mathrm{Mod}^a_R$. Then

1. We say that M or M^a is *almost flat* if $M^a \otimes (-)$ is exact on Mod^a_R; equivalently, $\mathrm{Tor}^R_{>0}(M, N)$ is almost zero for any R-module N.

2. We say that M or M^a is *almost projective* if $\mathrm{alHom}(M, -)$ is exact; equivalently, $\mathrm{Ext}^{>0}_R(M, N)$ is almost zero for any R-module N.

[2] The containment $\subset$ is clear as J is non-principal, and the reverse follows by noting that if $|a| < c$ with $a \notin J$, then $|a| > |x|$ for any $x \in J$ (as $|J| \subset |K|$ is closed under taking smaller elements, via scalar multiplication), whence $|a| \geq c$, contradicting the assumption $|a| < c$.

3. We say that M or M^a is *almost finitely generated* (resp. *almost finitely presented*) if for each $\epsilon \in I$, there exists a finitely generated (resp. finitely presented) R-module N_ϵ and a map $N_\epsilon \to N$ with kernel and cokernel killed by ϵ. If the number of generators of N_ϵ can be bounded independently of ϵ, we say that M is *uniformly almost finitely generated*.

The last definition given above depends *a priori* on the choice of the R-module M giving M^a on almostification. However, one checks readily that, in fact, the definition is independent of this choice.

rojective **Remark 4.2.5** (Categorical projectivity gives the incorrect notion)**.** The notion of almost projectivity is distinct from the categorical notion of projectivity in the abelian category Mod_R^a: the latter is far more restrictive. Indeed, the ring R is almost projective with the above definition. However, R^a need not be a projective object of Mod_R^a. In fact, using Exercise 4.2.3, one can show: with $R = K^\circ$ and $I = K^{\circ\circ}$ for a perfectoid field K with residue field k, the group $\mathrm{Ext}^1_{R^a}(R^a, R^a)$ identifies with $\mathrm{Ext}^2_R(k, R)$, and is nonzero if K is not spherically complete (such as $K = \widehat{\overline{\mathbf{Q}}_p}$).

Injective **Remark 4.2.6** (Injectives behave well)**.** The category Mod_R^a has enough injectives. In fact, if I is an injective R-module, then I^a is injective (as $(-)^a$ has an exact left-adjoint $(-)_!$). Conversely, if $J \in \mathrm{Mod}_R^a$ is injective,then J_* is an injective R-module (as $(-)_*$ has an exact left-adjoint $(-)^a$) such that $(J_*)^a \simeq J$. Thus, we may construct injective resolutions in Mod_R^a by simply computing them in Mod_R (under either $(-)_*$ or $(-)_!$) and applying almostification.

We give an example illustrating why the notion of almost finite generation is defined as above (instead of a stronger condition).

draticROI **Example 4.2.7** (A quadratic extension of a perfectoid field)**.** Let $K = \widehat{\mathbf{Q}_p[p^{\frac{1}{p^\infty}}]}$, and $L = K(\sqrt{p})$ with $p \neq 2$. Then we claim that L° is a uniformly almost finitely presented projective K°-module. For this, it suffices to show: for each n, there exists a finite free K°-module R_n of rank 2 and an injective map $R_n \to L^\circ$ with cokernel annihilated by $p^{\frac{1}{p^n}}$. (Indeed, then $p^{\frac{1}{p^n}}$ on either module will factor over this map, showing $\mathrm{Ext}^{>0}_{K^\circ}(L^\circ, -)$ is killed by $p^{\frac{1}{p^n}}$ for all n, and thus almost zero.)

Set $R_n = K^\circ \oplus K^\circ \cdot p^{\frac{1}{2p^n}} \subset L^\circ$, so we have $L^\circ \simeq \widehat{\mathrm{colim}_n R_n}$. We first claim that the cokernel of $R_n \to N_{n+1}$ is killed by $p^{\frac{1}{p^n}}$. To see this, observe that

$$p^{\frac{1}{p^n}} \cdot p^{\frac{1}{2p^{n+1}}} = p^{\frac{(p+1)/2}{p^{n+1}}} \cdot p^{\frac{1}{2p^n}}.$$

It follows that the cokernel of $R_n \to \mathrm{colim}_m R_m$ is killed by

$$p^{\sum_{m \geq n} \frac{1}{p^n}} = p^{\frac{p}{p^n \cdot (p-1)}}.$$

Strictly speaking, the element written above does not make sense: its norm does not live in $|K^*| = |p|^{\mathbf{Z}[\frac{1}{p}]}$. However, any element that is more divisible (i.e., has smaller norm) also works by the same reasoning. In particular, the cokernel of $R_n \to \mathrm{colim}_m R_m$ is killed by $p^{\frac{1}{p^{n-1}}}$. This shows that, as R_0 is p-adically complete, so is $\mathrm{colim}_n R_n$, and thus $L^\circ = \mathrm{colim}_n R_n$. But then each R_n is finite projective of rank 2, and the cokernel of the injective map $R_n \to L^\circ$ is killed by $p^{\frac{1}{p^{n-1}}}$. In particular, L° is an uniformly almost finitely presented projective K°-module.

We also pause to explain the construction of honest algebras from almost algebras in a fashion that respects faithful flatness. Strictly speaking, the rest of these notes can be developed without reference to this functor. However, this functor is quite convenient in extracting honest consequences out of an almost mathematical statement, so we shall use it.

tAlgebras **Remark 4.2.8** (Left adjoint to almostification for algebras)**.** As Mod_R^a has a symmetric monoidal structure, there is an evident notion of a commutative algebra object in this category: it is given by an object $S \in \mathrm{Mod}_R^a$ with maps $R^a \to S$ and $S \otimes S \to S$ satisfying some natural natural axioms). Write $\mathrm{CAlg}(\mathrm{Mod}_R^a)$ for the category of commutative algebras in almost R-modules. The almostification functor $M \mapsto M^a$ commutes with tensor products, and hence takes commutative algebras to commutative algebras, inducing a functor

$$(-)^a : \mathrm{CAlg}(\mathrm{Mod}_R) \to \mathrm{CAlg}(\mathrm{Mod}_R^a).$$

The functor $M \mapsto M_*$ is lax symmetric monoidal (i.e., there is a canonical map $M_* \otimes_R N_* \to (M \otimes N)_*$ for $M, N \in \mathrm{Mod}_R^a$), and hence takes commutative algebras to commutative algebras. In fact, the resulting functor

$$(-)_* : \mathrm{CAlg}(\mathrm{Mod}_R^a) \to \mathrm{CAlg}(\mathrm{Mod}_R)$$

is easily seen to be a right adjoint to the almostification functor $(-)^a$. In contrast, the left adjoint $M \mapsto M_!$ to almostification does not preserve commutative algebras: if A is an almost R-algebra, then $A_!$ does carry a multiplication as $(-)_!$ commutes with non-empty tensor products, but, as $R_!^a = I$ does not coincide with R, so one does not have a unit for $A_!$. To fix this, given such an A, define $A_{!!}$ as pushout of $R \leftarrow I \simeq R_!^a \to A_!$. There is then a unique way to make $A_{!!}$ into a commutative ring such that the defining map $R \to A_{!!}$ is a unit and the defining map $A_! \to A_{!!}$ is compatible with the multiplication. This construction gives a functor

$$(-)_{!!} : \mathrm{CAlg}(\mathrm{Mod}_R^a) \to \mathrm{CAlg}(\mathrm{Mod}_R)$$

that is left-adjoint to the almostification functor $(-)^a$, see [GR, §2.2.23]. The crucial properties are:

1. $(-)_{!!}$ commutes with colimits (clear by adjointness).

2. (Exercise) $(-)_{!!}$ preserves faithful flatness. However, it may not preserve flatness; see [GR, Remark 3.1.3].

Note that there is a natural map $A_{!!} \to A_*$ of almost algebras which is an almost isomorphism; this can often be used to move completeness properties between the two rings.

4.3 Almost étale extensions

The main point of making the definitions earlier is to enable discussion of the following key notion:

Definition 4.3.1. A map $A \to B$ of R^a-algebras is *almost finite etale* if

1. (Finite projectivity) B is an almost finite presented projective A-module.
2. (Unramifiedness) There exists a diagonal idempotent in $(B \otimes_A B)_*$, i.e., an element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu_*(e) = 1$, and $\ker(\mu)_* \cdot e = 0$, where $\mu : B \otimes_A B \to B$ is the multiplication map.

We write A_{afet} for the category of almost finite étale maps $A \to B$.

An instructive example of such maps arises from Example 4.2.7.

Example 4.3.2. Let $K = \widehat{\mathbf{Q}_p[p^{\frac{1}{p^\infty}}]}$, and $L = K(\sqrt{p})$ with $p \neq 2$. We shall show that L° / K° is almost finite étale. In fact, as K° is a valuation ring and L° is a torsionfree K°-module, flatness is clear. We have already shown that L° is a uniformly almost finitely presented projective K°-module in Example 4.2.7. It remains to verify almost unramifiedness. For this, note that L/K is a Galois extension of degree 2, and hence we have a canonical isomorphism

$$\mathrm{can} : L \otimes_K L \simeq L \times L \quad \text{via} \quad a \otimes b \mapsto (ab, a\sigma(b)),$$

where $\sigma : L \to L$ denotes the non-trivial Galois automorphism. Using this presentation, we see that the idempotent $e \in L \otimes_K L$ is given by the formula

$$e = \frac{1}{2\sqrt{p} \otimes 1}(1 \otimes \sqrt{p} + \sqrt{p} \otimes 1) \in L \otimes_K L.$$

It is then easy to see that $p \cdot e \in L^\circ \otimes_{K^\circ} L^\circ$. But note that using the isomorphism can above, we also have

$$e = \frac{1}{2p^{\frac{1}{2p^n}} \otimes 1}(1 \otimes p^{\frac{1}{2p^n}} + p^{\frac{1}{2p^n}} \otimes 1) \in L \otimes_K L.$$

This then gives $p^{\frac{1}{p^n}} \cdot e \in L^\circ \otimes_{K^\circ} L^\circ$. As this is true for all $n \geq 0$, we learn that $e \in (L^\circ \otimes_{K^\circ} L^\circ)_*$, as wanted.

To give further examples, we review a standard algebraic construction that explicitly exhibits finite étale algebras as finite projective modules in terms of the diagonal idempotent:

dempotent **Construction 4.3.3** (Finite étale algebras seen explicitly as finite projective modules)**.** Let R be a ring, and let $R \to S$ be a finite étale extension. Let $e \in S \otimes_R S$ be the diagonal idempotent cutting out the multiplication $\mu : S \otimes_R S \to S$, i.e., $e^2 = e$ and $\ker(\mu) = 0$. Using e, one has a product decomposition $\psi : S \otimes_R S \simeq S \times S'$ with the projection to S map corresponding to μ and $\psi(e) = (1, 0)$. Write $e = \sum_{i=1}^n a_i \otimes b_i$ for $a_i, b_i \in S$. Then we can explicitly realize S is a direct summand of R^n via the maps

$$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$$

given by

$$\alpha(f) = (\mathrm{Tr}_{S/R}(fa_i)) \quad \text{and} \quad \beta((g_i)) = \sum_{i=1}^n g_i b_i.$$

To see this works, we must show that $\beta \circ \alpha = \mathrm{id}$. In other words, we want to check that

$$\sum_{i=1}^{n} \mathrm{Tr}_{S/R}(f a_i) b_i = f \tag{4.1}$$ eq:TraceI

for any $f \in S$. To prove this, note that

$$\mathrm{Tr}_{i_2}(e) = \mathrm{Tr}_{S/S}(1) = 1,$$

where $i_2 : S \to S \otimes_R S$ is the second inclusion $s \mapsto 1 \otimes s$; this formula is proven by using the isomorphism ψ and the observation that trace maps are additive across products of finite étale S-algebras. Plugging in $e = \sum_i a_i \otimes b_i$ above and using the compatibility of trace maps with base change, we get

$$\sum_{i=1}^{n} \mathrm{Tr}_{S/R}(a_i) b_i = 1.$$

In particular, this verifies (4.1) for $f = 1$. In general, one repeats the same argument by replacing e with $(f \otimes 1) \cdot e$ (which equals $(1 \otimes f) \cdot e$ as $\ker(\mu) \cdot e = 0$).

Using this construction, we arrive at the almost purity theorem in characteristic p, which provides a large supply of almost finite étale covers:

:APTCharp **Proposition 4.3.4** (Almost purity in characteristic p: primitive version)**.** *Let $\eta : R \to S$ be an integral map of perfect rings. Assume that $\eta[\frac{1}{t}]$ is finite étale for some $t \in R$. Then η is almost finite étale with respect to the ideal $I = (t^{\frac{1}{p^\infty}})$.*

In other words, the assumption that $\eta[\frac{1}{t}]$ on the "generic fibre" spreads out to the conclusion that η is almost finite étale on the "almost integral fibre". This style of propogation of information from the generic fibre to the almost integral context will occur repeatedly in the sequel.

Proof. We first begin by reducing to the t-torsionfree case by observing that the t-power torsion ideals $R[t^\infty] \subset R$ and $S[t^\infty] \subset S$ are almost zero. Indeed, given $\alpha \in R$ with $t^c \cdot \alpha = 0$ for some $c \geq 0$, we have $t^c \alpha^{p^n} = 0$ in R for all $n \geq 1$, and therefore $t^{\frac{c}{p^n}} \alpha = 0$ in R by perfectness; this shows that $R[t^\infty]$ is almost zero, and similarly for $S[t^\infty]$. Replacing R with $R/R[t^\infty]$ and S with $S/S[t^\infty]$ (which does not change the corresponding almost rings), we may assume that both R and S are t-torsionfree.

Next, we reduce to the case where R is integrally closed in $R[\frac{1}{t}]$, and likewise for S. Let R_{int} be the integral closure of R in $R[\frac{1}{t}]$. Then, for any $f \in R_{int}$, the R-submodule of $R[\frac{1}{t}]$ spanned by $f^{\mathbf{N}}$ is finitely generated, and hence contained in $t^{-c}R$ for some $c \geq 0$. Thus, we have $t^c f^{p^n} \in R$ for all $n \geq 0$. By perfectness, this means that $t^{\frac{c}{p^n}} f \in R$ for all $n \geq 1$, so $f \in R_*$. In other words, the map $R \to R_{int}$ is an almost isomorphism. As the conclusion of the proposition is insenstive to passage to the almost world, we may assume that both R and S are integrally closed in $R[\frac{1}{t}]$ and $S[\frac{1}{t}]$ respectively.

We next check almost unramifiedness. Let $e \in (S \otimes_R S)[\frac{1}{t}]$ be the diagonal idempotent. Then $t^c e$ comes from $S \otimes_R S$ for some $c \geq 0$. As $e^{p^n} = e$ and all rings in sight are perfect, we conclude that

$t^{\frac{c}{p^n}} e$ comes from $S \otimes_R S$ for all $n \geq 0$. In other words, $e \in (S \otimes_R S)_*$. Thus, $R \to S$ is almost unramified.

It remains to show that S is almost finite projective over R. Fix some $m \geq 0$, and represent $t^{\frac{1}{p^m}} \cdot e \in (S \otimes_R S)[\frac{1}{t}]$ as an element $\sum_{i=1}^n a_i \otimes b_i \in S \otimes_R S$ (which is possible by the previous paragraph). Now consider the maps

$$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$$

given by

$$\alpha(f) = (\mathrm{Tr}_{S/R}(f a_i)) \quad \text{and} \quad \beta((g_i)) = \sum_{i=1}^n g_i b_i.$$

The maps make sense as R is integrally closed in $R[\frac{1}{t}]$. The analysis in Construction 4.3.3 shows that we have an equality of maps $\beta \circ \alpha = t^{\frac{1}{p^m}}$: this is true after inverting t, and thus on the nose as S is t-torsionfree. In particular, multiplication by $t^{\frac{1}{p^m}}$ on S factors through a finite free R-module. As this is true all m, we conclude that S is an almost finite projective R-module. □

versCharp **Remark 4.3.5** (Covers spread out to covers)**.** In the context of Proposition 4.3.4, assume further that $R[\frac{1}{t}] \to S[\frac{1}{t}]$ is injective. Then one can show that $R \to S$ is almost split; consequently, this map is an almost finite étale *cover*, i.e., it is almost faithfully flat. In fact, by the reductions in the proof above, we may assume that R is an integrally closed subring of $R[\frac{1}{t}]$. There is then an induced trace map $\mathrm{Tr} : S[\frac{1}{t}] \to R[\frac{1}{t}]$. We claim that this map satisfies $t^{\frac{1}{p^\infty}} \in \mathrm{Tr}(S)$; this immediately implies that $R \to S$ is almost split. To see this claim, observe that $t^c \in \mathrm{Tr}(S)$ for some $c \geq 0$ since $\mathrm{Tr}(S[\frac{1}{t}]) = R[\frac{1}{t}]$ as $R[\frac{1}{t}] \to S[\frac{1}{t}]$ is a finite étale cover. It is easy to see that trace maps are compatible with Frobenius; this is a version of the projection formula. So, if we write $t^c = \mathrm{Tr}(f)$ for some $f \in S$, then $t^{\frac{c}{p^n}} = \mathrm{Tr}(f^{\frac{1}{p^n}})$. As this holds true for all n, the claim follows.

We can upgrade the preceding objectwise statement to an equivalence of categories:

:APTCharp **Theorem 4.3.6** (Almost purity in characteristic p: categorical version)**.** *Let R be a perfect ring of characteristic p, and consider almost mathematics with respect to $I = (t^{\frac{1}{p^\infty}})$ for a fixed element $t \in R$. Then inverting t gives an equivalence of categories $R_{afet} \simeq R[\frac{1}{t}]_{fet}$.*

Proof. As in the proof of Proposition 4.3.4, we may assume that t is a nonzerodivisor on R. We must show that the functor $R_{afet} \to R[\frac{1}{t}]_{fet}$ obtained via $S \mapsto S_*[\frac{1}{t}]$ is an equivalence. As any integral extension of $R[\frac{1}{t}]$ is obtained by inverting t in an integral extension of R, Proposition 4.3.4 gives the essential surjecitivty. For full faithfulness, fix some $S \in R_{afet}$. We claim:

Lemma 4.3.7. *$S \simeq T^a$ for the integral closure T of R in $S_*[\frac{1}{t}]$*

Proof. We begin by noting that S is perfect: this follows from the almost version of Lemma 4.3.8 as almost finite étale maps are easily seen to be weakly étale. By functoriality, we learn that S_* is perfect. Moreover, as S is almost flat over R, the element $t \in S_*$ is a nonzerodivisor, so $S_* \subset S_*[\frac{1}{t}]$. It is also clear that T is perfect, that $t \in T$ is a nonzerodivisor, and that $R \to T$ is an integral extension of perfect rings that is identified with $R[\frac{1}{t}] \to S_*[\frac{1}{t}]$ on inverting t. To show $T^a = S$, we shall check $T_* = S_*$. For $\subset$, fix some $f \in T$. As $R \to T$ is integral, the set $f^{\mathbf{N}}$ spans a finitely generated R-submodule of $T \subset T[\frac{1}{t}] = S_*[\frac{1}{t}]$. Hence, it must lie inside some $t^{-c}S_*$. This gives $t^c f^{\mathbf{N}} \in S_*$. By perfectness of S_*, we conclude that $t^{\frac{c}{p^n}} \cdot f \in S_*$ for all $n \geq 0$, and hence $f \in (S_*)_* = S_*$. This gives $T \subset S_*$, and hence $T_* \subset S_*$. Conversely, for any $g \in S_*$, the set $t \cdot g^{\mathbf{N}}$ is contained in finitely generated R-submodule of S_* by almost finite generation of S over R. In particular, $t \cdot g^{\mathbf{N}}$ lies in a finitely generated T-submodule of S_*. As $T[\frac{1}{t}] = S_*[\frac{1}{t}]$, it follows that $t \cdot g^{\mathbf{N}} \in t^{-c}T$ for some $c \geq 0$. Thus, $t^{c+1}g^{\mathbf{N}} \in T$. But then $t^{\frac{c+1}{p^n}} g \in T$ for all $n \geq 0$ by perfectness of T, so $g \in T_*$. □

The lemma recovers S functorially from the map $R \to S_*[\frac{1}{t}]$, so the claim follows. □

The next result is a slightly non-standard variant of a standard result in commutative algebra and was used above. The proof given below may feel a bit contrived, but we have tried to make sure it adapts readily to the almost context; see also [GR, Theorem 3.5.13] for the original source of the argument.

Frobenius **Lemma 4.3.8.** *Let $A \to B$ be weakly étale map of $\mathbf{F}_p$-algebras, i.e., both $A \to B$ and $\mu : B \otimes_A B \to B$ are flat. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{Frob_A} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{Frob_B} & B \end{array}$$

is a pushout square of rings. In particular, if A is perfect, so is B.

Proof. We shall use the following stability properties of weakly étale maps without proof: weakly étale maps are stable under base change and composition, and any map between weakly étale maps is weakly étale.

Now let $B^{(1)} = B \otimes_{A,Frob_A} A$ be the actual pushout, and consider the induced relative Frobenius $F_{B/A} : B^{(1)} \to B$. We must show that $F_{B/A}$ is an an isomorphism. This is a map between weakly étale A-algebras, and is thus weakly étale. On the other hand, a diagram chase reveals that $F_{B/A}$ factors Frobenius on both $B^{(1)}$ and B. So it suffices to show that any weakly étale map $\alpha : R \to S$ of $\mathbf{F}_p$-algebras that factors a power of Frobenius on R and S is an isomorphism. We first claim that any such α is faithfully flat, i.e., if $R/I \otimes_R S \simeq 0$ for some ideal $I \subset R$, then $R/I = 0$. By base change, we reduce to the case $R = R/I$, so we must check that $S = 0$ implies $R = 0$. But Frobenius on R factors over S, so this is clear. To show that α is an isomorphism, note that the property of a map to factor a power of Frobenius on source and target is stable under composition, base change, and passes to sections. By base change along itself, we may assume α has a section $\beta : S \to R$. But then β is also weakly étale, and factors a power of Frobenius on both S and R (as the same holds true for α). It follows that β is a faithfully flat surjective map, and hence an isomorphism: the kernel is carried to 0 along β^* and must thus be 0. As β is a section to α, it follows that α is an isomorphism too.

□

Exercise 4.3.9. Let $(R, \mathfrak{m})$ be a complete noetherian local $\mathbf{F}_p$-algebra. Using a suitable Noether normalization, show that R_{perf} is *almost Cohen-Macaulay*, i.e., there exists some nonzerodivisor $g \in R$ such that $H^i_{\mathfrak{m}}(R_{perf})$ is almost zero with respect to $I = (g^{\frac{1}{p^\infty}})$ for $i < \dim(R)$.

4.4 Some more almost commutative algebra

We end this chapter by collecting some lemmas in almost commutative algebra that will be useful later. They are mainly concerned with the interaction of the completion operation with the functors relating the almost category to the honest modules.

ompletion **Lemma 4.4.1.** *Fix a perfectoid field K. Let $R = K^\circ$ and $I = K^{\circ\circ}$. Fix a pseudouniformizer $t \in I$ and $M \in \mathrm{Mod}_R^a$.*

1. *M is almost flat if and only if M_* is R-flat if and only if $M_!$ is almost flat.*

2. *Assume M is almost flat. Then M is t-adically complete if and only if M_* is so.*

3. *Assume M is almost flat. Then for each $t \in K^\circ$, we have $tM_* \simeq (tM)_*$, and $M_*/tM_* \subset (M/tM)_*$ by the canonical map. Moreover, for each $\epsilon \in I$, the images of*

$$(M/t\epsilon M)_* \to (M/tM)_* \leftarrow M_*/tM_*$$

are identical.

Proof. 1. For $(-)_*$: as R is a valuation ring, M_* is R-flat exactly when $M_*[t] = 0$. As $M \mapsto M_*$ is left exact, we have $M_*[t] = (M[t])_*$. Thus, if M is almost flat, then M_* is R-flat. The converse is clear as $M = (M_*)^a$.

For $(-)_!$: this follows from the observation that both $(-)_!$ and $(-)^a$ are exact and commute with tensor products.

2. As $(-)^a$ commutes with limits and colimits, there is nothing to prove in the forward direction. Conversely, assume M is flat and t-adically complete. Then M_* and $M_!$ are also flat by (1). Now consider the commutative diagram

$$\begin{array}{ccc} M_! & \xrightarrow{a} & \lim (M/t^nM)_! \simeq \lim M_!/t^nM_! =: \widehat{M_!} \\ \downarrow d & & \downarrow b \\ M_* & \xrightarrow{c} & \lim (M/t^nM)_*. \end{array}$$

Here the isomorphism on the top right arises from the commutation of $(-)_!$ with colimits. The map c is an isomorphism as $(-)_*$ commutes with limits and M is t-adically complete. Both b and d are almost isomorphisms as $(-)_! \to (-)_*$ is an almost isomorphism of functors. It particular, a is an almost isomorphism as well. As $M_!$ is torsionfree, it follows that a and d are injective with almost zero cokernel. As the target of a is clearly t-adically complete, Lemma 4.4.2 shows that $M_!$ must be t-adically complete. But then Lemma 4.4.2 again applied to d shows that M_* must be t-adically complete as well.

3. The assertion $tM_* = (tM)_*$ follows from the left-exactness of $(-)_*$. Exactness of $(-)^a$ and left-exactness of $(-)_*$ then give $M_*/tM_* \subset (M/tM)_*$.

For the rest, consider the canonical map of short exact sequences of R^a-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{t} & M & \longrightarrow & M/tM & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M/\epsilon M & \xrightarrow{x} & M/t\epsilon M & \longrightarrow & M/tM & \longrightarrow & 0. \end{array}$$

Applying $(-)_* = \mathrm{Hom}_{R^a}(R^a, -)$ and using Exercise 4.2.3, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_*/tM_* & \xrightarrow{a} & (M/tM)_* & \longrightarrow & \mathrm{Ext}^1_{R^a}(R^a, M)[t] & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & (M/t\epsilon M)_* & \xrightarrow{b} & (M/tM)_* & \longrightarrow & \mathrm{Ext}^1_{R^a}(R^a, M/\epsilon M) & & \end{array}$$

with exact rows. We wish to show that a and b have the same image. It thus suffices to show c is injective. Now note that $\mathrm{Ext}^1_{R^a}(R^a, M)$ is almost zero (as it vanishes once we pass back to the almost world), and thus

$$\mathrm{Ext}^1_{R^a}(R^a, M)[t] = \mathrm{Ext}^1_{R^a}(R^a, M).$$

Thus, we want the canonical map $M \to M/\epsilon M$ to induce an injection on $\mathrm{Ext}^1_{R^a}(R^a, -)$. Applying $\mathrm{Hom}_{R^a}(R^a, -)$ to the sequence

$$0 \to M \xrightarrow{\epsilon} M \to M/\epsilon M \to 0,$$

it is enough to show that multiplication by ϵ on $\mathrm{Ext}^1_{R^a}(R^a, M)$ is 0. But this group is almost 0, so multiplication by ϵ certainly vanishes.

2. We can also give a proof of (2) using (3), and avoiding the $(-)_!$ functor. If M_* is complete, so is $M \simeq (M_*)^a$: the functor $(-)^a$ commutes with limits and colimits. Conversely, assume M is complete. Then we have $M \simeq \lim M/t^n M$. As $(-)_*$ commutes with limits, this gives $M_* \simeq \lim (M/t^n M)_*$. Using the pro-isomorphism $\{M_*/t^n M_*\}$ and $\{(M/t^n M)_*\}$ from (3), this yields $M_* \simeq \lim M_*/t^n M_*$, as wanted. , we get the conclusion.

□

The following general lemma was used above:

ubobjects **Lemma 4.4.2.** *Let A be a ring equipped with a nonzerodivisor t. Let $\alpha : M \to N$ be a map of t-torsionfree A-modules. Assume that α is injective with t-torsion cokernel Q. Then M is t-adically complete if and only if N is so.*

Proof. Consider the short exact sequence

$$0 \to M \to N \to Q \to 0. \tag{4.2}$$ eq:Almost

Reducing modulo t^n, we get an exact sequence

$$0 \to Q[t^n] \to M/t^n \to N/t^n \to Q \to 0.$$

As n varies, these form a natural projective system of exact sequences, where the transition maps $Q[t^{n+1}] \to Q[t^n]$ are given by multiplication by t, and are thus 0 as Q is killed by t. Passing to inverse limits, we get an exact sequence

$$0 \to \widehat{M} \to \widehat{N} \to Q \to 0. \tag{4.3}$$ eq:Almost

The claim now follows by the snake lemma applied to the obvious map of short sequences from (4.2) to (4.3). □

In fact, it turns out that completions and $(-)_*$ interact well in general. The next proposition fleshes this out using the language of derived completions; this is not relevant to the sequel.

Proposition 4.4.3. *Fix (R, I) as in the rest of this section. Fix $J = (f_1, ..., f_r) \subset R$ a finitely generated ideal. Then an R^a-module M is J-adically complete if and only if M_* is so.*

Proof. As $(-)_*$ commutes with limits and colimits, the "if" direction is clear. Conversely, assume M is J-adically complete, and fix a representative N with $N^a = M$. The J-adic completeness of M means that $R\lim_{f_i} N$ is almost zero for all i, and thus R-complex $\mathrm{RHom}_R(I, N)$ is derived J-complete. Passing to cohomology shows that $M_* = \mathrm{Hom}_R(I, N)$ is also derived J-complete. The J-adic completeness of M and commutation of $(-)_*$ with limits shows that

$$M_* \simeq \lim(M/J^n M)_*$$

The canonical factorization

$$M_* \to \lim(M_*/J^n M_*) \to \lim(M/J^n M)_*,$$

shows that the first map must be injective. This implies that M_* is J-adically separated. As any derived J-complete and J-adically separated R-module is J-adically complete, we are done. □

Chapter 5

Non-archimedean Banach algebras via commutative algebra

The goal of this chapter is record a dictionary to translate the language of Banach algebras over a non-archimedean field into classical commutative algebra. Along the way, we also need to discuss the behaviour of integral closures (as well as variants, such as total integral closures) under completions.

5.1 Commutative algebras: completions and closures

We discuss the preservation of certain integral closedness properties under completions. We begin with the notion of p-root-closedness:

letionPRC **Lemma 5.1.1.** *Let A be a ring equipped with a nonzerodivisor $f \in A$. Assume that $A \subset A[\frac{1}{f}]$ is p-root closed, i.e., if $g \in A[\frac{1}{f}]$ and $g^p \in A$, then $g \in A$. Then:*

1. *$\widehat{A} \subset \widehat{A}[\frac{1}{f}]$ is p-root-closed (where the completion is f-adic).*

2. *Assume that f admits a compatible system of p-power roots. Then $A_* \subset A_*[\frac{1}{f}]$ is p-root-closed (where almost mathematics is performed with respect to $(f^{\frac{1}{p^\infty}})$).*

Proof. We first check that we can replace A with its maximal separated quotient A/I, where $I = \cap_n f^n A$. First, we need f to be a nonzerodivisor on A/I: if $g \in A$ such that $fg \in I$, then $fg \in f^n A$ for all n, and thus $g \in f^{n-1}A$ for all n as f is a nonzerodivisor, which gives $g \in I$. Next, we need $A/I \subset A/I[\frac{1}{f}]$ to be p-root-closed: if $a \in A/I[\frac{1}{f}]$ such that $a^p \in A/I$, then we can write $\tilde{a}^p = b + f^{-c}d$ for some lift $\tilde{a} \in A[\frac{1}{f}]$ of a, an integer $c \geq 0$, and elements $b, d \in A$ with $d \in I$. But $I = fI = f^c I$, so $f^{-c}d \in I$ as well, and thus $\tilde{a}^p \in A$, so $\tilde{a} \in A$ by the p-root-closure assumption. Thus, we have reduced to the case where A is f-adically separated. In particular, $A \to \widehat{A}$ is injective.

1. Choose $g \in \widehat{A}[\frac{1}{f}]$ such that $g^p \in \widehat{A}$. Choose some N such that $f^N \cdot g \in \widehat{A}$. Choose another integer m with $m \geq N \cdot (p-1)$. By the density of $A[\frac{1}{f}] \subset \widehat{A}[\frac{1}{f}]$, as $f^m \widehat{A} \subset \widehat{A}[\frac{1}{f}]$ is open, we can write
$$g = g_0 + f^m g_1,$$
where $g_0 \in A[\frac{1}{f}]$, and $g_1 \in \widehat{A}$. Note that $f^N \cdot g_0 \in \widehat{A}$ since $f^N \cdot g$ and $f^N \cdot f^m g_1$ both lie in $\widehat{A}$. Raising to the p-th power, we can write
$$g^p = g_0^p + \binom{p}{1} g_0^{p-1} f^m g_1 + \binom{p}{2} g_0^{p-2} (f^m g_1)^2 + .. + (f^m g_1)^p.$$
As $m \geq N \cdot (p-1)$, it is easy to see that all terms on the right except possibly g_0^p lie in $\widehat{A}$. The same is true for the left side by assumption. But then $g_0^p \in A$. As $A \subset A[\frac{1}{f}]$ is p-root closed, it follows that $g_0 \in A$, and thus $g \in \widehat{A}$.

2. Choose $g \in A[\frac{1}{f}]$ with $g^p \in A_*$. Then $f^{\frac{1}{p^n}} \cdot g^p \in A$ for all $n \geq 0$. But then $(f^{\frac{1}{p^{n+1}}} \cdot g)^p \in A$ for all $n \geq 0$. As A is p-root closed, we learn $f^{\frac{1}{p^{n+1}}} \cdot g \in A$ for $n \geq 0$, and thus $g \in A_*$.

□

Analogously, integral closures also behave well under completions:

pletionIC **Lemma 5.1.2.** *Let A be a ring equipped with a nonzerodivisor $f \in A$. Assume that $A \subset A[\frac{1}{f}]$ is integrally closed. Then:*

1. *$\widehat{A} \subset \widehat{A}[\frac{1}{f}]$ is integrally closed.*

2. *Assume that f admits a compatible system of p-power roots. Then $A_* \subset A[\frac{1}{f}]$ is integrally closed (where almost mathematics is performed with respect to $(f^{\frac{1}{p^\infty}})$).*

Proof. We first check that we can replace A with its maximal separated quotient A/I, where $I = \cap_n f^n A$. As in the proof of Lemma 5.1.1, we know that f is a nonzerodivisor on A/I, and that I is uniquely f-divisible. Choose some $g \in A/I[\frac{1}{f}]$ which satisfies a monic polynomial $h(X) \in A/I[X]$ over A/I. Choose lifts $\tilde{g} \in A[\frac{1}{f}]$ and $\tilde{h}(X) \in A[X]$. Then $\tilde{h}(\tilde{g}) \in I[\frac{1}{f}]$. But I is uniquely f-divisible, so $I = I[\frac{1}{f}]$, and thus $\tilde{h}(\tilde{g}) \in I \subset A$. Thus, $\tilde{g}$ is integral over A. The hypothesis on A then shows that $\tilde{g} \in A$, and thus $g \in A/I$ as well. We have now reduced to the case where A is f-adically separated. In this case, one checks that $A = \widehat{A} \cap A[\frac{1}{f}] \subset \widehat{A}[\frac{1}{f}]$.

1. Choose $g \in \widehat{A}[\frac{1}{f}]$ with g integral over $\widehat{A}$. Then $g = f^{-c} h$ with $h \in \widehat{A}$ and $c \geq 0$. By integrality, we can write
$$g^n = a_{n-1} g^{n-1} + a_{n-2} g^{n-2} + ... + a_0$$

for some $a_i \in \widehat{A}$. Scaling by f^{cn}, this gives

$$h^n = f^c a_{n-1} h^{n-1} + f^{2c} a_{n-2} h^{n-2} + ... + f^{cn} a_0.$$

for some $b_i \in \widehat{A}$. By approximation, we can choose $h_0 \in A$ with $h_0 \equiv h \mod f^{cn}$ and $b_i \in A$ with $b_i \equiv a_i \mod f^{cn}$. This allows us to write

$$h_0^n = f^c b_{n-1} h_0^{n-1} + f^{2c} b_{n-2} h_0^{n-2} + ... + f^{cn} b_0 + f^{cn} d$$

for some $d \in \widehat{A}$. But then $d \in \widehat{A} \cap A[\frac{1}{f}] = A$, so the previous equation is taking place in A. Dividing both sides by f^{cn} then shows that $g_0 = f^{-c} h_0 \in A[\frac{1}{f}]$ is integral over A, and hence $g_0 \in A$ by integral closedness. Thus, $h_0 \in f^c A$. As we have $h \equiv h_0 \mod f^{cn+1}$, it follows that $h \in f^c \widehat{A}$, and thus $g = f^{-c} h \in \widehat{A}$.

2. Choose $g \in A[\frac{1}{f}]$ with g integral over A_*. Then we have an equation

$$g^n = a_{n-1} g^{n-1} + a_{n-2} g^{n-2} + ... + a_0$$

with $a_i \in A_*$. Let $\epsilon = f^{\frac{1}{p^k}}$ for some $k \geq 0$. Scaling both sides by ϵ^n gives

$$(\epsilon g)^n = a_{n-1} \epsilon (\epsilon g)^{n-1} + a_{n-2} \epsilon^2 (\epsilon g)^{n-2} + ... + \epsilon^n a_0.$$

As $a_i \in A_*$, we have $a_{n-i} \epsilon^i \in A$ for each $i \geq 1$. In particular, the above equation shows that ϵg is integral over A, and thus $\epsilon g \in A$. As this is true for $\epsilon = f^{\frac{1}{p^k}}$ for any $k \geq 0$, we conclude that $g \in A_*$.

□

Finally, we discuss the preservation of total integral closedness[1] under completions

letionTIC **Lemma 5.1.3.** *Let A be a ring equipped with a nonzerodivisor $f \in A$ admitting a compatible system of p-power roots $f^{\frac{1}{p^n}}$ for all $n \geq 0$. Assume that $A \subset A[\frac{1}{f}]$ is totally integrally closed. Then we have:*

1. *$\widehat{A} \subset \widehat{A}[\frac{1}{f}]$ is totally integrally closed.*

2. *$A = A_*$ (where almost mathematics is performed with respect to $(f^{\frac{1}{p^\infty}})$).*

[1]Recall its definition: given an inclusion $A \subset B$ of rings, the *total integral closure* A_{tic} of A in B is the set of all $f \in B$ such that $f^{\mathbf{N}}$ is contained in a finitely generated A-submodule of B. This can easily be checked to be a ring; if A is noetherian, then this coincides with the integral closure, but is different in general. If $A = A_{tic}$, then we say that A is totally integrally closed in B. Unlike the case of integral closures, it need not be true that A_{tic} is itself totally integrally closed.

Proof. 1. Note that $A \subset A[\frac{1}{f}]$ is p-root closed, and thus the same holds for $\widehat{A} \subset \widehat{A}[\frac{1}{f}]$ by Lemma 5.1.1. Now fix some $g \in \widehat{A}[\frac{1}{f}]$ such that $f^k \cdot g^{\mathbf{N}} \subset \widehat{A}$ for some $k > 0$. By p-root-closedness of $\widehat{A} \subset \widehat{A}[\frac{1}{f}]$, we learn that $f^{\frac{k}{p^n}} \cdot g \in \widehat{A}$ for all $n \geq 0$, i.e., g gives an almost zero element $\overline{g}$ of $\widehat{A}[\frac{1}{f}]/\widehat{A}$. We must show $\overline{g} = 0$. But $A[\frac{1}{f}]/A \simeq \widehat{A}[\frac{1}{f}]/\widehat{A}$, so $\overline{g}$ gives an almost zero element of $A[\frac{1}{f}]/A$. Undoing the previous reasoning, we see that $\overline{g}$ lifts under $A[\frac{1}{f}] \to A[\frac{1}{f}]/A$ to an element in the total integral closure of A in $A[\frac{1}{f}]$. As the latter equals A, we must have $\overline{g}$.

2. We have $A_* \subset f^{-\frac{1}{p^k}} A$ for any $k \geq 0$ by definition of A_*. But then each $g \in A_*$ is totally integral over A, and thus lies in A by hypothesis.

□

5.2 The dictionary

We can now give the promised dictionary. Fix a NA field K with a nontrivial valuation. The basic objects we wish to describe are:

Definition 5.2.1 (K-Banach algebras)**.** Let K be a NA field. A *Banach K-algebra* R is a K-algebra R equipped with a map $|\cdot| : R \to \mathbf{R}_{\geq 0}$ extending the norm on K such that

1. (Norm) $|f| = 0$ only if $f = 0$.
2. (Submultiplicativity) $|fg| \leq |f||g|$, with equality if $f \in K$
3. (NA property) $|f + g| \leq \max(|f|, |g|)$.
4. R is complete in the metric d given by $d(f, g) = |f - g|$.

The category of K-Banach algebras has as objects Banach K-algebras, and morphisms given by continuous maps.

We wish to access K-Banach algebras in terms of (usual) algberas over K°. To this end, we formalize a procedure for extracting a K°-algebra out of a K-Banach algebra.

Definition 5.2.2. For a K-Banach algebra R, define the set $R^\circ \subset R$ of *power bounded* elements as

$$R^\circ := \{f \in R \mid \{|f^n|\} \text{ is bounded.}\} = \{f \in R \mid \{f^n\} \subset R \text{ is bounded.}\}$$

Some easy observations include that R° is a subring (by the NA property and submultiplicativity for the norm) which is open (as it contains unit ball $R_{\leq 1}$ of all elements with norm ≤ 1), and the construction $R \mapsto R^\circ$ only depends on R as a topological ring. A good source of examples arises as follows:

ctoBanach **Example 5.2.3** (Complete K°-algebras yield K-Banach algebras)**.** Assume K is discretely valued, and fix a uniformizer $t \in K$. Let A be a t-adically complete and t-torsionfree K°-algebra. Set $R = A[\frac{1}{t}]$. Define a seminorm on R via

$$|f| = \min\{|t|^n \mid f \in t^n A\}. \tag{5.1}$$ eq:Banach

One can check that this construction endows R with the structure of a Banach K-algebra, and that the resulting topology coincides[2] with the t-adic topology, i.e., the group topology on R with a basis of open subgroups given by $t^n A$. We trivially have $A \subset R_{\leq 1} \subset R^\circ$, and one checks that $R_{<1} \subset A$. In particular, the systems $\{t^n A\}$ and $\{R_{\leq |t|^n}\}$ are cofinal amongst each other. Note R° can be much larger than $R_{\leq 1}$ in practice: if $A = K^\circ[x]/(x^2)$, then $\frac{1}{t^n}x \in R^\circ$ for each n, but these elements have norm > 1 for $n > 0$, so they are not in $R_{\leq 1}$.

More generally, the same discussion applies to potentially non-discretely valued base fields with the following modification of the norm. For each $\gamma \in |K|$, choose $t_\gamma \in K$ with $|t_\gamma| = \gamma$. Then we can set

$$|f| = \inf\{\gamma \mid f \in t_\gamma A\}.$$

This seminorm turns R into a Banach K-algebra, and all the preceding properties hold true in this setting. In fact, if $|K^*| \subset \mathbf{R}_{>0}$ is dense, then we can describe $R_{\leq 1}$ as $A_* := \mathrm{Hom}(K^{\circ\circ}, A)$.

We shall often restrict to the following class of rings to avoid having a large ring of power bounded elements.

Definition 5.2.4. A K-Banach algebra R is *uniform* if R° is itself a bounded subset of R in the metric topology.

By a variant of the argument in Example 5.2.3, one sees that any uniform Banach K-algebra must be reduced[3]. In fact, this category can be described completely algebraically as follows:

chNAField **Proposition 5.2.5.** *Fix a pseudouniformizer $t \in K$. The following categories are equivalent:*

- *The category $\mathcal{C}$ of uniform Banach K-algebras R with continuous K-algebra maps.*
- *The category $\mathcal{D}_{tic}$ of t-adically complete and t-torsionfree K°-algebras A with A totally integrally closed*[4] *in $A[\frac{1}{t}]$.*

The functors between $\mathcal{C}$ and $\mathcal{D}_{tic}$ (for any i) are $R \mapsto R^\circ$ and $A \mapsto A[\frac{1}{t}]$ (equipped with the Banach algebra structure from Example 5.2.3).

[2] Fix some $\epsilon > 0$, and say $|f| \leq \epsilon$. Then $2\epsilon \geq |t|^n$ for some n with $f \in t^n A$. As $0 < |t| < 1$, this means that n must be very large. Conversely, it is clear that if $f \in t^m A$ for m large, then $|f|$ is small.

[3] Indeed, for any K-Banach algebra R, a nilpotent element is power bounded, so it lies in R°. On the other hand, if $\epsilon \in R^\circ$ is a nonzero nilpotent, then so is any multiple, so $\{\frac{1}{t^n}\epsilon\} \subset R^\circ$. But $|\frac{1}{t^n}\epsilon| = |t|^{-n}|\epsilon|$ is unbounded as $n \to \infty$, so R° cannot be bounded.

[4] This means that given $f \in A[\frac{1}{t}]$ with $f^{\mathbf{N}}$ lying in a finitely generated A-submodule of $A[\frac{1}{t}]$ (or, equivalently, inside $\frac{1}{t^c}A$ for some $c > 0$), we have $f \in A$.

Proof. We first construct the functor $\mathcal{C} \to \mathcal{D}_{tic}$. Let R be a uniform Banach K-algebra. Then $R^\circ \subset R$ contains the unit ball $R_{\leq 1} = \{f \in R \mid |f| \leq 1\}$ by the NA inequality, and is thus an open subring. Moreover, R° is bounded by assumption, so $R^\circ \subset R_{\leq r}$ for some $r > 0$. As R is a K-Banach algebra, we have have $|t^n| \to 0$ as $n \to \infty$, and thus $t^n \to 0$ as $n \to \infty$. In particular, $\cap_n t^n R_{\leq r} = 0$ as an element in the intersection must have norm 0. It follows that R° is t-adically separated. Similarly, using the completeness of R for the metric topology, it is easy to see that $R_{\leq r}$, and hence its open subgroup R°, is t-adically complete. To check that R° is totally integrally closed in R, fix some $f \in R$ such that $f^{\mathbf{N}}$ is contained in a finitely generated R°-submodule of R. As $R = R^\circ[\frac{1}{t}]$, it follows that $f^{\mathbf{N}} \subset \frac{1}{t^k} R^\circ \subset \frac{1}{t^k} R_{\leq r}$; but this immediately implies that f is power bounded, so $f \in R^\circ$. The extraction of R° from R is functorial in continuous maps of K-Banach algebras, so we obtain a functor $F : \mathcal{C} \to \mathcal{D}_{tic}$.

Conversely, fix some $A \in \mathcal{D}_{tic}$, and view $R = A[\frac{1}{t}]$ as a Banach K-algebra, as in Example 5.2.3, so $A \subset R_{\leq 1} \subset R^\circ$. We shall show that $A = R^\circ$; this will construct the functor $G : \mathcal{D}_{tic} \to \mathcal{C}$ and checks that $F \circ G \simeq \mathrm{id}$. Pick some $f \in R^\circ$. Then $f^{\mathbf{N}}$ is bounded. As $\{t^n A\}$ and $\{R_{\leq |t|^n}\}$ are cofinal amongst each other, there must be some $c > 0$ such that $t^c f^{\mathbf{N}} \subset A$ by boundedness of $f^{\mathbf{N}}$. But then $f^{\mathbf{N}} \subset \frac{1}{t^c} A$, so $f \in A$ as is totally integrally closed in $A[\frac{1}{t}]$.

To finish proving the theorem, we must show that $G \circ F \simeq \mathrm{id}$. Unwinding definitions, this amounts to showing the following: given $R \in \mathcal{C}$, the given Banach norm $|\cdot|_{given}$ on R is equivalent to the one $|\cdot|_{R^\circ}$ coming from Example 5.2.3 via $R = R^\circ[\frac{1}{t}]$. In other words, we must show that the identity map on R is bounded for either norm. The unit ball for $|\cdot|_{R^\circ}$ is exactly R°, and we always have a containment $R_{|\cdot|_{given}, \leq 1} \subset R^\circ$ for any K-Banach algebra. Conversely, by uniformity, we also have $R^\circ \subset R_{|\cdot|_{given}, \leq r}$ for some $r > 0$. Combining these gives the claim. □

When K is perfectoid, the category $\mathcal{D}_{tic}$ admits alternative descriptions.

toidField **Proposition 5.2.6.** *In the context of Proposition 5.2.5, assume K is a perfectoid field. Then the categories mentioned in Proposition 5.2.5 are equivalent to:*

- *The category $\mathcal{D}_{ic}$ of t-adically complete and t-torsionfree K°-algebras A with A integrally closed in $A[\frac{1}{t}]$ and $A \simeq A_*$.*
- *The category $\mathcal{D}_{prc}$ of t-adically complete and t-torsionfree K°-algebras A with A p-root closed in $A[\frac{1}{t}]$ and $A \simeq A_*$.*

Proof. For $A \in \mathcal{D}_{tic}$, we have $A = A_*$ by Lemma 5.1.3, so we have containments $\mathcal{D}_{tic} \subset \mathcal{D}_{ic} \subset \mathcal{D}_{prc}$. It suffices to show that any $A \in \mathcal{D}_{prc}$ is totally integrally closed in $A[\frac{1}{t}]$. Fix some $f \in A[\frac{1}{t}]$ with $f^{\mathbf{N}} \subset \frac{1}{t^k} A$ for some $k > 0$. Then $t^k \cdot f^{p^n} \in A$ for all $n \geq 0$. As A is p-root closed in $A[\frac{1}{t}]$, we learn that $t^{\frac{k}{p^n}} \cdot f \in A$ for all $n \geq 0$, and thus $f \in A_*$. But $A = A_*$, so we are done. □

Using these descriptions, we will show that uniform Banach K-algebras admit all limits and colimits.

itsBanach **Corollary 5.2.7.** *Assume K is a perfectoid field. The category of uniform Banach K-algebras has all colimits and limits.*

Proof. For colimits, we work with $\mathcal{D}_{ic}$. It is clear that the category $\mathcal{D}_{all}$ of all t-adically complete and t-torsionfree K°-algebras A has all colimits (given by the t-adic completion of the t-torsionfree quotient of the underlying colimit of rings), so it is enough to show that the fully faithful inclusion $\mathcal{D}_{ic} \subset \mathcal{D}_{all}$ has a left adjoint. In other words, given $A \in \mathcal{D}_{all}$, we seek a universal map $A \to A_u$ in $\mathcal{D}_{all}$ with $A_u \in \mathcal{D}_{ic}$. For this, set A_{int} to be the integral closure of A in $A[\frac{1}{f}]$, and set $A_u := (\widehat{A_{int}})_*$. Then $A_u \subset A_u[\frac{1}{t}]$ is integrally closed by Lemma 5.1.2: the act of completion and applying $(-)_*$ preserve integral closedness by the lemma. Moreover, as $M \to M_*$ is an almost isomorphism for any B, it is clear that A_u is t-adically complete as well. Finally, we have $A_u = (A_u)_*$ by construction. Thus, $A_u \in \mathcal{D}_{ic}$. We leave to the reader to check that the natural map $A \to A_u$ has the desired universal property.

For limits, we work with $\mathcal{D}_{prc}$. It is clear that $\mathcal{D}_{all}$ has all limits, and that they are computed by the underlying limit of sets. It thus suffices to show that $\{A_i\}$ is a diagram in $\mathcal{D}_{prc}$, then $A := \lim A_i$ is p-root closed in $A[\frac{1}{t}]$, and that $A \simeq A_*$. The former can be checked directly using the injectivity of $A[\frac{1}{t}] \to \lim_i A_i[\frac{1}{t}]$ (as both sides embed into $\prod_i A_i[\frac{1}{t}]$), while the latter follows as $M \mapsto M_*$ commutes with limits. $\square$

Chapter 6

Perfectoid algebras

In this chapter, we introduce perfectoid algebras over a perfectoid field, and prove the tilting correspondence equating the theory in characteristic 0 with characteristic p; for the latter, we follow Scholze's approach based on the cotangent complex, which is reviewed first. Finally, we formulate the almost purity theorem for perfectoid algebras, and explain why our previous work establishes this theorem in two special cases: in characteristic p in general, and for perfectoid fields. The general case will be established later once the theory of adic spaces has been introduced.

6.1 Reminders on the cotangent complex

Recall the following construction from non-abelian homological algebra:

Construction 6.1.1 (Free resolutions of rings)**.** For any ring A and a set S, we write $A[S]$ for the polynomial algebra over A on a set of variables x_s indexed by $s \in S$. The functor $S \mapsto A[S]$ is left adjoint to the forgetful functor from A-algebras to sets. In particular, for any A-algebra B, we have a canonical map $\eta_B : A[B] \to B$, which is evidently surjective. Repeating this construction, we obtain two A-algebra maps $\eta_{A[B]}, A[\eta_B] : A[A[B]] \to A[B]$. Iterating this process allows one to define a simplicial A-algebra $P^\bullet_{B/A}$ augmented over B that looks like

$$P^\bullet_{B/A} := \Big(...A[A[A[B]]] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} A[A[B]] \substack{\longrightarrow\\\longrightarrow} A[B] \Big) \longrightarrow B.$$

This map is a resolution of B in the category of simplicial A-algebras, and is called the canonical simplicial A-algebra resolution of B; concretely, this implies that chain underlying $P^\bullet_{B/A}$ (obtained by taking an alternating sum of the face maps as differentials) is a free resolution of B over A. Slightly more precisely, there is a model category of simplicial A-algebras, and the factorization $A \to P^\bullet_{B/A} \to B$ provides a functorial cofibrant replacement of B, and can thus be used to calculate non-abelian derived functors. We do not discuss this theory here, and will take certain results (such as the fact that such polynomial A-algebra resolutions are unique up to a suitable notion of homotopy) as blackboxes.

Using the previous construction, the main definition is:

Definition 6.1.2 (Quillen)**.** For any map $A \to B$ of commutative rings, we define its cotangent complex $L_{B/A}$, which is a complex of B-modules and viewed as an object of $D(B)$, as follows: set $L_{B/A} := \Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$, where $P^\bullet \to B$ is a simplicial resolution of B by polynomial A-algebras. Here we view the simplicial B-module $\Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ as a B-complex by taking an alternating sum of the face maps as a differential.

For concreteness and to obtain a strictly functorial theory, one may choose the canonical resolution $P^\bullet_{B/A}$ in the definition above. However, in practice, it is important to allow the flexibility of changing resolutions without changing $L_{B/A}$ (up to quasi-isomorphism). The following properties can be checked in a routine fashion, and we indicate a brief sketch of the argument:

1. If B is a polynomial A-algebra, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$: this follows because any two polynomial A-algebra resolutions of B are homotopic to each other, so we may use the constant simplicial A-algebra with value B to compute $L_{B/A}$.

2. If B and C are flat A-algebras, then $L_{B\otimes_A C/A} \simeq L_{B/A} \otimes_B C \oplus B \otimes_A L_{C/A}$: this reduces to the case of polynomial algebras by passage to resolutions. The flatness hypothesis gets used in concluding that if $P^\bullet \to B$ and $Q^\bullet \to C$ are polynomial A-algebra resolutions, then $P^\bullet \otimes_A Q^\bullet \to B \otimes_A C$ is also a polynomial A-algebra resolution.

3. Given a composite $A \to B \to C$ of maps, we have a canonical exact triangle
$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$$
in $D(C)$. To prove this, one first settles the case where $A \to B$ and $B \to C$ are polynomial maps (which reduces to a classical fact in commutative algebra). The general case then follows by passage to resolutions as the exact sequences constructed in the previous case were functorial.

4. Given a flat map $A \to C$ and an arbitrary map $A \to B$, we have $L_{B/A} \otimes_A C \simeq L_{B\otimes_A C/C}$. Again, one first settles the case of polynomial rings, and then reduces to this by resolutions, using flatness to reduce a derived base change to a classical one.

5. If $A \to B$ is étale, then $L_{B/A} \simeq 0$: for this, assume first that $A \to B$ is a Zariski localization. Then $B \otimes_A B \simeq B$, so (2) implies that $L_{B/A} \oplus L_{B/A} \simeq L_{B/A}$ via the sum map. This immediately gives $L_{B/A} = 0$ for such maps. In general, as $A \to B$ is étale, the multiplication map $B \otimes_A B \to B$ is a Zariski localization, and thus $L_{B/B\otimes_A B} \simeq 0$. By the transitivity triangle for $B \xrightarrow{i_1} B \otimes_A B \to B$, this yields $L_{B\otimes_A B/B} \otimes_{B\otimes_A B} B \simeq 0$. But, by (4), we have $L_{B\otimes_A B/B} \simeq L_{B/A} \otimes_A B$, so the base change of $L_{B/A}$ along $A \to B \to B \otimes_A B \to B$ vanishes. The latter is just the structure map $A \to B$, so $L_{B/A} \otimes_A B \simeq 0$. The standard map $L_{B/A} \to L_{B/A} \otimes_A B$ has a section coming from the B-action on $L_{B/A}$, so $L_{B/A} \simeq 0$.

6. If $B \to C$ is an étale map of A-algebras, then $L_{B/A} \otimes_B C \simeq L_{C/A}$: this follows from (3) and (5) as $L_{C/B} \simeq 0$.

7. For any map $A \to B$, we have $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$. This can be shown directly from the definition.

8. If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$. By (6), there is a natural map $L_{B/A} \to \Omega^1_{B/A}[0]$. To show this is an isomorphism, we may work locally on A by (6). In this case, there is an étale map $B' := A[x_1, ..., x_n] \to B$. We know that $L_{B'/A} \simeq \Omega^1_{B/A}[0]$ and $L_{B/B'} \simeq 0$. By (3), it follows that $L_{B/A} \simeq L_{B'/A} \otimes_{B'} B \simeq \Omega^1_{B/A}[0]$.

The main reason to introduce the cotangent complex is that it controls deformation theory in complete generality, analogous to how the tangent bundle controls deformations of smooth varieties. In particular, the following consequence is relevant to us:

ormations **Theorem 6.1.3** (Infinitesimal invariance of formally étale rings)**.** *For any ring A, write $\mathcal{C}_A$ for the category of flat A-algebras B such that $L_{B/A} \simeq 0$. Then for any surjective map $\tilde{A} \to A$ with nilpotent kernel, base change induces an equivalence $\mathcal{C}_{\tilde{A}} \simeq \mathcal{C}_A$.*

Any étale A-algebra B is an object of $\mathcal{C}_A$. In particular, for such maps, Theorem 6.1.3 captures the invariance of the étale site under infinitesimal thickenings. For our purposes, the following class of examples is crucial:

lyPerfect **Proposition 6.1.4** (Perfect rings have a trivial cotangent complex)**.** *Assume A has characteristic p. Let $A \to B$ be a flat that is relatively perfect, i.e., the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A,F_A} A \to B$ is an isomorphism. Then $L_{B/A} \simeq 0$.*

Proof. We first note that for any A-algebra B, the relative Frobenius induces the 0 map $L_{F_{B/A}} : L_{B^{(1)}/A} \to L_{B/A}$: this is clear when B is a polynomial A-algebra (as $d(x^p) = 0$), and thus follows in general by passage to the canonical resolutions. On the other hand, if $A \to B$ is relatively perfect, then $L_{F_{B/A}}$ is also an isomorphism by functoriality. The only way the 0 map can be an isomorphism is if both source and target are 0, so $L_{B/A} \simeq 0$. □

The following consequence is derived by passing to the limit from Theorem 6.1.3

ionsLimit **Corollary 6.1.5.** *Let R be a ring equipped with a nonzero divisor $f \in R$. Then reduction modulo f gives an equivalence between the category of f-adically complete and f-torsionfree R-algebras S with $R/f \to S/f$ flat and $L_{S/R} \otimes_R R/f \simeq 0$ and the category $\mathcal{C}_{R/f}$ from Theorem 6.1.3.*

Proof. We first observe that specifying an f-adically complete and f-torsionfree R-algebra S with $R/f \to S/f$ flat is the same as specifying a projective system $\{S_n\}$ of flat R/f^n-algebras with $S_n \otimes_{R/f^n} R/f^{n-1} \simeq S_{n-1}$: the functors are $S \mapsto \{S/f^n\}$ and $\{S_n\} \mapsto \lim_n S_n$. The corollary now follows immediately from Theorem 6.1.3 the observation that for such an R-algebra S, we have $L_{S/R} \otimes_R R/f \simeq L_{(S/f)/(R/f)}$. □

This leads to the following conceptual description of the Witt vector functor:

Example 6.1.6 (Witt vectors via deformation theory)**.** Let R be a perfect ring of characteristic p. Then R is relatively perfect over $\mathbf{Z}/p$. Proposition 6.1.4 tells us that $L_{R/\mathbf{F}_p} \simeq 0$, so Theorem 6.1.3 implies that R has a flat R_n to $\mathbf{Z}/p^n$ for any $n \geq 1$, and that this lift is unique up to unique isomorphism. In fact, this lift is simply given by the Witt vector construction $W_n(R)$. Setting $W(R) = \lim_n W_n(R)$ gives the Witt vectors of R, which can also be seen as the unique p-adically complete p-torsionfree $\mathbf{Z}_p$-algebra lifting R by Corollary 6.1.5. This perspective also allows one to see some additional structures on $W(R)$. For example, the identity map $R \to R$ of multiplicative monoids lifts uniquely across the map $W_n(R) \to R$: the monoid R is uniquely p-divisible, while the fiber over $1 \in R$ of $W_n(R) \to R$ is p-power torsion. Explicitly, one simply sends $r \in R$ to $\widetilde{r_n}^{p^n}$, where $\widetilde{r_n} \in W_n(R)$ denotes some lift of $r_n := r^{\frac{1}{p^n}}$. The resulting multiplicative maps $R \to W_n(R)$ and $R \to W(R)$ are called the Teichmuller lifts, and denoted by $r \mapsto [r]$. From the universal property describing $W(R)$, it is clear that if R is f-adically complete for some elemnt $f \in R$, then $W(R)$ is $(p, [f])$-adically complete.

taineAinf **Remark 6.1.7** (Fontaine's map θ and A_{inf})**.** Fix a map $A \to B$ in $\mathcal{C}_A$. With a bit more care in analyzing deformation theory via the cotangent complex, one can show the following lifting feature: if $C' \to C$ is surjective with nilpotent kernel, then every A-algebra map $B \to C$ lifts unique to an A-algebra map $B \to C'$. In particular, given a p-adically complete $\mathbf{Z}_p$-algebra C, a perfect ring R, and a map $R \to C/p$, we obtain unique lifts $W(R) \to C/p^n$ for all n, and thus a unique map $W(R) \to C$. In the perfectoid theory, this observation shows:

Proposition 6.1.8 (The kernel of θ)**.** *Given a perfectoid field K, the canonical map $\overline{\theta} : K^{\circ\flat} \to K^\circ/p$ lifts to a unique map $\theta : A_{inf}(K^{\circ\flat}) := W(K^{\circ\flat}) \to K^\circ$. The kernel of θ is a principal ideal generated by a nonzerodivisor. In fact, for K having characteristic 0, one may choose $\xi \in \ker(\theta)$ to be any element such that ξ generates the kernel of $K^{\circ\flat} \to K^\circ/p$; when K has characteristic p, we have* $\ker(\theta) = (p)$.

Proof. The first part follows from deformation theory. For the second part, the characteristic p case is clear (as θ coincides with the structure map $W(K^{\circ\flat}) \to K^{\circ\flat} = K^\circ$). In characteristic 0, choose $\xi = pu - [t]$, where $t \in K^{\circ\flat}$ is a pseudouniformizer with $t^\sharp = p \cdot u_0$ for a unit $u_0 \in K^{\circ *}$ (possible by Lemma 3.2.2), and the element $u \in W(K^{\circ\flat})$ is some unit lifting u_0 along θ (possible by surjectivity of θ), and $[t]$ is the Teichmuller lift of t. Now claim follows by observing that the inclusion $(\xi) \subset \ker(\theta)$ of p-adically complete flat $W(K^{\circ\flat})$ is bijective after reduction modulo p, and must thus be bijective by Nakayama. $\square$

From the proposition, we obtain a pushout square

$$\begin{array}{ccc} A_{inf}(K^\circ) & \xrightarrow{\theta} & K^\circ \\ \downarrow & & \downarrow \\ K^{\circ\flat} & \longrightarrow & K^\circ/p. \end{array} \tag{6.1}$$ eq:AinfDe

In characteristic p, the right vertical map and the bottom horizontal map are isomorphisms, while the remaining two maps coincide with reduction modulo p. In characteristic 0, all maps are quotients by nonzerodivisors along which the source ring is complete. In particular, in both cases, all rings involved can be viewed as pro-infinitesimal thickenings of K°/p. This perspective shall be useful later in providing an alternate description of the tilting correspondence.

6.2 Perfectoid algebras

Fix a perfectoid field K with tilt $K^\flat$. Let $t \in K^\flat$ be a pseudouniformizer such that $\pi := t^\sharp$ satisfies $|p| \leq |\pi| < 1$. Then K°/π has characteristic p, and we have a distinguished collection $\pi^{\frac{1}{p^k}} = (t^{\frac{1}{p^k}})^\sharp$ of p-power roots of π. All occurrences of almost mathematics in this section are with respect to the ideal $K^{\circ\circ}$ in the ring K°. The main players in the theory are the following algebras:

ectoidAlg **Definition 6.2.1** (Three flavours of perfectoid rings)**.** Fix K, π as above.

1. A Banach K-algebra R is *perfectoid* if $R^\circ \subset R$ is bounded, and the Frobenius map $R^\circ/\pi \to R^\circ/\pi$ is surjective. With continuous morphisms as morphisms, this gives the category Perf_K of perfectoid K-algebras.

2. A $K^{\circ a}$*-algebra* A is *perfectoid* if:

   - A is t-adically complete and flat over K°.
   - The map $K^\circ/\pi \to A/\pi$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\pi^{\frac{1}{p}} \simeq A/\pi$.

   With the evident notion of morphisms, this gives the category $\mathrm{Perf}_{K^{\circ a}}$ of perfectoid $K^{\circ a}$-algebras.

3. A $K^{\circ a}/\pi$*-algebra* A is *perfectoid* if:

   - A is flat over K°/π.
   - The map $K^\circ/\pi \to A$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\pi^{\frac{1}{p}} \simeq A$.

   With the evident notion of morphisms, this gives the category $\mathrm{Perf}_{K^{\circ a}/\pi}$ of perfectoid $K^{\circ a}/\pi$-algebras.

Remark 6.2.2 ("Strict" integral perfectoid algebras)**.** Perfectoid $K^{\circ a}$-algebras, as defined above, live in the almost category. Alternately, one may also consider "strict" perfectoid K°-algebras: these are K°-algebras A satisfying the two conditions appearing above (in Mod_{K° and not just $\mathrm{Mod}^a_{K^\circ}$) with $A \simeq A_*$. Unraveling the proof of the equivalence $\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}}$ in the proof of Theorem 6.2.5, one may check that the category of such "strict" perfectoid K°-algebras is equivalent to $\mathrm{Perf}_{K^{\circ a}}$. We prefer working with the latter as the condition $A \simeq A_*$ is somewhat subtle: the functor $(-)_*$ has non-trivial right derived functors (see Exercise 4.2.3). In particular, this condition does not work well with reduction modulo π.

fullyFlat **Remark 6.2.3** (Faithful flatness of nonzero perfectoids)**.** Nonzero flat $K^{\circ a}/\pi$-algebras A are automatically faithfully flat, and similarly over $K^{\circ a}$ for t-adically complete flat algebras. In particular, any nonzero object in $\mathrm{Perf}_{K^{\circ a}}$ or $\mathrm{Perf}_{K^{\circ a}/\pi}$ is automatically faithfully flat over the corresponding base. To see the first assertion, assume that A is a nonzero flat $K^{\circ a}/\pi$-algebra. If the structure map $K^\circ/\pi \to A$ is not faithfully flat, then there exists some ideal $I \subset K^\circ/\pi$ such that K°/I is not zero, but $A/I \simeq 0$. As we are working in the almost category, the first condition forces I to be strictly contained inside $K^{\circ\circ}$. Thus, there exists some $\varpi \in K^{\circ\circ} - I$, so $I \subset (\varpi)$ by valuation considerations. Our assumption $A/I \simeq 0$ implies that $A/\varpi \simeq 0$ as well. But $|\varpi^n| < |\pi|$ for $n \gg 0$, so K°/π is filtered by finitely many copies of K°/ϖ. As $K^\circ/\varpi \otimes A \simeq 0$, it follows that $A = K^\circ/\pi \otimes A \simeq 0$, which is a contradiction. The case of nonzero flat t-adically complete $K^{\circ a}$-algebra A is proven similarly, once one observes that $A/\varpi \simeq 0$ for a pseudouniformizer ϖ implies $A \simeq 0$ by completeness.

Example 6.2.4. We give some examples of perfectoid algebras.

1. If K has characteristic p, then a K-Banach algebra R is perfectoid if and only if it is uniform and perfect. In fact, the uniformity is implied by perfectness, see Lemma 7.1.6. Likewise, a π-adically complete and π-torsionfree $K^{\circ a}$-algebra is perfectoid if and only if it is perfect.

2. Set A to be the π-adic completion of $A^{nc} := K^\circ[x_1^{\frac{1}{p^\infty}}, ..., x_n^{\frac{1}{p^\infty}}]$. Then A^a is a perfectoid $K^{\circ a}$-algebra. Endowing $R = A[\frac{1}{\pi}]$ with the K-Banach algebra structure coming from Example 5.2.3, we get a perfectoid K-algebra: by it Lemma 5.1.3, as A^{nc} is totally integrally closed in $A^{nc}[\frac{1}{\pi}]$, we learn that $R^\circ = A$, so everything follows.

One of the first miraculous results about these objects is that the definitions above capture the same properties. More precisely, the resulting categories are naturally identified:

g0toMixed **Theorem 6.2.5** (Tilting from characteristic 0 to mixed characteristic)**.** *There are canonical equivalences*

$$\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi}.$$

The functors implementing the first equivalence are $R \mapsto R^{\circ a}$ and $A \mapsto A_*[\frac{1}{t}]$, as in Proposition 5.2.5. The second equivalence is given by reduction modulo π, i.e., $A \mapsto A/\pi$.

Proof of the equivalence $\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}}$. We first check that given $R \in \mathrm{Perf}_K$, the $K^{\circ a}$-algebra $A = R^{\circ a}$ is perfectoid. By Proposition 5.2.5 and the easy direction of Lemma 4.4.1 (3), we already know that $A := R^{\circ a}$ is t-adically complete and flat over K°. Moreover, the perfectoidness of R ensures that the Frobenius $A/\pi \to A/\pi$ is surjective. To understand the kernel, as almostification is exact, it is enough to show that $R^\circ/\pi \to R^\circ/\pi$ has kernel $(\pi^{\frac{1}{p}})$. For this, fix some $\alpha \in R^\circ$ such that $\alpha^p \in \pi R^\circ$. Then $\pi^{-\frac{1}{p}} \cdot \alpha \in R$ has its p-th power in R°, and thus must itself lie in R° by the definition of powerboundedness. It follows that $\alpha \in \pi^{\frac{1}{p}} R^\circ$, as wanted. Thus, we have constructed a functor $\mathrm{Perf}_K \to \mathrm{Perf}_{K^{\circ a}}$.

Next, we check that given $A \in \mathrm{Perf}_{K^{\circ a}}$, the ring A_* is t-adically complete, t-torsionfree, p-root closed in $A_*[\frac{1}{\pi}]$, and has a surjective Frobenius modulo π; this will construct a left-inverse to the functor from the previous paragraph by Proposition 5.2.6 (2). The completeness results from Lemma 4.4.1 (3), while the t-torsionfreeness is clear.

For p-root closedness: observe that $A_*/\pi^{\frac{1}{p}} \subset (A/\pi^{\frac{1}{p}})_* \xrightarrow{Frob} (A/\pi)_*$ is injective, and thus so is $A_*/\pi^{\frac{1}{p}} \xrightarrow{Frob} A_*/\pi$. Thus, if $y \in A_*$ with $y^p \in \pi A_*$, then $y \in \pi^{\frac{1}{p}} A_*$. Now fix some $x \in A_*[\frac{1}{\pi}]$ with $x^p \in A_*$. Then we can write $y = \pi^{\frac{k}{p}} x \in A_*$ for a positive integer $k \geq 1$; we shall check that it is always possible to lower k by 1, which proves that $x \in A_*$ by induction. Taking powers gives $y^p = \pi^k x^p \in \pi^k A_*$. As $k \geq 1$, we have $y^p \in \pi A_*$, and thus $y \in \pi^{\frac{1}{p}} A_*$ by the injectivity of Frobenius. Thus, $\pi^{\frac{k-1}{p}} x = \frac{y}{\pi^{\frac{1}{p}}} \in A_*$, which proves the assertion about lowering k.

For surjectivity of Frobenius on A_*/π: as almost surjectivity is clear, it is enough to show surjectivity of Frobenius on $A_*/K^{\circ\circ}A_*$. Take some $x \in A_*$. Choose some $c < 1$. Then almost surjectivity implies that $\pi^c x = y^p \mod \pi A_*$ for some $y \in A_*$. But then $z := \frac{y}{\pi^{\frac{c}{p}}} \in A_*[\frac{1}{\pi}]$ satisfies $z^p \in A_*$ (as $c < 1$), and thus $z \in A_*$ by the previous paragraph. We conclude that $y \in \pi^{\frac{c}{p}} A_*$. Dividing the previous equality by π^c (which is allowed as A_* is π-torsionfree), we get $x = z^p \mod \pi^{1-c} A_*$. As $c < 1$, this shows $x = z^p \mod K^{\circ\circ} A_*$, as wanted.

Finally, to check that this construction also provides a right inverse, we must show the following: given $A \in \mathrm{Perf}_{K^{\circ a}}$, we have $A \simeq R^{\circ a}$ for R being the K-Banach algebra associated to A_*. But Proposition 5.2.5 shows that $A_* \simeq R^\circ$, so the claim follows by almostification. □

Proof of the equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi}$. There is an obvious functor $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\pi}$ given by reduction modulo π. To construct the inverse, we recall some deformation theory.

Write $\mathcal{C}_n$ for the category of flat K°/π^n-algebras B_n with such that the relative Frobenius $K^\circ/\pi \to B_n/\pi$ is an isomorphism. By deformation theory (see Theorem 6.1.3 and Proposition 6.1.4), we have $\mathcal{C}_{n+1} \simeq \mathcal{C}_n$ via the reduction map. Moreover, by taking inverse limits, these categories are also identified with $\mathcal{C}$, the category of t-adically complete and flat K°-algebras B such that the relative Frobenius for $K^\circ/\pi \to B$ is an isomorphism. Write $B \mapsto \widetilde{B}$ for the inverse equivalence $\mathcal{C}_1 \simeq \mathcal{C}$.

Now say $A \in \mathrm{Perf}_{K^{\circ a}/\pi}$. We shall show that A deforms uniquely to a perfectoid $K^{\circ a}$-algebra. For this, we may assume A is nonzero, and thus faithfully flat over $K^{\circ a}/\pi$ by Remark 6.2.3. Using the functor $(-)_{!!}$ from Remark 4.2.8, we have $A_{!!} \in \mathcal{C}_1$: the functor $(-)_{!!}$ preserves faithful flatness, pushout diagrams (or all colimits), and carries Frobenius to Frobenius (in characteristic p). We have the corresponding lift $\widetilde{A_{!!}} \in \mathcal{C}$ to K°. Write $\widetilde{A} := \widetilde{A_{!!}}^a$ for the corresponding almost algebra. Then $\widetilde{A_{!!}}$ is K°-flat and t-adically complete, so the same holds true for $\widetilde{A}$ (as almostification preserves limits, colimits and flatness). Moreover, we have $\widetilde{A}/\pi = \widetilde{A_{!!}}^a/\pi \simeq (\widetilde{A_{!!}}/\pi)^a \simeq A_{!!}^a \simeq A$. Thus, the construction $A \mapsto \widetilde{A}$ sending A to the almostification of the unique lift to $\mathcal{C}$ of $A_{!!}$ provides a right-inverse to the canonical projection $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\pi}$.

It remains to check that the functor in the previous paragraph also gives a left-inverse. Fix some $A \in \mathrm{Perf}_{K^{\circ a}}$. We want to show that $A \simeq \widetilde{(A/\pi)}$. We may assume $A \neq 0$, and thus A is faithfully flat by Remark 6.2.3. By the preservation of colimits and faithful flatness under $(-)_{!!}$, the ring $A_{!!}$ is a faithfully flat K°-algebra with $K^\circ/\pi \to A_{!!}/\pi \simeq (A/\pi)_{!!}$ relatively perfect. By Lemma 4.4.1, the ring A_* is complete, and hence so is $A_{!!}$: the canonical map $A_{!!} \to A_*$ is injective with almost zero cokernel, so we can apply Lemma 4.4.2 to conclude that $A_{!!}$ is complete. Thus $A_{!!} \in \mathcal{C}$. The corresponding object of $\mathcal{C}_1$ is given by $A_{!!}/\pi \simeq (A/\pi)_{!!}$ as $(-)_{!!}$ commutes with colimits. The construction in the previous paragraph then shows that $\widetilde{A/\pi}$ is the almostification of the unique lift to $\mathcal{C}$ of $(A/\pi)_{!!}$, and thus $\widetilde{A/\pi} \simeq A^a_{!!} \simeq A$, as wanted. $\square$

Putting everything together, we get the tilting and untilting functors:

Criterion

Corollary 6.2.6 (Tilting from characteristic 0 to characteristic p)**.** *We have a chain of equivalences*

$$\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi} \simeq \mathrm{Perf}_{K^{\circ \flat a}/t} \simeq \mathrm{Perf}_{K^{\circ \flat a}} \simeq \mathrm{Perf}_{K^\flat}.$$

Under this equivalence $R \in \mathrm{Perf}_K$ corresponds to the unique $S \in \mathrm{Perf}_{K^\flat}$ such that we have an identification $R^{\circ a}/\pi \simeq S^{\circ a}/t$ living over the identification $K^\circ/\pi \simeq K^{\flat\circ}/t$. In this case, we call $S := R^\flat$ the tilt of R, and $R := S^\sharp$ untilt of S.

Proof. This follows from Theorem 6.2.5 by observing that $K^\circ/\pi \simeq K^{\flat\circ}/t$. $\square$

Unraveling the proof of the previous theorem, we get an explicit formula for the functor $\mathrm{Perf}_K \to \mathrm{Perf}_{K^\flat}$ via Fontaine's functor from Definition 2.0.1.

ngFormula

Theorem 6.2.7 (Tilting via Fontaine's functors)**.** *Let $R \in \mathrm{Perf}_K$.*

1. *The tilt $R^\flat$ is naturally isomorphic to $R^{\circ\flat}[\frac{1}{t}]$.*

2. *The multiplicative identification $R^{\circ\flat} \simeq \lim_{x\mapsto x^p} R^\circ$ of Lemma 2.0.6 extends to a multiplicative bijection $R^\flat \simeq \lim_{x\mapsto x^p} R$. Write $\sharp : R^\flat \to R$ for the resulting map.*

3. *We have $R^{\flat\circ} = R^{\circ\flat}$ under the identification in (1).*

Proof. For (1), set a_n to be the composite map $K^{\flat\circ}/t^{p^n} \overset{\phi^{-n}}{\simeq} K^{\flat\circ}/t \simeq K^\circ/\pi \to R^\circ/\pi$, where he last map is the structure map. This fits into a commutative diagram

$$\begin{array}{ccccc} K^{\flat\circ}/t^{p^n} & \xrightarrow[\simeq]{\phi^{-n}} & K^{\flat\circ}/t \simeq K^\circ/\pi & \longrightarrow & R^\circ/\pi \\ & \searrow^{std} & \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \phi^n} \\ & & K^{\flat\circ}/t \simeq K^\circ/\pi & \longrightarrow & R^\circ/\pi. \end{array}$$

The square on the right is a pushout square as R is perfectoid. The horizontal arrow on the bottom is a_0, while the one on top is a_n. It follows that a_n is flat and relatively perfect, and that it is the unique flat and relatively perfect lift of a_0 along $K^{\flat\circ}/t^{p^n} \to K^{\flat\circ}/t$ (by Theorem 6.1.3). In particular, a_{n+1} is the unique lift of a_n along $K^{\flat\circ}/t^{p^{n+1}} \to K^{\flat\circ}/t^{p^n}$. Taking inverse limits over n, we see that the structure map $K^{\flat\circ} \simeq K^{\circ\flat} \to R^{\circ\flat}$, which is just the map $\lim a_n$, is (the necessarily unique) flat and relatively perfect lift of a_0 to a t-adically complete and t-torsionfree $K^{\flat\circ}$-algebra. The characterization of the correspondence in Corollary 6.2.6 then implies that $R^{\circ\flat a} \simeq R^{\flat\circ a}$, and thus $R^\flat = R^{\circ\flat}[\frac{1}{t}]$.

For (2), as $t^\sharp = \pi$, the identification $R^{\circ\flat} \simeq \lim_{x\mapsto x^p} R^\circ$ induces, after inverting t, a multiplicative map

$$R^\flat \simeq (\lim_{x\mapsto x^p} R^\circ)[\frac{1}{t}] \to \lim_{x\mapsto x^p}(R^\circ[\frac{1}{\pi}]) \simeq \lim_{x\mapsto x^p} R.$$

We must check that this map is bijective. Injectivity is clear as t is a nonzerodivisor on either side. For surjectivity, given $(f_n) \in \lim_{x\mapsto x^p} R$, we want to find some $c \geq 0$ such that $\pi^{\frac{c}{p^n}} f_n \in R^\circ$ for all n. But $R^\circ \subset R^\circ[\frac{1}{\pi}]$ is p-root-closed, and we have $f_{n+1}^p = f_n$ for all n. So any $c \geq 0$ such that $\pi^c f_0 \in R^\circ$ solves the problem.

For the last assertion, we must check that $R^{\circ\flat} \subset R^{\circ\flat}[\frac{1}{t}] = R^\flat$ is totally integrally closed. Fix some $f \in R^\flat$ such that $t^c f^{\mathbf{N}} \in R^{\circ\flat}$. Applying $\sharp$, we learn that $f^\sharp \in R$ satisfies $\pi^c (f^\sharp)^{\mathbf{N}} \in R^\circ$. This means $f^\sharp$ is powerbounded, so $f^\sharp \in R^\circ$. As $R^\circ \subset R$ is p-root closed, all p^n-th roots of $f^\sharp$ also lie in $R^\sharp$. But then the sequence $(f_n) \in \lim_{x\mapsto x^p} R$ corresponding to f under $\sharp$ lies in $\lim_{x\mapsto x^p} R^\circ$, so $f \in R^\circ$, as wanted. □

Remark 6.2.8 (Untilting via A_{inf})**.** Remark 6.1.7 can be used to give an alternate perspective on the tilting corresondence, and explicit description of the untilting functor, Theorem 6.2.7. Remark 6.1.7 gave us the pushout square

$$\begin{array}{ccc} A_{inf}(K^\circ) & \xrightarrow{\theta} & K^\circ \\ \downarrow & & \downarrow \\ K^{\circ\flat} & \longrightarrow & K^{\circ\flat}/t \simeq K^\circ/\pi, \end{array}$$

where all rings could be viewed as pro-infinitesimal thickenings of K°/π. In particular, by Proposition 6.1.5, any relatively perfect (or, equivalentlly, perfect) $K^{\circ\flat}$-algebra A has unique lift $W(A)$ along $A_{inf}(K^\circ) \to K^{\circ\flat}$, and the base change $W(A) \otimes_{A_{inf}(K^\circ)} K^\circ$ provides a K°-algebra that lifts $A \otimes_{K^{\circ\flat}} K^\circ/\pi$ by the above diagram. It follows from this observation and Corollary 6.2.6 that if $S \in \mathrm{Perf}_{K^\flat}$, then its untilt $S^\sharp \in \mathrm{Perf}_K$ is given by

$$S^\sharp := \big(W(S^\circ) \otimes_{A_{inf}(K^\circ)} K^\circ\big)[\frac{1}{\pi}].$$

erfectoid **Remark 6.2.9** (Limits and colimits)**.** Any of the equivalent categories in Corollary 6.2.6 has all limits and colimits. It is enough to show this for $\mathrm{Perf}_{K^{\circ\flat a}}$. This is clear for limits: the propertes of being t-adically complete, t-torsionfree, and perfect all pass through limits. For colimits, given a diagram $\{A_i\}$ in $\mathrm{Perf}_{K^{\circ\flat a}}$, the colimit A in computed by simply t-adically completing the perfection of the colimit of the A_i's in the category of all $K^{\circ\flat a}$-algebras. This will give a perfect t-adically complete ring; on any such ring, the t-power torsion is almost zero (by the argument of Proposition 4.3.4), so we also get flatness. Likewise, one checks that filtered colimits in $\mathrm{Perf}_{K^{\circ a}}$ are computed by simply π-adically completing the underlying filtered colimit of $K^{\circ a}$-algebras.

We can now formulate the almost purity theorem:

thm:APT **Theorem 6.2.10** (Almost purity theorem)**.** *Fix a perfectoid K-algebra R with tilt S.*

1. *Almost purity in characteristic p: Inverting t gives an equivalence $S^\circ_{afet} \simeq S_{fet}$.*

2. *Almost purity in characteristic* 0*: Inverting* p *gives an equivalence* $R^\circ_{afet} \simeq R_{fet}$.

3. *Tilting and untilting functors induce equivalences* $R^\circ_{afet} \simeq S^\circ_{afet}$ *and* $R_{fet} \simeq S_{fet}$.

We do not prove the full statement at the moment. Instead, we shall construct construct fully faithful functors

$$S^\flat_{fet} \xleftarrow{a} S^{\circ a}_{afet} \xrightarrow{b} (S^{\circ a}/t)_{afet} \simeq (R^{\circ a}/\pi)_{afet} \xleftarrow{c} R^{\circ a}_{afet} \xrightarrow{d} R_{fet},$$

and show that all but (d) are equivalences. We also prove (d) is an equivalence when R is a perfectoid field; the general case of (d) shall be reduced to the field case by localizing on the perfectoid space attached to a perfectoid algebra.

Construction of functors. In Theorem 4.3.6, we have already seen that $S_{fet} \simeq S^\circ_{afet}$ by passing to powerbounded elements and inverting t respectively. This gives the equivalence (a).

To construct the equivalences (b) and (c), note that tilting and untilting give equivalences

$$\mathrm{Perf}_{R^{\circ a}} \simeq \mathrm{Perf}_{R^{\circ a}/\pi} \simeq \mathrm{Perf}_{S^{\circ a}/t} \simeq \mathrm{Perf}_{S^{\circ a}}$$

by passing to comma categories along the analogous equivalences for K from Corollary 6.2.6. Now if $T \in R^{\circ a}_{afet}$, then T is itself a perfectoid $K^{\circ a}$-algebra: the t-adic completeness and flatness of $R^{\circ a}$ passes to almost finite projective modules over it, and the weakly étale map $R^{\circ a} \to T^{\circ a}$ is relatively perfect modulo π by the almost analog of Lemma 4.3.8, so $K^{\circ a}/\pi \to T/\pi$ is also relatively perfect. Applying the same reasoning to $S^{\circ a}$, $S^{\circ a}/t$ and $R^{\circ a}/t$, we get the fully faithful functors b and c. Moreover, these are equivalences by deformation theory: this is the almost analog of the deformation invariance of the category of finite étale maps, and is proven in [GR, Theorem 5.3.27].

The functor (d) is given $A \mapsto A_*[\frac{1}{\pi}]$, with the Banach algebra structure as prescribed in Example 5.2.3: as any $A \in R^{\circ a}_{afet}$ is a perfectoid $K^{\circ a}$-algebra (see previous paragraph), this recipe turns $A_*[\frac{1}{\pi}]$ into a perfectoid R-algebra with the powerbounded ring being A_* by Theorem 6.2.5. The resulting functor d is clearly faithful. For fullness, we shall recover A_* from $A_*[\frac{1}{\pi}]$ as the total integral closure A_{tic} of R° in $A_*[\frac{1}{\pi}]$; this implies the fullness as the formation of the total integral closure of R° in an R-algebra T is functorial in the R-algebra T (without any topology). For the claim, as A_* is the powerbounded subring of the uniform Banach K-algebra $A_*[\frac{1}{\pi}]$, we know that A_* is an R°-algebra that is totally integrally closed in $A_*[\frac{1}{\pi}]$ by Proposition 5.2.5, and hence contains A_{tic}. Conversely, fix some $f \in A_*$. As A is almost finitely generated over R°, the set $\pi \cdot f^{\mathbf{N}}$ lies in a finitely generated R°-submodule of A_*. But this means that f lies in the total integral closure of R° in A_*, so $f \in A_{tic}$. □

Proof of almost purity over a field. We now specialize to the case $R = K$ is a perfectoid field of characteristic 0. By the previous arguments, it suffices to show that the untilting functor $\sharp : K^\flat_{fet} \to K_{fet}$ is an equivalence. This functor is fully faithful (as this is true on all perfectoid algebras), preserves degrees (by construction), and preserves automorphism groups (by full faithfulness). In particular, it preserves Galois extensions. By Galois theory, this reasoning shows the following: for any Galois extension $L/K^\flat$, we have a bijective correspondence between K-subfields of $L^\sharp$

and $K^\flat$-subfields of L. In particular, any subfield of such an $L^\sharp$ is obtained by untilting. It thus suffices to show that every finite extension of K embeds into some $L^\sharp$. For this, let $M = \widehat{\overline{K^\flat}}$ be a completed algebraic closure of $K^\flat$, so M is an algebraically closed perfectoid field. Its untilt $M^\sharp$ is then algebraically closed by Proposition 3.2.10. Now $\widehat{\overline{K^\flat}}$ is the filtered colimit in perfectoid $K^\flat$-algebras of all finite Galois extensions $L/K^\flat$ contained in M. By tilting and Remark 6.2.9, it follows that $M^\sharp$ is the filtered colimit in uniform Banach K-algebras of finite Galois extensions of the form $L^\sharp/K$ for a finite Galois extension $L/K^\flat$. Write $N \subset M^\sharp$ for the subfield obtained by taking the filtered colimit (as abstract rings) of all the $L^\sharp$'s. Then N is algebraic over K (clear), and $N \subset M$ is dense (by construction of the colimit, the valuation ring $M^{\sharp,\circ}$ is the completion of filtered colimit of the valuation rings $L^{\sharp,\circ}$). By Krasner's lemma and the algebraic closedness of M, we learn that N is also algebraically closed. In particular, every finite Galois extension of K embeds into N, and hence into some $L^\sharp$, as wanted. □

tExplicit **Example 6.2.11** (Explicitly untilting "perfectly finitely presented" quotients)**.** Fix a perfectoid K-algebra R with tilt $R^\flat$. Let $P = R^\circ$, and $P^\flat = R^{\flat\circ}$, so $P^\flat$ is also the tilt of P in the sense of Definition 2.0.1 via Theorem 6.2.7 (3). Now fix finitely many elements $f_1, ..., f_r \in P^\flat$. Each $f_i \in P^\flat$ gives rise to a perfect element $f_i^\sharp \in P$. Set

$$A^\sharp := \left(P/((f_1^\sharp)^{\frac{1}{p^\infty}}, ..., (f_r^\sharp)^{\frac{1}{p^\infty}})\right)^\wedge \quad \text{and} \quad A := \left(P^\flat/(f_1^{\frac{1}{p^\infty}}, ..., f_r^{\frac{1}{p^\infty}})\right)^\wedge,$$

where the completions are π-adic and t-adic respectively. We shall check that these are both perfectoid and related under tilting. More precisely:

Proposition 6.2.12. *With the notation as above, we have:*

1. *The $K^{\flat\circ a}$-algebra A^a is a perfectoid $K^{\flat\circ a}$-algebra.*
2. *The $K^{\circ a}$-algebra $A^{\sharp a}$ is a perfectoid $K^{\circ a}$-algebra that tilts to A^a.*

Proof. For (1), it is enough to check that A is perfect, t-adically complete, and almost t-torsionfree. The perfectness and completeness are clear (as the completion of a perfect ring is perfect). Also, A is almost t-torsionfree by perfectness via the standard argument (see the proof of Proposition 4.3.4). This gives (1).

To proceed further, we give a different description of A. Let

$$K_m := \mathrm{Kos}(P^\flat; f_1^{\frac{1}{p^m}}, ..., f_r^{\frac{1}{p^m}}) := \otimes_{i=1}^r \mathrm{Kos}(P^\flat, f_i^{\frac{1}{p^m}}) := \otimes_{i=1}^r \left(P^\flat \xrightarrow{f_i^{\frac{1}{p^m}}} P^\flat\right)$$

be the displayed Koszul complex, normalized so that the lowest (homological) degree term sits in degree 0. As $f_i^{\frac{1}{p^{m+1}}} \mid f_i^{\frac{1}{p^m}}$, there are obvious transition maps $K_m \to K_{m+1}$ inducing the identity on the degree 0 terms, and we set $K_\infty := \mathrm{colim}\, K_m$ to be the direct limit. This is a complex of flat $P^\flat$-modules, and can also be described

$$K_\infty = \otimes_{i=1}^r \operatorname*{colim}_m \left(P^\flat \xrightarrow{f_i^{\frac{1}{p^m}}} P^\flat\right).$$

When $r = 1$, the calculation in Example 4.1.3 shows that K_∞ is discrete, i.e., $H^i(K_\infty) = 0$ for $i \neq 0$; using induction, one can then show the same for any r, though we do not do so here[1]. Moreover, $H^0(K_\infty)$ is the perfect $P^\flat$-algebra $P^\flat/(f_1^{\frac{1}{p^\infty}}, ..., f_r^{\frac{1}{p^\infty}})$ (by construction), and hence its π-torsion is almost zero. Set $\widehat{K_\infty}$ to be the t-adic completion of K_∞ (either at the level of complexes, or equivalently in the derived sense). Passing to the almost category, we get an almost quasi-isomorphism

$$\widehat{K_\infty} \overset{a}{\simeq} A \quad \text{and thus} \quad \widehat{K_\infty}/t \overset{a}{\simeq} A/t; \tag{6.2}$$

eq:Explic

here we use that derived t-completions commute with almostification, and that the derived t-completion of an almost t-torsionfree module is almost isomorphic to its classical t-adic completion.

We now prove (2). Let

$$M_m := \mathrm{Kos}(P; (f_1^\sharp)^{\frac{1}{p^m}}, ..., (f_r^\sharp)^{\frac{1}{p^m}}) = \otimes_{i=1}^r \mathrm{Kos}(P, (f_i^\sharp)^{\frac{1}{p^m}})$$

be the analog over P of the Koszul complexes used above. Again, we have obvious transition maps $M_m \to M_{m+1}$. Set $M_\infty = \mathrm{colim}_m M_m$ to be the direct limit, and let $\widehat{M_\infty}$ be its π-adic completion. As the identification $P/\pi \simeq P^\flat/t$ carries $f_i^\sharp$ to f_i, we have an obvious quasi-isomorphism

$$\widehat{M_\infty}/\pi \simeq \widehat{K_\infty}/t. \tag{6.3}$$

eq:Explic

Using (6.2), this implies that $\widehat{M_\infty}/\pi$ is almost discrete, i.e., $H^i(\widehat{M_\infty}/\pi) \overset{a}{\simeq} 0$ for $i \neq 0$. The same then holds true for $\widehat{M_\infty}/\pi^n$ as well. Thus, we get an almost quasi-isomorphism

$$\widehat{M_\infty} := R\lim_n M_\infty/\pi^n \overset{a}{\simeq} R\lim_n P/(\pi^n, (f_1^\sharp)^{\frac{1}{p^\infty}}, ..., (f_r^\sharp)^{\frac{1}{p^\infty}}) =: A^\sharp, \tag{6.4}$$

eq:Explic

where the second almost quasi-isomorphism arises by observing that M_∞/π^n is almost discrete and explicitly calculating $H^0(M_\infty/\pi^n)$. This formula shows that

$$\left(A^\sharp \xrightarrow{\pi} A^\sharp\right) \simeq A^\sharp \otimes^L_{K^\circ} K^\circ/\pi \overset{a}{\simeq} \widehat{M_\infty}/\pi$$

is almost discrete, and thus $A^\sharp$ is almost π-torsionfree. Thus, $A^\sharp$ is a π-adically complete and almost π-torsionfree K°-algebra. Equations (6.2), (6.3), (6.4) give an identification

$$A^\sharp/\pi \overset{a}{\simeq} A/t.$$

This shows that $K^\circ/\pi \to A^\sharp/\pi$ is relatively perfect in the almost sense, thus proving that $A^\sharp$ is a perfectoid $K^{\circ a}$-algebra; secondly, this formula then shows that $A^\sharp$ is related to A via tilting, thus proving (2). □

[1] In more fancy language, the perfection of any simplicial commutative $\mathbf{F}_p$-algebra, or even any E_∞-$\mathbf{F}_p$-algebra, is discrete (see [BS, §11]). In particular, given a diagram $B \leftarrow A \to C$ of perfect $\mathbf{F}_p$-algebras, we have $\mathrm{Tor}_i^A(B, C) = 0$ for $i > 0$. This recovers the case at hand as K_∞ is the perfection of K_0, which is the simplicial commutative $P^\flat$-algebra obtained by freely setting $f_i = 0$ for $i = 1, ...r$.

We shall later need the following criterion for detecting when a perfectoid K-algebra is a perfectoid field.

PerfField **Lemma 6.2.13.** *A perfectoid K-algebra R is a perfectoid field if and only if $R^\flat$ is a perfectoid field.*

Proof. It is clear that if R is a perfectoid field, so is $R^\flat$. For the converse, write $|\cdot|$ for the norm on K. Consider the spectral norm on R defined as follows:

$$||f||_R = \inf\{|t| \mid t \in K^*, f \in tR^\circ\}.$$

For any uniform Banach K-algebra R, this is a submultiplicative continuous map $R \to \mathbf{R}_{\geq 0}$ that defines the topology on R, i.e., a neighbourhood basis of 0 is given by sets of the form $||\cdot||_R^{-1}([0,\epsilon))$ for $\epsilon \in \mathbf{R}_{\geq 0}$. Thus, we must check that if $R^\flat$ is a perfectoid field, then the following hold:

1. $||\cdot||_R$ gives a NA valuation on R, i.e., $||\cdot||_R$ is multiplicative.
2. R is a field.

Indeed, (1) and (2) immediately show that R is a NA field for its given topology, as wanted. To prove these, we observe that since $R^\flat$ is a perfectoid field, its NA valuation is necessarily given by its spectral norm $||\cdot||_{R^\flat}$. The latter coincides with the map $R^\flat \xrightarrow{\sharp} R \xrightarrow{||\cdot||_R} \mathbf{R}_{\geq 0}$: for an element $f \in R^\flat$, we have $f \in R^{\flat\circ}$ exactly when $f^\sharp \in R^\circ$.

We now show (1). Say $f, g \in R$. As $||\cdot||$ is commutes with scalar multiplication and extends the norm on K, we may rescale to assume that $f, g \in R^\circ - \pi^{\frac{1}{p}}R^\circ$. Choose $a, b \in R^{\flat\circ}$ such that both $a^\sharp - f$ and $b^\sharp - g$ lie in πR°; this is possible as $R^{\flat\circ} \simeq R^{\circ\flat}$ surjects onto R°/π (see Theorem 6.2.7 (3)). As $R^\flat$ is a perfectoid field and $\sharp$ is multiplicative, using our hypothesis that $f, g \notin \pi^{\frac{1}{p}}R^\circ$, we have $a, b \notin t^{\frac{1}{p}}R^{\flat\circ}$, so $ab \notin tR^{\flat\circ}$, and thus $fg - (ab)^\sharp \in \pi R^\circ$ with $fg \notin \pi R^\circ$. In particular, f, g, and fg are all nonzero modulo πR°. Now, as $f \notin \pi R^\circ$, for any $c < 1$ in the value group of K, one has $f \in \pi^c R^\circ$ exactly when $a^\sharp \in \pi^c R^\circ$; the latter happens exactly when $a \in t^c R^{\circ\flat}$. Thus, we learn that $||f||_R = ||a||_{R^\flat}$. Similarly, we learn that $||g||_R = ||b|_{R^\flat}$ and $||fg||_R = ||ab||_{R^\flat}$. The claim now follows from the multiplicativity of $||\cdot||_{R^\flat}$.

For (2), fix some $f \in R - \{0\}$. We may scale f to assume $f \in R^\circ - \pi^{\frac{1}{p}}R^\circ$. Choose $a \in R^\flat$ such that $f = a^\sharp + \pi g$ with $a \in R^\flat - \{0\}$ and $g \in R^\circ$. As $R^\flat$ is a field, there is some $b \in R^\flat$ with $ab = 1$. Our hypothesis $f \notin \pi R^\circ$ implies that $||f||_R = ||a||_{R^\flat}$. Now we have

$$||\pi||_R < ||\pi^{\frac{1}{p}}||_R \leq ||f||_R = ||a||_{R^\flat} \leq 1.$$

Multiplying by bg shows that $||\pi b^\sharp g||_R < 1$. One then checks that the following formula gives an inverse to f

$$b^\sharp \cdot \frac{1}{a^\sharp + \pi g} := b^\sharp \cdot \frac{1}{1 + \pi b^\sharp g} := b^\sharp \cdot \Big(\sum_{i=0}^{\infty} (-1)^i (\pi b^\sharp g)^i\Big),$$

as wanted. ∎

Chapter 7

Adic spaces

In this chapter, we pause the development of the perfectoid theory to introduce the adic spectrum as defined by Huber [Hu1]. The input here (defined in §7.2) is a pair (A, A^+) where A is a special type of topological ring called a Tate ring (defined in §7.1), and $A^+ \subset A$ is an open and integrally closed subring; the hypotheses ensure that $\mathrm{Spec}(A) \to \mathrm{Spec}(A^+)$ is an open immersion with complement defined by the radical $A^{\circ\circ} \subset A^+$ of a principal ideal, and the main goal is to study a space that can be thought of as "a punctured tubular neighbourhood" of the Zariski closed set $\mathrm{Spec}(A^+/A^{\circ\circ})$ inside $\mathrm{Spec}(A^+)$. More precisely, the output is a spectral space $X := \mathrm{Spa}(A, A^+)$ equipped with a distinguished basis of quasi-compact open subsets $U \subset X$ called rational sets (defined and studied in §7.3 and §7.4), as well as a *presheaf* $\mathcal{O}_X$ on X (defined in §7.5).

With the exception of recourse to some basic facts about spectral spaces, we have attempted to keep the exposition essentially self-contained, relying only on standard commutative algebra facts about valuation rings in lieu of the somewhat more advanced and subtle constructions with value groups employed by Huber.

7.1 Tate rings

TateRings

We shall restrict attention to the following class of topological rings:

Definition 7.1.1. A topological ring A is called *Tate* if there exists an open subring A_0 such that the induced topology on A_0 is t-adic for some $t \in A_0$ that becomes a unit in A; any such A_0 is called *a ring of definition*, the element t is called *a pseudouniformizer*, and the pair (A, t) is called *a couple of definition*. A morphism of Tate rings is just a continuous morphism of topological rings.

Example 7.1.2. Let K be a NA field, and let R be a K-Banach algebra. Then R is Tate: a couple of definition is given by $(R_{\leq 1}, t)$, where $t \in K$ is a pseudouniformizer.

Remark 7.1.3. Somewhat more generally, a topological ring A is called *Huber* if there exists an open subring A_0 such that the induced topology on A_0 is I-adic for some finitely generated ideal I. A Huber ring A is Tate if and only if there exists a unit $t \in A$ such that $t^n \to 0$ as $n \to \infty$. It follows that given a continuous morphism of Huber rings $A \to B$, if A is Tate, so is B. As we shall restrict attention to perfectoid geometry over a perfectoid field, this forces all rings under consideration to be Tate. Thus, we do not develop the theory in the more general context of Huber rings in these notes.

Fix a Tate ring A. We enumerate some basic properties and structures for such a ring; the proof of any unproven assertion is straightforward from the definitions, and left to the reader.

1. *Algebraic description of A.* For any couple of definition (A_0, t), we have $A = A_0[\frac{1}{t}]$. Indeed, the ideals $(t^n A_0)$ of A_0 give a basis of open neighbourhoods of the induced topology on A_0, and thus also for A as $A_0 \subset A$ is open. It follows that for any $f \in A$, by continuity of left multiplication by f, we must have $t^n f \in A_0$ for $n \gg 0$, and thus $f \in A_0[\frac{1}{t}]$. In particular, given *any* pair (B, f) comprising a commutative ring B_0 with an element $f \in B_0$, we get a Tate ring $B_0[\frac{1}{f}]$ with couple of definition $(B_0/(\text{f-torsion}), f)$.

2. *Bounded sets.* A subset $S \subset A$ is *bounded* if for some couple of definition (A_0, t), we have $S \subset t^{-n} A_0$ for some $n \geq 0$. Bounded subsets have easily verified stability properties: finite unions of bounded subsets are bounded, and the A_0-submodule of A generated by a bounded subset is bounded. It is also easy to see that any ring of definition of bounded[1], so one may replace "some" with "any" in the previous definition. Moreover, given a couple of definition (A_0, t), any open bounded subgroup $B \subset A$ satisfies $t^n A_0 \subset B \subset t^{-n} M$ for some $n \geq 0$. It follows that if B is an open bounded ring, then B is a ring of definition; conversely, any ring of definition is open and bounded. Using these properties, one checks that the collection of all rings of definition of A is filtered.

3. *Powerbounded elements.* An element $f \in A$ is *powerbounded* if $f^{\mathbf{N}}$ is bounded, i.e., given a couple of definition (A_0, t), we have $t^c f^{\mathbf{N}} \subset A_0$ for some $c \geq 0$. The collection A°, called the ring of *powerbounded elements*, of all such elements is then an A_0-subalgebra of A. As each ring of definition is bounded, it follows that A° contains all rings of definition of A. Conversely, for any $f \in A^\circ$ and couple of definition (A_0, t), the subring $A_0[f] \subset A$ is immediately seen to be open (obvious) and bounded (as the A_0-submodule of A generated by a bounded set is bounded), and is thus also a ring of definition. It follows that A° is the filtered direct limit of all rings of definition (or, equivalently, open bounded subrings) of A. Using this description, one checks that A° is integrally closed in A: if $f \in A$ is integral over A°, then f must be integral over some ring of definition A_0 of A, whence the subring $A_0[f] \subset A$ is bounded (being a finitely generated A_0-submodule of A), and thus $A_0[f] \subset A^\circ$, so $f \in A^\circ$. We shall say that A is *uniform* if A° is bounded.

4. *Topologically nilpotent elements.* An element $f \in A^{\circ\circ}$ is *topologically nilpotent* if $f^n \to 0$, i.e., for any couple of definition (A_0, t) and any $n \geq 0$, we have $f^m \in t^n A_0$ for $m \gg 0$.

[1] Given couples of definition (A_0, t_0) and (A_1, t_1), we know that $t_1^\ell A_1 \subset A_0$, and thus $A_1 \subset t_1^{-\ell} A_0$, for $\ell \gg 0$. As multiplication by t_1 is a homeomorphism that preserves the notion of being a neighbourhood basis 0, this proves that A_1 is bounded with respect to A_0; by symmetry, the converse is also true, so the notion of boundedness is independent of the couple of definition.

Given a couple of definition (A_0, t), the pseudouniformizer t is clearly a topologically nilpotent unit of A. Conversely, any $t \in A$ that is a topologically nilpotent unit is a pseudouniformizer[2]. We remark that any open and integrally closed subring $B \subset A^\circ$ must contain $A^{\circ\circ}$; this gives a bijective correspondence between open and integrally closed subrings of A° and integrally closed subrings of $A^\circ/A^{\circ\circ}$.

5. *Completion.* We say that A is *complete* if A is complete in its topology; equivalently, some (or, equivalently, any) ring of definition A_0 of A is required to be complete in the t-adic topology, where $t \in A_0$ is a pseudouniformizer. The inclusion of complete Tate rings inside all Tate rings has a left adjoint $A \mapsto \widehat{A}$, called the completion of A. Explicitly, if (A_0, t) is a couple of definition of A, then we set $\widehat{A_0} := \lim A_0/t^n A_0$ to be the t-adic completion of A_0, and $\widehat{A} = \widehat{A}_0[\frac{1}{t}]$, viewed as a topological ring by declaring $t^n \widehat{A_0}$ to be a neighbourhood basis of 0. One checks that $\widehat{A}$ is a complete[3] Tate ring that coincides with $\lim_n A/t^n A_0$ as a group, and also coincides with the completion of A in the sense of Cauchy sequences.

6. *Compatibility of rings of definition under morphisms.* Let $\phi : A \to B$ be a continuous map of Tate rings. For any pseudouniformizer $t \in A$, the element $\phi(t) \in B$ is also a pseudouniformizer: the condition of a being topologically nilpotent unit can be transported across ring homomorphisms. Moreover, we can also choose compatible rings of definition in either order. Fix rings of definition $A_0 \subset A$ and $B_0 \subset B$ and a pseudouniformizer $t \in A_0$. The preimage $\phi^{-1}(B_0)$ is an open subring of A, so $A_0' = A_0 \cap \phi^{-1}(B_0)$ is open and bounded in A, and thus a ring of definition; this gives an induced map $A_0' \to B_0$ between rings of definition. Conversely, we have $t^n A_0 \subset A_0'$ for $n \gg 0$. Setting $B_0' \subset B$ to be the subring generated by $\phi(A_0)$ and B_0, we get $\phi(t)^n B_0' \subset B_0$, so B_0' is open and bounded in B, and thus a ring of definition; this gives us a map $A_0 \to B_0'$ between rings of definition.

TateRings **Exercise 7.1.4.** Let A be a Tate ring.

1. If A_0 is a ring of definition, show that A° is the total integral closure of A_0 in A.

2. If A is uniform, show that A° is totally integrally closed in A. Find an example where the converse fails.

3. If A is complete and uniform, show A is reduced.

4. Find an example of a uniform Tate ring that is not reduced.

5. Find an example of a reduced complete Tate ring that is not uniform.

[2] If (A_0, f) is a couple of definition, then $t^m \in fA_0$ for $m \gg 0$ by topological nilpotence. After replacing t with t^m (which does not change the property of being pseudouniformizer or a topologically nilpotent unit), we may assume $t \in fA_0$, so t is an element of A_0 divisible by f. But t also becomes a unit in $A = A_0[\frac{1}{f}]$, so there exists some $g \in A_0$ such that $tg = f^m$ for $m \gg 0$, so t divides f^m. Thus, we have $f^m A_0 \subset tA_0 \subset fA_0$, so the t-adic topology coincides with the f-adic topology on A_0, and thus t is a pseudouniformizer.

[3] This relies on the following algebraic fact: if R is a ring, $I \subset R$ a finitely generated ideal, and M is any R-module, then setting $\widehat{M} := \lim M/I^n M$ to be the I-adic completion of M, we have $\widehat{M}/I^n\widehat{M} \simeq M/I^n M$ for all n. See [SP, Tag 05GG] for a proof.

Example 7.1.5. Let K be a NA field. Then any Banach K-algebra R is a a complete Tate ring. Moreover, if K and R are perfectoid, then we have $R^{\circ\circ} = K^{\circ\circ}R^\circ$: if $\alpha \in R$ is topologically nilpotent, then $\alpha^n \in tR^\circ$ for a pseudouniformizer t, so we get the desired equality up to radicals. For the rest, it is enough to show that $K^{\circ\circ}R^\circ$ is radical. But, by perfectoidness, the quotient $R^\circ/K^{\circ\circ}R^\circ$ is a perfect k-algebra, where $k = K^\circ/K^{\circ\circ}$ is the residue field of k, so the ideal $K^{\circ\circ}R^\circ$ must be radical.

The next lemma provides a useful test for uniformity:

esUniform **Lemma 7.1.6.** *Let A be a complete Tate ring of characteristic p that is perfect. Then A is uniform.*

The statement of the above lemma was pointed out to us by Scholze, who learnt it from André.

Proof. Let (A_0, t) be a ring of definition of A. Write $A_n = A_0^{\frac{1}{p^n}} \subset A$ for the subring of A generated by the p^n-th roots of elements of A_0, so the inclusion $A_0 \to A_n$ is isomorphic to the n-fold Frobenius $A_0 \to A_0$. We have inclusions $A_0 \subset A_1 \subset A_2 \subset ...$, and write $A_\infty = \mathrm{colim}_n A_n$, so A_∞ is the perfection of A_0. We shall check $t^{\frac{1}{p^n}} A^\circ \subset A_\infty$ for any $n \geq 0$, and $t^c A_\infty \subset A_0$ for some $c \geq 0$. Together, these imply that A° is bounded, and thus A is uniform.

To see $t^{\frac{1}{p^n}} A^\circ \subset A_\infty$, fix some $f \in A^\circ$. Then $t^a f^{\mathbf{N}} \in A_0 \subset A_\infty$ for some $a \geq 0$. As $A_\infty \subset A$ is closed under taking p-th roots, it follows that $t^{\frac{a}{p^n}} \cdot f \in A_\infty$ for all $n \geq 0$, which easily gives the desired claim.

To see $t^c A_\infty \subset A_0$, we use the Banach open mapping theorem: the Frobenius map $F : A \to A$ is a continuous bijection of Banach spaces, and is thus open. In terms of the sequence A_i introduced above, this means that $t^m A_1 \subset A_0$ for some $m \geq 0$. By Frobenius twisting, this means $t^{\frac{m}{p^n}} A_{n+1} \subset A_n$. Thus, $t^{\sum_{i=0}^{n} \frac{m}{p^i}} A_{n+1} \subset A_0$ for all n. It is then clear that $t^c A_\infty \in A_0$ for any $c \geq \frac{mp}{p-1}$, as wanted. □

7.2 Affinoid Tate rings

TateRings

The basic input for defining adic spaces comprises the following data:

Definition 7.2.1. For a Tate ring A, a *ring of integral elements* of A is any open and integrally closed subring of A°. An *affinoid (Tate) ring* is a pair (A, A^+) where A is a Tate ring, and A^+ is a ring of integral elements of A. A morphism $(A, A^+) \to (B, B^+)$ of affinoid algebras is just a continuous map $A \to B$ that carries A^+ into B^+.

We often choose $A^+ = A^\circ$ in applications. However, in setting up the theory, it is more convenient to not impose this assumption. Algebraically, the difference between the two is roughly the same as the difference between integral closures and total integral closures. Topologically, as we shall see, the difference between A^+ and A° is also closely linked to the presence of higher rank points in the adic spectrum. For future reference, we list some properties of an affinoid Tate ring (A, A^+):

1. We have $A^{\circ\circ} \subset A^+$ as an ideal: any $t \in A^{\circ\circ}$ has a power that lies in the open subring A^+, and hence $t \in A^+$ by integral closedness, which gives the containment. As $A^{\circ\circ}$ is an ideal

in the larger ring A°, it is automatically an ideal in A^+. Note further that we have shown: for any pseudouniformizer $t \in A$, we have $tA^\circ \subset A^{\circ\circ} \subset A^+ \subset A^\circ$, so A° is bounded if and only if A^+ is so.

2. The ideal $A^{\circ\circ} \subset A^+$ is the radical of the ideal generated by any pseudouniformizer (by using topological nilpotence of elements of $A^{\circ\circ}$). In particular, the zero locus of pseudouniformizer on $\mathrm{Spec}(A^+)$ coincides with either $\mathrm{Spec}(A^+) - \mathrm{Spec}(A)$ or with $\mathrm{Spec}(A^+/A^{\circ\circ})$, and is thus intrinsic to the map $A^+ \to A$ of abstract rings.

We shall be interested in affinoid Tate rings (A, A^+) which are "local" along the closed set $\mathrm{Spec}(A^+/A^{\circ\circ})$. The notion of "locality", which is formulated next algebraically in multiple inequivalent ways, is trying to capture the fact we are ultimately interested only in "small neighbourhoods" (or, better, "*punctured* small neighbourhoods") of $\mathrm{Spec}(A^+/A^{\circ\circ})$ inside $\mathrm{Spec}(A^+)$.

ingsLocal **Definition 7.2.2.** Fix an affinoid Tate ring (A, A^+). This ring is called:

1. *complete* if A is so as a Tate ring.

2. *henselian* if the pair $(A^+, A^{\circ\circ})$ is henselian[4], i.e., an étale map $A^+ \to B$ admits a section provided it does so modulo $A^{\circ\circ}$.

3. *Zariski* if $A^{\circ\circ}$ lies in the Jacobson radical of A^+.

Note at the property of being either henselian or Zariski is intrinsic to the map $A^+ \to A$ of abstract rings[5]. In fact, the entire discussion of this chapter, with the exception of anything involving completions, can be formulated in terms of the map $A^+ \to A$ of abstract rings. There are some obvious relations between the thee notions above:

ompletion **Lemma 7.2.3** (Equivalent descriptions of Zariski, henselian, and complete pairs)**.** *Fix an affinoid Tate ring (A, A^+) with a couple of definition (A_0, t) with $A_0 \subset A^+$.*

1. *A^+ is a filtered colimit of open bounded subrings of A.*

2. *(A, A^+) is Zariski if and only if t lies in the Jacobson radical of A_0*

3. *(A, A^+) is henselian if and only if the pair (A_0, tA_0) is henselian.*

4. *(A, A^+) is complete if and only if A_0 is t-adically complete.*

[4]More classically, a pair (R, I) comprising a ring R with an ideal $I \subset R$ is *henselian* if for any finite R-algebra S, the map $S \to S/IS$ induces a bijection on idempotents; we sometimes also say R is I-adically henselian in this case. It is a non-trivial fact that this condition is equivalent to the one formulated above in Definition 7.2.2. A *henselian local ring* is a local ring $(R, \mathfrak{m})$ which is henselian. Examples can be constructed using the following stability properties: (a) if R is I-adically complete then R is I-adically henselian, and (b) if (R, I) is henselian and $J \subset I$ is an ideal, then $(R/J, I/J)$ and (R, J) are both henselian. See [SP, Tag 09XD] for more.

[5]This statement is *not* true for completeness, and is the main reason to carry the topology around. For example, set $A_0 = \mathbf{Z}_p[\epsilon]/(\epsilon^2)$, $t = p$, $A = A_0[\frac{1}{p}]$, and A^+ to be the integral closure of A_0 in A. Then (A, A^+) is a complete affinoid Tate ring with couple of definition (A_0, t), but the ring A^+ equals $\mathbf{Z}_p \cdot 1 \oplus \mathbf{Q}_p \cdot \epsilon \subset A$, and is thus not t-adically complete.

5. *If (A, A^+) is complete, then it is henselian.*

6. *If (A, A^+) is henselian, then it is Zariski.*

Proof. 1. We already know that $A^\circ = \mathrm{colim}_i A_i$ is a filtered colimit of open bounded subrings $A_i \subset A$. Intersecting with $A^+ \subset A^\circ$ shows that $A^+ = \mathrm{colim}_i A_i'$ is a filtered colimit of the A_i's; now each A_i' is open (as A_i and A^+ are open) and bounded (as it is contained in A_i, which is bounded), so the claim follows.

2. Assume t belongs to the Jacobson radical of A_0, so each maximal ideal of A_0 contains t. Say $A_0 \subset B_0$ is an inclusion of rings of definition. We claim that t also lies in the Jacobson radical of B_0. This claim implies that t (and hence $A^{\circ\circ}$) lies in the Jacobson radical of A^+ by (1) as the property that "t lies in the Jacobson radical" passes through a filtered colimit. To prove the claim, fix a maximal ideal $\mathfrak{m}$ of B_0 with residue field $k = B_0/\mathfrak{m}$. We must show that $t \in \mathfrak{m}$. If not, then t must map to a unit of k. Choose some $n \geq 0$ such that $t^n B_0 \subset A_0$, and choose some $b \in B$ that maps to t^{-n-1} in k. Then the element $a = t^n b$ lies in A_0, and maps to $t^{-1} \in k$. But then the composite $A_0 \to B_0 \to k$ must also be surjective: any element of the latter is the image of some $b \in B_0$, and hence also the image of $a^n \cdot (t^n b) \in A_0$. Thus, we have constructed a maximal ideal of A_0 that does not contain t, which is a contradiction.

   Conversely, if t (and hence $A^{\circ\circ}$) lies in the Jacobson radical of A^+, every point of $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$ specializes into a point of $\mathrm{Spec}(A^+/tA^+)$. As $A_0[\frac{1}{t}] = A = A^+[\frac{1}{t}]$, it follows that every point of $\mathrm{Spec}(A) \subset \mathrm{Spec}(A_0)$ also specializes into a point of $\mathrm{Spec}(A_0/tA_0)$, so t lies in the Jacobson radical of A_0.

3. Say (A_0, t) is henselian, and $A_0 \subset B_0$ is an inclusion of rings of definition. By (2), t lies in the Jacobson radical of B_0. The cokernel B_0/A_0 is killed by t^n for some n by boundedness. Hence, $t^n B_0 \subset A_0$, and so $I := t^{n+1} B_0 \subset tA_0$. Now observe that I is an ideal of both A_0 and B_0, and sits in the Jacobson radical of either ring. Moreover, we have $t^{n+1} A_0 \subset I \subset tA_0$. In particular, the pair (A_0, I) is henselian. As this property only depends on the ideal I viewed as a non-unital ring (see [GR, Remark 5.1.9 (ii)]), it follows that (B_0, I) is also henselian, and hence (B_0, tB_0) is henselian. Writing A^+ as a filtered colimit of all such B_0's then shows that (A^+, tA^+) is henselian, and hence also $(A^+, A^{\circ\circ})$ is henselian.

   Conversely, assume that $(A^+, A^{\circ\circ})$ is henselian. Set $A_1 = A_0 + A^{\circ\circ} \subset A^+$ to be the subgroup of elements of the form $a + \epsilon$ with $a \in A_0$ and $\epsilon \in A^{\circ\circ}$. As $A_0 \cdot A^{\circ\circ} \subset A^+ \cdot A^{\circ\circ} = A^{\circ\circ}$, one checks: (a) A_1 is a subring of A^+, and (b) $A^{\circ\circ} \subset A_1$ is an ideal. Moreover, $A^{\circ\circ} = \sqrt{tA_1}$ as ideals in A_1: if $f \in A^{\circ\circ}$, then $f^n \in tA_0 \subset tA_1$ for $n \gg 0$ by topological nilpotence. The proof of the second half of (2) shows that t (and hence $A^{\circ\circ}$) lies in the Jacobson radical of A_1. We then conclude that $(A_1, A^{\circ\circ})$ is also henselian: the ideal $A^{\circ\circ}$ is a common ideal in both A_1 and A^+, is contained in the Jacobson radical of both rings, and the pair $(A^+, A^{\circ\circ})$ is henselian.

   Next, we claim that the canonical map $\phi : A_0 \to A_1$ is integral, and that both $\phi[\frac{1}{t}]$ and ϕ/t are universal homeomorphisms on spectra; this implies $\mathrm{Spec}(\phi)$ is a universal homeomorphism (as it is universally specializing by integrality, and universally bijective by the assertion about topological spaces). For integrality, if $f = a + \epsilon \in A_1$ with $a \in A_0$ and $\epsilon \in A^{\circ\circ}$, then there exists some $N \geq 0$ such that $b := \epsilon^N \in A_0$ by topological nilpotence, so f satisfies the monic polynomial $(X - a)^N - b \in A_0[X]$. For the rest, it is clear that $\phi[\frac{1}{t}]$ is an isomorphism, so it is enough to check that $\overline{\phi} : A_0/\sqrt{tA_0} \to A_1/\sqrt{tA_1}$ is a homeomorphism on spectra. But $A_1 = A_0 + A^{\circ\circ}$ and $A^{\circ\circ} = \sqrt{tA_1}$ as ideals in A_1. It is then easy to check that $\overline{\phi}$ is actually an isomorphism of rings.

Summarizing, the map $A_0 \to A_1$ is a universal homeomorphism on spectra, and the pair (A_1, tA_1) is henselian. By the topological invariance of the étale site and the characterization of henselianness in terms of étale maps, it follows that (A_0, tA_0) is also henselian.

4. Clear from the definition.

5. As filtered colimits of henselian rings are henselian, this follows from (1) as open bounded subrings of A containing A_0 are t-adically complete and hence t-adically henselian.

6. Left to the reader.

□

In particular, the property of being Zariski for an affinoid Tate ring (A, A^+) is the mildest of the three properties introduced above, and only depends on the map $A^+ \to A$ of abstract rings. Thus, for maximal generality and technical ease, in the sequel, we shall try to use this property as a hypothesis when necessary (see Proposition 7.3.10 for an example); the reader should feel free to substitute one of the others, such as completeness, at first pass. The most important example for our purposes shall be:

Example 7.2.4. Let K be a perfectoid field, and let R be a perfectoid K-algebra. Then (R, R°) is a complete affinoid Tate ring, and it admits a natural structure map from (K, K°). More generally, let $\overline{R^+} \subset R^\circ/R^{\circ\circ}$ be any integrally closed $K^\circ/K^{\circ\circ}$-algebra. Then its preimage $R^+ \subset R^\circ$ is a ring of integral elements, and the result (R, R^+) is a complete affinoid Tate ring.

We remark that there is a good notion of completion for affinoid Tate algebras.

ompletion **Lemma 7.2.5** (Completion, henselization, Zariski localization)**.** *The inclusion of complete affinoid Tate rings into all affinoid Tate rings has a left-adjoint* $(A, A^+) \mapsto (\widehat{A}, \widehat{A}^+)$*; we call* $(\widehat{A}, \widehat{A}^+)$ *the completion of* (A, A^+)*. A similar statement holds true for "complete" replaced by "henselian" or "Zariski", and we denote the result as* $(A, A^+)_{hens} := (A_{hens}, A^+_{hens})$ *and* $(A, A^+)_{Zar} := (A_{Zar}, A^+_{Zar})$*. Moreover, these are related by canonical maps*

$$(A, A^+) \xrightarrow{a} (A, A^+)_{Zar} \xrightarrow{b} (A, A^+)_{hens} \xrightarrow{c} (\widehat{A}, \widehat{A}^+)$$

with a *being an isomormphism on Zariski localizaton,* a, b *being isomorphisms on henselizations, and* a, b, c *being isomorphisms on completions.*

The construction, together with Lemma 7.2.3, shows that the assocations $(A, A^+) \mapsto (A, A^+)_{hens}$ and $(A, A^+) \mapsto (A, A^+)_{zar}$ factor through the functor that ignores topology on either side, i.e., $(A, A^+)_{hens}$ as an abstract pair of rings can be calculated from (A, A^+) as an abstract pair, and likewise for $(A, A^+)_{zar}$; this is *not* true for the completion.

Proof. For completion: choose a couple of definition (A_0, t) of A. Then the t-adic completion $\widehat{A}$ is a ring of definition for the completion $\widehat{A}$ of the Tate ring A. Set $\widehat{A}^+$ to be the integral closure of the image of $\widehat{A_0} \otimes_{A_0} A^+ \to \widehat{A}$. Then $\widehat{A}^+$ contains $\widehat{A_0}$, and is thus an open and integrally closed subring of $\widehat{A}$. Moreover, note that the image of $A^\circ \otimes_{A_0} \widehat{A_0} \to \widehat{A}$ is contained in $\widehat{A}^\circ$. Hence, each element of $\widehat{A}^+$ is integral over $\widehat{A}^\circ$. As the latter is integrally closed, we then have $\widehat{A}^+ \subset \widehat{A}^\circ$. Thus, $(\widehat{A}, \widehat{A}^+)$ is a complete affinoid Tate ring. We leave it to the reader to check the universal property.

For henselization (resp. Zariski localization), one simply replaces $\widehat{A_0}$ with the henselization (resp. Zariski localization) of A_0 along tA_0 in the construction above; to see this satisfies the desired properties, one uses Lemma 7.2.3.

The last part follows from general nonsense about adjoint functors. □

The next exercise explains why the category of uniform affinoid Tate rings can be described purely in terms of commutative algebra (i.e., without any topology on the rings); this description will be used in the sequel to invoke certain results or constructions in commutative algebra.

rmization

Exercise 7.2.6 (Describing uniform Tate rings algebraically)**.** Write Tate for the category of affinoid Tate rings, and wirte $\mathrm{Tate}_u \subset \mathrm{Tate}$ for the full subcategory of uniform ones, i.e., those (A, A^+) where A is uniform.

1. Show that $(A, A^+) \in \mathrm{Tate}$ is uniform if and only if A^+ is a ring of definition.

2. Show that the inclusion $i : \mathrm{Tate}_u \hookrightarrow \mathrm{Tate}$ has a left adjoint L such that $L(A, A+)$, called the *uniformization* of (A, A^+), is just (A, A^+) as an abstract ring, but the topology is determined by making (A^+, t) a couple of definition for any pseudouniformizer t. Thus, we may regard Tate_u as a localization of Tate. Explicitly, the functor L inverts all maps $(A, A^+) \to (B, B^+)$ that give bijections $A^+ \simeq B^+$ of underlying (abstract) rings.

3. Write Tate_{alg} for the category of pairs (R, I) where R is a commutative ring, $I \subset R$ is the radical of an ideal generated by a nonzerodivisor, and R is integrally closed in the ring $R[f^{-1}]$ where $f \in I$ is a generator (up to radicals); a map $(R, I) \to (S, J)$ in this category is an adic map, i.e., a map $f : R \to S$ such that $\sqrt{f(I)S} = J$. Show that $(A, A^+) \mapsto (A^+, A^{\circ\circ})$ gives a functor $\mathrm{Tate} \to \mathrm{Tate}_{alg}$.

4. Given $(R, I) \in \mathrm{Tate}_{alg}$ with a generator (up to radicals) $f \in I$, we may construct an affinoid Tate ring $(R[\frac{1}{f}], R)$ with couple of definition (R, f). Show that this gives a well-defined functor $\mathrm{Tate}_{alg} \to \mathrm{Tate}_u$.

5. Show that the functors in (3) restricts to an equivalence $\mathrm{Tate}_u \simeq \mathrm{Tate}_{alg}$ with inverse the functor from (4).

6. Show that Tate_u admits pushouts and filtered colimits.

7. Fix a uniform affinoid Tate ring $(A, A^+) \in \mathrm{Tate}_u$ and a pseudouniformizer $t \in A^+$. Then show that the completion $\widehat{(A, A^+)}$ of the underlying affinoid Tate ring (A, A^+) is computed as $(\widehat{A^+}[\frac{1}{t}], \widehat{A^+})$, where $\widehat{A^+}$ is the t-adic completion of A^+ and the couple of definition is $(\widehat{A^+}, t)$. In particular, this completion is also uniform.

7.3 Affinoid adic spaces: definition and basic properties

ectrumDef Our goal is to attach a space $\mathrm{Spa}(A, A^+)$ to an affinoid Tate ring (A, A^+). To a first approximation, this space parametrizes a point of $\mathrm{Spec}(A)$ equipped with a totally ordered set of specializations in $\mathrm{Spec}(A^+) - \mathrm{Spec}(A)$. More precisely, one works with valuations realizing this picture. For this, recall that a *valuation* on a a ring A is given by a map $x : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group written multiplicatively with an identity element $1 \in \Gamma$, and x is a multiplicative map that carries 0 and 1 to 0 and 1 respectively, and satisfies the NA inequality. Any such x determines a prime ideal $\mathfrak{p}_x := x^{-1}(0)$ called the *support* of x as well as a valuation ring $R_x \subset \kappa(\mathfrak{p}_x)$. Two valuations are equivalent exactly when they have identical supports and valuation rings. We shall construct the space $\mathrm{Spa}(A, A^+)$ as a suitable subspace of the space of all valuations on A:

cSpectrum **Definition 7.3.1** (The adic spectrum)**.** Let (A, A^+) be an affinoid Tate ring. The *adic spectrum* $\mathrm{Spa}(A, A^+)$ is defined as the set of equivalence classes of valuations $x : A \to \Gamma \cup \{0\}$ (for varying Γ) such that

1. $x(f) \leq 1$ for $f \in A^+$.

2. x is continuous with respect to the order topology on the target, i.e. $x^{-1}(\Gamma_{<\gamma} \cup \{0\})$ is open for all $\gamma \in \Gamma$.

Given a valuation x as above and $f \in A$, we often write $x(f)$ as $|f(x)|$ instead. Given $f, g \in A$, we obtain subsets

$$\mathrm{Spa}(A, A^+)\Big(\frac{f}{g}\Big) := \{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}.$$

We endow $\mathrm{Spa}(A, A^+)$ with the coarsest topology where all such sets are open.

It is easy to see that $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ gives a contravariant functor from affinoid Tate rings to topological spaces: we simply precompose valuations with the map of underlying rings to pull them back. It is also clear that if $A^+ \subset A^{+'}$ is an inclusion of rings of integral elements, then the induced map $\mathrm{Spa}(A, A^{+'}) \to \mathrm{Spa}(A, A^+)$ is injective; in particular, $\mathrm{Spa}(A, A^\circ)$ sits inside $\mathrm{Spa}(A, A^+)$ for any choice of A^+. We remark next that the adic spectrum is essentially an algebraic object:

Remark 7.3.2 (The adic spectrum does not depend on the topology)**.** The space $\mathrm{Spa}(A, A^+)$ only depends on the map $A^+ \to A$ of abstract rings, and not on the topology of A. More formally, the functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ factors through the uniformization functor from Exercise 7.2.6. To see this: as the opens defining the topology obviously only depend on A, it suffices to show that the two conditions on the valuation x appearing in Definition 7.3.1 can be formulated algebraically. The ideal $A^{\circ\circ} \subset A^+$ can be recovered as the radical ideal defining complement of the open immersion $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$; any $t \in A^+$ such that $\sqrt{(t)} = A^{\circ\circ}$ is a pseudouniformizer. It now suffices to observe that a given valuation $x : A \to \Gamma \cup \{0\}$ lies in $\mathrm{Spa}(A, A^+)$ exactly[6] when $x(A^+) \leq 1$, and $x(t^n) \to 0$ for one (or, equivalently, any) pseudouniformizer t.

[6]Indeed, if x is continuous, then certainly $x(t^n) \to 0$ as $t^n A_0$ is a neighbourhood basis of 0 for a couple of definition (A_0, t). Conversely, if (A_0, t) is a couple of definition and x is a valuation on A with $x(A^+) \leq 1$, then $tA_0 \subset A^{\circ\circ} \subset A^+$ so $x(tA_0) \leq 1$, so $x(t^n A_0) \leq x(t^{n-1})$, and hence $x(t^n A_0) \to 0$ if $x(t^n) \to 0$, giving continuity.

A basic piece of structure used to study $\mathrm{Spa}(A, A^+)$ is the kernel map:

nelMapSpa **Remark 7.3.3** (The kernel map)**.** Given a point $x \in \mathrm{Spa}(A, A^+)$, taking the support $\mathfrak{p}_x := \ker(x) \subset A$ gives a map

$$\ker : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A).$$

In fact, this map is continuous: given $f \in A$, the preimage of $D(f) \subset \mathrm{Spec}(A)$ is $\mathrm{Spa}(A, A^+)\left(\frac{f}{f}\right)$. We shall call inverse images of open subsets of $\mathrm{Spec}(A)$ along $\ker$ the *Zariski open subsets* of $\mathrm{Spa}(A, A^+)$.

We want to give a purely ring-theoretic desription of the adic spectrum. We already know that any valuation x on A is completely determined by its support $\mathfrak{p}_x$ together with the valuation ring $R_x \subset \kappa(\mathfrak{p}_x)$. These must satisfy some additional conditions when $x \in \mathrm{Spa}(A, A^+)$. To formulate these conditions algebraically, we need the following definition, encoding essential features of the valuation rings of the form R_x for $x \in \mathrm{Spa}(A, A^+)$.

Definition 7.3.4 (Microbial valuation rings)**.** Let V be a valuation ring. We say that $f \in V$ is a *pseudouniformizer* if f is nonzero and contained in a height 1 prime. If a pseudouniformizer exists (or, equivalently, if V has a height 1 prime), we say that V is *microbial*

Using the fact that any radical ideal in a valuation ring is prime[7], one checks that the height 1 prime of a microbial valuation ring V can be computed as $\sqrt{(f)}$ for any pseudouniformizer f. Any valuation ring of finite Krull dimension is automatically microbial, so one has to work hard to find a non-example (but they do exist). There are several equivalent characterizations of microbiality that will be useful, and are recorded in the next exercise.

Exercise 7.3.5 (Characterizations of microbial valuation rings)**.** Let V be a nonzero valuation ring. Prove that the following are equivalent:

1. V is microbial.
2. There exists some $0 \neq f \in V$ such that, for any $0 \neq g \in V$, we have $|f^n| < |g|$ for $n \gg 0$, i.e., f is topologically nilpotent in the valuation topology.
3. V is f-adically separated for some nonzero element f (which then necessarily is a pseudouniformizer).
4. There exists some non-unit $f \in V$ such that $\mathrm{Frac}(V) = V[\frac{1}{f}]$.
5. $(\mathrm{Frac}(V), V)$ is an affinoid Tate ring for the valuation topology on $\mathrm{Frac}(V)$.
6. $\mathrm{Frac}(V)$ is a Tate ring for the valuation topology.

[7]Indeed, as all ideals in a valuation ring V are totally ordered, a radical ideal can be written as the intersection $I := \cap_s \mathfrak{p}_s$ of primes indexed by a totally ordered set S such that $\mathfrak{p}_s \subset \mathfrak{p}_t$ for $s \leq t$ in S. Now given $f, g \in V$ with $fg \in I$, consider the subsets $S_f, S_g \subset S$ defined by $S_f := \{s \in S \mid f \in \mathfrak{p}_s\}$ and likewise for S_g. We want to show that $S_f = S$ or $S_g = S$. We have $S_f \cup S_g = S$ by hypothesis. Also, both S_f and S_g are closed under taking larger elements of S. It is then formal that $S_f = S$ or $S_g = S$.

7. $\mathrm{Frac}(V)$ is a NA field in the valuation topology.

The following observation will be very useful later in converting an arbitrary valuation ring into a microbial one.

Microbial **Lemma 7.3.6** (Constructing microbial valuation rings)**.** *If V is a valuation ring and $f \in V$ is a nonzero nonunit, then the f-adically separated quotient $\overline{V} := V/\cap_n f^n V$ and its f-adic completion $\widehat{V}$ are both microbial valuation rings with pseudouniformizer f, and the natural map $\overline{V} \to \widehat{V}$ is a fathfully flat extension of microbial valuation rings that preserves pseudouniformizers.*

Proof. We first check that if a valuation ring W is g-adically separated for a nonzero element $g \in W$, then W is microbial with pseudouniformizer g: the separatedness ensures that g is not a unit, and that for any $0 \neq h \in W$, we have $h \notin (g^n)$ for $n \gg 0$, so $g^n \in (h)$ for $n \gg 0$, so $|g^n| < |h|$ for $n \gg 0$.

Next, we check that $\overline{V}$ is a valuation ring. For this, as $V \to \overline{V}$ is surjective, it suffices to check that $I = \cap_n f^n V$ is a radical (and hence prime) ideal. If $g \in V$ with $g^k \in I$ for some $k \geq 0$, then $g^k \in f^{kn} V$ for any n, so $\frac{g}{f^n} \in \mathrm{Frac}(V)$ has its k-th power in V, and must thus lie in V as V is normal. This gives $g \in f^n V$ for all n, so $g \in I$, proving $\overline{V}$ is a valuation ring. One can also check that the f-adic completion $\widehat{V}$ is a valuation ring (but we leave this to the reader).

Applying this to $\overline{V}$ and $\widehat{V}$ proves the first half of the lemma. The faithful flatness of $\overline{V} \to \widehat{V}$ is the consequence of the general fact that any any injective local map $S \to T$ of valuation rings is faithfully flat: injectivity ensures flatness via torsionfreeness so $\mathrm{Spec}(T) \to \mathrm{Spec}(S)$ has image closed under generalization, while locality ensures that $\mathrm{Spec}(T) \to \mathrm{Spec}(S)$ hits the closed point, and hence is surjective by the generalizing property. □

Using this, we show:

tionRings **Proposition 7.3.7** (Adic spectrum via valuation rings)**.** *Fix an affinoid Tate ring (A, A^+). There is a natural bijection between $\mathrm{Spa}(A, A^+)$ and the set S of equivalence classes of maps $\phi : A^+ \to V$ where V is a microbial valuation ring and ϕ carries pseudouniformizers to pseudouniformizers; here the equivalence relation is generated by requiring that if $\psi : V \to W$ is a faithfully flat extension of microbial valuation rings that preserves pseudouniformizers, then $\phi : A^+ \to V$ and $\psi \circ \phi : A^+ \to W$ are equivalent.*

We make two observations about the conclusion of this proposition.

- As any two pseudouniformizers define the same closed set (namely, the complement of $\mathrm{Spec}(A)$ in $\mathrm{Spec}(A^+)$), a map $\phi : A^+ \to V$ with V a valuation ring gives an element of the set S if and only if $\phi(t)$ is a pseudouniformizer for a single pseudouniformizer $t \in A^+$.

- Using Lemma 7.3.6 and replacing the rings V appearing above with their $\phi(t)$-adic completions, one can restrict to valuation rings V which are $\phi(t)$-adically complete in the proposition.

Proof. Fix a pseudouniformizer $t \in A^+$. Given a point $x \in \mathrm{Spa}(A, A^+)$, we have its support $\mathfrak{p}_x \subset A$ as well as the induced valuation ring $R_x \subset \kappa(\mathfrak{p}_x)$. Condition (1) in Definition 7.3.1 ensures that we have a map $\phi_x : A^+ \to R_x$, while condition (2) ensures that the image $\phi_x(t)$ is a pseudouniformizer. Thus, the map ϕ_x gives an element of the set S, and so we have constructed a map $\mathrm{Spa}(A, A^+) \to S$.

Conversely, fix a map $\phi : A^+ \to V$ representing an element of S. Thus, $|\phi(f)| \leq 1$ for $f \in A^+$, and $\phi(t)$ is a pseudouniformizer, so $\phi(t) \neq 0$ and that for any $g \in V$, we have $|\phi(t)^n| < g$ for $n \gg 0$. Let $\mathfrak{p}$ denote the kernel of ϕ, so ϕ defines a valuation x_ϕ on $\kappa(\mathfrak{p})$ satisfying the two conditions in the previous sentence; explicitly, for $f, g \in A^+$, we have $|f(x_\phi)| \leq |g(x_\phi)|$ if and only if $\phi(g) \mid \phi(f)$. Thus, x_ϕ gives a point of $\mathrm{Spa}(A, A^+)$. It is easy to see that ϕ and $\phi \circ \psi$ define the same valuation for any faithfully flat map $\psi : V \to W$ of valuation rings. Thus, we obtain a well-defined map $S \to \mathrm{Spa}(A, A^+)$.

We leave it to the reader to check that these constructions give mutually inverse bijections. □

As an upshot, each point $x \in \mathrm{Spa}(A, A^+)$ determines (and is determined by) the map $(A, A^+) \to (\kappa(\mathfrak{p}_x), R_x)$ of affinoid Tate rings.

Remark 7.3.8 (Affinoid fields)**.** An affinoid Tate ring (K, K^+) is called an *affinoid field* if K is a NA field (for its topology) and $K^+ \subset K$ is an open valuation ring. In this case, the ring K° is a rank 1 valuation ring, and the space $\mathrm{Spa}(K, K^+)$ is a totally ordered set (see Proposition 7.3.10 and Remark 7.3.11 below). The discussion above shows that for any affinoid Tate ring (A, A^+) and a point $x \in \mathrm{Spa}(A, A^+)$, the affinoid Tate (A, A^+)-algebra $(\kappa(\mathfrak{p}_x), R_x)$ is an affinoid field. Using this, one may also formulate Proposition 7.3.7 as describing the set $\mathrm{Spa}(A, A^+)$ in terms of equivalence classes of maps $(A, A^+) \to (K, K^+)$ to affinoid fields.

Remark 7.3.9 (Visual desription of points)**.** Proposition 7.3.7 gives us a visual description of points of $\mathrm{Spa}(A, A^+)$. Given a map $A^+ \to V$ as in Proposition 7.3.7, the induced map $\mathrm{Spec}(V) \to \mathrm{Spec}(A^+)$ carries the generic point into $\mathrm{Spec}(A) = \mathrm{Spec}(A^+) - \mathrm{Spec}(A/t)$, while all other points are carried into $\mathrm{Spec}(A^+/t) = \mathrm{Spec}(A^+) - \mathrm{Spec}(A)$. As any such V has a height 1 prime, this gives a point of $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$ specializing into a point of $\mathrm{Spec}(A/t) \subset \mathrm{Spec}(A^+)$ together with a totally ordered set of specializations of the latter. (Draw picture.)

We record some of the basic properties of the adic spectrum.

oidBasics **Proposition 7.3.10.** *Let (A, A^+) be an affinoid Tate ring.*

1. *The canonical map $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ induces a homeomorphism $\mathrm{Spa}(A, A^+) \simeq \mathrm{Spa}(\widehat{A}, \widehat{A}^+)$.*
2. *$\mathrm{Spa}(A, A^+) = \emptyset$ if and only if A_{Zar} (or A_{hens} or $\widehat{A}$) vanishes.*
3. *We have $A^+ = \{f \in A \mid |f(x)| \leq 1$ for all $x \in \mathrm{Spa}(A, A^+)\}$.*
4. *If $x \rightsquigarrow y$ is a specialization in $\mathrm{Spa}(A, A^+)$, then the supports coincide (i.e., $\mathfrak{p}_x = \mathfrak{p}_y =: \mathfrak{p}$). Moreover, we have a containment $R_y \subset R_x$ of the corresponding valuation rings of $\kappa(\mathfrak{p})$, so R_x is a localization of R_y.*

5. *Fix $y \in \mathrm{Spa}(A, A^+)$. A prime ideal $\mathfrak{q} \subset R_y$ contains a pseudouniformizer exactly when $\mathfrak{q} \neq 0$. In this case, the valuation attached to $R_{y,\mathfrak{q}} \subset \kappa(\mathfrak{p}_y)$ gives a point $x \in \mathrm{Spa}(A, A^+)$ specializing to y.*

6. *Assume (A, A^+) is Zariski. Then $f \in A$ is a unit if and only if $|f(x)| \neq 0$ for each $x \in \mathrm{Spa}(A, A^+)$.*

Proof. Fix a couple of definition (A_0, t) with $A_0 \subset A^+$.

1. By functoriality, we have a continuous map $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$. We shall use the description of points in terms of valuation rings given at the end of Remark 7.3.3.

   We first check surjectivity. A point $x \in \mathrm{Spa}(A, A^+)$ determines a map $\phi : A^+ \to V$, where V is a microbial valuation ring with pseudouniformizer such that V is $\phi(t)$-adically complete. Then $(\mathrm{Frac}(V), V)$ is a complete Tate ring for the valuation topology (which coincides with the $\phi(t)$-adic topology on V). Thus, the map $(A, A^+) \to (\mathrm{Frac}(V), V)$ extends uniquely to a map $(\widehat{A}, \widehat{A}^+) \to (\mathrm{Frac}(V), V)$. The resulting map $\widehat{A}^+ \to V$ then gives a point of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ lifting x, proving surjectivity.

   Injectivity follows from the density of A inside $\widehat{A}$: the maps $\widehat{A}^+ \to V$ that occur in characterization of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ given in Proposition 7.3.7 are determined by the induced map $\widehat{A} \to V[\frac{1}{t}] = \mathrm{Frac}(V)$.

   For homeomorphy: fix $f, g \in \widehat{A}$, and consider the subset $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)\Big(\frac{f}{g}\Big)$. As t is a pseudouniformizer, we can write

   $$\mathrm{Spa}(\widehat{A}, \widehat{A}^+)\Big(\frac{f}{g}\Big) = \cup_n \mathrm{Spa}(\widehat{A}, \widehat{A}^+)\Big(\frac{f, t^n}{g}\Big);$$

   here we use that $|g(x)| \neq 0$ for any x in the left side, and that $|t^n(x)| \to 0$ for any x. Each of the sets appearing on the right is open (as it is an intersection of two opens), so it is enough to show that each of sets appearing on the right can be defined using functions that come from A. By scaling with powers of t, we may assume $f, g, t \in \widehat{A}_0$. We shall check the following stronger statement: for fixed n and any $N > n$, every $f' \in f + t^N \widehat{A_0}$ and $g' \in g + t^N \widehat{A_0}$ satisy

   $$\mathrm{Spa}(\widehat{A}, \widehat{A}^+)\Big(\frac{f, t^n}{g}\Big) = \mathrm{Spa}(\widehat{A}, \widehat{A}^+)\Big(\frac{f', t^n}{g'}\Big). \tag{7.1}$$ eq:AdicSp

   Note that this implies the desired claim because $A_0/t^N \simeq \widehat{A}_0/t^N$, so some such f' and g' come from A_0. To see this equality, fix some x in the left hand side, so $|f(x)| \leq |g(x)|$ and $|t^n(x)| \leq |g(x)|$. For any $h \in A_0$ and $N \geq n$, we then have

   $$|(f + t^N h)(x)| \leq \max(|f(x)|, |t^N(x)|) \leq |g(x)|.$$

   Now note that $|g(x)| \geq |t^n(x)| > |t^N(x)|$ for $N > n$ as $0 \neq |t(x)| < 1$. By the strict NA inequality, for any $k \in A_0$, we then get

   $$|(g + t^N k)(x)| = |g(x)|.$$

Thus, we learn that for any $h, k \in A_0$ and $N > n$, we have

$$|(f + t^N h)(x)| \leq |(g + t^N k)(x)| \quad \text{and} \quad |t^n(x)| \leq |(g + t^N k)(x)|.$$

This gives the containment $\subset$ in the equality (7.1) claimed above. The reverse inequality is proven similarly.

2. If $A_{Zar} = 0$, then $A_{hens} = \widehat{A} = 0$ by Lemma 7.2.5, and thus $\mathrm{Spa}(A, A^+) = \emptyset$ by (1). Conversely, it is enough to show that if $\widehat{A} \neq 0$, then $\mathrm{Spa}(A, A^+) \neq 0$. By (1), we may thus assume that (A, A^+) is a complete affinoid Tate ring with $A \neq 0$. We shall construct a point of $\mathrm{Spa}(A, A^+)$. We first claim that t is not a unit of A^+: if it were, then t would not lie in the Jacobson radical of A^+ (as $A^+ \neq 0$), which contradicts the fact that (A, A^+) is Zariski. We can thus choose a maximal ideal $\mathfrak{m} \subset A^+$ containing t. Via the dense open immersion $\mathrm{Spec}(A_{\mathfrak{m}}) = \mathrm{Spec}(A^+_{\mathfrak{m}}[\frac{1}{t}]) \subset \mathrm{Spec}(A_{\mathfrak{m}})$, we can choose a prime $\mathfrak{p}$ of $A_{\mathfrak{m}}$ specializing to $\mathfrak{m}$. By working with the domain $\bar{A} := A^+/(\mathfrak{p} \cap A^+) \subset A/\mathfrak{p}$, we can choose a valuation ring $V \subset \kappa(\mathfrak{p})$ containing $\bar{A}$ such that the induced map $\mathrm{Spec}(V) \to \mathrm{Spec}(\bar{A}) \subset \mathrm{Spec}(A^+)$ sends the generic point to $\mathfrak{p}$ and the special point to $\mathfrak{m}$. In particular, t is nonzero in V. Write W for the t-adic completion of V, so W is a microbial valuation ring with pseudouniformizer t by Lemma 7.3.6. The resulting map $A^+ \to W$ gives a point of $\mathrm{Spa}(A, A^+)$ via Proposition 7.3.7, as wanted.

3. The containment $\subset$ is clear from the definition. For $\supset$, fix some $f \in A$ such that $|f(x)| \leq 1$ for each $x \in \mathrm{Spa}(A, A^+)$. We must show that $f \in A^+$. Consider the subring $A^+[f^{-1}] \subset A[f^{-1}]$. Note that if $f^{-1} \in A[f^{-1}]$ is a unit, then f is integral over A^+, and hence in A^+: writing f as a polynomial in f^{-1} and clearing denominators gives the required monic equation that f satisfies. We may thus assume towards contradiction that the element $f^{-1} \in A^+[f^{-1}]$ is not a unit. Then we can choose some maximal ideal $\mathfrak{m} \subset A^+[f^{-1}]$ that contains f^{-1}. Let $\mathfrak{p}$ be a minimal prime of $A^+[f^{-1}]$ contained in $\mathfrak{m}$. We may then choose a valuation ring V and a map $A^+[f^{-1}] \to V$ such that the generic point of $\mathrm{Spec}(V)$ is carried to $\mathfrak{p}$, while the special point is carried to $\mathfrak{m}$. This gives a valuation x on A^+ by the composition $A^+ \to A^+[f^{-1}] \to V$.

   We first check that the image of t in V is nonzero. By the injectivity of $A^+[f^{-1}] \to A[f^{-1}]$ (obtained from injectivity of $A^+ \to A$ by inverting f), the open immersion $\mathrm{Spec}(A[f^{-1}]) \to \mathrm{Spec}(A^+[f^{-1}])$ is dense, and hence hits all generic points. In particular, the prime $\mathfrak{p}$ lies in $\mathrm{Spec}(A[f^{-1}]) \to \mathrm{Spec}(A^+[f^{-1}])$, so $t \in A^+$ maps to a non-zero element of V (as it maps to a unit in A and hence in $\mathrm{Frac}(V)$).

   Next, we check that t maps into the maximal ideal of V. We have $t^n f \in A^+$ for $n \gg 0$ as t is topologically nilpotent and $A^+ \subset A$ is open. As x carries A^+ into V, it follows that $|t^n(x) f(x)| \leq 1$. But x also carries f^{-1} into the maximal ideal of V by construction, i.e., $|f^{-1}(x)| < 1$. These conditions force $|t(x)| < 1$, i.e., t maps into the maximal ideal of V.

   By replacing V with its t-adic completion, we may assume that V is microbial with pseudouniformizer t. The resulting map $A^+ \to A^+[f^{-1}] \to V$ then gives a point of $x \in \mathrm{Spa}(A, A^+)$. After inverting t, we get $A \to A[f^{-1}] \to \mathrm{Frac}(V)$, inducing the valuation

x on A. Then we have $|f^{-1}(x)| < 1$ (by construction) and $|f(x)| \leq 1$ as $f \in A^+$ maps into V. As $|f(x)| \cdot |f^{-1}(x)| = 1$, this is impossible, so we have a contradiction.

4. For $\mathfrak{p}_x \subset \mathfrak{p}_y$: given $f \in \mathfrak{p}_x$, we have $x \notin \mathrm{Spa}(A, A^+)\left(\frac{f}{f}\right)$. The specialization hypothesis ensures $y \notin \mathrm{Spa}(A, A^+)\left(\frac{f}{f}\right)$, so $|f(y)| = 0$, and thus $f \in \mathfrak{p}_y$.

   For $\mathfrak{p}_y \subset \mathfrak{p}_x$: if $f \in \mathfrak{p}_y$, then $y \in \mathrm{Spa}(A, A^+)\left(\frac{f}{t^n}\right)$ for all $n \geq 0$. The specialization hypothesis ensures that $x \in \mathrm{Spa}(A, A^+)\left(\frac{f}{t^n}\right)$ as well, so $|f(x)| = 0$.

   As the supports are the same, we can view both R_y and R_x as valuation rings with the same fraction field $\kappa(\mathfrak{p})$. To see $R_y \subset R_x$, fix some $f \in R_y \subset \kappa(\mathfrak{p})$. We can write $f = \frac{g}{h}$ for $g, h \in A$ with $h \notin \mathfrak{p}$. As $f \in R_y$, we have $y \in \mathrm{Spa}(A, A^+)\left(\frac{g}{h}\right)$. But then the same holds true for x by the specialization hypothesis, so we learn that $f \in R_x$. The fact that $R_y \to R_x$ is a localization is then a general fact: any injective map $S \subset T$ of valuation rings with the same fracton field must be a localization: it factors as $S \xrightarrow{a} S_{\mathfrak{q}} \xrightarrow{b} T$ where $\mathfrak{q} \subset S$ is the prime corresponding to the image of the closed point $\mathrm{Spec}(T)$, the map a is the obvious localization, and b is a faithfully flat extension of valuation rings with the same fraction field, and thus an isomorphism.

5. As R_y is microbial, it admits a (unique) height 1 prime $\mathfrak{q}_0$, characterized as the radical of the ideal generated by any pseudouniformizer. Thus, a prime $\mathfrak{q} \subset R_y$ is nonzero exactly when $\mathfrak{q} \supset \mathfrak{q}_0$, which gives the first assertion. For such a prime $\mathfrak{q}$, the valuation ring $R_{y,\mathfrak{q}} \subset \kappa(\mathfrak{p}_y)$ gives a point $x \in \mathrm{Spa}(A, A^+)$ with $R_x = R_{y,\mathfrak{q}}$ by Proposition 7.3.7. The inclusion $R_y \subset R_x$ can be used to show that $x \rightsquigarrow y$, as in the proof of (4).

6. One direction is clear, so assume $|f(x)| \neq 0$ for all $x \in \mathrm{Spa}(A, A^+)$. We must show that f is invertible in every residue field of A. Although this can be deduced from (2), we give a direct argument. Fix a pseudouniformizer $t \in A^+$. As (A, A^+) is Zariski, the element t lies in the Jacobson radical of A^+. In particular, every point of $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$ specializes into a point of $\mathrm{Spec}(A^+/t) \subset \mathrm{Spec}(A^+)$. Then, for every prime ideal $\mathfrak{p} \subset A$, we can choose a valuation ring V and a map $\phi : A^+ \to V$ such that $\mathrm{Spec}(\phi)$ carries the generic point of $\mathrm{Spec}(V)$ to $\mathfrak{p}$, while the special point goes to a point in $\mathrm{Spec}(A^+/t)$. As $t \in \kappa(\mathfrak{p})$ is a unit, Lemma 7.3.6 shows that the $\phi(t)$-adic completion $\widehat{V}$ of V is a microbial valuation ring with pseudouniformizer $\phi(t)$. By Proposition 7.3.7, this defines a point $x \in \mathrm{Spa}(A, A^+)$ with support $\mathfrak{q}$ being a specialization of $\mathfrak{p}$ (corresponding to the image of the prime $\cap_n \phi(t)^n V \in \mathrm{Spec}(V)$) The hypothesis $|f(x)| \neq 0$ then forces f to be invertible in $\kappa(\mathfrak{q})$, and hence also in $\kappa(\mathfrak{p})$. As this is true for all $\mathfrak{p} \in \mathrm{Spec}(A)$, it follows that f is a unit.

□

Structure

Remark 7.3.11 (Structure of generalizations)**.** Proposition 7.3.10 (4) and (5) ensure that all generalizations in $\mathrm{Spa}(A, A^+)$ are easily classified: given $y \in \mathrm{Spa}(A, A^+)$, the set of all generalizations of y is in bijection $\operatorname{Spec}(R_y/t)$ as a poset, and is thus totally ordered. Moreover, as R_y is microbial, the scheme $\operatorname{Spec}(R_y/t)$ has a unique generic point y_1 (corresponding to the height 1 prime of R_y). The corresponding generalization $y_1 \rightsquigarrow y$ is the unique rank 1 generalization of X. The map $y \to y_1$ thus defines a retraction of $\operatorname{Spa}(A, A^+)$ onto its subset $\operatorname{Spa}(A, A^+)_{gen}$ of generic points. We also remark that the inclusion $\operatorname{Spa}(A, A^\circ) \subset \operatorname{Spa}(A, A^+)$ is closed under generalization (which is clear from the structure of generalizatons) and we have $\operatorname{Spa}(A, A^\circ)_{gen} = \operatorname{Spa}(A, A^+)_{gen}$: this amounts to the observation that for any $x \in \operatorname{Spa}(A, A^+)$, the map $A \to \kappa(\mathfrak{p}_x)$ carries A° into $\kappa(x)^\circ = R_{x_1}$, where $\kappa(x)$ is given the valuation topology attached to x, and x_1 is the unique rank 1 generalization of x. In particular, each point of $\operatorname{Spa}(A, A^+)$ generalizes to a point of $\operatorname{Spa}(A, A^\circ) \subset \operatorname{Spa}(A, A^+)$, so the difference between the two is indeed a "higher rank" phenomenon.

7.4 Spectrality of affinoid adic spaces

mSpectral

Let (A, A^+) be an affinoid Tate ring. We shall prove the following theorem:

mSpectral

Theorem 7.4.1 (Adic spectra are spectral)**.** *The adic spectrum* $\operatorname{Spa}(A, A^+)$ *is a spectral space. Moreover, a basis* $\mathcal{B}$ *of quasi-compact opens is given by "rational subsets", i.e., subsets of the form*

$$\operatorname{Spa}(A, A^+)\Big(\frac{f_1, ..., f_n}{g}\Big) := \{x \in \operatorname{Spa}(A, A^+) \mid |f_i(x)| \leq |g(x)| \textit{ for all } i\}$$

where $f_i \in A$ *generate the unit ideal in* A*, and* $g \in A$*. The construction* $(A, A^+) \mapsto \operatorname{Spa}(A, A^+)$ *naturally defines a functor from affinoid Tate rings to spectral spaces, i.e., pulling back valuations along a map of affinoid Tate rings gives rise to a spectral map on adic spectra.*

Remark 7.4.2 (Compatibility with Definition 7.3.1)**.** For any $x \in \operatorname{Spa}(A, A^+)\big(\frac{f_1, ..., f_n}{g}\big)$ (with notation as in Theorem 7.4.1), we must have $|g(x)| \neq 0$: if $|g(x)| = 0$, then $|f_i(x)| = 0$ for all i by definition of the set, so $f_i \in \mathfrak{p}_x$, whence $1 \in (f_i) \subset \mathfrak{p}_x$ by assumption on the f_i's, which forces the abusrd equality $1 = |1| = 0$. Thus, the notation introduced in Theorem 7.4.1 is compatible with the one in Definition 7.3.1.

cSpectrum

Remark 7.4.3 (Modifying presentations)**.** Fix a pseudouniformizer $t \in A$. As t is a unit with $|t(x)| \neq 0$ for any $x \in \operatorname{Spa}(A, A^+)$, we are free to scale the parameters f_i and g appearing in Theorem 7.4.1 by powers of t. In particular, we can choose the functions f_i, g appearing in Theorem 7.4.1 to lie inside A^+ (or any ring of definition); the condition that the f_i's generate a unit ideal of A then amounts to the condition that the f_i's generate an open ideal of A^+, i.e., they contain the pseudouniformizer t^N for $N \gg 0$. Moreover, in this case, we remark that

$$\operatorname{Spa}(A, A^+)\Big(\frac{f_1, ..., f_n}{g}\Big) = \operatorname{Spa}(A, A^+)\Big(\frac{f_1, ..., f_n, t^N}{g}\Big).$$

Indeed, $\supset$ is clear, while $\subset$ follows from the observation that $t^N = \sum_i a_i f_i$ with $a_i \in A^+$, so

$$|t^N(x)| \leq \max(|a_i(x)| \cdot |f_i(x)|) \leq \max(|f_i(x)|) \leq |g(x)|,$$

where the second inequality uses $|a_i(x)| \leq 1$ as $a_i \in A^+$ and $x \in \mathrm{Spa}(A, A^+)$. Thus, after possibly modifying our original choice of the f_i's, we may assume that f_n is a pseudouniformizer.

Intersect **Remark 7.4.4** (Stability under intersections)**.** The collection $\mathcal{B}$ appearing in Theorem 7.4.1 is actually stable under intersections. Fix two opens $B_1 := \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right)$ and $B_2 := \mathrm{Spa}(A, A^+)\left(\frac{a_1, \ldots, a_n}{b}\right)$ as in Theorem 7.4.1. We clearly have

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n, g}{g}\right),$$

and similarly

$$\mathrm{Spa}(A, A^+)\left(\frac{a_1, \ldots, a_m}{b}\right) = \mathrm{Spa}(A, A^+)\left(\frac{a_1, \ldots, a_m, b}{b}\right).$$

But then we can write

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n, g}{g}\right) \cap \mathrm{Spa}(A, A^+)\left(\frac{a_1, \ldots, a_m, b}{b}\right) = \mathrm{Spa}(A, A^+)\left(\frac{T}{gb}\right),$$

where T is the finite set of products of elements appearing in the numerator on the left, i.e., we set

$$T := \{f_i a_j, f_i b, a_j g, gb\}.$$

The verification of the preceding equality is left to the reader, and relies crucially on the fact that $|g(x)| \neq 0$ for $x \in B_1$, $|b(x)| \neq 0$ for $x \in B_2$, and they are both nonzero for any $x \in \mathrm{Spa}(A, A^+)\left(\frac{T}{gb}\right)$.

onqcopens **Warning 7.4.5** (Defining opens are not quasi-compact)**.** Theorem 7.4.1 does *not* assert that the sets $\mathrm{Spa}(A, A^+)\left(\frac{f}{g}\right)$ appearing in Definition 7.3.1 are quasi-compact. In fact, after fixing a pseudouniformizer t, we can write

$$\mathrm{Spa}(A, A^+)\left(\frac{f}{g}\right) := \cup_n \mathrm{Spa}(A, A^+)\left(\frac{f, t^n}{g}\right),$$

as $|g(x)| \neq 0$ implies that $|g(x)| > |t^n(x)|$ for $n \gg 0$. This expresses the open set on the left as a (typically infinite) union of the opens on the right. In particular, Zariski open subsets (corresponding to the condition $f = g$) are typically not quasi-compact. Note also that each set appearing on the right is quasi-compact by Theorem 7.4.1. This is the main reason the basis for the topology provided in Theorem 7.4.1 is more useful than open sets used to define the topology. For future reference, we remark that the displayed formula above shows that the topology on $\mathrm{Spa}(A, A^+)$ from Definition 7.3.1 coincides with the one generated by using the sets appearing in Theorem 7.4.1 as a sub-base.

Remark 7.4.6 (Adic spaces do not look like classical schemes)**.** Theorem 7.4.1 ensures that the topological space $\mathrm{Spa}(A, A^+)$ is homeomorphic to the spectrum of a ring, and is thus a familiar object from algebraic geometry. However, we caution the reader against placing much faith in the resulting intuition: the spectral spaces arising from adic spectra are quite different from the spectral spaces arising in classical algebraic geometry. Here are two (related) aspects in which they differ significantly:

1. As explained in Remark 7.3.11, the set of generalizations of any $y \in \mathrm{Spa}(A, A^+)$ is totally ordered. This structure is essentially never found in the spectral spaces coming from algebraic varieties over a field: there are multiple (in fact, infinitely many) non-comparable irreducible algebraic sets containing a given closed point on any variety of dimension > 1, so the set of generalizations of any such point is never totally ordered.

2. In Proposition 7.4.13, we shall show that the quotient of $X := \mathrm{Spa}(A, A^+)$ by the equivalence relation generated by specialization gives a compact Hausdorff space $\overline{X}$. When A arises from a NA Banach algebra, the space $\overline{X}$ is closely linked to the Berkovich space associated to A, and is a very interesting invariant of A. In contrast, doing the analogous construction when $X := \mathrm{Spec}(R)$ for a noetherian ring R simply collapses X to its finite set of connected components, viewed as a discrete topological space.

We now prove Theorem 7.4.1. The first step of the proof is to show that the valuation spectrum of any ring A is spectral; we recall briefly how this is shown, as the method will be crucial to the proof of Theorem 7.4.1 as well.

mSpectral **Theorem 7.4.7** (Valuation spectrum is spectral)**.** *Let A be a ring. The valuation spectrum* $\mathrm{Spv}(A)$ *is the set of equivalence classes of valuations on A, topologized using the sub-base $\mathcal{B}$ of open sets of the form*

$$\mathrm{Spv}(A)\left(\frac{f}{g}\right) = \{x \in \mathrm{Spv}(A) \mid |f(x)| \leq |g(x)| \neq 0\}$$

for varying $f, g \in A$. The resulting space $\mathrm{Spv}(A)$ *is spectral, and each $B \in \mathcal{B}$ is a quasi-compact open subset.*

As explained in Warning 7.4.5, it follows that for an affinoid Tate ring (A, A^+), the defining continuous inclusion $\mathrm{Spa}(A, A^+) \to \mathrm{Spv}(A)$ is *not* spectral in general.

Proof. One checks that each valuation x on $\mathrm{Spv}(A)$ determines and is determined by a binary relation $|_x$ on A via $a \mid_x b$ if and only if $|a(x)| \leq |b(x)|$. Sending x to the subset of $A \times A$ determined by this relation gives an injection $j : \mathrm{Spv}(A) \to \mathcal{P}(A \times A)$. An elementary argument shows that the image can be described as all relations satisfying the following for all $a, b, c \in A$:

1. $a \mid b$ or $b \mid a$

2. If $a \mid b$ and $b \mid c$ then $a \mid c$.

3. If $a \mid b$ and $a \mid c$ then $a \mid b + c$.

4. If $a \mid b$ then $ac \mid bc$.

5. If $ac \mid bc$ and $0 \nmid c$ then $a \mid b$.

6. $0 \nmid 1$.

Note that $\mathcal{P}(A \times A) := \mathrm{Map}(A \times A, \{0,1\}) = \{0,1\}^{A \times A}$ is naturally a profinite set. One checks that each of the conditions above describes a closed subset[8] of $\mathcal{P}(A \times A)$, so the image of j is closed. As the target is a quasi-compact, endowing $\mathrm{Spv}(A)$ with the subspace topology via j gives a quasi-compact space; call this space $\mathrm{Spv}(A)_{cons}$.

Now consider the topology $\mathrm{Spv}(A)$ from the theorem, i.e., the one generated by the sub-base $\mathcal{B}$. We claim this topology is T_0. Indeed, if $x, y \in \mathrm{Spv}(A)$ have different supports, then a separating open set can be pulled back from $\mathrm{Spec}(A)$. If the supports are the same prime $\mathfrak{p}$ but $x \neq y$, then there exist elements $a, b \in A$, not both in $\mathfrak{p}$, such that $|a(x)| \leq |b(x)|$ but $|a(y)| > |b(y)|$. If $|b(x)| = 0$, then $b \in \mathfrak{p}$, and the first inequality would force $a \in \mathfrak{p}$, which is not allowed. Thus, $|b(x)| \neq 0$, so the open $\mathrm{Spv}(A)\left(\frac{a}{b}\right)$ contains x but not y.

Finally, we remark that each $B \in \mathcal{B}$ is clopen in $\mathrm{Spv}(A)_{cons}$: we have $\mathrm{Spv}(A)\left(\frac{f}{g}\right) = \pi_{f,g}^{-1}(1) \cap \pi_{g,0}^{-1}(0)$ in the notation of the footnote.

Combining the assertions in the previous three paragraphs, we may apply Theorem 7.4.8 to $X = \mathrm{Spv}(A)_{cons}$ using the basis $\mathcal{B}$ to prove the theorem. □

We used the following criterion for spectrality:

Criterion

Theorem 7.4.8 (Hochster's criterion for spectrality)**.** *Let X be a quasi-compact topological space. Let $\mathcal{B}$ be a collection of clopen subsets $B \subset X$. Assume that the topology Σ generated by using $\mathcal{B}$ as a sub-base is T_0. Then (X, Σ) is a spectral space, and each $B \in \mathcal{B}$ is a quasi-compact open subset of (X, Σ).*

We can now prove that the adic spectrum is spectral.

Proof of Theorem 7.4.1. Fix a pseudouniformizer $t \in A$; all constructions will be independent of this choice. We proceed in a series of steps.

1. *Strategy of the proof:* Let $\mathcal{B}$ be the collection of subsets $B \subset \mathrm{Spa}(A, A^+)$ appearing in the statement of the theorem, i.e., $B = \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right)$ for $f_i, g \in A^+$ with $t^N \in (f_i)$ for some $N > 0$ (see Remark 7.4.3 to see why we can impose these additional restrictions on the f_i's and g). Let Σ be the topology on $\mathrm{Spa}(A, A^+)$ generated by using $\mathcal{B}$ as a sub-base; by the last statement of Warning 7.4.5, this coincides with defining topology on $\mathrm{Spa}(A, A^+)$, so it suffices to show that $(\mathrm{Spa}(A, A^+), \Sigma)$ is spectral with each $B \in \mathcal{B}$ being a quasi-compact open. Note that since $\mathrm{Spv}(A)$ is T_0, the same is true for the subspace $(\mathrm{Spa}(A, A^+), \Sigma)$. By Hochster's criterion, it suffices to construct another topology Σ_{quot} on $\mathrm{Spa}(A, A^+)$ such that $(\mathrm{Spa}(A, A^+), \Sigma_{quot})$ is quasi-compact with each $B \in \mathcal{B}$ being clopen in Σ_{quot}. We shall do so by realizing $(\mathrm{Spa}(A, A^+), \Sigma_{quot})$ as the quotient topology for a surjective map $r : X \to \mathrm{Spa}(A, A^+)$ with $X \subset \mathrm{Spv}(A)_{cons}$ a closed subset.

[8] We give one example. For $f, g \in A$, write $\pi_{f,g} : \mathcal{P}(A \times A) \to A \times A$ the projection onto the (f, g)-component of $\{0,1\}^{A \times A}$, so $\pi_{f,g}^{-1}(1)$ is the clopen subset of relations $\mid$ with $f \mid g$. Then the relations satisfying (4) are given by $\left(\pi_{a,b}^{-1}(1) \cap \pi_{ac,bc}^{-1}(1)\right) \cup \pi_{a,b}^{-1}(0)$. Fox fixed a, b, c, this set is clopen. Thus, intersecting over all choices of $a, b, c \in A$ gives a closed set.

2. *Construction of an auxiliary profinite set X:* Write $X \subset \mathrm{Spv}(A)$ for the subset spanned by those valuations x where $|f(x)| \leq 1$ for each $f \in A^+$, and $0 \neq |t(x)| \neq 1$. Using the embedding j from the proof of Theorem 7.4.7, we get an embedding $i : X \hookrightarrow \mathcal{P}(A \times A)$. One then checks[9] that X is a closed subset of $\mathcal{P}(A \times A)$. Thus, X is naturally a profinite set.

3. *The key step: relating X to* $\mathrm{Spa}(A, A^+)$. We clearly have $\mathrm{Spa}(A, A^+) \subset X$ as subsets of $\mathrm{Spv}(A)$, but we shall go in the other direction. By definition of X, for any $x \in X$, the valuation ring R_x naturally receives a map from A^+ with the image of $t \in R_x$ being a nonzero nonunit. Then $\mathrm{Spa}(A, A^+)$ is given by exactly those $x \in X$ where t actually gives a pseudouniformizer in R_x. For any $x \in X$, we may consider the t-adically separated quotient $\overline{R}_x := R_x / \cap_n t^n R_x$ of R_x. This is a microbial valuation ring with pseudouniformizer t by Lemma 7.3.6, so the resulting map $A^+ \to R_x \to \overline{R}_x$ gives a point $r(x) \in \mathrm{Spa}(A, A^+)$. This defines a set-theoretic map $r : X \to \mathrm{Spa}(A, A^+)$. As R_x is already microbial with pseudouniformizer t when $x \in \mathrm{Spa}(A, A^+)$, the map r is a section to the obvious inclusion $\mathrm{Spa}(A, A^+) \subset X$. Write Σ_{quot} for the quotient topology on $\mathrm{Spa}(A, A^+)$ induced by the surjective map r, so $(\mathrm{Spa}(A, A^+), \Sigma_{quot})$ is quasi-compact.

4. *Behaviour of the map r on the sub-base $\mathcal{B}$:* Now consider some $B \in \mathcal{B}$, i.e., $B := \mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right)$ for $f_i, g \in A^+$ with $t^N \in (f_i)$ for some $N > 0$. We claim that

$$r^{-1}(\mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right)) = X \cap \cap_{i=1}^n \mathrm{Spv}(A)\left(\frac{f_i}{g}\right),$$

and thus each of these sets is a clopen subset of X (as the basic opens $\mathrm{Spv}(A)\left(\frac{f_i}{g}\right)$ are clopen in $\mathrm{Spv}(A)$). To verify $\supset$: if x lies in the right hand side, then we have $|f_i(x)| \leq |g(x)| \neq 0$ for al i. By considering the composite $A^+ \to R_x \to \overline{R}_x$ as in the previous paragraph, this clearly implies $|f_i(r(x))| \leq |g(r(x))|$ for all i. Thus, x lies in the left hand side. For $\subset$: if x lies in the left hand side, we have $|f_i(r(x))| \leq |g(r(x))|$ for all i. Consider the map $A \to R_x \to \overline{R}_x$ defining r_x as above. As $t^N \in \sum_i f_i A^+$ and t is a pseudouniformizer in $\overline{R}_x$, it follows that at least one $|f_i(r(x))|$ is nonzero, and thus $|g(r(x))| \neq 0$, and so $|g(x)| \neq 0$ as well. As $R_x \to \overline{R}_x$ is a surjection of valuation rings, the divisibility $g \mid f_i$ in $\widehat{R}_x$ forces the same divisibility in R_x: if not, we must have $f_i \mid g$ for some i in R_x, but then $f_i \mid g$ and $g \mid f_i$ in $\overline{R}_x$, so f_i/g is a unit in $\overline{R}_x$, and hence also in R_x, so we get the desired divisibility anyways. So $|f_i(x)| \leq |g(x)| \neq 0$ for all i, and hence x gives a point on the right.

5. *Putting everything together:* We have already checked that $(\mathrm{Spa}(A, A^+), \Sigma)$ is T_0, and that $(\mathrm{Spa}(A, A^+), \Sigma_{quot})$ is quasi-compact. Moreover, we also saw that any $B \in \mathcal{B}$ is clopen in $(\mathrm{Spa}(A, A^+), \Sigma_{quot})$: its preimage under r was shown to be clopen in X. Theorem 7.4.8 then proves the theorem as Σ is generated by $\mathcal{B}$ as a sub-base.

[9] In addition to the intersection of clopens used to define $\mathrm{Spv}(A)$, we must intersect with the following additional closed sets: the intersection $\cap_{f \in A^+} \pi_{f,1}^{-1}(1)$ to capture $|f(x)| \leq 1$ for $f \in A^+$, the clopen $\pi_{t,0}^{-1}(0)$ to capture $|t(x)| > 0$, and the clopen $\pi_{1,t}^{-1}(0)$ to capture $1 > |t(x)|$.

□

We give two sample applications of the spectrality of the adic spectrum. First, using quasi-compactness, we can detect topological nilpotence of elements of A by their behaviour on points, analogous to Proposition 7.3.10 (3):

Corollary 7.4.9 (Detecting nilpotence locally). *Fix an affinoid Tate ring (A, A^+), and an element $f \in A$. Let $X := \mathrm{Spa}(A, A^+)$. Then $f \in A^{\circ\circ}$ if and only if $|f(x)|^n \to 0$ for all $x \in X$.*

Proof. Fix a pseudouniformizer $t \in A$. If f is topologically nilpotent, then $f^N \in tA^+$ for $N \gg 0$. But then $|f(x)|^N \leq |t(x)|$ for all $x \in X$, so $|f(x)|^n \to 0$ as $|t(x)|^n \to 0$. Conversely, assume that $|f(x)|^n \to 0$. Then we have

$$X = \cup_n X\left(\frac{f^n}{t}\right)$$

by hypothesis. As X is quasi-compact, this means $X = X\left(\frac{f^n}{t}\right)$ for some n. But then $|f(x)|^n \leq |t(x)|$ for all $x \in X$, so the element $\frac{f^n}{t} \in A$ lies in A^+. This implies $f^n \in tA^+$. Writing A^+ as a filtered colimit of open bounded subrings containing t, we learn that $f^n \in tA_0$ for some open bounded subring $A_0 \subset A$. This immediately implies $f^n \in A^{\circ\circ}$ and thus $f \in A^{\circ\circ}$. □

Colimits in the category of Tate rings are somewhat subtle, due to topological issues. For example, given a filtered directed system $\{A_i\}$ of Tate rings, there is no obvious topology on the colimit $\mathrm{colim}_i A_i$ that makes it into a Tate ring (as there might not be compatible rings of definition in the system). This creates some problems in discussing inverse systems of adic specta. However, if one restricts to the uniform setting, there is a canonical choice of ring of definition (namely, the powerbounded elements), so the previous problem disappears:

Affinoids **Corollary 7.4.10** (Directed limits of uniform affinoids). *Let (A_i, A_i^+) be a filtered system of uniform affinoid Tate rings. Then*

1. *The direct limit (A, A^+) of (A_i, A_i^+) exists in the category of all uniform affinoid Tate rings. Moreover, the A^+ is identified as the $\mathrm{colim}_i A_i^+$, computed in the category of abstract rings.*
2. *The natural map gives a homeomorphism $\mathrm{Spa}(A, A^+) \simeq \lim \mathrm{Spa}(A_i, A_i^+)$. Moreover, each rational subset of $\mathrm{Spa}(A, A^+)$ is pulled back from a rational subset of some $\mathrm{Spa}(A_i, A_i^+)$.*

Part (1) was already asserted earlier in Exercise 7.2.6.

Proof. The proof of (2) shall use the construction of (1).

1. We may assume the index set has some minimal element i_0. Choose a pseudouniformizer $t \in A_{i_0}^+$. By uniformity, each $A_i^+ \subset A_i$ is a ring of definition with pseudouniformizer t. Set $A = \mathrm{colim}_i A_i$, viewed as a Tate ring with couple of definition $(\mathrm{colim}_i A_i^+, t)$, and let $A^+ = \mathrm{colim}_i A_i^+ \subset A$. We claim that (A, A^+) is a uniform affinoid Tate ring, and is the colimit of (A_i, A_i^+) in uniform affinoid Tate rings. In fact, the first assertion is clear from the definition as the ring of integral elements is the ring of definition.

For the assertion about colimits: say $f_i : (A_i, A_i^+) \to (B, B^+)$ is a compatible system of maps with (B, B^+) being a uniform affinoid Tate ring. We must factor the f_i's through a unique map $f : (A, A^+) \to (B, B^+)$ of affinoid Tate rings. At the level of (non-topological) rings, we clearly get an induced map $f : A \to B$ that carries A^+ into B^+. As (B, B^+) is uniform, we may use B^+ as a ring of definition for f. But we clearly that $f^{-1}(t^n B^+) \supset t^n A^+$, so f is continuous. Thus, f defines a continuous map $(A, A^+) \to (B, B^+)$ of affinoid Tate rings. The uniqueness is clear.

2. Given a point $x \in \mathrm{Spa}(A, A^+)$, we obtain a valuation ring $R_x \subset \kappa(\mathfrak{p}_x)$. Then $(\kappa(\mathfrak{p}_x), R_x)$ is a uniform affinoid Tate ring for the valuation topology on $\kappa(\mathfrak{p}_x)$, and the point x determines (and is determined by) the map $(A, A^+) \to (\kappa(\mathfrak{p}_x), R_x)$ of affinoid Tate rings. The point x also determines points $x_i \in \mathrm{Spa}(A_i, A_i^+)$ together with the corresponding maps $(A_i, A_i^+) \to (\kappa(\mathfrak{p}_{x_i}), R_{x_i})$. Unraveling definitions shows that $(\kappa(\mathfrak{p}_x), R_x) = \mathrm{colim}_i(\kappa(\mathfrak{p}_{x_i}), R_{x_i})$ (where the colimit is computed in uniform affinoid Tate rings, as in (1)). It is now easy to see that $\mathrm{Spa}(A, A^+) \to \lim_i \mathrm{Spa}(A_i, A_i^+)$ is a continuous bijection. For the rest, we simply observe that $A = \mathrm{colim}_i A_i$ as rings, so the defining open subsets $\mathrm{Spa}(A, A^+)\left(\frac{f}{g}\right)$ for $f, g \in A$ arise via pullback from some $\mathrm{Spa}(A_i, A_i^+)$.

□

The following exercise shall be useful later in studying perfectoid algebras in terms of finitely presented ones:

cSpectrum **Exercise 7.4.11** (Adic spectrum under perfection)**.** Let (A, A^+) be an affinoid Tate ring of characteristic p.

1. Show that the Frobenius map on underlying rings gives a map $(A, A^+) \to (A, A^+)$ of affinoid Tate rings that induces a homeomorphism on adic spectra that preserves rational subsets.

2. Assume (A, A^+) is uniform. Show that there is an initial object (A_{perf}, A_{perf}^+) amongst all uniform affinoid Tate (A, A^+)-algebras (B, B^+) such that B is perfect. Moreover, check that A_{perf} is the perfection of A in the algebraic sense, and likewise for A^+.

3. With notation as in (2), show that the canonical map $\mathrm{Spa}(A_{perf}, A_{perf}^+) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism preserving rational subsets.

The next remark gives an alternative to the support map as a method of probing the topology of the adic spectrum using spectra of commutative rings.

alization **Remark 7.4.12** (The specialization map)**.** Let (A, A^+) be an affinoid Tate ring. We have seen in Warning 7.4.5 that the kernel map $\ker : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ is continuous, but not spectral, i.e., the preimage of quasi-compact open subsets of $\mathrm{Spec}(A)$ under $\ker$ need not be quasi-compact. In this remark, we explain why there is another natural continuous map

$$\mathrm{sp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/A^{\circ\circ}),$$

called the *reduction* or *specialization* map, which is spectral. To explain this, note first that $\operatorname{Spec}(A^+) - \operatorname{Spec}(A) = \operatorname{Spec}(A^+/A^{\circ\circ})$ as subsets of $\operatorname{Spec}(A^+)$. Now given a point $x \in \operatorname{Spa}(A, A^+)$ corresponding to the map $\phi_x : A^+ \to R_x$ as in Proposition 7.3.7, we set $\operatorname{sp}(x)$ to be the image in $\operatorname{Spec}(A^+)$ of the closed point of $\operatorname{Spec}(R_x)$; as ϕ carries pseudouniformizers to pseudouniformizers, it is clear that $\operatorname{sp}(x)$ lies in $\operatorname{Spec}(A^+) - \operatorname{Spec}(A) = \operatorname{Spec}(A^+/A^{\circ\circ})$. For continuity and spectrality, fix some $\overline{f} \in A^+/A^{\circ\circ}$ defining an open set $D(\overline{f}) \subset \operatorname{Spec}(A^+/A^{\circ\circ})$. Representing $\overline{f}$ by some $f \in A^+$, the preimage $\operatorname{sp}^{-1}(D(\overline{f}))$ consists of exactly those $x \in \operatorname{Spa}(A, A^+)$ for which $\phi_x(f) \in R_x$ is a unit, i.e., it is exactly those x for which $|f(x)| = 1$. As $|f(x)| \leq 1$ for all x, we have shown

$$\operatorname{sp}^{-1}(D(\overline{f})) = \operatorname{Spa}(A, A^+)\Big(\frac{1}{f}\Big) = \{x \in \operatorname{Spa}(A, A^+) \mid |f(x)| = 1\}.$$

As the right side above is a rational subset, this proves both continuity and spectrality.

We briefly comment on the Hausdorffness of the space obtained collapsing specializations on X; this is closely related to the Hausdorffness of of Berkovich spaces; this discussion is not relevant to the sequel.

fQuotient **Proposition 7.4.13** (Maximal Hausdorff quotient)**.** *Let* $X := \operatorname{Spa}(A, A^+)$ *for an affinoid Tate ring* (A, A^+)*, and let* $\overline{X}$ *be the quotient of* X *by the equivalence relation generated by specializations, endowed with the quotient topology. Then* $\overline{X}$ *is Hausdorff, and is the maximal Hausdorff quotient of* X.

Note that by Remark 7.3.11, the subspace X_1 of generic points of X (corresponding to rank 1 valuations) maps bijectively to $\overline{X}$; the subspace topology on X_1, however, is distinct from the quotient topology on $\overline{X}$. The proof of this proposition relies exclusively on the structure of generalizations from Remark 7.3.11 and the fact that X is spectral.

Proof. If we prove that $\overline{X}$ is Hausdorff, it automatically follows that $X \to \overline{X}$ is the maximal Hausdorff quotient of X: any map $f : X \to Y$ with Y Hausdorff has to identify points related by a specialization in X (as points on X in distinct fibers of f must have disjoint open neighbourhoods by Hausdorffness of Y), and thus factors through $\overline{X}$.

To show $\overline{X}$ is Hausdorff, fix $x, y \in X$ with distinct images in $\overline{X}$, so x and y are not related by specializations in $\overline{X}$. By passing to the maximal rank 1 generalization as in Remark 7.3.11, we may assume x and y are both generic. As the closure $\overline{\{x\}}$ is the set of specializations of x (as X is spectral), we have $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$ by hypothesis on x and y. Unraveling definitions, we must find disjoint open neighbourhoods of x and y that are stable under specializations. Let $V_y = X - \overline{\{x\}}$, so V_y is an open neighbourhood of $\overline{\{y\}}$. Let $W_y \subset V_y$ be a quasi-compact open containing $\overline{\{y\}}$ (which is possible because $\overline{\{y\}}$ is spectral and thus quasi-compact), and let $T_x = X - W_y$ be the complementary constructible closed subset, so $\overline{\{x\}} \in T_x$. Then $U_x := \operatorname{Int}(T_x)$ does not contain y. Moreover, applying (1) in the next lemma twice and using that x has no non-trivial generalizations, we learn that U_x is an open neighbourhood of x stable under specializations. The quasi-compact open neighbourhood W_y of $\overline{\{y\}}$ also contains a smaller open neighbourhood U_y of y that is closed under specializations by (2) in the next lemma. These open neighbourhoods U_x and U_y provide the desired disconnection.

Lemma 7.4.14. *Let $X := \mathrm{Spa}(A, A^+)$.*

1. *Let $T \subset X$ be a constructible closed set. Then the interior $\mathrm{Int}(T)$ is exactly those $t \in T$ such that all generalizations of t in X are also contained in T. Moreover, this set is closed under specializations in X.*

2. *Let $y \in X$ be a generic point. Then for any quasi-compact open neighbourhood W of the closure $\overline{\{y\}}$, we can find a smaller open neighbourhood W' of $\overline{\{y\}}$ such that W' is closed under specializations in X.*

Proof. For (1): the first part is a general fact about spectral spaces, topologically dual to the characterization of the closure of a quasi-compact open subset of a spectral space as the set of specializations of points in the open. To show that $\mathrm{Int}(T)$ is closed under specializations in X, fix some $t \in \mathrm{Int}(T)$. Any specialization s of t in X must lie in T as $T \subset X$ is closed. We must show that each generalization u of s lies in T. But the set of generalizations of s forms a totally ordered set by Remark 7.3.11. So either u is a specialization of t or u is a generalization of t; in the former case, we have $u \in \overline{\{t\}} \subset T$, and in the latter case we have $u \in \mathrm{Int}(T) \subset T$ by the characterization of points in $\mathrm{Int}(T)$ as $t \in \mathrm{Int}(T)$. In either case, we get $u \in T$, as wanted.

For (2): the closure $\overline{\{y\}}$ (or any closed set of a spectral space) can be written as $\cap_i Z_i$ where Z_i runs through all constructible closed subsets of X containing y. Then $Z_i \subset W$ for large i by quasi-compactness. But $\mathrm{Int}(Z_i)$ is closed under specializations by (1), and contains $\overline{\{y\}}$ by the characterization of interiors mentioned previous paragraph as y is generic. So we may use $W' = \mathrm{Int}(Z_i)$. □

□

7.5 The structure presheaf and adic spaces

mPresheaf We shall attach a natural structure *presheaf* to an affinoid adic space; this presheaf is not always a sheaf. The values of this presheaf on rational open subsets are described next.

nPresheaf **Theorem 7.5.1** (Functions on rational subsets)**.** *Let (A, A^+) be an affinoid Tate ring. Let $U \subset X := \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a unique complete affinoid Tate (A, A^+)-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ satisfying:*

1. *The map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ has image contained in U.*

2. *$(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is universal with the property in (1) amongst all complete affinoid Tate (A, A^+)-algebras.*

Moreover, in this case, the canonical map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to U$ is a homeomorphism identifying rational subsets of the source with rational subsets of X contained in U.

Proof. Choose a couple of definition (A_0, t). We can write $U = \mathrm{Spa}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right)$ for $f_i, g \in A_0$ with $t^N \in (f_i) \subset A_0$ for some $N \geq 0$. Let B_0 be the subring of $A[\frac{1}{g}]$ generated by the image of $A_0[\frac{f_i}{g}]$, and set $B = A[\frac{1}{g}]$. Note that $B = B_0[\frac{1}{t}]$: we clearly that $B_0[\frac{1}{t}] = A_0[\frac{f_i}{g}, \frac{1}{t}] \subset A[\frac{1}{g}]$, and, since the ideal $(f_i) \subset A_0$ contains a power of t, we can write $\frac{1}{g} = \frac{1}{t^N} \cdot (\sum_i a_i \frac{f_i}{g})$ with $a_i \in A_0$. Thus, B can be viewed as a Tate A-algebra with couple of definition (B_0, t). Let B^+ be the integral closure of the subring of B generated by $A^+[\frac{f_i}{g}]$. Then (B, B^+) is an affinoid Tate (A, A^+)-algebra. We set $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be its completion.

By construction, the image of $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ is contained in U: for any x on the the left hand side, we must have $|g(x)| \neq 0$ as g is a unit in $\mathcal{O}_X(U)$ and $|\frac{f_i(x)}{g(x)}| \leq 1$ as $\frac{f_i}{g} \in \mathcal{O}_X^+(U)$.

Let (C, C^+) be any complete affinoid Tate (A, A^+)-algebra such that the induced map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ has image contained in U. For any $x \in \mathrm{Spa}(C, C^+)$, we have $|f_i(x)| \leq |g(x)| \neq 0$ by hypothesis. Proposition 7.3.10 (6) shows that the image of g in C is a unit, so $\frac{f_i}{g} \in C$ must lie in C^+ by Proposition 7.3.10 (3). As $C^+ \subset C^\circ$, and because C° is the direct limit of all rings of definition of C, we can then choose a ring of definition $C_0 \subset C$ that contains the image of A_0 as well as the elements $\frac{f_i}{g} \in C$. This gives a map $B_0 \to C_0$ that produces a map $B \to C$ of Tate A-algebras. Passing to t-adic completions defines a map $\mathcal{O}_X(U) \to C$ of Tate A-algebras. As $\frac{f_i}{g} \in C^+$ and $C^+ \subset C$ is integrally closed, the map $B \to C$ carries B^+ into C^+, and hence the map $\mathcal{O}_X(U) \to C$ carries $\mathcal{O}_X^+(U)$ into C^+, thus giving the desired map $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (C, C^+)$ of complete affinoid Tate (A, A^+)-algebras.

For the last assertion, by Proposition 7.3.10 (1), it suffices to show that $\mathrm{Spa}(B, B^+) \to U$ is a homoemorphism preserving rational subsets. The injectivity is clear as $A \to B$ is a localization; the surjectivity can be shown directly using the valuative description of points given in Proposition 7.3.7 and the universal property proven above. Continuity is clear, so we have a bijective continuous map

$$Y := \mathrm{Spa}(B, B^+) \xrightarrow{\Psi} U$$

of spectral spaces. It suffices to show Ψ carries rational subsets of Y into rational subsets of X contained in U. Consider a rational subset V of Y. We can write $V = Y\left(\frac{b_1,\dots,b_n}{c}\right)$ for some $b_i \in B_0$ generating an ideal that contains t^m for some $m \geq 0$, and $c \in B_0$. As g is a unit on B, we have $|g(y)| \neq 0$ for any $y \in Y$. So we are free to scale our parameters defining V by a power of g; note that the property $t^m \in (b_i) \subset B_0$ for some $m \geq 0$ is equivalent to the property that $1 = (b_i) \subset B = B_0[\frac{1}{t}]$, and (as $g \in B$ is a unit) is thus preserved when we scale the b_i's by a power of g. In particular, as $B_0 \subset A_0[\frac{1}{g}]$, we may thus assume that b_i and c lift to some $\tilde{b_i}$ and $\tilde{c}$ in A_0. We claim that

$$\Psi(Y\left(\frac{b_1,\dots,b_n}{c}\right)) = U \cap X\left(\frac{\tilde{b_i}, t^k}{c}\right)$$

for some $k \gg 0$; this suffices to prove the theorem by Remark 7.4.4. We clearly have $\supset$ for any $k \geq 0$ by simply using the valuative description of points from Proposition 7.3.7. For $\subset$, we need to check that there exists some $k \geq 0$ such that $|t^k(y)| \leq |c(y)|$ for all $y \in V := Y\left(\frac{b_1,\dots,b_n}{c}\right)$. As

$|c(y)| \neq 0$ for all $y \in V$ and t is a pseudouniformizer, we get

$$V = \cup_k Y\Big(\frac{b_1, ..., b_n, t^k}{c}\Big).$$

As V is quasi-compact by Theorem 7.4.1, we get $V = Y\Big(\frac{b_1,...,b_n,t^k}{c}\Big)$ for some $k \gg 0$, proving the claim. □

iskiLocal **Remark 7.5.2** (Relaxing completeness)**.** The proof of Theorem 7.5.1 goes through if we relax the condition of being "complete" in (2) with the conditon of being "Zariski" or "henselian". Indeed, in the proof above, the completeness was invoked via Proposition 7.3.10 (6), but the latter holds as long as (C, C^+) is Zariski. Thus, we learn that for any rational open U, there exists a universal Zariski affinoid (A, A^+)-algebra $(\mathcal{O}_{X,zar}(U), \mathcal{O}_{X,zar}(U))$ with the image of $\mathrm{Spa}(\mathcal{O}_{X,zar}(U), \mathcal{O}^+_{X,zar}(U)) \to \mathrm{Spa}(A, A^+)$ lying in U; similarly, there exists a universal henselian (A, A^+)-algebra $(\mathcal{O}_{X,hens}(U), \mathcal{O}^+_{X,hens}(U))$ with the image of $\mathrm{Spa}(\mathcal{O}_{X,hens}(U), \mathcal{O}^+_{X,hens}(U)) \to \mathrm{Spa}(A, A^+)$ lying in U.

The values of Huber's presheaf behave well under base change:

rPresheaf **Corollary 7.5.3.** *Let $(A, A^+) \to (B, B^+)$ be a map of complete affinoid Tate rings, and let $f : X := \mathrm{Spa}(B, B^+) \to Y := \mathrm{Spa}(A, A^+)$ be the induced map on adic spectra. Let $U \subset Y$ be a rational subset, and let $V := f^{-1}(U)$ be the corresponding rational subset of X. Then the diagram*

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \downarrow \\ (\mathcal{O}_Y(U), \mathcal{O}^+_Y(U)) & \longrightarrow & (\mathcal{O}_X(V), \mathcal{O}^+_X(V)) \end{array}$$

is a pushout of complete affinoid Tate rings.

Proof. This follows immediately from the universal property in Theorem 7.5.1. □

The stalks for the structure presheaf (to be defined) are:

ePresheaf **Definition 7.5.4** (Stalks)**.** Let (A, A^+) be an affinoid Tate ring, and let $x \in X := \mathrm{Spa}(A, A^+)$ be a point. We define the stalks

$$\mathcal{O}^+_{X,x} := \operatorname*{colim}_{x \in U} \mathcal{O}^+_X(U) \quad \text{and} \quad \mathcal{O}_{X,x} := \operatorname*{colim}_{x \in U} \mathcal{O}_X(U)$$

where both colimits run through rational open subsets $U \subset X$ containing x. Note that the colimits above are computed in the category of rings (i.e., we ignore the topology on $\mathcal{O}_X(U)$ and $\mathcal{O}^+_X(U)$).

As the Tate rings $\mathcal{O}_X(U)$ attached to the rational open sets $U \subset X$ were defined to be complete, the stalks defined above only depend on the completion of (A, A^+).

aceStalks **Proposition 7.5.5** (The local rings of adic spectra)**.** *Let (A, A^+) be an affinoid Tate ring, and let $x \in X := \mathrm{Spa}(A, A^+)$ be a point. Fix also a pseudouniformizer $t \in A^+$.*

1. *The valuation $f \mapsto |f(x)|$ extends to the stalk $\mathcal{O}_{X,x}$. We then have $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\}$.*

2. *The stalk $\mathcal{O}_{X,x}$ is local with maximal ideal given by the support $\mathfrak{m}_x := \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$.*

3. *The stalk $\mathcal{O}_{X,x}^+$ is local with maximal ideal $\{f \in \mathcal{O}_{X,x} \mid |f(x)| < 1\}$. In particular, the set $\mathfrak{m}_x$ from (2) is an ideal in both $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$.*

4. *Write $k(x)$ for the residue field of $\mathcal{O}_{X,x}$, and let $k(x)^+ \subset k(x)$ be the image of $\mathcal{O}_{X,x}^+$. The valuation on $\mathcal{O}_{X,x}$ endows $k(x)$ with the structure of a valued field, and the corresponding valuation ring is $k(x)^+$. Thus, $(k(x), k(x)^+)$ is an affinoid field under (A, A^+). Moreover, we have a natural map $R_x \to k(x)^+$ which is an isomorphism after t-adic completion.*

5. *The ring $\mathcal{O}_{X,x}^+$ is t-adically henselian and $\mathcal{O}_{X,x}^+ \to k(x)^+$ induces an isomorphism on t-adic completion.*

6. *The pairs $(\mathcal{O}_{X,x}^+, \mathfrak{m}_x)$ and $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ are henselian.*

Proof. Write $\mathcal{U}$ for the collection of all rational open subsets $U \subset X$ containing x.

1. Consider the t-adic completion $\widehat{R_x}$ of the valuation ring $R_x \subset \kappa(\mathfrak{p}_x)$ attached to x. Then $\widehat{R_x}$ is a microbial valuation ring with fraction field $\widehat{\kappa(\mathfrak{p}_x)}$, so $(\widehat{\kappa(\mathfrak{p}_x)}, \widehat{R_x})$ is an affinoid Tate (A, A^+)-algebra corresponding to x under the valuative description of Proposition 7.3.7. Using the universal property from Theorem 7.5.1, there is a unique map $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (\widehat{\kappa(\mathfrak{p}_x)}, \widehat{R_x})$ of affinoid Tate (A, A^+)-algebras for each $U \in \mathcal{U}$. By passage to the limit, this defines maps $\mathcal{O}_{X,x} \to \widehat{\kappa(\mathfrak{p}_x)}$ and $\mathcal{O}_{X,x}^+ \to \widehat{R}_x$. The first map induces the desired valuation on $\mathcal{O}_{X,x}$, whence the second map gives the containment $\subset$ in the desired equality $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\}$. For $\supset$, if $\overline{f} \in \mathcal{O}_{X,x}$ with $|\overline{f}(x)| \leq 1$, then we can represent it by some $f \in \mathcal{O}_X(U)$ with $|f(x)| \leq 1$. But then $x \in V := U\left(\frac{f,1}{1}\right) \subset U$, so $f \in \mathcal{O}_X^+(V)$, and thus $\overline{f} \in \mathcal{O}_{X,x}^+$.

2. It suffices to check that any $g \in \mathcal{O}_{X,x}$ outside $\mathfrak{m}_x$ is invertible in $\mathcal{O}_{X,x}$. Given such a g, we have $|g(x)| \neq 0$, so $|g(x)| \geq t^n$ for $n \gg 0$. By shrinking X if necessary, we may assume $g \in A$ is globally defined. But then $U := X\left(\frac{t^n}{g}\right) \in \mathcal{U}$, and g is invertible in $\mathcal{O}_X(U)$, and thus also in $\mathcal{O}_{X,x}$.

3. It suffices to check that any $g \in \mathcal{O}_{X,x}^+$ outside the ideal $I := \{f \in \mathcal{O}_{X,x} \mid |f(x)| < 1\}$ is invertible. By shrinking X if necessary, we may assume $g \in A^+$ is globally defined. We then have $|g(x)| \leq 1$ as g is integral, and thus $|g(x)| = 1$ as g lies outside I. But then $x \in U := X\left(\frac{1}{g}\right)$. It is easy to see that $g \in \mathcal{O}_X^+(U)$ is invertible, and thus g is invertible.

4. It is clear that the valuation on $\mathcal{O}_{X,x}$ induces a valuation on $k(x)$, and that the valuation ring V contains $k(x)^+$. Conversely, given any $\bar{f} \in V$ represented by some $f \in \mathcal{O}_{X,x}$, we have $|f(x)| \leq 1$, so $f \in \mathcal{O}^+_{X,x}$ by (1), and hence $\bar{f} \in k(x)^+$; thus the valuation ring is exactly $k(x)^+$. For the last part, it is clear that $0 < |t(x)| < 1$, so we must show that t is topologically nilpotent in $k(x)^+$. This immediately reduces to the corresponding statement for $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$, where it is clear.

   The last part follows from the construction of the valuation in (1).

5. For the first part: as direct limits of henselian rings are henselian, it suffices to show that for each $U \in \mathcal{U}$, the ring $\mathcal{O}^+_X(U)$ is t-adically henselian, which follows from Lemma 7.2.3 (5). For the second part, observe that the support ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is uniquely t-divisible (as it is an ideal in a ring where t is a unit) and is contained in $\mathcal{O}^+_{X,x}$ by (3). The quotient $\mathcal{O}^+_{X,x}/J$ identifies with $k(x)^+$, so $\widehat{\mathcal{O}^+_{X,x}} \simeq \widehat{k(x)^+}$.

6. The claim for $(\mathcal{O}^+_{X,x}, \mathfrak{m}_x)$ follows from (5) as $\mathfrak{m}_x = t \cdot \mathfrak{m}_x \subset t\mathcal{O}^+_{X,x}$ by (1) and (2). For $\mathcal{O}_{X,x}$, recall the following fact: given a pair (R, I) comprising of a ring R with an ideal I with I contained in the Jacobson radical of R, the pair (R, I) is henselian if and only if the pair $(\mathbf{Z} \oplus I, I)$ is henselian (see [GR, Remark 5.1.9 (iii)]). Thus, the property of a pair (R, I) being henselian only depends on the ideal I viewed as a non-unital ring. In particular, $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is henselian as $(\mathcal{O}^+_{X,x}, \mathfrak{m}_x)$ is so. In the interest of completeness, we also give a more direct argument.

   Fix a finite $\mathcal{O}_{X,x}$-algebra R, and write $\overline{R} := R \otimes_{\mathcal{O}_{X,x}} k(x)$. We must show that $R \to \overline{R}$ is a bijection on idempotents. Write R^+ for the integral closure of $\mathcal{O}^+_{X,x}$ inside R, and write $\overline{R^+} := R \otimes_{\mathcal{O}^+_{X,x}} k(x)^+$ for the base change to the valuation ring. Then R^+ is an integral $\mathcal{O}^+_{X,x}$-algebra, so $R^+ \to \overline{R^+}$ is a bijection on idempotents as $\mathcal{O}^+_{X,x}$ is henselian along $\mathfrak{m}_x$. Moreover, as R^+ is the integral closure of $\mathcal{O}^+_{X,x}$ in R, the map $R^+ \to R$ is a bijection on idempotents (as idempotents are integral over $\mathbf{Z}$). We thus have the following diagram

$$\begin{array}{ccc} R^+ & \xrightarrow{a} & \overline{R^+} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ R & \xrightarrow{d} & \overline{R} \end{array}$$

   where the horizontal maps are surjective, the vertical maps invert t, and both a and b are bijection on idempotents. We shall show that the map c realizes $\overline{R^+}$ as an integrally closed subring of $\overline{R}$; this will show that c, and hence d, is a bijection on idempotents, as wanted. To see this claim, consider the following commutative diagram of short exact sequences resulting from the analysis in (1) and (2):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{m}_x & \longrightarrow & \mathcal{O}^+_{X,x} & \longrightarrow & k(x)^+ & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathfrak{m}_x & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & k(x) & \longrightarrow & 1. \end{array}$$

Tensoring over $\mathcal{O}_{X,x}^+$ with R^+, we get

$$\begin{array}{ccccccc}
\mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}^+} R^+ & \xrightarrow{e} & R^+ & \longrightarrow & \overline{R^+} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}^+} R^+ & \xrightarrow{f} & R & \longrightarrow & \overline{R} & \longrightarrow & 0.
\end{array}$$

The middle vertical map is injective. A diagram chase then shows that $\ker(e) \simeq \ker(f)$ is an ideal in both R^+ and R. Calling this common ideal I, we get a diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I & \longrightarrow & R^+ & \longrightarrow & \overline{R^+} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & \overline{R} & \longrightarrow & 0.
\end{array}$$

In particular, $\overline{R^+} \to \overline{R}$ is injective by the snake lemma. It remains to show integral closedness. Pick some $\overline{f} \in \overline{R}$ that is integral over $\overline{R^+}$. Then we have an equation in $\overline{R}$ of the form

$$\overline{f}^n = \sum_{i=0}^{n-1} \overline{a_i}\,\overline{f}^i,$$

where $\overline{a_i} \in \overline{R^+}$. Writing f and a_i for lifts of $\overline{f}$ and $\overline{a_i}$ to R, this becomes an equation of the form

$$f^n + \delta = \sum_{i=0}^{n-1} a_i f^i,$$

where $\delta \in I = \ker(R \to \overline{R}) \subset R$. But $I \subset R^+$, so we can rename a_0 to $a_0 - \delta \in R^+$ to learn that f is integral over R^+, and hence in R^+ by definition of R^+. This shows that $\overline{f} \in \overline{R^+}$, as wanted.

□

Remark 7.5.6 (Relaxing completeness)**.** The conclusions of Proposition 7.5.5 are also valid if we replace the presheaf $\mathcal{O}_X$ with its henselian version $\mathcal{O}_{X,hens}$ (but note that parts (5) and (6) will fail for the Zariski version).

idueField **Remark 7.5.7** (Completed residue field)**.** The NA field $k(x)$ from Proposition 7.5.5 is called the *residue field* of X at x; we write $\widehat{k(x)}$ for its completion in the valuation topology, and call it the *completed residue field*. Thus, attached to $x \in \mathrm{Spa}(A, A^+)$, we have three NA fields $\kappa(\mathfrak{p}_x) \subset k(x) \subset \widehat{k(x)}$ with identical completion. In practice, it is often convenient to use $\widehat{k(x)}$ instead of $k(x)$ or $\kappa(\mathfrak{p}_x)$ while making arguments, for the following reasons:

1. $\widehat{k(x)}$ is a *complete* NA field, unlike $k(x)$ or $\kappa(\mathfrak{p}_x)$.

2. $\widehat{k(x)}$ can be calculated directly from the ring A as the completion of $\kappa(\mathfrak{p}_x)$ for valuation topology defined by x. In contrast, calculating $k(x)$ involves contemplating $\mathcal{O}_X(U)$ for rational open subsets $U \subset X$ containing x.

3. $\widehat{k(x)}$ is manifestly independent of the choice of X: replacing X by a rational open subset $U \subset X$ does not change $k(x)$ as the latter only depends on the local ring $\mathcal{O}_{X,x}$. In contrast, the field $\kappa(\mathfrak{p}_x)$ depends on the ring A appearing in the definition of X, and can change when we shrink X around x.

Likewise, we have a complete microbial valuation ring $\widehat{k(x)^+} \subset \widehat{k(x)}$ obtained as the t-adic completion of $k(x)^+$. This gives a complete affinoid field $(\widehat{k(x)}, \widehat{k(x)^+})$ attached to x, and this often plays the role in adic geometry of the residue field in classical scheme theory.

Proposition 7.5.5 (5) is surprising from an algebro-geometric perspective. As the next corollary explains, this essentially leads to the conclusion that Zariski closed subsets of adic spaces are limits of open ones; this assertion is perhaps even more jarring from a naive algebro-geometric perspective, but becomes much more intuitive upon observing that an analogous statement holds true in complex analytic geometry as well:

Immersion **Corollary 7.5.8** (Zariski closed subsets)**.** *Let (A, A^+) be an affinoid Tate ring. Let $I \subset A$ be an ideal, and $t \in A^+$ a pseudouniformizer.*

1. *The category of affinoid Tate (A, A^+)-algebras (R, R^+) such that $IR = 0$ has an initial object $(A/I, A/I^+)$ with underlying ring A/I, and A/I^+ being the integral closure of the image A^+ in A/I.*

2. *The natural map $i : Z := \mathrm{Spa}(A/I, A/I^+) \to X := \mathrm{Spa}(A, A^+)$ is a closed immersion of topological spaces whose image is exactly those valuations on A that contain I in their support. Moreover, $i(Z)$ is an intersection of quasi-compact open subsets of X. From now on, we identify Z with its image $i(Z)$ in X.*

3. *For any $z \in Z \subset X$, the natural yields an isomorphism $\widehat{\mathcal{O}^+_{X,z}} \simeq \widehat{\mathcal{O}^+_{Z,z}}$ of t-adic completions.*

4. *Say $I = (f_1, ..., f_r)$ is finitely generated, and $U \subset X$ is an open set containing Z. Then U contains the rational open set $X\left(\frac{f_1,...,f_r,t^N}{t^N}\right)$ for $N \gg 0$.*

5. *Let $\widehat{A/I^+}$ be the t-adic completion of the ring from (1), where $t \in A^+$ is a pseudouniformizer. For each rational $U \subset X$ that contains Z, we have a map $\mathcal{O}^+_X(U) \to \widehat{A/I^+}$. The induced map*

$$\operatorname*{colim}_{Z \subset U \text{ rational}} \mathcal{O}^+_X(U) \to \widehat{A/I^+}$$

identifies the target with the t-adic completion of the source.

Proof. 1. Fix a couple of definition (A_0, t) of A with $A_0 \subset A^+$. Set $I_0 = A_0 \cap I \subset A_0$. Then $A/I = A_0/I_0[\frac{1}{t}]$, so we can endow A/I with the structure of a Tate ring by setting $(A_0/I_0, t)$ to be a couple of definition. The integral closure $A/I^+ \subset A/I$ of the image of $A^+ \to A/I$ is open (it contains A_0/I_0) and integrally closed (by construction), so we have an affinoid Tate ring $(A/I, A/I^+)$. The natural map $(A, A^+) \to (A/I, A/I^+)$ is also continuous: the preimage of $t^n A_0/I_0 \subset A/I$ contains $t^n A_0$. Thus, $(A/I, A/I^+)$ is an affinoid Tate (A, A^+)-algebra. For the universal property, let $f : (A, A^+) \to (R, R^+)$ be a map of affinoid Tate rings with $f(I) = 0$. Then the underlying map $A \to R$ clearly factors over $A \to A/I$. The resulting map $A/I \to R$ carries the image $A^+ \to A/I$ into R^+, and hence also carries A/I^+ into R^+. This gives a map of abstract pairs $\overline{f} : (A/I, A/I^+) \to (B, B^+)$. Using a ring of definition for B that contains $f(A_0)$, one easily checks that $\overline{f}$ is a map of affinoid Tate (A, A^+)-algebras. The uniqueness of $\overline{f}$ is clear.

2. It is immediate from the definitions that i is injective, and its image is exactly those $x \in \mathrm{Spa}(A, A^+)$ such that I lies in the support $\mathfrak{p}_x$. Thus, $i(Z)$ is the preimage of $\mathrm{Spec}(A/I) \subset \mathrm{Spec}(A)$ under the support map, and is thus closed. Fix a pseudouniformizer $t \in A$. For each $f \in I$, we have the containment

$$i(Z) \subset \bigcap_n X\Big(\frac{f, t^n}{t^n}\Big)$$

as $|f(x)| = 0$ if $x \in i(Z)$. Varying f, this gives

$$i(Z) \subset \bigcap_{f \in I} \bigcap_n X\Big(\frac{f, t^n}{t^n}\Big).$$

It is enough to show this is an equality. But if x lies on the right hand side, then $|f(x)| \leq |t^n(x)|$ for all $f \in I$ and $n \geq 0$, so $|f(x)| = 0$ as t is a pseudouniformizer; it follows that $x \in i(Z)$, as wanted.

3. This follows by combining Proposition 7.5.5 (5), the last assertion in Proposition 7.5.5 (4), and the observation that z defines the same pair $(\kappa(\mathfrak{p}_z), R_z)$, independent of whether we view z as a valuation on A or on A/IA.

4. By the proof of (3), we have $Z = \bigcap_N U_N$, where $U_N := X\Big(\frac{f_1, \ldots, f_r, t^N}{t^N}\Big)$. But if $\bigcap_N U_N \subset U$, then $U_N \subset U$ for $N \gg 0$: in X_{cons}, the sets $X - U_N$ are open (even clopen), while $X - U$ is closed (and hence compact).

5. Write $R_Z^+ := \widehat{\mathrm{colim}_{Z \subset U} \mathcal{O}_X^+}(U)$, and let $R_Z = R_Z^+[\frac{1}{t}]$. We may view (R_Z, R_Z^+) and $(\widehat{A/I^+}[\frac{1}{t}], \widehat{A/I^+})$ as complete uniform affinoid Tate (A, A^+)-algebras. We claim that both these (A, A^+)-algebras corepresent the same functor on the category of complete uniform affinoid Tate (A, A^+)-algebras (B, B^+). To see this, fix one such (B, B^+). Then, by the universal property in (1) and the t-adic completeness of B^+ (ensured by uniformity), the structure map $(A, A^+) \to (B, B^+)$ factors (uniquely) over $(A, A^+) \to (\widehat{A/I^+}[\frac{1}{t}], \widehat{A/I^+})$ exactly when

$IB = 0$. If we write $J = I \cap A^+ \subset A^+$, then $I = JA$, so the previous condition is also equivalent to requiring that $JB^+ = 0$. As B^+ is t-adically complete (by uniformity and completeness), this happens if and only if for each $f \in J$ and $n \geq 0$, we have $t^n \mid f$ in B^+. But, again thanks to the completeness of B^+, the latter condition is equivalent to requiring the structure map factor uniquely over $(A, A^+) \to (R_Z, R_Z^+)$ by the description of the basis of rational open neighbourhoods of Z in X given in the proof of (2) (and the observation that we may replace I with J in the intersection arising in (2)).

□

Remark 7.5.9 (Prime ideals in the stalk)**.** For $X := \mathrm{Spa}(A, A^+)$ and $x \in X$, write $\mathfrak{m}_x := \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$ for the maximal ideal of $\mathcal{O}_{X,x}$. Then have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{m}_x & \longrightarrow & \mathcal{O}^+_{X,x} & \longrightarrow & k(x)^+ & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathfrak{m}_x & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & k(x) & \longrightarrow & 0 \end{array}$$

of short exact sequences. In particular, geometrically, we get a pushout diagram of schemes

$$\begin{array}{ccc} \mathrm{Spec}(k(x)) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,x}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(x)^+) & \longrightarrow & \mathrm{Spec}(\mathcal{O}^+_{X,x}) \end{array}$$

where the horizontal maps are closed immersions while the vertical maps are open immersions. Thus, topologically, $\mathrm{Spec}(\mathcal{O}^+_{X,x})$ is obtained from the local scheme $\mathrm{Spec}(\mathcal{O}_{X,x})$ by formally adjoining the totally ordered set $\mathrm{Spec}(k(x)^+/t)$ as the poset of non-trivial specializations of the closed point of $\mathrm{Spec}(\mathcal{O}_{X,x})$.

Remark 7.5.10 (Finite étale covers via residue fields)**.** Proposition 7.5.5 (5) and (6) are surprising from a scheme-theoretic perspective: the local rings of schemes are usually neither henselian, nor identified with their residue fields upon completion. These features imply that passage to the residue field is much milder operation in the context of adic spaces than in the case of schemes; for example, we have

$$\operatorname*{colim}_{x \in U} U_{fet} \simeq (\mathcal{O}_{X,x})_{fet} \simeq k(x)_{fet},$$

where the colimit runs through rational open subsets containing x, the first equivalence is formal once the appropriate notions have been defined, while the second arises from the henselian property. In particular, this shall allow us to prove properties about the étale topology of X by reduction to the case of a field. Ultimately, a similar reduction shall be used to deduce the almost purity theorem for general perfectoid algebras to the case of perfectoid fields.

The assignment $U \mapsto \mathcal{O}_X(U)$ from Theorem 7.5.1 gives a presheaf on the category of rational open subsets of X. As the collection of all rational opens forms a basis, this presheaf extends formally to all open subsets as follows:

Definition 7.5.11 (Structure presheaf)**.** Fix an affinoid Tate ring (A, A^+), and let $X := \mathrm{Spa}(A, A^+)$ be its adic spectrum. We define the *strucrure presheaf* $\mathcal{O}_X$ on X by setting

$$\mathcal{O}_X(W) = \lim_{U \subset W} \mathcal{O}_X(U),$$

where the limit runs over all rational open subsets U of X contained in W. We define the *integral structure presheaf* $\mathcal{O}_X^+$ on X by a similar procedure. Note that both these presheaves are naturally valued in topological rings.

It is an easy exercise to see that both presheaves above agree with the ones coming from Theorem 7.5.1 when evaluated on rational open subsets. As rational open subsets form a basis, it follows that the stalks of the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ as defined above coincide with those from Definition 7.5.4. In particular, for any open $W \subset X$ and $x \in W$, we get a valuation $f \mapsto |f(x)|$ on $\mathcal{O}_X(W)$. Using this, one checks that

$$\mathcal{O}_X^+(W) = \{f \in \mathcal{O}_X(W) \mid |f(x)| \leq 1 \text{ for all } x \in \mathrm{Spa}(A, A^+)\}. \tag{7.2}$$

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This formula tells us that $\mathcal{O}_X^+$ is completely determined by $\mathcal{O}_X$ given the valuations on the stalks. Conversely, the knowledge of $\mathcal{O}_X^+ \subset \mathcal{O}_X$ is enough to reconstruct the valuation on the stalks. In particular, one readily checks that if $\mathcal{O}_X$ is a sheaf, then the same is true for $\mathcal{O}_X^+$. It is an unfortunate fact of life that $\mathcal{O}_X$ is *not* a sheaf in general[10]. Of course, we could just pass to the sheafification; however, we would then lose the universal property from Theorem 7.5.1, so we avoid doing so. Instead, we name those affinoid Tate rings where $\mathcal{O}_X$ is already a sheaf:

Definition 7.5.12 (Sheafy Tate rings)**.** An affinoid Tate ring (A, A^+) is *sheafy* if the structure presheaf $\mathcal{O}_X$ on $X := \mathrm{Spa}(A, A^+)$ is a sheaf; in this case, (7.2) implies that $\mathcal{O}_X^+$ is also a sheaf.

Such objects naturally live in the following category:

Definition 7.5.13 (Huber's category $\mathcal{V}$)**.** The category $\mathcal{V}$ has as objects triples $(X, \mathcal{O}_X, \{v_x\}_{x \in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings such that $(X, \mathcal{O}_X)$ is a locally ringed space (ignoring the topology on sections of $\mathcal{O}_X$), and v_x is a continuous valuation on the stalk $\mathcal{O}_{X,x}$ for each $x \in X$. The morphisms are evidently defined.

Note that limits in topological rings are computed as limits of the underlying set; in particular, given a presheaf $\mathcal{O}_X$ of topological rings[11]. For any affinoid Tate ring (A, A^+), we have constructed an triple $(X := \mathrm{Spa}(A, A^+), \mathcal{O}_X, \{v_x\}_{x \in X})$ as above, except that $\mathcal{O}_X$ need not be a sheaf; we call this triple an *affinoid pre-adic space*. If (A, A^+) is sheafy, this triple is an object of $\mathcal{V}$, and we call it an *affinoid adic space*. More general objects are constructed by glueing:

Definition 7.5.14 (Adic space)**.** An *adic space* is an object of $\mathcal{V}$ that is locally isomorphic to an affinoid adic space, i.e., is locally of the form $\mathrm{Spa}(A, A^+)$ for a sheafy affinoid Tate ring (A, A^+).

[10] The structure presheaf *is* a sheaf in most examples that come up in nature. In particular, it is a sheaf in the following cases: (a) there exists a noetherian ring of definition $A_0 \subset A$, (b) the Tate ring A is *stably uniform*, i.e., the Tate ring $\mathcal{O}_X(U)$ is uniform for all rational opens U, and (c) the ring A is perfectoid. We shall restrict attention to (c) in these notes.

[11] Recall that the underlying set of a limit of topological rings is the limit of the underlying abstract rings. Thus, a sheaf of topological rings also gives a sheaf of abstract rings.

Chapter 8

The adic spectrum via algebraic geometry

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Let (A, A^+) be an affinoid Tate ring. In this chapter, we describe the adic spectrum $\mathrm{Spa}(A, A^+)$ as an inverse limit certain modifications of $\mathrm{Spec}(A^+)$, and then use this to show that the henselian variant of Huber's presheaf is always a sheaf; this material is closely related to Raynaud's approach to rigid geometry. It is not relevant to the sequel.

8.1 Topological spaces

Fix an affinoid Tate ring (A, A^+). The key player is the following category of "integral models" of $\mathrm{Spec}(A)$.

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Construction 8.1.1 (The inverse limit of all models)**.** Let I be the category of all proper[1] maps of schemes such that f is an isomorphism over $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$. By taking fiber products, it is easy to see that I is cofiltered, so $\{X_i\}$ is a cofiltered diagram of spectral spaces indexed by I. For each such f_i, write $\overline{X_i} \subset X_i$ for the Zariski closed set defined by preimage of the complement $\mathrm{Spec}(A^+) - \mathrm{Spec}(A)$; this is also simply the closed set of X_i defined as the vanishing locus of any pseudouniformizer in A^+. Again, as i varies, we obtain a cofiltered system $\{\overline{X_i}\}$ of spectral spaces; the natural maps $\overline{X_i} \to X_i$ are constructible closed subsets pulled back from $\mathrm{Spec}(A^+)$. We define

$$X := \lim_i X_i \quad \text{and} \quad \overline{X} = \lim_i \overline{X}.$$

Note that both X and $\overline{X}$ are spectral spaces by [SP, Tag 0A2Z]. Moreover, the closed subset $\overline{X} \subset X$ is constructible and pulled back from $\mathrm{Spec}(A^+)$.

The main result of this section is:

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Theorem 8.1.2 (The adic spectrum as an inverse limit of all models)**.** *There is a natural homeomorphism*

$$\Phi : \mathrm{Spa}(A, A^+) \xrightarrow{\simeq} \overline{X}$$

[1]Recall that a proper map of schemes can be defined as a finite type map that satisfies the valuative criterion for properness. In particular, we do not impose a finite presentation constraint, so any closed immersion is proper.

given by: for $x \in \operatorname{Spa}(A, A^+)$ and index i, the image of $\Phi(x) \in \overline{X}$ under $\overline{X} \subset X \to X_i$ is the image of the closed point under the unique lift $\operatorname{Spec}(R_x) \to X_i$ of the natural map $\operatorname{Spec}(R_x) \to \operatorname{Spec}(A^+)$ along $f_i : X_i \to \operatorname{Spec}(A^+)$. Note that such a lift exists and is compatible as i varies by the valuative criterion, so Φ is a well-defined map.

The rest of the section is devoted to the proof. We shall freely use the interpretation of points of $\operatorname{Spa}(A, A^+)$ provided in Proposition 7.3.7. For any $x \in X$ and map $f_i : X_i \to \operatorname{Spec}(A^+)$ in I, we write $x_i \in X_i$ for the image of x under f_i. Fix a pseudouniformizer $t \in A$. If $\pi_i : X \to X_i$ denotes the natural map, then the space X has a natural sheaf $\mathcal{O}_X = \operatorname{colim}_i \pi_i^{-1}\mathcal{O}_{X_i}$, and the pair $(X, \mathcal{O}_X)$ is the inverse limit of the schemes X_i in the category of locally ringed spaces. The next proposition attaches valuation rings to points of $\overline{X}$, and is the key construction of this section:

Proposition 8.1.3. *For each $x \in \overline{X} \subset X$, the t-adically separated quotient $S_x := \mathcal{O}_{X,x}/\cap_n t^n \mathcal{O}_{X,x}$ is a microbial valuation ring with pseudouniformizer t.*

Proof. We have $\mathcal{O}_{X,x} = \operatorname{colim}_i \mathcal{O}_{X_i,x_i}$. In particular, any section $f \in \mathcal{O}_{X,x}$ can be represented by a section of some $\mathcal{O}_{X_i}(U_i)$ where $U_i \subset X_i$ is an affine open containing x_i. In the following proof, we shall implicitly use that S_x is nonzero. To see this, note that if $S_x = 0$, then $1 \in t\mathcal{O}_{X,x}$, which is impossible because $\mathcal{O}_{X,x}$ is a filtered colimit of local rings $\mathcal{O}_{X_i,x_i}$ (where $x_i \in X_i$ is the image of $x \in X$) containing t in their maximal ideal (as $x \in \overline{X} \subset X$).

We begin by observing that the sheaf $\mathcal{O}_X$, and hence the local ring $\mathcal{O}_{X,x}$, are t-torsionfree. Indeed, the collection of all $X_i \in I$ with $\mathcal{O}_{X_i}$ being t-torsionfree span a cofinal subcategory (as we can simply kill the t-power torsion to get another object in I). As filtered colimits of t-torsionfree modules are t-torsionfree, the claim follows.

Next, we show that for any $\overline{f} \in S_x$, we either have $\overline{f} = 0$ or $\overline{f} \mid t^n$ for n large. Lift $\overline{f}$ to some $f \in \mathcal{O}_{X,x}$. We shall check that either $f \mid t^n$ for some $n \geq 0$ or $t^n \mid f$ for all $n \geq 0$. Represent f by some element $f \in \mathcal{O}_{X_i}(U_i)$ for some $i \in I$ and an affine open $U_i \subset X_i$. Fix some $n \geq 0$, and consider the ideal $J = (f, t^n) \subset \mathcal{O}_{X_i}(U_i)$. By [SP, Tag 01PF], we can extend this to a finte type quasi-coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_{X_i}$. As $t^n \in J$, we may also assume that $t^n \in \mathcal{J}$ (by adding it if necessary). Then $\mathcal{J}$ becomes invertible on inverting t, so the blowup of X_i at the ideal sheaf $\mathcal{J}$ is an object of I, and thus gives a map $X_j \to X_i$ in I. The preimage $V_j \subset X_j$ of $U_i \subset X_i$ under this blowup is exactly the blowup of $\mathcal{O}_{X_i}(U_i)$ at J. The point x defines a point $x_j \in V_j \subset X_j$. The point $x_j \in V_j$ lies in one of the two standard charts for the blowup. In particular, in the local ring $\mathcal{O}_{X_j,x_j}$ (and hence also in $\mathcal{O}_{X,x}$), we have $f \mid t^n$ or $t^n \mid f$, depending on the chart x_j ends up inside. In the former case, we are done. In the latter case, we increase n by 1 and repeat the argument to get the desired conclusion.

Next, choose $\overline{f}, \overline{g} \in S_x$. We shall check that either $\overline{f} \mid \overline{g}$ or $\overline{g} \mid \overline{f}$. We may assume both elements are nonzero. Pick lifts $f, g \in \mathcal{O}_{X,x}$. As $\overline{f} \neq 0$, we must have $f \mid t^n$ for some $n \geq 0$ by the previous paragraph, and similarly for g. Thus, the ideal $(f, g) \subset \mathcal{O}_{X,x}$ contains t^n for some $n \geq 0$. By approximation, we can choose some $i \in I$ and an affine open $U_i \subset X_i$ such that f and g come from U_i, and that $t^n \in J = (f, g) \subset \mathcal{O}_{X_i}(U_i)$. Repeating the argument in the previous paragraph with blowups then gives us the desired claim.

Next, we check that S_x is a domain if it is nonzero. Pick nonzero elements $\overline{f}, \overline{g} \in S_x$ with $\overline{f}\overline{g} = 0$. We can represent them by $f, g \in \mathcal{O}_{X,x}$. As $\overline{f}$ and $\overline{g}$ are nonzero, we can choose non-negative integers m and n such that $\overline{f} \mid t^m$ and $\overline{g} \mid t^n$ by the third paragraph. But then we also have $f \mid t^m$ and $g \mid t^n$: the kernel of $\mathcal{O}_{X,x} \to S_x$ is t-divisible. On the other hand, as $\overline{f}\overline{g} = 0$, we have $fg \in \cap_k t^k \mathcal{O}_{X,x}$. As $t^{m+n} \in (fg) \subset \mathcal{O}_{X,x}$, we learn that $t^{m+n} \in \cap_k t^k \mathcal{O}_{X,x}$ for all k. As t is a nonzerodivisor on $\mathcal{O}_{X,x}$, it follows that $1 \in \cap_k t^k \mathcal{O}_{X,x}$, so $S_x = 0$, which is a contradiction.

Combining the assertions of the last two paragraphs shows that S_x is a valuation ring with $t \neq 0$ in S_x. The claim in the third paragraph shows that t is a pseudouniformizer on S_x, so we get the proposition. □

As a result, for each $x \in \overline{X}$, we get a map $\mathrm{Spec}(S_x) \to \mathrm{Spec}(\mathcal{O}_{X_i,x_i}) \to \mathrm{Spec}(A^+)$ which carries the closed point of $\mathrm{Spec}(S_x)$ to the image of x under the canonical map $\overline{X} \to \mathrm{Spec}(A^+)$. As S_x is microbial with pseudouniformizer t, this gives a point $\Psi(x)$ of the adic spectrum. We have thus constructed maps

$$\Phi : \mathrm{Spa}(A, A^+) \to \overline{X} \quad \text{and} \quad \Psi : \overline{X} \to \mathrm{Spa}(A, A^+).$$

Our construction immediately shows (using the valuative criterion for properness) that $\Phi \circ \Psi = \mathrm{id}$. Conversely:

hValRings **Lemma 8.1.4.** *The composition $\Psi \circ \Phi$ equals the identity. Moreover, for any $x \in \mathrm{Spa}(A, A^+)$ with image $y = \Phi(x) \in \overline{X}$, the maps $A^+ \to R_x$ and $A^+ \to S_y$ are isomorphic.*

Proof. Pick a point $x \in \mathrm{Spa}(A, A^+)$. This gives a map $\phi_x : \mathrm{Spec}(R_x) \to \mathrm{Spec}(A^+)$. The point $y = \Phi(x) \in \overline{X}$ is defined by lifting ϕ_x via the valuative criterion across each object $f_i : X_i \to \mathrm{Spec}(A^+)$ in I. In particular, if $y_i \in X_i$ denotes the image of y, then we have a compatible system of local maps $\mathcal{O}_{X_i,y_i} \to R_x$, and hence a local map $\mathcal{O}_{X,y} \to R_x$. The target is t-adically separated, so this defines a local map $S_y \to R_x$ of valuation rings. This map is also injective: the kernel is a prime ideal, and would thus contain t by microbiality if it were nonzero, which is impossible. Thus, $S_y \to R_x$ is a faithfully flat map of microbial valuation rings. But then the points of $\mathrm{Spec}(A^+)$ corresponding to the maps $A^+ \to S_y$ and $A^+ \to R_x$ must coincide, so we get the first part of the lemma.

For the second part: consider the map

$$A \xrightarrow{a} \mathrm{Frac}(S_y) \xrightarrow{b} \mathrm{Frac}(R_x)$$

obtained by inverting t from the maps $A^+ \to S_y \to R_x$ considered in the previous paragraph. By faithful flatness of $S_y \to R_x$, the kernel of a and $b \circ a$ coincide, and they are both thus the prime ideal $\mathfrak{p}_x$. As the targets of a and $b \circ a$ are both fields, this gives injective maps

$$\kappa(\mathfrak{p}_x) \to \mathrm{Frac}(S_y) \to \mathrm{Frac}(R_x)$$

of fields. But the composite is an isomorphism by definition of R_x as a valuation ring in $\kappa(\mathfrak{p}_x)$. It follows that $\mathrm{Frac}(S_y) = \mathrm{Frac}(R_x)$, so the faithfully flat map $S_y \to R_x$ can be viewed as an inclusion of valuation rings with the same fraction field. Any such map is an isomorphism, so we are done. □

It remains to prove Φ and Ψ match up the topologies.

nalBlowup **Lemma 8.1.5.** *The map Ψ is continuous and spectral.*

Proof. Fix $f_1, ..., f_n, g \in A^+$ such that $t^N \in (f_1, .., f_n)$ for some $N > 0$. Consider the rational open set $U = \mathrm{Spa}(A, A^+)\left(\frac{f_1,...,f_n}{g}\right)$. We shall identify $\Psi^{-1}(U)$ explicitly as a quasi-compact open in $\overline{X}$. For this, consider the ideal $J = (g, f_1, ..., f_n) \in A^+$. This ideal contains t^N, so the blowup $Y := \mathrm{Bl}_J(\mathrm{Spec}(A^+)) \to \mathrm{Spec}(A^+)$ is an object in I. By construction of the blowup, our choice of generators for J gives a closed immersion $i : Y \hookrightarrow \mathbf{P}^n_{A^+}$ of A^+-schemes via by $x_0 \mapsto g$ and $x_i \mapsto f_i$ for $i \neq 0$; here the x_i's are the standard homogeneous co-ordinates on $\mathbf{P}^n_{A^+}$. Let $V \subset Y$ be the preimage of the standard affine open $\mathbf{A}^n_{A^+} \subset \mathbf{P}^n$ defined the complement of the hyperplane $x_0 \neq 0$; thus, $g \mid f_i$ in $\mathcal{O}_Y(V)$. Write $W \subset \overline{X}$ for the preimage of V under $\overline{X} \to X \to Y$. We claim that $W = \Psi^{-1}(U)$, which would prove both continuity and spectrality.

The containment $\subset$ is clear: for any $x \in \overline{X}$ lying over V, we have $g \mid f_i$ in $\mathcal{O}_Y(V)$, hence also in $\mathcal{O}_{X,x}$ and S_x as x lies over V, so the point $\Psi(x) \in \mathrm{Spa}(A, A^+)$ corresponding to the map $A^+ \to S_x$ must lie in U.

For $\supset$, fix a point $x \in \overline{X}$ with $\Psi(x) \in U$. Thus, in the valuation ring S_x, we have $g \mid f_i$ for all i. We must check that the image y of x under $\overline{X} \to Y$ lies in the open set V. Composing with i, we get a map $k : \mathrm{Spec}(S_x) \to \mathbf{P}^n_{A^+}$ determined by the same formula as i: the co-ordinate functions x_i pull back to f_i for $i \neq 0$ and g for $i = 0$. More canonically, using the universal property of blowups, we see that the line $k^*\mathcal{O}_{\mathbf{P}^n_{A^+}}(1)$ is the ideal $(g, f_1, ..., f_n)$ with the displayed sections being the generators. As $g \mid f_i$, the map k is equivalent (as a point of $\mathbf{P}^n(S_x)$) to the map defined by the line bundle underlying the ideal sheaf $(1, \frac{f_1}{g}, ..., \frac{f_n}{g})$ with the displayed sections being the generators. But this point clearly lies in the distinguished $\mathbf{A}^n_{A^+} \subset \mathbf{P}^n_{A^+}$ considered above to define V, so the claim follows. □

Thus, we have constructed a continuous spectral bijection $\Psi : \overline{X} \to \mathrm{Spa}(A, A^+)$.

Lemma 8.1.6. *The map Ψ is generalizing, i.e., given $y \in \overline{X}$, each generalization of $x = \Psi(y)$ lifts to a generalization of x.*

Proof. The spectral space of generalizations of x in $\mathrm{Spa}(A, A^+)$ is homeomorphic to $\mathrm{Spec}(R_x/t)$ by Remark 7.3.11. By the second half of Lemma 8.1.4 (and chasing the maps Φ and Ψ), it suffices to check that the set of generalizations of $y \in \overline{X}$ is bijection with $\mathrm{Spec}(S_y/t)$ as a poset. If we set $y_i \in \overline{X_i}$ to be the image of y, then the set of generalizations of $y \in \overline{X}$ is the inverse limit over i of the sets of generalizations of y_i in $\overline{X_i}$, i.e., it's the inverse limit of spectral spaces $\mathrm{Spec}(\mathcal{O}_{\overline{X_i},y_i})$, where $\overline{X_i}$ is viewed as as a scheme by setting $t = 0$ in X_i. This inverse limit coincides with $\lim \mathrm{Spec}(\mathcal{O}_{X_i,y_i}/t) \simeq \mathrm{Spec}(\mathcal{O}_{X,y}/t)$. But the map $\mathcal{O}_{X,y} \to S_y$ is an isomorphism modulo t (as the kernel is t-divisible), so $\mathrm{Spec}(\mathcal{O}_{X,y}/t) \simeq \mathrm{Spec}(S_y/t)$, so we are done. □

We can now finish the proof of the theorem:

Proof of Theorem 8.1.2. We have already shown that Ψ is a continuous bijection between spectral spaces that is generalizing. But any such map is a homeomorphism by [SP, Tag 09XU], so we are done. □

Via the valuative intepretation, we obtain the following compatibility; this includes, in particular, the compatibility of the construction $(A, A^+) \mapsto \overline{X}$ with Zariski localizations of A^+.

aseChange **Corollary 8.1.7.** *Let $X_0 \to \mathrm{Spec}(A^+)$ be a map in I, i.e., a proper map that is an isomorphism after inverting t. Let $V_0 = \mathrm{Spec}(B_0) \subset X_0$ be an affine open subset. Then the preimage of V_0 in $\overline{X}$ coincides with $\overline{Y}$, the inverse limit of all proper maps $Y_i \to \mathrm{Spec}(B_0)$ that are isomorphisms after inverting t.*

Proof. Let B^+ be the integral closure of B_0 in $B_0[\frac{1}{t}]$, so (B, B^+) gives a uniform affinoid Tate ring. Note that the space $\overline{Y}$ defined above can also be defined by replace B_0 with B^+, so we may assume $B_0 = B^+$. Moreover, the map $A \to B$ is a Zariski localization as $V_0 \to \mathrm{Spec}(A^+)$ gives an open immersion on inverting t. Using the identification of Theorem 8.1.2, we are reduced to checking that the natural map $i : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ identifies with the left side with the preimage of V_0 under the specialization map $\mathrm{sp}_{X_0} : \mathrm{Spa}(A, A^+) \to X_0$. As $A \to B$ is a Zariski localization, the injectivity of i is clear. We are thus reduced to checking that an element $x \in \mathrm{Spa}(A, A^+)$ lies in $\mathrm{Spa}(B, B^+)$ exactly when $\mathrm{sp}_{X_0}(x) \in V_0$. The containment $\subset$ is clear. Conversely, given $x \in \mathrm{Spa}(A, A^+)$ with $\mathrm{sp}_{X_0}(x) \in V_0$, the induced map $A^+ \to R_x$ factors over $A^+ \to \mathcal{O}_{X_0}(V_0) = B^+$ by the definition of the specialization map; this then gives the desired point of $\mathrm{Spa}(B, B^+)$. □

Remark 8.1.8 (Replacing proper with projective)**.** The proof of Theorem 8.1.2 also goes through if we replace the word "proper" by "projective" in Construction 8.1.1. Indeed, the only properties of proper maps used above are:

- Proper maps satisfy the valuative criterion for properness.
- The fiber product and compositions of proper maps is proper.
- Closed immersions and blowups are proper.

As all of these also hold for projective maps, the arguments go through.

8.2 Comparing sheaves

We show next that identification of Theorem 8.1.2 can also be used to compare the (henselian version of the) structure presheaf on $\mathrm{Spa}(A, A^+)$ with a natural structure *sheaf* on $\overline{X}$, thus proving that the former is a sheaf. For this, we need the following lemma allowing us calculate the effect on global sections of pulling back along the maps $\overline{X}_i \subset X_i$ and $\overline{X} \subset X$ appearing above.

ckZariski **Lemma 8.2.1.** *Let R be a ring that is I-adically henselian for some ideal I. Let $f : Z \to \mathrm{Spec}(R)$ be a proper morphism, and let $i : Z_0 \hookrightarrow Z$ be the closed immersion defined by I. Then, working everywhere with étale sheaves, we have*

$$H^0(Z, \mathcal{O}_Z) \simeq H^0(Z_0, i^{-1}\mathcal{O}_Z)$$

via the restriction map.

Proof. This follows from [SP, Tags 09ZF, 09ZG]. ☐

Now consider the tower $\{\overline{X_i} \subset X_i\}$ defining $\overline{X} \subset X$ in the limit. On each X_i, we have the structure sheaf $\mathcal{O}_{X_i}$, which is an étale sheaf. Restricting this along the inclusion $k_i : \overline{X_i} \hookrightarrow X$ as étale sheaves, we obtain an étale sheaf $k_i^{-1}\mathcal{O}_{X_i,et}$ on $\overline{X_i}$, viewed as a Zariski sheaf via pushforward from the étale site to the Zariski site[2]. As i varies, these pullback to a filtered system $\{\pi_i^{-1}k_i^{-1}\mathcal{O}_{X_i,et}\}$ of sheaves on $\overline{X}$, where $\pi_i : \overline{X} \to \overline{X_i}$ is the projection. Write $\mathcal{F}$ for the direct limit. This sheaf turns out to equal the presheaf $\mathcal{O}^+_{\mathrm{Spa}(A,A^+),hens}$ defined in Remark 7.5.2.

Proposition 8.2.2. *Assume (A, A^+) is a henselian affinoid Tate ring. For any rational set $U \subset \mathrm{Spa}(A, A^+)$, we have a natural identification $\mathcal{F}(U) \simeq \mathcal{O}^+_{\mathrm{Spa}(A,A^+),hens}(U)$.*

Proof. Fix a pseudouniformizer $t \in A^+$. In the proof below, for any A^+-algebra R, we write R_{hens} for the henselization of R along tR.

1. *The structure of* $\mathcal{F}$. Fix a quasi-compact open $V \subset \overline{X}$ arising as the preimage of some affine open $V_0 \subset X_0$. Write $V_i \subset X_i$ for the preimage of V_0 along any map $X_i \to X_0$ in I and write $\overline{V_i} := V_i \cap \overline{X_i} \subset X_i$, so $V = \lim \overline{V_i}$. Then we can write

$$\mathcal{F}(V) = \operatorname*{colim}_{i \geq 0} H^0(\overline{V_i}, k_i^{-1}\mathcal{O}_{X_i,et}) \simeq \operatorname*{colim}_{i \geq 0} \mathcal{O}_{X_i}(V_i) \otimes_{\mathcal{O}_{X_0}(V_0)} \mathcal{O}_{X_0}(V_0)_{hens}.$$

   Indeed, the first isomorphism is clear from generalities about cofiltered limits of spectral spaces, while the second follows from flat base change for coherent cohomology and Lemma 8.2.1 applied to $R = \mathcal{O}_{X_0}(V_0)_{hens}$ and $Z = V_i \times_{V_0} \mathrm{Spec}(R)$ for each $i \geq 0$. Commuting the tensor product with a filtered colimit, we can write

$$\mathcal{F}(V) \simeq \mathcal{O}_{X_0}(V_0)_{hens} \otimes_{\mathcal{O}_{X_0}(V_0)} \operatorname*{colim}_{i \geq 0} \mathcal{O}_{X_i}(V_i).$$

   We claim that $\mathrm{colim}_{i\geq 0}\, \mathcal{O}_{X_i}(V_i)$ is the integral closure of $\mathcal{O}_{X_0}(V_0)$ in $\mathcal{O}_{X_0}(V_0)[\frac{1}{t}]$. To see this, observe that the former is an integral $\mathcal{O}_{X_0}(V_0)$-algebra (as each $V_i \to V_0$ is proper) that is t-torsionfree (as the sheaf $\mathcal{F}$ is t-torsionfree) and contained in $\mathcal{O}_{X_0}(V_0)[\frac{1}{t}]$ (as $V_i \to V_0$ is an isomorphism after inverting t). It remains to show that $\mathrm{colim}_{i\geq 0}\, \mathcal{O}_{X_i}(V_i)$ is integrally closed in $\mathcal{O}_{X_0}(V_0)[\frac{1}{t}]$. This is clear when $V_0 = X_0 = \mathrm{Spec}(A^+)$, and the general case is reduced to this one by Corollary 8.1.7.

   As the formation of integral closures commutes with étale localization, we learn that $\mathcal{F}(V)$ is the integral closure of $\mathcal{O}_{X_0}(V_0)_{hens}$ in $\mathcal{O}_{X_0}(V_0)_{hens}[\frac{1}{t}]$. In particular, this has the following consequences:

   - The pair $(\mathcal{F}(V)[\frac{1}{t}], \mathcal{F}(V))$ is a henselian affinoid Tate (A, A^+)-algebra with couple of definition $(\mathcal{F}(V), t)$.

[2] Even though $\mathcal{O}_{X_i}$ is both a Zariski sheaf and an étale sheaf, the pullbacks in the two different topologies are different, so $k_i^{-1}\mathcal{O}_{X_i,et}$ cannot be simply defined as the pullback of $\mathcal{O}_{X_i}$ viewed as a Zariski sheaf.

- The map $(A, A^+) \to (\mathcal{F}(V)[\frac{1}{t}], \mathcal{F}(V))$ is an epimorphism in the category of henselian affinoid Tate rings. Indeed, this map factors as

  $$(A, A^+) \to (\mathcal{O}_{X_0}(V_0)[\frac{1}{t}], \mathcal{O}_{X_0}(V_0)) \to (\mathcal{O}_{X_0}(V_0)_{hens}[\frac{1}{t}], \mathcal{O}_{X_0}(V_0)_{hens}) \to (\mathcal{F}(V)[\frac{1}{t}], \mathcal{F}(V))$$

  The first map is an epimorphism in the category larger category of abstract pairs as $A \to \mathcal{O}_{X_0}(V_0)[\frac{1}{t}]$ is a localization: the map $X_0 \to \mathrm{Spec}(A^+)$ is an isomorphism after inverting t, so $V_0 \to \mathrm{Spec}(A^+)$ is an affine open immersion after inverting p. The second map is a henselization, and hence clearly an epimorphism with respect to henselian pairs. The last map is an isomorphism on the rational rings (i.e., $\mathcal{F}(V)[\frac{1}{t}] = \mathcal{O}_{X_0}(V_0)_{hens}[\frac{1}{t}]$) by the discussion above, and hence an epimorphism of abstract pairs.

We shall use the above two pieces of structure in two special cases next.

2. *$\mathcal{F}$ applied to the whole space.* Assume first that $V_0 = \mathrm{Spec}(A^+)$ so $V = \overline{X}$. In this case, the formula for $\mathcal{F}(-)$ in (1) shows that $\mathcal{F}(V) = A^+$. Indeed, we may take $V_0 = X_0 = \mathrm{Spec}(A^+)$, so our formula tells us that $\mathcal{F}(V)$ is the integral closure of A^+_{hens} in $A^+_{hens}[\frac{1}{t}]$. But (A, A^+) is a henselian affinoid Tate ring, so we immediately get that $\mathcal{F}(V) = A^+$, as wanted.

3. *$\mathcal{F}$ applied to a rational subset.* Choose $f_1, ..., f_n, g \in A^+$ such that $t^N \in (f_i) \subset A^+$ for some $N \geq 0$ and $U = \mathrm{Spa}(A, A^+)\left(\frac{f_1,...,f_n}{g}\right)$. Let $X_0 \to \mathrm{Spec}(A^+)$ be the blowup at the ideal $J = (g, f_1, ..., f_n)$. Write $V_0 \subset X_0$ for the standard chart where $g \mid f_i$ for all i. We have seen in Lemma 8.1.5 that the identification $\mathrm{Spa}(A, A^+) \simeq \overline{X}$ carries U to the preimage V of V_0 under $\overline{X} \subset X \to X_0$. Set $C^+ = \mathcal{F}(V)$ and $C = C^+[\frac{1}{t}]$. By (1), this yields a henselian affinoid Tate (A, A^+)-algebra (C, C^+). The induced map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ has image contained in U simply because $g \mid f_i$ in C^+. If we set

   $$(B, B^+) := (\mathcal{O}_{\mathrm{Spa}(A,A^+),hens}(U), \mathcal{O}^+_{\mathrm{Spa}(A,A^+),hens}(U)),$$

   then the universal property gives us a map

   $$\Phi : (B, B^+) \to (C, C^+)$$

   of henselian affinoid Tate (A, A^+)-algebras. We will show this map is an isomorphism by constructing the inverse.

4. *Constructing maps out of $\mathcal{F}(V)$.* Say (R, R^+) is any henselian affinoid Tate (A, A^+). The assocation $(A, A^+) \mapsto (\overline{X}, \mathcal{F})$ is functorial in the input pair (A, A^+). When applied to (R, R^+), this produces a locally ringed space $(\overline{Y}, \mathcal{G})$ equipped with a local map to $(\overline{X}, \mathcal{G})$. The underlying map $\overline{Y} \to \overline{X}$ coincides with the canonical map $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ under the identification of Theorem 8.1.2. In particular, if the latter has image in the rational open set $U \subset \mathrm{Spa}(A, A^+)$ from (3), then passing to sections over U gives a map

   $$\Psi : (C, C^+) \to (\mathcal{G}(\overline{Y})[\frac{1}{t}], \mathcal{G}(\overline{Y})) \simeq (R, R^+)$$

   of henselian affinoid Tate (A, A^+)-algebras, where the last isomorphism uses (2) above.

5. *Conclusion.* By (3) and (4) applied to $(R, R^+) = (B, B^+)$, we obtain maps

$$\Phi : (B, B^+) \to (C, C^+) \quad \text{and} \quad \Psi : (C, C^+) \to (B, B^+)$$

of affinoid Tate (A, A^+)-algebras. To check these are inverses to each other, it suffices to observe that both $(A, A^+) \to (B, B^+)$ and $(A, A^+) \to (C, C^+)$ are epimorphisms of henselian affinoid Tate rings. The map $(A, A^+) \to (B, B^+)$ is an epimorphism by the universal property, while the map $(A, A^+) \to (C, C^+)$ is an epimorphism by (1).

□

Corollary 8.2.3 (The henselian structure presheaf is a sheaf)**.** *Under the identification* $\mathrm{Spa}(A, A^+) \simeq \overline{X}$ *of Theorem 8.1.2, we have*

$$\mathcal{O}^+_{\mathrm{Spa}(A,A^+),hens} \simeq \mathcal{F} \quad \textit{and} \quad \mathcal{O}_{\mathrm{Spa}(A,A^+),hens} \simeq \mathcal{F}[\frac{1}{t}]$$

for any pseudouniformizer $t \in A$*. In particular, both these presheaves are sheaves.*

8.3 A fully faithful embedding of affinoid Tate rings into a geometric category

Finally, we have a suitable geometric category where the adic spectrum of any affinoid Tate ring lives. Let $\mathcal{V}_{mic}$ denote the category[3] of triples $(X, \mathcal{O}_X, \{v_x\}_{x \in X})$ where $(X, \mathcal{O}_X)$ is a locally ringed space, and v_x is a microbial valuation on the stalks $\mathcal{O}_{X,x}$ (i.e., the corresponding valuation ring is microbial); the morphisms are given by maps of locally ringed spaces that are compatible with the valuations, and such that the induced map on valuation rings at any point preserves pseudouniformizers. The association $(A, A^+) \mapsto (\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+),hens}, \{v_x\})$ gives a functor $\underline{\mathrm{Spa}}(-)$ from affinoid Tate rings into $\mathcal{V}_{mic}$.

Proposition 8.3.1. *The functor* $\underline{\mathrm{Spa}}(-)$ *is fully faithful on the full subcategory of uniform henselian affinoid Tate rings.*

Proof. Fix a henselian affinoid Tate ring (A, A^+). For notational ease, write $X_A := \mathrm{Spa}(A, A^+)$, $\mathcal{F}_A = \mathcal{O}^+_{\mathrm{Spa}(A,A^+),hens}$, and $\mathcal{G}_A = \mathcal{O}_{\mathrm{Spa}(A,A^+),hens}$. Using the valuations v_x, we can functorially recover the inclusion $\mathcal{F}_A \subset \mathcal{G}_A$ from $\underline{\mathrm{Spa}}(A, A^+)$: we can obviously recover $\mathcal{G}_A$, and we have

$$\mathcal{F}_A(V) = \{f \in \mathcal{G}_A(V) \mid |f(x)| \leq 1 \text{ for all } x \in X_A\}.$$

Moreover, we have also seen $\mathcal{F}_A(X_A) = A^+$ and $\mathcal{G}_A = A$ as (A, A^+) is henselian. Thus, we recover the abstract pair (A, A^+) from $\underline{\mathrm{Spa}}(A, A^+)$. By uniformity, this also recovers (A, A^+) as

[3] This is similar to Huber's category $\mathcal{V}$ with some differences: (a) we only work with sheaves of abstract rings instead of topological rings, (b) we require the valuations at each point to be microbial, and (c) the continuity condition on the morphisms in Huber's category is replaced by the requirement that the maps on valuation rings preserves pseudouniformizers.

an affinoid Tate ring. Moreover, an element $f \in A$ is a pseudouniformizer if and only if f is a unit and f gives a pseudouniformizer in the valuation ring attached to the each stalk $\mathcal{G}_{A,x}$ for $x \in X_A$.

Now fix two uniform henselian affinoid Tate rings (A, A^+) and (B, B^+), and consider the maps

$$\Phi : \mathrm{Hom}((A, A^+), (B, B^+)) \xrightarrow{f \mapsto \underline{\mathrm{Spa}}(f)} \mathrm{Hom}(\underline{\mathrm{Spa}}(B, B^+), \underline{\mathrm{Spa}}(A, A^+)).$$

and

$$\Psi : \mathrm{Hom}(\underline{\mathrm{Spa}}(B, B^+), \underline{\mathrm{Spa}}(A, A^+)) \to \mathrm{Hom}((A, A^+), (B, B^+)),$$

where $\Psi(\phi)$ is the map $(A, A^+) \to (B, B^+)$ of abstract pairs induced via $\mathcal{G}_A(X_A) \to \mathcal{G}_B(X_B)$ and $\mathcal{F}_A(X_A) \to \mathcal{F}_B(X_B)$ using the identifications in the previous paragraph. Note that $\Psi(\phi)$ is indeed a continuous[4] map of affinoid Tate rings: using uniformity, it suffices to observe that $\Psi(\phi) : A \to B$ preserves a ring of definition (namely, it carries A^+ to B^+) and pseudouniformizers (by the characterization mentioned in the previous paragraph). It is then immediate from the definitin that $\Psi \circ \Phi$ is the identity.

To check $\Phi \circ \Psi$ is the identity, fix a map $\phi : \underline{\mathrm{Spa}}(B, B^+) \to \underline{\mathrm{Spa}}(A, A^+)$, and let $\phi^\sharp : (A, A^+) \to (B, B^+)$ be the induced map on affinoid Tate rings via Ψ. We must check that $\phi = \underline{\mathrm{Spa}}(f)$.

Fix a point $x \in \mathrm{Spa}(B, B^+)$ with image $\phi(x) \in \mathrm{Spa}(A, A^+)$. We want to show that $\phi(x)$ is the valuation on A induced by x by composition with $\phi^\sharp$. But this is immediate from the compatibility of ϕ with valuations (as in the definition of $\mathcal{V}_{mic}$). In particular, ϕ is a spectral map, and ϕ^{-1} carries rational open subsets to rational open subsets.

Now fix a rational open subset $U \subset \mathrm{Spa}(A, A^+)$ with preimage ϕ^{-1}. To finish proving $\phi = \underline{\mathrm{Spa}}(f)$, we must check that the map of (abstract) pairs $(\mathcal{G}_A(U), \mathcal{F}_A(U)) \to (\mathcal{G}_B(V), \mathcal{F}_B(V))$ induced by ϕ agrees with the map induced by $\underline{\mathrm{Spa}}(\phi^\sharp)$. But both these maps fit as the bottom horizontal arrow into the commutative diagram

$$\begin{array}{ccc} (A, A^+) & \xrightarrow{\phi^\sharp} & (B, B^+) \\ \downarrow{\scriptstyle std} & & \downarrow{\scriptstyle std} \\ (\mathcal{G}_A(U), \mathcal{F}_A(U)) & \longrightarrow & (\mathcal{G}_B(V), \mathcal{F}_B(V)). \end{array}$$

The vertical maps are epimorphisms of henselian pairs, so there is a unique choice for the bottom horizontal arrow, so our claim follows.

□

[4] Given uniform affinoid Tate rings (R, R^+) and (S, S^+), a map $R \to S$ carrying $R^+ \to S^+$ is not always continuous; it is continuous exactly when it is strict, i.e., preserves pseudouniformizers.

Chapter 9

Perfectoid spaces

We now return to the perfectoid theory. The first goal of this chapter is to attach an adic space (called an affinoid perfectoid space) to each perfectoid algebra R; concretely, we shall check that the affinoid Tate ring (R, R°) is sheafy. In fact, whilst establishing this result, we shall simultaneously also prove that the higher cohomology of the structure sheaf is 0. With this result in place, we introduce the notion of a perfectoid space over a perfectoid field, and explain why its invariant under tilting. Finally, with an eye towards future applications, we spend some time on analyzing Zariski closed subsets of an affinoid perfectoid space; surprisingly, these admit a natural structure as an affinoid perfectoid space.

9.1 Perfectoid affinoid algebras

Fix a perfectoid field K, and write $\mathfrak{m} \subset K^\circ$ and $\mathfrak{m}^\flat \subset K^{\flat\circ}$ for the corresponding maximal ideals. Choose a pseudouniformizer $t \in K^\flat$ with $|t|^\flat \geq |p|$, and let $\pi = t^\sharp$. We simply use the phrase *affinoid K-algebra* to describe affinoid Tate (K, K°)-algebras.

Definition 9.1.1. An affinoid K-algebra (R, R^+) is a *perfectoid affinoid K-algebra* if R is perfectoid.

Note that $\mathfrak{m}R^\circ = R^{\circ\circ} \subset R^+ \subset R^\circ$, so the map $R^+ \to R^\circ$ is an almost isomorphism. Moreover, as one easily checks from the definitions, a ring of integral elements $R^+ \subset R^\circ$ determines and is determined by specifying the integrally closed subring $\overline{R^+} := R^+/\mathfrak{m}R^\circ \subset R^\circ/\mathfrak{m}R^\circ$; note that an integrally closed subring of a perfect ring is perfect. As $R^\circ/\mathfrak{m}R^\circ \simeq R^{\flat\circ}/\mathfrak{m}R^{\flat\circ}$, the tilting correspondence implies:

dAffinoid **Proposition 9.1.2** (Tilting correspondence)**.** *The categories of perfectoid affinoid algebras over K and $K^\flat$ are equivalent to each other. Under this equivalence, (R, R^+) corresponds to $(R^\flat, R^{\flat+})$ if and only if $R^\flat$ is the tilt of R and the isomorphism $R^\circ/\mathfrak{m}R^\circ \simeq R^{\flat\circ}/\mathfrak{m}^\flat R^\circ$ identifies $R^+/\mathfrak{m}R^\circ$ with $R^{\flat+}/\mathfrak{m}^\flat R^{\flat+}$. Moreover, R^+/p is semiperfect, and we have $R^{\flat+} \simeq R^{+\flat}$ as subrings of $R^{\circ\flat} \simeq R^{\flat\circ}$.*

Proof. We already know the corresponding statement when $R^+ = R^\circ$ from Theorem 6.2.5 and Theorem 6.2.7 (3). The equivalence for perfectoid affinoid K-algebras then follows from the description of R^+ in terms of $\overline{R^+}$ mentioned above. Explicitly, given a perfectoid affinoid K-algebra (R, R^+), the corresponding ring of integral elements $R^{\flat +} \subset R^{\circ\flat}$ is defined via the pullback square

$$\begin{array}{ccc} R^{\flat +} & \longrightarrow & R^{\circ\flat} \\ \downarrow & & \downarrow \\ R^+/\mathfrak{m}R^\circ & \longrightarrow & R^\circ/\mathfrak{m}R^\circ \end{array}$$

where the bottom horizontal and right vertical maps are the evident ones.

For the semiperfectness of R^+/p, observe that the integrally closed subring $R^+/\mathfrak{m}R^\circ$ of the perfect ring $R^\circ/\mathfrak{m}R^\circ$ is perfect. Now surjectivity of an almost surjective map of K°-modules can be detected after reducing modulo $\mathfrak{m}$, so the semiperfectness of R^+/p follows from that of R°/p and $R^+/\mathfrak{m}R^\circ$.

Finally, to check $R^{+\flat} = R^{\flat +}$, it is enough to show that the square of natural maps

$$\begin{array}{ccc} R^{+\flat} & \longrightarrow & R^{\circ\flat} \\ \downarrow & & \downarrow \\ R^+/\mathfrak{m}R^\circ & \longrightarrow & R^\circ/\mathfrak{m}R^\circ \end{array}$$

is a pullback square. We leave it to the reader to deduce this using the perfectness of $R^+/\mathfrak{m}R^\circ$. □

As a matter of notation, we denote the tilt of a perfectoid affinoid K-algebra (R, R^+) by $(R^\flat, R^{\flat +})$.

oidFields **Remark 9.1.3** (Perfectoid affinoid fields)**.** Say (R, R^+) is a perfectoid affinoid K-algebra with R being a perfectoid field. Then (R, R^+) is an affinoid field exactly when $R^+ \subset R$ is an open valuation ring. This happens exactly when the integrally closed subring $R^+/\mathfrak{m}R^\circ \subset R^\circ/\mathfrak{m}R^\circ$ is a valuation ring; to see this, one uses that $R^+ \subset R^\circ$ is the preimage of $R^+/\mathfrak{m}R^\circ \subset R^\circ/\mathfrak{m}R^\circ$, and that a reduced ring V is a valuation ring if and only if for any $a, b \in V$, we have $a \mid b$ or $b \mid a$. In particular, the property of being an affinoid field is preserved under, and can be detected after, tilting. For future reference, we shall refer to perfectoid affinoid K-algebras (R, R^+) which are affinoid fields as *perfectoid affinoid fields*.

The following exercise shall be used implicitly in the sequel.

Exercise 9.1.4. Fix a perfectoid field K as above.

1. Show that the category of perfectoid affinoid K-algebra admits all filtered colimits, and that these are also filtered colimits in the larger category of complete uniform affinoid Tate rings.

2. Deduce from (1) and uniformity that the functor $(R, R^+) \mapsto R^+$ from perfectoid affinoid K-algebras to π-adically complete K°-algebras preserves filtered colimits.

9.2 Tilting rational subsets

We reintroduce standard notation that shall be followed throughout this chapter.

dAffinoid **Notation 9.2.1.** Fix a perfectoid field K, and write $\mathfrak{m} \subset K^\circ$ and $\mathfrak{m}^\flat \subset K^{\flat\circ}$ for the corresponding maximal ideals. Choose a pseudouniformizer $t \in K^\flat$ with $|t|^\flat \geq |p|$, and let $\pi = t^\sharp$. Fix a perfectoid affinoid K-algebra (R, R^+) with tilt $(R^\flat, R^{\flat+})$. Write $X := \operatorname{Spa}(R, R^+)$ and $X^\flat := \operatorname{Spa}(R^\flat, R^{\flat+})$ for the attached adic spectra.

With notation as above, recall from Remark 2.0.9 that we have a multiplicative map

$$\sharp : R^\flat \to R.$$

Using this map, we can identify the adic spectra in a satisfying fashion:

tAnalytic **Theorem 9.2.2.** *Fix notation as above.*

1. *For any $x : R \to \Gamma \cup \{0\}$ in $\operatorname{Spa}(R, R^+)$, the composition $R^\flat \xrightarrow{\sharp} R \xrightarrow{x} \Gamma \cup \{0\}$ gives a point $x^\flat$ of $\operatorname{Spa}(R^\flat, R^{\flat+})$. This construction gives a homeomorphism $\operatorname{Spa}(R, R^+) \simeq \operatorname{Spa}(R^\flat, R^{\flat+})$ preserving rational subsets. For a rational $U \subset \operatorname{Spa}(R, R^+)$, write $U^\flat \subset \operatorname{Spa}(R^\flat, R^{\flat+})$ for its image, and call it the* tilt *of U.*

2. *For any rational open subset $U \subset X$ with tilt $U^\flat \subset X^\flat$, the complete affinoid Tate K-algebra $(\mathcal{O}_X(U), \mathcal{O}_{X^\flat}(U))$ is perfectoid with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}^+_{X^\flat}(U^\flat))$.*

The rest of this section will be dedicated to the proof. We begin by observing that the argument in Remark 3.2.5 applies *mutatis mutandis* to show that $x \mapsto x^\flat$ gives a well-defined set-theoretic map $X \to X^\flat$. This map is also continuous: one readily checks that the preimage of $X^\flat\left(\frac{f_1,\ldots,f_n}{g}\right)$ is given by $X\left(\frac{f_1^\sharp,\ldots,f_n^\sharp}{g^\sharp}\right)$; here we assume, as in Remark 7.4.3, that $f_n = t^N$ is a pseudouniformizer (to ensure that the $f_i^\sharp$ generate a unit ideal of R). To proceed further, we begin by describing Huber's presheaf on the rational open subsets of X in characteristic p.

heafCharp **Lemma 9.2.3** (Huber's presheaf in characteristic p)**.** *Assume K has characteristic p. Let $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ be a rational subset of X defined by $f_i, g \in R^+$ with $f_n = \pi^N$. Then:*

1. *Let $R^+\langle\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}\rangle$ be the π-adic completion of the subring*

$$R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}] \subset R[\frac{1}{g}].$$

Then $R^+\langle\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}\rangle$ is a perfectoid $K^{\circ a}$-algebra.

2. *The canonical R^+-algebra map*

$$\psi : R^+[X_i^{\frac{1}{p^\infty}}] \to R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}] \quad \textit{determined by} \quad X_i^{\frac{1}{p^m}} \mapsto \left(\frac{f_i}{g}\right)^{\frac{1}{p^m}}$$

is surjective with kernel containing and almost equal to $I = (\forall m : g^{\frac{1}{p^m}} X_i^{\frac{1}{p^m}} - f_i^{\frac{1}{p^m}})$.

3. *The Tate K-algebra $\mathcal{O}_X(U)$ is a perfectoid K-algebra. Moreover, we have a natural almost isomorphism $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle \stackrel{a}{\simeq} \mathcal{O}_X(U)^{\circ a}$ of $K^{\circ a}$-algebras.*

Proof. 1. The ring $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$ is perfect and π-torsionfree by construction. Hence, its π-adic completion is also perfect and π-torsionfree. This immediately implies that $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle$ gives a perfectoid $K^{\circ a}$-algebra.

2. It is clear that ψ is surjective and $I \subset \ker(\psi)$. We also have $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}][\frac{1}{\pi}] = R[\frac{1}{g}]$ as $f_n = \pi^N$. It is then clear that $I[\frac{1}{\pi}] = \ker(\psi[\frac{1}{\pi}])$. Now consider the map

$$\overline{\psi} : P := R^+[X_i^{\frac{1}{p^\infty}}]/I \to R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$$

induced by ψ. Using the definition of I, it is clear that $I = I^{[p]}$, so the ring P is perfect. In particular, $\overline{\psi}$ is a surjective map between perfect K°-algebras that is an isomorphism after inverting π, so $\ker(\overline{\psi})$ is π^∞-torsion. As the π^∞-torsion in a perfect K°-algebra is always almost zero, the claim follows.

3. Consider the inclusions

$$R^+[\frac{f_i}{g}] \stackrel{a}{\hookrightarrow} R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}] \stackrel{b}{\hookrightarrow} R[\frac{1}{g}].$$

We claim that $\mathrm{coker}(a)$ is killed by π^{nN}. Indeed, as $f_n = \pi^N$, we can write

$$\pi^{nN} \cdot \prod_{i=1}^n (\frac{f_i}{g})^{\frac{1}{p^{a_i}}} = \prod_{i=1}^n \pi^N \cdot (\frac{f_i}{g})^{\frac{1}{p^{a_i}}} = \prod_{i=1}^n \big(f_i^{\frac{1}{p^{a_i}}} g^{1-\frac{1}{p^{a_i}}}\big)\frac{f_n}{g} \in R^+[\frac{f_i}{g}],$$

which proves π^{nN} kills all generators of the R^+-module $\mathrm{coker}(a)$, and hence also the cokernel. Passing to π-adic completions shows that the perfectoid $K^{\circ a}$-algebra $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle$ and the π-adically complete K°-algebra $\widehat{R^+[\frac{f_i}{g}]}$ give the same Tate K-algebra on inverting π. But the former construction clearly gives a perfectoid K-algebra, while the latter construction produces $\mathcal{O}_X(U)$. This gives the first part. The second part then follows from the fact that inverting π gives an equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_K$ with inverse given by $R \mapsto R^{\circ a}$.

□

Remark 9.2.4. Lemma 9.2.3 (3) is quite remarkable from the perspective of classical rigid geometry. Indeed, in the latter theory, it is quite hard to describe the rings $\mathcal{O}_X(U)^\circ$ explicitly (or even almost so, if working over a perfectoid field) due to various integral closures that intervene in the definition. In contrast, in the perfectoid setting, Lemma 9.2.3 gives an explicit "generators and relations" style description of $\mathcal{O}_X(U)^\circ$ that essentially says that these integral closures are (almost) unnecessary, thanks to perfectness; more precisely, part (2) says that $\mathcal{O}_X(U)^\circ$ is almost computed by *formally adjoining* to R^+ the "obvious" functions that ought to be defined on U. The tilting correspondence will then allow us to propogate this to a similar description in characteristic 0 as well, see Lemma 9.2.5.

Using the result above in characteristic p, we can now fully describe Huber's presheaf on rational open subsets of X that are pulled back from $X^\flat$:

heafChar0 **Lemma 9.2.5** (Huber's presheaf in characteristic 0)**.** *Let $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ be a rational subset of X defined by $f_i, g \in R^+$ with $f_n = \pi^N$. Assume that $f_i = a_i^\sharp$ and $g = b^\sharp$ for $a, b \in R^{\flat+}$, so $f_i^{\frac{1}{p^n}}$ and $g^{\frac{1}{p^n}}$ makes sense for all n. Write $U^\flat = X^\flat\left(\frac{a_1,\ldots,a_n}{b}\right)$ for the corresponding rational subset of $X^\flat$, so U is the preimage of $U^\flat$ under $X \mapsto X^\flat$.*

1. *Let $R^+\langle\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}\rangle$ be the π-adic completion of the subring*
$$R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}] \subset R[\frac{1}{g}].$$
Then $R^+\langle\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}\rangle$ is a perfectoid $K^{\circ a}$-algebra.

2. *The canonical R^+-algebra map*
$$\psi : R^+[X_i^{\frac{1}{p^\infty}}] \to R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}] \quad \text{determined by} \quad X_i^{\frac{1}{p^m}} \mapsto \left(\frac{f_i}{g}\right)^{\frac{1}{p^m}}$$
is surjective with kernel containing and almost equal to $I = (\forall m : g^{\frac{1}{p^m}} X_i^{\frac{1}{p^m}} - f_i^{\frac{1}{p^m}})$.

3. *The Tate K-algebra $\mathcal{O}_X(U)$ is a perfectoid K-algebra. Moreover, we have a natural almost isomorphism $R^+\langle\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}\rangle \overset{a}{\simeq} \mathcal{O}_X(U)^{\circ a}$ of $K^{\circ a}$-algebras.*

4. *The perfectoid affinoid K-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to the perfectoid affinoid $K^\flat$-algebra $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.*

Proof. 1. Write $P_0 = R^+[X_i^{\frac{1}{p^\infty}}]/I$, where $I = (\forall m : g^{\frac{1}{p^m}} X_i^{\frac{1}{p^m}} - f_i^{\frac{1}{p^m}})$. The association $X_i^{\frac{1}{p^m}} \mapsto \left(\frac{f_i}{g}\right)^{\frac{1}{p^m}}$ determines an R^+-algebra map
$$a_0 : P_0 \to R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}].$$

By definition of Huber's presheaf, there is an obvious R^+-algebra map
$$b_0 : R^+[\left(\frac{f_i}{g}\right)^{\frac{1}{p^\infty}}] \to \mathcal{O}_X^+(U).$$

Write (S, S^+) for the untilt of the perfectoid $(R^\flat, R^{\flat+})$-algebra $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$, viewed as an (R, R^+)-algebra. As $f_i = a_i^\sharp$ and $g = b^\sharp$, the map $\mathrm{Spa}(S, S^+) \to X$ has image contained in U: we have $g \mid f_i$ in S^+ since $b \mid a_i$ in $\mathcal{O}_{X^\flat}^+(U^\flat)$. By the universal property of Huber's presheaf, we get an induced R^+-algebra map
$$c : \mathcal{O}_X^+(U) \to S^+.$$

Write $d_0 = c \circ b_0$. Putting everything together, we get maps

$$\begin{array}{ccccc} P_0 & \xrightarrow{a_0} & R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}] & \xrightarrow{d_0} & S^+ \\ \downarrow & & \downarrow & & \| \\ P & \xrightarrow{a} & R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle & \xrightarrow{d} & S^+, \end{array}$$

where the second row is obtained by π-adic completion from the first row (and makes sense because S^+ is π-adically complete). Now the map a_0 is surjective, and hence a is also surjective. The map $d_0 \circ a_0$ modulo π is an almost isomorphism by Lemma 9.2.3 (2), and hence the same holds true for $d \circ a$ modulo π as well. As the source of $d \circ a$ is π-adically separated and the target is π-torsionfree, it follows[1] that $d \circ a$ is an almost isomorphism. The surjectivity of a then implies that both a and d are almost isomorphisms. In particular, both P and $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle$ give perfectoid $K^{\circ a}$-algebras.

For future reference, we remark that this argument proves that the perfectoid K-algebra $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle[\frac{1}{\pi}]$ identifies with S, the untilt of $\mathcal{O}_{X^\flat}(U^\flat)$.

2. This was already proven in the proof of (1).

3. This is proven exactly as in Lemma 9.2.3 (3).

4. In course of proving (1), we have seen that the perfectoid $K^{\circ a}$-algebra $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle$ tilts to the perfectoid $K^{\flat\circ a}$-algebra $\mathcal{O}_{X^\flat}(U^\flat)^{\circ a}$. It now follows from (3) that the perfectoid K-algebra $\mathcal{O}_X(U)$ tilts to the perfectoid $K^\flat$-algebra $\mathcal{O}_{X^\flat}(U^\flat)$.

   For the rest, observe that as in (1), we have a unique

$$\mu : (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))^\sharp =: (S, S^+)$$

   of affinoid Tate (R, R^+)-algebras as $g \mid f_i$ in $\mathcal{O}_{X^\flat}^+(U^\flat)^\sharp$ (as $b \mid a_i$ before tilting). As the structure map from (R, R^+) to either ring above is an epimorphism of complete affinoid Tate rings (by Huber's theorem for the source, and Huber's theorem as well as tilting for the target), it suffices to build a map of affinoid Tate (R, R^+)-algebras in the other direction. For this, we remark that the proof of (1) gives a map

$$(R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle[\frac{1}{\pi}], R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

[1] Say $\alpha : M \to N$ is an almost surjective map of K°-modules with M being π-adically separated and N being π-torsionfree. Assume that α modulo π is an almost isomorphism. Then we claim that α is an almost isomorphism as well. To see this, we may replace N with the image of α to assume that α is surjective. Now if $L = \ker(\alpha)$, then the π-torsionfreeness of N ensures that L/π is the kernel of α modulo π, and hence is almost zero. This implies that L is almost π-divisible. On the other hand, $L \subset M$, so L is π-adically separated. The combination of almost π-divisibility and π-adic separatedness force L to be almost zero, as wanted.

of affinoid Tate (R, R^+)-algebras. Write (T, T^+) for the tilt of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. Tilting the preceding map shows $b \mid a_i$ in T^+. Thus, the induced map $\mathrm{Spa}(T, T^+) \to \mathrm{Spa}(R^\flat, R^{\flat +}) = X^\flat$ has image contained $U^\flat$. By Huber's theorem, this gives a map

$$\nu : (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)) \to (T, T^+)$$

of affinoid Tate $(R^\flat, R^{\flat +})$-algebras. Untilting ν then provides the desired inverse to μ.

□

We shall also need the following approximation lemma, which is crucial in many applications. It roughly says we can approximate elements of a perfectoid K-algebra by perfect elements in controlled fashion. In fact, we first prove this for perfectoid polynomial rings; the general case will follow easily from this case.

tionLemma **Lemma 9.2.6** (Approximation lemma)**.** *Assume $R = K\langle T_0^{\frac{1}{p^\infty}}, ..., T_n^{\frac{1}{p^\infty}}\rangle$. Let $f \in R^\circ$ be homogeneous of degree $d \in \mathbf{N}[\frac{1}{p}]$. For any rational $c \geq 0$ and any $\epsilon > 0$, there exists some $g_{c,\epsilon} \in R^{\flat\circ}$ homogeneous of degree d such that*

$$|(f - g_{c,\epsilon}^\sharp)(x)| \leq |\pi|^{1-\epsilon} \max(|f(x)|, |\pi|^c)$$

for any $x \in \mathrm{Spa}(R, R^\circ)$.

Proof. Coming later □

Using this lemma, we get an analogous approximation statement for any perfectoid ring:

erfectoid **Proposition 9.2.7.** *Fix notation as in Notation 9.2.1.*

1. *Given $f \in R$, a rational $c \geq 0$, and an $\epsilon > 0$, there exists some $g_{c,\epsilon} \in R^\flat$ such that*

$$|(f - g_{c,\epsilon}^\sharp)(x)| \leq |\pi|^{1-\epsilon} \max(|f(x)|, |\pi|^c) \qquad (9.1)$$ eq:Approx

   for all $x \in X$.

2. *Given $f, g \in R$ and an integer $c \geq 0$, there exist $a, b \in R^\flat$ such that*

$$X\left(\frac{f, \pi^c}{g}\right) = X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right)$$

   as subsets of X.

3. *Every rational subset of X is the preimage of a rational subset of $X^\flat$.*

4. *For each $U \subset X$ rational, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid (R, R^+)-algebra.*

5. *For each $x \in X$, the NA field $\widehat{k(x)}$ is perfectoid.*

6. *The map $X \to X^\flat$ is a homeomorphism preserving rational subsets.*

Proof. 1. It suffices to solve the problem for any $f \in R^+$. Indeed, if this problem has been solved, then given any $f \in R$ and $c \geq 0$, we can simply set $g_c = t^{-n} h_{c+n}$, where n is chosen so that $\pi^n f \in R^+$, and $h_{c+n} \in R^{\flat +}$ is chosen so that $|\pi^n f - h_{c+n}^\sharp| < \max(|\pi^n f(x)|, |\pi|^{c+n})$. Thus, we reduce to the case $f \in R^+$.

Next, we may assume c is an integer. Indeed, this is simply because

$$\max(|f(x)|, |\pi|^d) \leq \max(|f(x)|, |\pi|^c)$$

if $c \leq d$.

Now assume $f \in R^+$ and $c \geq 0$ is an integer. Choose $g_0, ..., g_c \in R^{\flat +}$ and $f_{c+1} \in R^+$ such that we have

$$f = g_0^\sharp + g_1^\sharp \pi + ... + g_c^\sharp \pi^c + f_{c+1}\pi^{c+1}.$$

Write $f_0 = \sum_{i=0}^c g_i^\sharp \pi^i$, so $f = f_0 + \pi^{c+1} f_{c+1}$. Using the strict NA inequality, we see that right side of (9.1) does not change under replacing f by f_0; one then checks any choice g_c that solves (9.1) for f_0 also solves it for f. Thus, we may assume $f = f_0$ and $f_{c+1} = 0$. Now consider the map

$$\mu : P := K\langle T_0^{\frac{1}{p^\infty}}, ..., T_c^{\frac{1}{p^\infty}}\rangle \to R$$

of perfectoid K-algebras carrying T_i to $g_i^\sharp$; this map satisfies $\mu(P^\circ) \subset R^+$. By Lemma 9.2.6 applied to $\sum_{i=0}^c T_i \pi^i \in P^\circ$ with some $\epsilon < 1$, there exists some $h_0 \in P^\circ$ such that $h = \mu(h_0) \in R^+$ satisfies

$$|(f - h^\sharp)(x)| < \max(|f(x)|, |\pi|^c)$$

for all $x \in \mathrm{Spa}(R, R^+)$, as wanted.

2. Using (1), we can choose $a, b \in R^\flat$ such that

$$|(g - b^\sharp)(x)| < \max(|g(x)|, |\pi|^c) \quad (9.2)$$ eq:Approx

and

$$\max(|f(x)|, |\pi|^c) = \max(|a^\sharp(x)|, |\pi|^c). \quad (9.3)$$ eq:Approx

Now say $x \in X\left(\frac{f, \pi^c}{g}\right)$. We shall check that $x \in X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right)$. As $|\pi|^c \leq |g(x)|$, (9.2) gives $|g(x) - b^\sharp(x)| < |g(x)|$. By the strict NA inequality, this can only happen if $|b^\sharp(x)| = |g(x)|$, so we get $|\pi|^c \leq |b^\sharp(x)|$. Also, (9.3) immediately shows that for such x, we have either $|a^\sharp(x)| \leq |\pi|^c$ or $|a^\sharp(x)| = |f(x)|$; the former implies $|a^\sharp(x)| \leq |\pi|^c \leq |g(x)| = |b^\sharp(x)|$ by the assumption on x and the previous deduction, while the latter implies $|a^\sharp(x)| = |f(x)| \leq |g(x)| = |b^\sharp(x)|$ by the assumption on x and the previous deduction. In either case, we get the desired equality $|a^\sharp(x)| \leq |b^\sharp(x)|$, proving $x \in X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right)$.

Conversely, say $x \in X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right)$. We shall check that $x \in X\left(\frac{f, \pi^c}{g}\right)$. First, we check $|\pi^c| \leq |g(x)|$. If this failed, then we would have $|g(x)| > |\pi|^c$; by (9.2) and the strict NA inequality, this would mean $|g(x)| = |b^\sharp(x)|$, but the latter implies $|g(x)| = |b^\sharp(x)| \leq |\pi|^c$

by assumption on x, which is a contradiction. Thus, we must have $|\pi|^c \leq |g(x)|$. Next, as in the previous paragraph, this implies that $|g(x)| = |b^\sharp(x)|$ for such x. It remains to check that $|f(x)| \leq |g(x)|$. If not, we must have $|f(x)| > |g(x)| \geq |\pi|^c$. By (9.3), this implies that $|f(x)| = |a^\sharp(x)|$, and thus $|a^\sharp(x)| > |g(x)|$ as well. But we just checked that $|g(x)| = |b^\sharp(x)|$, so we get $|a^\sharp(x)| > |b^\sharp(x)|$, which contradicts the assumption on x.

3. Let $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ be a rational subset of X. We may assume after scaling that $f_i \in R^+$ and $f_n = \pi^c$ for some integer $c \geq 1$. We can then write $U = \cap_{i=1}^{n-1} X\left(\frac{f_i,\pi^c}{g}\right)$. The claim now follows by applying (2) $(n-1)$-times.

4. This follows from Lemma 9.2.5.

5. The open valuation ring $\widehat{k(x)^+} \subset \widehat{k(x)}$ is the π-adic completion of the direct limit of the K°-algebras $\mathcal{O}_X^+(U)$ as U ranges through rational open subsets of X. By (4), this gives a perfectoid $K^{\circ a}$-algebra (see Remark 6.2.9 for a description of filtered colimits in the world of perfectoid $K^{\circ a}$-algebras); note that if K has characteristic p, then the instance of (4) being invoked here does not require Lemma 9.2.6. Inverting π shows that $\widehat{k(x)}$ is a perfectoid K-algebra, as wanted.

6. We have already shown in (3) that $\psi : X \to X^\flat$ is a continuous map such that each rational subset of X is a pullback of a rational subset of $X^\flat$; as X is T_0, this formally gives injectivity of ψ. It now suffices to prove surjectivity of ψ: using (3), this will imply that ψ carries rational subsets to rational subsets, proving continuity for the inverse. For surjectivity, pick $x \in X^\flat$. This point defines a map $(R^\flat, R^{\flat+}) \to (\widehat{k(x)}, \widehat{k(x)^+})$ to the corresponding perfectoid affinoid field by easier part of (5). By tilting and Lemma 6.2.13, this untilts to a map $(R, R^+) \to (L, L^+)$ where (L, L^+) is a perfectoid affinoid field (see Remark 9.1.3). This corresponds to a point $y \in \mathrm{Spa}(R, R^+)$. It is then easy to see that $\psi(y) = x$: the valuation $\psi(y)$ is defined by the map $R^\flat \xrightarrow{\sharp} R \to L$, and the latter coincides with the map $R^\flat \to \widehat{k(x)} \xrightarrow{\sharp} L$ defining x.

□

Remark 9.2.8 (A direct proof of the homeomorphy of $X \to X^\flat$)**.** In the context of Proposition 9.2.7, it is possible to prove that $X \to X^\flat$ is a homeomorphism by a relatively "soft" argument that does not use the approximation lemma: one merely needs the surjectivity modulo π of the map $\sharp : R^{+\flat} \to R^+$. We briefly sketch how to do this in the following sequence of steps:

1. The argument given in Proposition 9.2.7 for surjectivity of $X \to X^\flat$ does not use Lemma 9.2.6, so we are allowed to use it. In particular, if a subset $U \subset X$ is the preimage of some $V \subset X^\flat$, then V is uniquely determined as the image of U. In the special case where $U \subset X$ and $V \subset X^\flat$ are known to be rational, we also know from Lemma 9.2.5 that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to $(\mathcal{O}_{X^\flat}(V), \mathcal{O}_{X^\flat}^+(V))$. This fact will be used below to pass from X to U.

2. For $f \in R^+$, the rational set $X\left(\frac{f}{\pi}\right) \subset X$ is the preimage of the rational set $X^\flat\left(\frac{g}{t}\right) \subset X^\flat$, where $g \in R^{+\flat}$ is any element with $g^\sharp \equiv f \mod \pi R^+$. Indeed, this amounts to showing: given $x \in X$, we have $|f(x)| \leq |\pi(x)|$ if and only if $|g^\sharp(x)| \leq |\pi(x)|$, which follows from the NA inequality.

3. For $f \in R^+$ and $n \geq 1$, the rational set $U_n := X\left(\frac{f}{\pi^n}\right) \subset X$ is the preimage of a rational set $V_n \subset X^\flat$. Using the containment $U_{n+1} \subset U_n$, we prove this by induction on n. The case $n = 1$ follows from (2). In general, we have $U_{n+1} = U_n\left(\frac{h}{\pi}\right)$, where $h = \frac{f}{\pi^n} \in \mathcal{O}_X^+(U_n)$. In particular, we may apply (2) to U_n to see that $U_{n+1} \subset U_n$ is the preimage of a rational set $V_{n+1} \subset V_n$. Using (1), it follows that $U_{n+1} \subset X$ is also the preimage of $V_{n+1} \subset X^\flat$. As rational subsets of rational subsets are rational, the claim follows.

4. For $f \in R^+$ and $\epsilon \in \mathbf{N}[\frac{1}{p}] > 0$, the rational set $X\left(\frac{\pi^{1-\epsilon}}{f}\right)$ is the preimage of the rational set $X^\flat\left(\frac{t^{1-\epsilon}}{g}\right)$, where $g \in R^{+\flat}$ is any element with $g^\sharp \equiv f \mod \pi$. This follows by the same argument as in (1).

5. Fix some $c \in \mathbf{N}[\frac{1}{p}]$ with $0 < c < 1$ (such as $c = \frac{1}{p}$). For $f \in R^+$ and any integer $n \geq 1$, the rational set $X\left(\frac{\pi^{nc}}{f}\right)$ is the preimage of a quasi-compact open subset of $X^\flat$. To see this, we write

$$X\left(\frac{\pi^{nc}}{f}\right) = \bigcup_{i=1}^{n} U_i$$

as a union of "annuli", where

$$U_i := \{x \in X \mid |\pi^{nc-(i-1)c}| \leq |f(x)| \leq |\pi^{nc-ic}|\} \subset U_i' := \{x \in X \mid |f(x)| \leq |\pi^{nc-ic}|\}.$$

Then each U_i and U_i' are rational. Moreover, $U_i' \subset X$ is the preimage of a rational subset $V_i' \subset X^\flat$ by (3). As in the proof of (3), it then suffices to check $U_i \subset U_i'$ is the preimage of a rational subset $V_i \subset V_i'$. But $g := \frac{f}{\pi^{nc-ic}} \in \mathcal{O}_X^+(U_i')$, so we can also write

$$U_i = \{x \in U_i' \mid |\pi^c| \leq |g(x)|\}.$$

As $c < 1$, it follows from (3) that $U_i \subset U_i'$ is the preimage of a rational subset of V_i', as wanted.

6. As in Proposition 9.2.7 (3), it is enough to show that rational subsets $U \subset X$ of the form $U := X\left(\frac{f,\pi^N}{g}\right)$ are preimages of quasi-compact open subsets of $X^\flat$. The rational set $U' := X\left(\frac{\pi^N}{g}\right)$ is the preimage of a quasi-compact open V' in $X^\flat$ by (5). If we write $V' = \cup_i V_i'$ with V_i' rational, then $U' = \cup_i U_i'$ with $U_i' \subset X$ being the rational subset that is the preimage on V_i'. Now we have a well-defined element $h := \frac{f\pi^N}{g} \in \mathcal{O}_X^+(U')$, and hence also corresponding elements in each $\mathcal{O}_X^+(U_i')$. Applying (3), the rational sets $U_i := U_i'\left(\frac{h}{\pi^N}\right) \subset U_i'$ are the

preimages of rational subsets $V_i \subset V_i'$. As in (3), this shows that $U_i \subset X$ is the preimage of the rational set $V_i \subset X^\flat$. It remains to observe that $U = \cup_i U_i$, which is clear.

Note that this argument does not prove that rational subsets are preimages of rational subsets in the cases tackled in (5) or (6): one merely proves that they are preimages of quasi-compact opens. I do not know if there is a way to prove the stronger statement without the approximation lemma.

9.3 Tate acyclicity and other sheaf-theoretic properties

Our main goal is to prove the following theorem, stating roughly that perfectoid affinoid algebras behave like affine schemes in algebraic geometry:

paceSheaf **Theorem 9.3.1** (Tate acyclicity for perfectoids)**.** *Fix a perfectoid field K, and a perfectoid affinoid K-algebra (R, R^+) with adic spectrum $X := \mathrm{Spa}(R, R^+)$. Then:*

1. *(R, R^+) is sheafy, i.e., $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves.*
2. *We have $\mathcal{O}_X^+(X) = R^+$, and $H^i(X, \mathcal{O}_X^+) \overset{a}{\simeq} 0$ for $i > 0$.*
3. *We have $\mathcal{O}_X(X) = R$, and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

The strategy of the proof is to prove all three statements at once by showing that the Cech complex attached to the values of $\mathcal{O}_X^+$ on a cover of X by rational subsets is almost acyclic. Thanks to Theorem 9.2.2, this statement can be checked after tilting to characteristic p. In this case, we reduce by an analog of noetherian approximation (and the functoriality of perfections) to the classical form of Tate's acyclicity theorem in rigid geometry. To implement this approach, we first handle the "finitely presented" case. Thus, for the rest of this section, let L is the perfectoid field $\widehat{\mathbf{F}_p[\![t]\!]_{perf}}[\frac{1}{t}]$. We shall be interested in the following class of rings:

Definition 9.3.2. An $\mathbf{F}_p[t]$-algebra A^+ is *algebraically admissible*[2] if it is finitely presented, reduced, t-torsionfree, and integrally closed in $A^+[\frac{1}{t}]$. A perfectoid affinoid L-algebra (R, R^+) is *p-finite* if it is the completion of the perfection of a uniform affinoid Tate ring of the form $(A^+[\frac{1}{t}], A^+)$, where A^+ is algebraically admissible.

To prove Theorem 9.3.1 for p-finite perfectoid L-algebras, we shall use the the following result, which more properly belongs to rigid geometry:

initeType **Proposition 9.3.3** (Tate acyclicity for classical affinoid algebras)**.** *Let A^+ be an algebraically admissible $\mathbf{F}_p[t]$-algebra. Set $A = A^+[\frac{1}{t}]$, so (A, A^+) gives a uniform affinoid Tate ring. Let $X = \mathrm{Spa}(A, A^+)$.*

[2]This is nonstandard terminology, and is meant to be reminiscent of the admissible algebras showing up in Raynaud's approach to rigid geometry via formal schemes. Instead of introducing the latter, we have tried to work with the simplest class of rings that suffices for the perfectoid applications. Corresponding to this choice, the notion of p-finiteness introduced here is also more restrictive than the analogous notion from [Sc1].

1. *For any rational subset $U \subset X$, the affinoid Tate ring $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is also uniform. Moreover, $\mathcal{O}_X^+(U)$ is the t-adic completion of an algebraically admissible $\mathbf{F}_p[t]$-algebra, and $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^\circ$.*

2. *For any covering $X = \cup_i U_i$ of X by rational subsets, each cohomology group of the Cech complex*
$$K := \left(\mathcal{O}_X(X)^\circ \to \prod_i \mathcal{O}_X(U_i)^\circ \to \prod_{i<j} \mathcal{O}_X(U_i \cap U_j)^\circ ... \right)$$
is killed by t^N for some $N \gg 0$.

3. *The uniform affinoid ring (A, A^+) is sheafy and $H^i(X, \mathcal{O}_X^+)$ is killed by t^∞-torsion for $i > 0$.*

Proof. 1. We may assume that rational set U has the form $U := X\left(\frac{f_1,...,f_n}{g}\right)$ for $f_1, ..., f_n, g \in A^+$ with $f_n = t^N$. Let B_0 be the subring $A^+[\frac{f_i}{g}] \subset A[\frac{1}{g}]$. As $f_n = t^N$, we have $B = B_0[\frac{1}{t}] = A[\frac{1}{g}]$. Write B^+ for the integral closure of B_0 in B. Then (B, B^+) is an affinoid Tate (A, A^+)-algebra with couple of definition (B_0, t), and its completion is $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ by construction. Standard facts in algebraic geometry show that B^+ is an algebraically admissible $\mathbf{F}_p[t]$-algebra. As $B_0 \to B^+$ is an isomorphism after inverting t, it follows from the finite presentation of B^+ that B^+/B_0 is killed by t^N for some N. But then (B, B^+) is uniform, and hence so is its completion. Moreover, by Exercise 7.2.6 (7), the latter is computed as $(\widehat{B^+}[\frac{1}{t}], \widehat{B^+})$ with $\widehat{B^+}$ being the t-adic completion of B^+. Thus, we have shown everything except $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^\circ$. For the latter equality, note that Exercise 7.1.4 identifies $\mathcal{O}_X(U)^\circ$ is the total integral closure of $\mathcal{O}_X^+(U)$ in $\mathcal{O}_X(U)$. As $\mathcal{O}_X^+(U) = \widehat{B^+}$ is noetherian, its total integral closure coincides with the integral closure, so we are done as $\widehat{B^+}$ is integrally closed in $\widehat{B}$.

2. The Tate acyclicity theorem[3] implies that $K[\frac{1}{t}]$ is acyclic. As each term of K is open and bounded in the corresponding term for $K[\frac{1}{t}]$ by (1), the claim follows by the Banach open mapping theorem. Indeed, $\ker(d_i[\frac{1}{t}]) \subset K^i[\frac{1}{t}]$ is a closed subset of a Banach space, and hence is itself a Banach space. The differential $d_i[\frac{1}{t}]$ is a continuous surjection $K^{i-1}[\frac{1}{t}] \to \ker(d_i[\frac{1}{t}])$ of Banach spaces, and hence must have open image. But this means exactly that $d(K^{i-1})$ contains $t^N \ker(d_i)$ for some $N \geq 0$, which immediately gives the claim.

3. The sheafyness follows from (2) by inverting t, while the rest follows from (2) by taking a direct limit of the complexes M as the rational cover changes (as the vanishing of Cech cohomology for a basis implies the vanishing of cohomology, see [SP, Tag 01EW]).

□

We give an example showing that $H^i(X, \mathcal{O}_X^+)$ may be unbounded torsion in general. On the other hand, as we shall see later, this phenomenon cannot occur in the perfectoid setting: the corresponding group is almost zero.

[3] Insert proof in previous chapter.

edTorsion **Remark 9.3.4** (Unbounded torsion in $\mathcal{O}_X^+$-cohomology)**.** The integer N in Proposition 9.3.3 (3) cannot be chosen independently of the cover $\{U_i\}$. If this were the case, then it would follow that $H^i(X, \mathcal{O}_X^+)$ has bounded t-torsion for any i. However, we claim that $H^1(X, \mathcal{O}_X^+)$ has unbounded t-torsion for $X = \mathrm{Spa}(A, A^+)$ where $A^+ = \mathbf{F}_p[t, x, y]/(y^2 - z^3)$. To see this, by the long exact sequence

$$0 \to \mathcal{O}_X^+ \xrightarrow{t^n} \mathcal{O}_X^+ \to \mathcal{O}_X^+/t^n \to 0$$

it suffices to find a compatible system of elements $f_n \in H^0(X, \mathcal{O}_X^+/t^n)$ that does not come from an element of $H^0(X, \mathcal{O}_X^+)$.

Let $0 \in X$ be the closed point defined by setting $z = y = 0$; this is the unique point in X that contains z (or equivalently y) in its support. One easily checks that both y and z are invertible outside 0, and thus there is a well-defined global function $\frac{y}{z} \in H^0(X - \{0\}, \mathcal{O}_X)$. In fact, thanks to the equation $y^2 = z^3$, we immediately see that $|\frac{y}{z}(x)| = \sqrt{|z(x)|} \leq 1$ for any $x \in X - \{0\}$, so $\frac{y}{z} \in H^0(X - \{0\}, \mathcal{O}_X^+)$.

Now consider the rational open neighbourhood $U_n := \{x \in X \mid |z(x)| \leq |t^{2n}|\}$ of 0. We claim that $\frac{y}{z}$ restricts to the 0 function on $U_n - \{0\}$ modulo t^n, i.e., that the image of $\frac{y}{x}$ in $H^0(U_n - \{0\}, \mathcal{O}_X^+/t^n)$ is 0. To see this, as both $y(x)$ and $z(x)$ are nonzero for $x \in U_n - \{0\}$, it is enough to check that $|y(x)| \leq |z(x)||t^n|$ for any $x \in U_n - \{0\}$. But we have $|z(x)|^{\frac{1}{2}} \leq |t^n|$ for such an x by definition of U_n, and hence

$$|y(x)| = |z(x)|^{\frac{3}{2}} \leq |z(x)||t^n|.$$

as well. In particular, it follows that we can glue the 0 element in $H^0(U_n, \mathcal{O}_X^+/t^n)$ to the element $\frac{y}{z} \in H^0(X - \{0\}, \mathcal{O}_X^+/t^n)$ to obtain an element $f_n \in H^0(X, \mathcal{O}_X^+/t^n)$. It is clear from the construction that the $\{f_n \in H^0(X, \mathcal{O}_X^+/t^n)\}$ form a compatible system. By checking on the open subset $X - \{0\}$, one can also see that this system does not come from the t-adic completion $\widehat{A^+}$ of A^+. To finish the argument, it remains to observe that $\widehat{A^+} = H^0(X, \mathcal{O}_X^+)$ as A^+ is integrally closed in $A = A^+[\frac{1}{t}]$.

We can now prove Theorem 9.3.1 in the p-finite case.

typfinite **Corollary 9.3.5** (Tate acyclicity for p-finite perfectoid algebras)**.** *Let (R, R^+) be a p-finite perfectoid affinoid L-algebra, arising as the completed perfection of some (A, A^+) with A^+ algebraically admissible.*

1. *The map $X := \mathrm{Spa}(R, R^+) \to Y := \mathrm{Spa}(A, A^+)$ is a homeomorphism preserving rational subsets.*

2. *For any rational $V \subset Y$ with preimage $U \subset X$, the completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ identifies with $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$.*

3. *For any covering $X = \cup_i U_i$ of X by rational subsets, the Cech complex*

$$M := \left(\mathcal{O}_X(X)^\circ \to \prod_i \mathcal{O}_X(U_i)^\circ \to \prod_{i<j} \mathcal{O}_X(U_i \cap U_j)^\circ ...\right)$$

is almost exact.

4. *(R, R^+) is sheafy, and $H^i(X, \mathcal{O}_X^+) \overset{a}{\simeq} 0$ for $i > 0$.*

Proof. 1. The adic spectrum is insenstive to passage to the perfection (by Exercise 7.4.11 (1)) and the completion (by Proposition 7.3.10 (3)).

2. Note that all affinoid Tate rings under consideration are uniform by Proposition 9.3.3. The completed perfection (S, S^+) of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ is the initial object in the category of complete and perfect uniform affinoid Tate (A, A^+) algebras (B, B^+) such that the induced map $\mathrm{Spa}(B, B^+) \to Y$ factors over V. By construction, (S, S^+) is itself a perfectoid affinoid L-algebra. Moreover, (R, R^+) is the completed perfection of (A, A^+) by assumption. The claim now follows from (1) as the functor that (S, S^+) corepresents on the category of perfectoid affinoid L-algebras coincides with that for $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$.

3. Let $Y = \cup_i V_i$ be the corresponding covering of Y by rational subsets. Using Proposition 9.3.3 (2), the complex

$$M' := \left(\mathcal{O}_Y(Y)^\circ \to \prod_i \mathcal{O}_Y(V_i)^\circ \to \prod_{i<j} \mathcal{O}_Y(V_i \cap V_j)^\circ ... \right)$$

has cohomology groups annihilated by t^N for some $N \geq 0$. Moreover, by (2) above and Proposition 9.3.3 (1), this complex gives M on passage to the completed perfection of each term. Passing to the perfection has the effect of making the complex almost acyclic (see Lemma 9.3.6 below), and this property is preserved under completions.

4. This is immediate from (3) as $\mathcal{O}_X^+(U) \overset{a}{\simeq} \mathcal{O}_X(U)^\circ$ for any rational $U \subset X$.

□

The following lemma is an algebraic incarnation of the "contracting" nature of Frobenius:

eModuleRH **Lemma 9.3.6.** *Let A be a commutative ring of characteristic p. Let $t \in A$ be an element, and let M be a t^N-torsion A-module equipped with a map $\beta_M : M \to F_* M$, where F is the Frobenius on A. Then the colimit of the diagram*

$$M \xrightarrow{\beta_M} F_* M \xrightarrow{F_* \beta_M} F_*^2 M \xrightarrow{F_*^2 \beta_M} ....$$

is naturally a module over $A_{perf} = \mathrm{colim}_e F_*^e M$, and is annihilated by $t^{\frac{1}{p^n}}$ for all $n \geq 0$.

Proof. It is clear that the direct limit has an A_{perf}-module structure. By replacing N with a larger quantity, we may assume $N = p^m$ for some $m \geq 0$. As M is killed by t^{p^m}, the A-module $F_*^e M$ is killed by $t^{p^{m-e}} \in F_*^e A$. Letting $e \to \infty$ then proves the lemma. □

To handle the general case, we use the following lemma allowing us to "approximate" an arbitrary perfectoid algebra in characteristic p in terms of p-finite ones; this result is analogous to noetherian approximation results in algebraic geometry used to reduce (certain) problems about arbitrary schemes to analogous questions about schemes that are finitely presented over **Z**.

idpfinite **Lemma 9.3.7** (Approximating perfectoid algebras in characteristic p). *Assume K is a perfectoid field of characteristic p with pseudouniformizer t, and view K as an extension L via $t \mapsto t$. Fix a t-adically complete, perfect, flat K°-algebra A that is integrally closed in $A[\frac{1}{t}]$; equivalently, A is a ring of integral elements in the perfectoid K-algebra $A[\frac{1}{t}]$.*

1. *We can write A as a completed filtered colimit $(\mathrm{colim}_i B_i)\widehat{}$, where B_i is the completed perfection of an algebraically admissible $\mathbf{F}_p[t]$-subalgebra A_i of A. For the rest of the lemma, we fix such a description.*

2. *There is a compatible system of maps $(B_i[\frac{1}{t}], B_i) \to (A[\frac{1}{t}], A)$ of uniform affinoid Tate rings that induces a homeomorphism*
$$\mathrm{Spa}(A[\frac{1}{t}], A) \simeq \lim_i \mathrm{Spa}(B_i[\frac{1}{t}], B_i).$$
Each rational subset on the left side is the preimage of a rational subset from some $\mathrm{Spa}(B_i[\frac{1}{t}], B_i)$.

3. *With notation as in (2), write $X := \mathrm{Spa}(A[\frac{1}{t}], A)$ and $X_i := \mathrm{Spa}(B_i[\frac{1}{t}], B_i)$. Fix some rational subset $U_i \subset X_i$ with preimage $U_j \subset X_j$ for $j \geq i$ and $U \subset X$. Then $(\mathcal{O}_{X_j}(U_j), \mathcal{O}^+_{X_j}(U_j))$ and $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ are perfectoid affinoid L-algebras, and the natural map gives an isomorphism*
$$\mathrm{colim}_j(\mathcal{O}_{X_j}(U_j), \mathcal{O}^+_{X_j}(U_j)) \simeq (\mathcal{O}_X(U), \mathcal{O}^+_X(U))$$
in the category of perfectoid affinoid L-algebras.

Proof. 1. We trivially have
$$\mathrm{colim}_i A_i \simeq A,$$
where the colimit ranges through all finitely presented $\mathbf{F}_p[t]$-subalgebras A_i of A. As A is reduced and t-torsionfree, the same is true for A_i. Moreover, as A is integrally closed in $A[\frac{1}{t}]$, we may also replace A_i with its integral closure in $A_i[\frac{1}{t}]$ by passing to a cofinal subsystem appearing above; here we use the finiteness of integral closures for finitely presented reduced $\mathbf{F}_[t]$-algebras. Thus, we may assume that each A_i appearing above is admissible. Applying the perfection functor gives
$$\mathrm{colim}_i A_{i,perf} \simeq A$$
as A is already perfect. Set $B_i = \widehat{A_{i,perf}}$ to be the t-adic completion of $A_{i,perf}$. Applying the t-adic completion functor to the previous isomorphism gives
$$(\mathrm{colim}_i B_i)\widehat{} \simeq A$$
as A is already t-adically complete (and because completing a filtered colimit can also be accomplished by first completing the terms, and then completing their colimit).

2. This is immediate from (1) and Corollary 7.4.10.

3. By Corollary 7.4.10 (1) and Exercise 7.2.6 (7), we have $(A[\frac{1}{t}], A) \simeq \mathrm{colim}_j(B_j[\frac{1}{t}], B_j)$ in the category of perfectoid affinoid L-algebras (or even all complete uniform affinoid Tate rings). The universal property of Huber's presheaf shows that the diagrams

$$\begin{array}{ccc} (B_i[\frac{1}{t}], B_i) & \longrightarrow & (B_j[\frac{1}{t}], B_j) \\ \downarrow & & \downarrow \\ (\mathcal{O}_{X_i}(U_i), \mathcal{O}^+_{X_i}(U_i)) & \longrightarrow & (\mathcal{O}_{X_j}(U_j), \mathcal{O}^+_{X_j}(U_j)) \end{array}$$

for $j \geq i$ and

$$\begin{array}{ccc} (B_i[\frac{1}{t}], B_i) & \longrightarrow & (A[\frac{1}{t}], A) \\ \downarrow & & \downarrow \\ (\mathcal{O}_{X_i}(U_i), \mathcal{O}^+_{X_i}(U_i)) & \longrightarrow & (\mathcal{O}_X(U), \mathcal{O}^+_X(U)) \end{array}$$

are pushouts of perfectoid affinoid L-algebras. Indeed, this pushout property for the larger category of all complete affinoid Tate rings is proven in Corollary 7.5.3, and this implies the analogous property in the subcategory of perfectoid affinoid L-algebras as all objects involved live in the smaller subcategory. The claim in the lemma follows as pushouts commute with filtered colimits.

□

We can now put everything together to prove Theorem 9.3.1.

Corollary 9.3.8 (Tate acyclicity for perfectoid algebras). *Fix notation as in Notation 9.2.1.*

1. *For every covering $X = \cup_i U_i$ by rational subsets, the Cech complex*

$$M := \left(\mathcal{O}_X(X)^\circ \to \prod_i \mathcal{O}_X(U_i)^\circ \to \prod_{i<j} \mathcal{O}_X(U_i \cap U_j)^\circ ... \right)$$

is almost exact.

2. *(R, R^+) is sheafy and $H^i(X, \mathcal{O}_X^+) \overset{a}{\simeq} 0$ for $i > 0$.*

In particular, the phenomenon encountered in Remark 9.3.4 cannot occur in the perfectoid world: the t-torsion in $H^i(X, \mathcal{O}_X^+)$ is always almost zero for X as above.

Proof. It is enough to prove (1). Moreover, as a complex M of π-adically complete and flat $\mathcal{O}_C$-modules is almost exact if and only if M/π is so, it is enough to show that M/π is almost exact. But M/π is almost isomorphic to the analogous complex constructed using the covering $X^\flat = \cup_i U_i^\flat$ of the tilt by Theorem 9.2.2. Thus, we may assume that $X = X^\flat$, i.e., the field K has characteristic p. Moreover, by replacing K with the subfield L, we may also assume $K = L$. Proposition 9.3.7 (as well as the fact that the functor $(A, A^+) \mapsto A^+$ from perfectoid affinoid K°-algebra to π-adically complete K°-algebras preserves filtered colimits) then allows us to reduce to the case where (R, R^+) is p-finite, and the latter is settled by Corollary 9.3.5. □

In particular, the adic spectrum of (R, R^+) is an affinoid adic space. This allows us to define:

Definition 9.3.9 (Perfectoid spaces)**.** The adic space $\mathrm{Spa}(R, R^+)$ attached to a perfectoid affinoid K-algebra (R, R^+) is called an *affinoid perfectoid space over* K. More generally, a *perfectoid space over* K is an adic space over $\mathrm{Spa}(K, K^\circ)$ that is locally isomorphic to an affinoid perfectoid space.

The tilting correspondence in Proposition 9.1.2 now immediately globalizes to give:

Theorem 9.3.10 (Tilting correspondence)**.** *Fix a perfectoid field K with tilt $K^\flat$.*

1. *For any perfectoid space X over K, there is a unique (up to unique isomorphism) perfectoid space $X^\flat$ over $K^\flat$ characterized as follows: for any perfectoid affinoid K-algebra (R, R^+), we have a functorial bijection*

$$X(R, R^+) \simeq X^\flat(R^\flat, R^{\flat +}).$$

   Moreover, there is a natural homeomorphism $|X| \simeq |X^\flat|$. We call $X^\flat$ the tilt *of X.*

2. *The association $X \mapsto X^\flat$ gives an equivalence between the categores of perfectoid spaces over K and $K^\flat$; we call this the* tilting *equivalence.*

3. *The tilting equivalence restricts to an equivalence on the subcategories of affinoid perfectoid spaces, and the latter equivalence is compatible with the one from Propositiion 9.1.2.*

Remark 9.3.11. It is not clear if a perfectoid space that is affinoid as an adic space is an affinoid perfectoid space, i.e., if $X := \mathrm{Spa}(R, R^+)$ for a sheafy affinoid Tate ring (R, R^+), and X has a rational cover $X := \cup_i U_i$ with each U_i being affinoid perfectoid, it is not clear if X is itself affinoid perfectoid. See the papers by Mihara and Buzzard-Verberkmoes for more.

Exercise 9.3.12 (Mihara)**.** Let $X := \mathrm{Spa}(A, A^+)$ be an affinoid adic space for a sheafy complete Tate ring (A, A^+) of characteristic p. Assume that X is a perfectoid space. Show that X is affinoid perfectoid. (Hint: use the sheaf axiom and Lemma 7.1.6.)

We conclude this section by discussing one pleasant feature of perfectoid spaces:

erProduct **Corollary 9.3.13.** *The category of perfectoid spaces over K admits fiber products.*

Proof. It is enough to prove this in characteristic p when all objects in sight are affinoid perfectoid. As our test objects are also obtained by glueing affinoid perfectoids, we are reduced to checking that the category of perfectoid affinoid K-algebras admits pushouts. Consider the following diagram

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \\ (C, C^+) & & \end{array}$$

of perfectoid affinoid K-algebras. Set $D_0 = B \otimes_A C$, and set D_0^+ to be the integral closure of the image of $B^+ \otimes_{A^+} C^+ \to D_0$. Then (D_0, D_0^+) gives a uniform affinoid K-algebra with D_0^+ being perfect and t-torsionfree. Passing to the t-adic completion then gives a perfectoid affinoid K-algebra (D, D^+). One easily checks that (D, D^+) has the desired universal property. $\square$

Exercise 9.3.14. Given a pushout diagram

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \downarrow \\ (C, C^+) & \longrightarrow & (D, D^+) \end{array}$$

of perfectoid affinoid K-algebras in any characteristic, show that the natural map $B^+ \otimes_{A^+} C^+ \to D^+$ is an almost isomorphism after π-adic completion.

9.4 Zariski closed subsets

We discuss one construction of perfectoid spaces. Namely, we explain why Zariski closed subsets of an affinoid perfectoid space support a unique perfectoid structure; this is in stark contrast to the case of schemes, where the analogous statement has no chance of being true due to the existence of nilpotents. We continue using Notation 9.2.1.

erfectoid

Proposition 9.4.1 (Zariski closed subsets of perfectoids)**.** *Let (R, R^+) be a perfectoid affinoid K-algebra. Let $I \subset R$ be an ideal. Let $X := \mathrm{Spa}(R, R^+)$, and let $Z \subset X$ be the Zariski closed set defined by I. Then:*

1. *There exists an initial object (R_Z, R_Z^+) in the category of complete uniform affinoid (R, R^+)-algebras (S, S^+) such that $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ has image contained in Z.*

2. *The map $(R, R^+) \to (R_Z, R_Z^+)$ is a filtered colimit of rational localizations in the category of complete uniform affinoid Tate rings. In particular, (R_Z, R_Z^+) is a perfectoid affinoid K-algebra.*

3. *The map $\mathrm{Spa}(R_Z, R_Z^+) \to Z$ is a homeomorphism. Thus, the closed subset Z supports a unique perfectoid structure.*

4. *When K has characteristic p, the map $R^+ \to R_Z^+$ is almost surjective, and hence $R \to R_Z$ is surjective.*

Proof. Set $(R_Z, R_Z^+) := \mathrm{colim}_{Z \subset U \text{ rational}}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$, where the colimit is computed in the category of perfectoid affinoid algebras; thus, R_Z^+ is the π-adic completion of $\mathrm{colim}_{Z \subset U} \mathcal{O}_X^+(U)$, and $R_Z = R_Z^+[\frac{1}{\pi}]$, viewed as a Tate ring with couple of definition (R_Z^+, π). Then (2) is true by construction and Theorem 9.2.2, while (1) and (3) follow from Corollary 7.5.8.

For (4), using (1) and (2), one checks that the map $R^+ \to R_Z^+$ is universal amongst maps $R^+ \to S^+$ to t-adically complete and t-torsionfree perfect K°-algebras S^+ that are integrally closed in $S^+[\frac{1}{t}]$ such that $I \cap R^+$ maps to 0 in S^+. From this, one immediately deduces that R_Z^+ is computed as follows: take the perfection $(R^+/I \cap R^+)_{perf}$ of $R^+/I \cap R^+$, take its integral closure R/I^+ in $(R/I)_{perf}$, and set R_Z^+ to be the t-adic completion of R/I^+. Now recall that for any perfect t-torsionfree K°-algebra A, the cokernel of the map from A to its integral closure (or even its total integral closure) in $A[\frac{1}{t}]$ is almost zero; see the second paragraph of the proof of Proposition 4.3.4. Thus, in the almost category, the preceding construction collapses to identify R_Z^+ almost as the t-adic completion of the perfection of $R^+/I \cap R^+$. As $R^+/I \cap R^+$ is a quotient of R^+, it has a surjective Frobenius. It follows that $R^+ \to R_Z^+$ is almost surjective modulo t, and thus almost surjective by completeness of both rings. □

Warning 9.4.2 (Zariski closed maps do not correspond to surjective maps)**.** The conclusion of Proposition 9.4.1 (4) is false if K has characteristic 0. For example, define (R, R^+) by setting $R^+ = \widehat{K^\circ[T^{\frac{1}{p^\infty}}]}$ and $I = (T-1)$, with K being a perfectoid field of characteristic 0. Then the images in R_Z of the elements $\frac{T^{\frac{1}{p}}-1}{p^{\frac{1}{p}}} \in R$ actually lies in R_Z^+. However, there is no element $h \in R^+$ that maps to this element of R_Z^+: if there were such an h, we would have $p^{\frac{1}{p}}h = T^{\frac{1}{p}} - 1 \bmod (T-1)$ in R^+ (as $R^+/(T-1)$ is π-torsionfree, so $R^+/(T-1) \to R_Z^+$ is injective), which implies that $T^{\frac{1}{p}} - 1$ lies in the ideal generated by $T-1$ in the ring $K^\circ/\mathfrak{m}[T^{\frac{1}{p^\infty}}]$, which is clearly false.

We can use the theory introduced above to prove a theorem in commutative algebra. Roughly, it says the following: given an perfectoid $K^{\circ a}$-algebra S and an element $g \in S$, one can freely extract a compatible system of p-power roots of g to obtain an extension $S \to S'$ of perfectoid $K^{\circ a}$-algebras that is almost faithfully flat modulo π. In fact, the precise statement is a bit stronger as it allows us to extract p-power roots of solutions of any monic polynomial (with the previous case corresponding to degree 1 polynomials):

ractRoots **Theorem 9.4.3** (André)**.** *Let (A, A^+) be a perfectoid affinoid K-algebra. Let $g(T) \in A^+[T]$ be a monic polynomial of positive degree. Let B^+ be the π-adic completion of the integral closure of A^+ in $A[T^{\frac{1}{p^\infty}}]/(g(T))$, and set $B := B^+[\frac{1}{\pi}]$, so (B, B^+) is a complete uniform affinoid Tate (A, A^+)-algebra.*

1. *(B, B^+) is a perfectoid affinoid K-algebra. Moreover, the map $(A, A^+) \to (B, B^+)$ is initial amongst maps $(A, A^+) \to (C, C^+)$ of complete uniform affinoid rings equipped with a solution $h_0 \in C^+$ of $g(T) = 0$ together with a compatible system of p-power roots of h_0.*

2. *The map $A^+ \to B^+$ is almost faithfully flat modulo π.*

Note that part (2) above is a purely algebraic statement that can be formulated without any of the theory of perfectoid spaces. The proof, however, crucially uses the tilting correspondence for rational subsets of affinoid perfectoid spaces. The main idea is to access (B, B^+) in terms of rational localizations of (A, A^+), and study the latter using characteristic p techniques (via Theorem 9.2.2).

Proof. 1. Consider the perfectoid affinoid (A, A^+)-algebra (R, R^+) obtained by freely adjoining a variable T together with all of its p-power roots, i.e., $R^+ = \widehat{K^\circ[T^{\frac{1}{p^\infty}}]}$ and $R = R^+[\frac{1}{\pi}]$. Thus, given a complete uniform affinoid Tate (A, A^+)-algebra (D, D^+), specifying a map $(R, R^+) \to (D, D^+)$ of (A, A^+)-algebras is the same as specifying an element $h_0 \in D^+$

together with a compatible sequence of p-power roots (h_n) of h_0; equivalently, one must specify an element $h \in D^{+\flat}$.

Let $X := \mathrm{Spa}(R, R^+)$, and let $Z \subset X$ be the Zariski closed set attached to the ideal $I = (g(T)) \subset R$. Let (R_Z, R_Z^+) be the perfectoid affinoid K-algebra coming from Proposition 9.4.1 applied to Z. Then the map $(R, R^+) \to (R_Z, R_Z^+)$ is universal amongst all complete uniform affinoid (R, R^+)-algebras (D, D^+) such that $\mathrm{Spa}(D, D^+) \to X$ has image in Z; the latter is equivalent to requiring that $g(T) \in R^+$ maps to 0 in D^+.

By combining the universal properties of $(A, A^+) \to (R, R^+)$ and $(R, R^+) \to (R_Z, R_Z^+)$, we learn the following: given a complete uniform affinoid Tate (A, A^+)-algebra (D, D^+), specifying a map $(R_Z, R_Z^+) \to (D, D^+)$ of (A, A^+)-algebras is the same as specifying an element $h \in D^{+\flat}$ such that $g(h^\sharp) = 0$; equivalently, one must specify a root $h_0 \in D^+$ of $g(T)$ together with a compatible system of p-power roots h_n of h_0. It is then easy to see that (B, B^+) has the same universal property, and thus $(B, B^+) \simeq (R_Z, R_Z^+)$.

2. We must check that $A^+ \to R_Z^+$ is almost faithfully flat modulo π. Proposition 9.4.1 and the description of filtered colimits in the category of complete uniform affinoid Tate rings shows that

$$\operatorname*{colim}_{Z \subset U \text{ rational}} \mathcal{O}_X^+(U)/\pi^\delta \simeq R_Z^+/\pi^\delta.$$

for any positive $\delta \in \mathbf{N}[\frac{1}{p}]$. In fact, a cofinal system of open neighbourhoods of Z is given by the rational sets $U_n = X\left(\frac{g(T), \pi^n}{\pi^n}\right) \subset X$, so we have

$$\operatorname*{colim}_{n} \mathcal{O}_X^+(U_n)/\pi^\delta \simeq R_Z^+/\pi^\delta$$

for any positive $\delta \in \mathbf{N}[\frac{1}{p}]$. To prove the lemma, as both $\mathcal{O}_X^+(U_n)$ and R_Z^+ are π-torsion free, it suffices[4] to show the following: for each $n \geq 0$, there exists some positive $\delta \in \mathbf{N}[\frac{1}{p}]$ such that $\mathcal{O}_X^+(U_n)/\pi^\delta$ is almost faithfully flat over A^+/π^δ.

Using Lemma 9.2.7 (1) with $1 - \epsilon = \frac{1}{p}$, we can find some $f \in R^{+\flat}$ such that

(a) $X\left(\frac{g(T), \pi^n}{\pi^n}\right) = X\left(\frac{f^\sharp, \pi^n}{\pi^n}\right)$ as subsets of X. Thus, $U_n \subset X$ is the preimage of $U_n^\flat := X^\flat\left(\frac{f, t^n}{t^n}\right) \subset X^\flat$ under $X \simeq X^\flat$.

(b) $f^\sharp = g(T) \bmod \pi^{\frac{1}{p}} R^+$.

Part (a) and Theorem 9.2.2 (2) show that

$$\mathcal{O}_X^+(U_n)/\pi^{\frac{1}{p}} \simeq \mathcal{O}_{X^\flat}^+(U_n^\flat)/t^{\frac{1}{p}}.$$

[4]Here we are implicitly using the following characterization of flatness: if M is an almost π-torsionfree A^+-module, then M/π is almost flat over A^+/π if and only if M/π^δ is almost flat over A^+/π^δ for some positive $\delta \in \mathbf{N}[\frac{1}{p}]$. This follows immediately from the long exact sequences of Tor by filtering for the π-adic filtration.

as algebras over $A^+/\pi^{\frac{1}{p}} \simeq A^{\flat+}/t^{\frac{1}{p}}$. Thus, we are reduced to proving the analogous assertion for the tilt. By Lemma 9.2.3 (2), the $A^{\flat+}$-algebra $\mathcal{O}^+_{X^\flat}(U_n^\flat)$ is almost isomorphic to the t-adic completion of the perfection of the $A^{\flat+}$-algebra $A^{\flat+}[T,X]/(t^nX - f)$. The latter is t-torsionfree: the elements $t^{\frac{1}{p}}$ and $t^nX - f$ form a Koszul regular sequence in $A^{\flat+}[T,X]$, as f is a monic polynomial in T modulo $t^{\frac{1}{p}}$ by (b) above. Thus, it is enough[5] to show $A^{\flat+}[T,X]/(t^nX - f)$ is faithfully flat over $A^{\flat+}$ modulo some power of t. Reducing modulo $t^{\frac{1}{p}}$ kills t^n and, thus, using (b) above reduces us to checking that the $A^{\flat+}/t^{\frac{1}{p}}$-algebra $(A^{\flat+}/t^{\frac{1}{p}})[T,X]/g(T)$ is faithfully flat, which is clear as $g(T)$ is monic of positive degree.

□

The following two examples capture some of the features of this construction:

Example 9.4.4. Let (A, A^+) be a perfectoid affinoid K-algebra, and consider the monic polynomial $g(T) := T \in A^+[T]$. Let (B, B^+) be affinoid Tate ring provided by Theorem 9.4.3. As complete uniform affinoid Tate rings are reduced, the universal property of (B, B^+) ensures that $(A, A^+) \simeq (B, B^+)$. This can also be shown directly: B^+ is the π-adic completion of the integral closure B_0^+ of $B_0 = A^+[T^{\frac{1}{p^\infty}}]/(T)$ in $A[T^{\frac{1}{p^\infty}}]/(T)$. Now B_0^+ contains $\pi^{-n}T^{\frac{1}{p^m}}$ for all n, m by nilpotence. It follows that $T^{\frac{1}{p^m}} \in \cap_n \pi^n B_0^+$ for all m, and thus it does in the completion. It is then not difficult to check that $A^+ \to B_0^+$ is an isomorphism after π-adic completion.

Exercise 9.4.5. Let K be a perfectoid field of characteristic 0 that contains μ_{p^∞}. Show that the affinoid Tate ring (B, B^+) obtained by applying the construction of Theorem 9.4.3 with $(A, A^+) = (K, K^+)$ and $g(T) = T - 1$ satisfies $B^+ \simeq \prod_{\mathbf{Z}_p(1)} K^+$.

Remark 9.4.6 (Finding perfectoid covers of syntomic extensions)**.** The preceding construction can be extended to a slightly more general context. Say (A, A^+) is a perfectoid affinoid K-algebra, and let B_0 be an A^+. Choose a presentation $B_0 := A^+[T_i]/I$. Let (B, B^+) be the perfectoid affinoid (A, A^+)-algebra attached to the Zariski closed set defined by I inside $\mathrm{Spa}(R, R^+)$, where $R^+ := \widehat{A^+[T_i^{\frac{1}{p^\infty}}]}$. Then there is an obvious map $B_0 \to B^+$. The argument above can then be adapted to show the following: if $A^+ \to B_0$ is (faithfully) flat and lci, then $(A, A^+) \to (B, B^+)$ is almost (faithfully) flat modulo π.

One may iterate the preceding construction to get functorial extensions of a perfectoid algebra where every fppf extension is split:

Corollary 9.4.7. *Let (A, A^+) be a perfectoid affinoid K-algebra. Then we can construct a map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid K-algebras that is functorial in the input pair (A, A^+), and satisfies:*

1. *$A^+ \to B^+$ is almost faithfully flat modulo π.*

[5] We are implicitly using the following abstract fact. If R is a perfect ring of characteristic p and $f_0 : R \to S_0$ is a map of rings that is faithfully flat modulo some $t \in R$ that is a nonzerodivisor on both R and S, then the map $f : R \to S := S_{0,perf}$ is also faithfully flat modulo any power of t. To see this, observe that f can be written as a filtered colimit of the maps $R \xrightarrow{Frob^e} R \xrightarrow{f_0} S_0$, and the latter is faithfully flat modulo any power of t.

2. *The ring B^+ is absolutely integrally closed, i.e., each monic polynomial has a root.*

In particular, each $b \in B^+$ admits a compatible system of p-power roots, so $\sharp : B^{+\flat} \to B^+$ is surjective.

The idea of the proof is to simply iterate the construction of Theorem 9.4.3 infinitely many times; to circumvent the issue that the one does not have good control on the fppf covers over the completion of a filtered colimit, we iterate the construction transfinitely many times (for a large ordinal).

Proof. Let $I \subset A^+[T]$ be the subset of all monic polynomials of positive degree. For each $i = i(T) \in I$, we have the associated map $A^+ \to B_{0,i} := A^+[T]/i(T)$. Applying Theorem 9.4.3 to each such map gives a perfectoid affinoid (A, A^+)-algebra (B_i, B_i^+) such that B_i^+ is almost faithfully modulo π over A^+, and the map $A^+ \to B_i^+$ factors $B_{0,i}$. Let (A_1, A_1^+) be the coproduct of (B_i, B_i^+) in the category of perfectoid affinoid (A, A^+)-algebras, so A_1^+ is almost isomorphic to the completion of the tensor product $\otimes_i B_i^+$ over A^+. This construction has the following properties:

1. The map $A^+ \to A_1^+$ is almost faithfully flat modulo π. One sees this by writing an infinite coproduct as a filtered colimit of finite ones.

2. Each monic polynomial over A^+ has a root in A_1^+. Indeed, this is simply because the map $A^+ \to A_1^+$ factors each extension $A^+ \to B_{0,i}$ by construction.

3. The construction that carries (A, A^+) to the map $(A, A^+) \to (A_1, A_1^+)$ is functorial in the input perfectoid affinoid pair (A, A^+) (as monic polynomials map to monic polynomials under change of scalars).

By transfinite induction, for each ordinal μ, we can define a perfectoid affinoid (A, A^+)-algebra (A_μ, A_μ^+) with a transitive system of maps $(A_\lambda, A_\lambda^+) \to (A_\mu, A_\mu^+)$ for $\lambda \leq \mu$ such that:

- $A_\lambda^+ \to A_\mu^+$ is almost faithfully flat modulo π if $\lambda \leq \mu$.
- Each monic polynomial over A_λ^+ has a root in A_μ^+ if $\lambda < \mu$.

Explicitly, for successor ordinals $\mu = \lambda + 1$, one defines (A_μ, A_μ^+) be applying the construction in the first paragraph to (A_λ, A_λ^+), and for a limit ordinal μ, we set

$$(A_\mu, A_\mu^+) = \operatorname*{colim}_{\lambda < \mu} (A_\lambda, A_\lambda^+),$$

where the colimit is computed in the category of perfectoid affinoid K-algebras. It is easy to see that this construction has the required properties.

Let $\mu := \omega_1$ be the first uncountable ordinal. As the functor $(R, R^+) \to R^+$ from perfectoid affinoid K-algebras to π-adically complete K°-algebras commutes with filtered colimits, we can calculate A_μ^+ as the π-adic completion of the underlying colimit $\operatorname{colim}_{\lambda < \mu} A_\lambda^+$ of abstract rings. But, since μ is countably filtered, so is the preceding colimit. It is easy to see that a countably filtered colimit of π-adically complete rings is already π-adically complete: any failure to completeness of the colimit can be witnessed using countably many elements, and can thus be witnessed using one of the terms defining the colimit by countable filteredness. In other words, the formula

$$A_\mu^+ \simeq \operatorname*{colim}_{\lambda<\mu} A_\lambda^+$$

also holds true in the world of all commutative rings (i.e., the colimit need not be completed). But then any monic polynomial over A_μ^+ is defined over some A_λ^+ for $\lambda < \mu$, and must thus have a root in A_μ^+. Also, the transition maps in the above system are almost faithfully flat modulo π by construction. Thus, setting $(B, B^+) = (A_\mu, A_\mu^+)$ for $\mu = \omega_1$ solves the problem. □

Chapter 10

The almost purity theorem

Fix a perfectoid field K. The main goal of this chapter is to finish proving the almost purity theorem 6.2.10 by establishing the following:

:APTRedux **Theorem 10.0.1.** *Fix a perfectoid affinoid K-algebra (R, R^+).*

1. *If $S \in R_{fet}$, then the integral closure S^+ of R^+ in S lies in R^{+a}_{afet}.*

2. *Inverting π gives an equivalence $R^+_{afet} \simeq R_{fet}$, and an inverse is given by forming integral closures as in (1).*

We have already proven Theorem 10.0.1 in two special cases: either when K has characteristic p (see Theorem 4.3.6), or when (R, R^+) is a perfectoid affinoid field (see discussion following Theorem 6.2.10). To prove the statement in characteristic 0, the strategy is to "localize" the theorem on the adic spectrum. This reformulation will ultimately reduce Theorem 10.0.1 to the case of a perfectoid field, which was already explained earlier. To make this discussion flow, we adopt the following definitions:

Definition 10.0.2 (Finite étale maps of adic spaces)**.** Define

1. A map $(A, A^+) \to (B, B^+)$ of affinoid Tate rings is *finite étale* if $A \to B$ is finite étale, and B^+ is the integral closure of A^+ in B. Write $(A, A^+)_{fet}$ for the category of all such maps.

2. A map $f : X \to Y$ of adic spaces is *finite étale* if there exists a cover of Y by affinoids $V \subset Y$ such that $U = f^{-1}(V)$ is affinoid, and $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V)) \to (\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ is finite étale. Write Y_{fet} for the category of such maps.

It is easy to see that the category the forgetful functor gives an equivalence $(A, A^+)_{fet} \simeq A_{fet}$. We shall eventually show that if $(A, A^+) \to (B, B^+)$ is a finite étale map with A perfectoid, then B is also perfectoid, and the map $A^+ \to B^+$ is almost finite étale. En route to this result, however, it is convenient to give this conclusion a name. Thus, we define:

Definition 10.0.3 (Strongly finite étale maps of perfectoids)**.** Define

1. A map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid K-algebras is *strongly finite étale* if it is finite étale, and B^{+a} is almost finite étale over A^{+a}. Write $(A, A^+)_{sfet}$ for the category of all such maps.

2. A map $f : X \to Y$ of perfectoid spaces is *strongly finite étale* if there exists a cover of Y by affinoid perfectoids $V \subset Y$ such that $U = f^{-1}(V)$ is affinoid perfectoid, and $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is strongly finite étale in the sense of (1). Write Y_{sfet} for the category of such maps.

For a perfectoid space X over K, the affinoid perfectoid subspaces of X and $X^\flat$ match up under tilting. The already proven portion of the almos purity theorem 6.2.10 shows that almost finite étale maps are compatible under tilting. It then follows from chasing definitions that tilting induces an equivalence $X_{sfet} \simeq X^\flat_{sfet}$. Thus, the notion of strongly finite étale morphisms is compatible with tilting. To proceed further, the following theorem of Gabber-Ramero, generalizing work of Elkik, will be crucial to our arguments.

aleApprox **Theorem 10.0.4** (Gabber-Ramero)**.** *Let A be a flat K°-algebra that is henselian along the ideal π, and let $\widehat{A}$ be its completion. Then $A[\frac{1}{\pi}]_{fet} \simeq \widehat{A}[\frac{1}{\pi}]_{fet}$.*

Proof. See [GR, Proposition 5.4.53]. □

In fact, it is the following corollary that is most important for our purposes.

niformFet **Corollary 10.0.5** (Finite étale covers and direct limits of complete uniform rings)**.** *Let (A_i, A_i^+) be a filtered system of complete uniform affinoid Tate K-algebras, and let (A, A^+) be their colimit in complete uniform affinoid Tate rings. Then $\mathrm{colim}_i A_{i,fet} \simeq A_{fet}$.*

Proof. Recall that (A, A^+) is computed by letting A^+ be the π-adic completion of the algebraic colimit $B^+ := \mathrm{colim}_i A_i^+$, and setting $A = A^+[\frac{1}{\pi}]$. As each A_i^+ is π-adically complete and π-torsionfree, the ring B^+ is π-adically henselian and π-torsionfree. In particular, Theorem 10.0.4 implies $B^+[\frac{1}{\pi}]_{fet} \simeq A_{fet}$. It now remains to observe that $\mathrm{colim}_i A_{i,fet} \simeq B^+[\frac{1}{\pi}]_{fet}$, which is immediate from the description $\mathrm{colim}_i A_i \simeq B^+[\frac{1}{\pi}]$ as the functor of "finite étale maps" commutes with filtered colimits of rings. □

The main component of the proof of Theorem 10.0.1 is the following proposition, which states that strongly finite étale maps can be described in terms of algebra if one works over affinoids.

tAffinoid **Proposition 10.0.6** (Strongly finite étale maps form a stack)**.** *Let $f : X \to Y$ be a strongly finite étale map with Y being affinoid perfectoid. Then X is also affinoid perfectoid, and the map*

$$(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$$

is strongly finite étale.

The proof below relies on "noetherian approximation" techniques to reduce the claim to an analogous statement for noetherian adic spaces, which is classical.

Proof. As strongly finite étale maps behave well under tilting, we may assume K has characteristic p. Set $L = \widehat{\mathbf{F}_p[\![t]\!]_{perf}}[\frac{1}{t}]$, and view K as an extension of L via $t \mapsto t$; we can now forget K and work directly over L. Write $Y = \mathrm{Spa}(A, A^+)$ for a perfectoid affinoid L-algebra (A, A^+). We can then write $(A, A^+) = \mathrm{colim}_i(B_i, B_i^+)$ as a filtered colimit of p-finite perfectoid affinoid L-algebras (B_i, B_i^+), as in Lemma 9.3.7. As both rational subsets and finite étale covers pass through such filtered colimits (by Corollary 7.4.10 and Corollary 10.0.5 respectively), we may assume that f arises via base change from some (B_i, B_i^+). The construction of fiber products in Corollary 9.3.13 shows that it suffices to solve the problem over (B_i, B_i^+) instead. Thus, we may assume that (A, A^+) is itself p-finite, i.e., $(A, A^+) = \mathrm{colim}_\phi(R, R^+)$, where R^+ itself being the t-adic completion of an algebraically admissible ring over $\mathbf{F}_p[t]$, and the colimit is computed in complete uniform affinoid rings; a similar description then applies to rational subsets as well by Proposition 9.3.3. By Corollary 7.4.10, Corollary 10.0.5, and the observation that Frobenius does not affect the notion of étale morphisms of rings or rational subsets, it follows that the finite étale map $X \to \mathrm{Spa}(A, A^+)$ arises as the base change of a finite étale map $Z \to \mathrm{Spa}(R, R^+)$. It is a classical theorem (INSERT BACKWARDS REFERENCE) that all finite étale maps $Z \to \mathrm{Spa}(R, R^+)$ are of the form $Z = \mathrm{Spa}(S, S^+)$ with $R \to S$ finite étale and S^+ being the integral closure of R^+ in S. By the description of pushouts in the category of complete uniform affinoid rings, it follows that $X = \mathrm{Spa}(D, D^+)$ where D^+ is the t-adic completion of the integral closure of the image of $S^+ \otimes_{R^+} A^+$ in $S \otimes_R A$.

We shall check that D^+ is almost finite étale over A^+. Note that S^+ is a finite R^+-algebra that is finite étale after inverting p. As A is perfect and $R \to S$ is étale, the ring $S \otimes_R A$ is also perfect. So we can also describe D^+ as the t-adic completion of the integral closure D_1 of the image of $D_0 := S^+_{perf} \otimes_{R^+_{perf}} A^+$ in $S \otimes_R A$. But the map $R^+_{perf} \to S^+_{perf}$ is almost finite étale by Proposition 4.3.4, and hence the same holds true for $A^+ \to D_0$ by base change. As D_0 is almost finite projective over A^+, it is also t-adically complete as A^+ is so. Moreover, the integral closure D_1 of D_0 in $D_0[\frac{1}{t}] = S \otimes_R A$ is almost isomorphic to D_0 by Theorem 4.3.6, and hence is already t-adically complete as D_0 is so. Finally, since D^+ is the t-adic completion of D_1, it follows that $D_1 \simeq D^+$, so the claim follows. □

As a consequence of this proposition, the notion of strongly finite étale maps has good geometric properties:

sFetSheaf **Corollary 10.0.7** (Algebraic description of strongly finite étale maps)**.** *For an affinoid perfectoid space $Y = \mathrm{Spa}(R, R^+)$, the functor $X \mapsto \mathcal{O}_X^+(X)$ induces an equivalence $Y_{sfet} \simeq R^{+a}_{afet}$, and the functor $X \mapsto \mathcal{O}_X(X)$ gives a fully faithful functor $Y_{sfet} \to R_{fet}$.*

Proof. The first part follows from Proposition 10.0.6, and the second then results from the already proven part of Theorem 6.2.10. □

The final ingredient needed to prove Theorem 10.0.1 is the perfectoid analog of the familiar fact in algebraic geometry that modules over a commutative ring R can be viewed as (certain) sheaves of $\mathcal{O}_X$-modules on $X := \mathrm{Spec}(R)$ in a lossless fashion:

eVectWeak **Lemma 10.0.8** (Realizing finite projective modules as sheaves). *Let* $X := \mathrm{Spa}(A, A^+)$ *for a perfectoid affinoid* K*-algebra* (A, A^+)*. For a finite projective* A*-module* M*, write* $\widetilde{M} := M \otimes_A \mathcal{O}_X$ *for the associated sheaf of* $\mathcal{O}_X$*-modules.*

1. *For any rational subset* $U \subset X$*, we have* $H^0(U, \widetilde{M}) = M \otimes_A \mathcal{O}_X(U)$*.*
2. *The functor* $M \mapsto \widetilde{M}$ *is fully faithful.*

The stronger statement that every locally free $\mathcal{O}_X$-module has the form $\widetilde{M}$ for a (necessarily unique) finite projective A-module M is also true, but we do not prove it here.

Proof. For (1): as $\widetilde{M}$ is the sheafification of the presheaf $U \mapsto M \otimes_A \mathcal{O}_X(U)$ on the category of rational subsets of X, it is enough to check that this presheaf is already a sheaf. By writing M as a summand of a finite free A-module, this reduces to checking $U \mapsto \mathcal{O}_X(U)^{\oplus n}$ is a sheaf, which follows from Theorem 9.3.1.

For (2): again, by passage to summands, we reduce to checking full faithfulness on the category of finite free A-modules. In this case, the assertion amounts to showing that $A \simeq H^0(X, \mathcal{O}_X)$, which follows from Theorem 9.3.1. $\square$

Proof of Theorem 10.0.1. We must show that the fully faithful functor $R^{+a}_{afet} \to R_{fet}$ is an equivalence; equivalently, we must check essential surjectivity. Fix a finite étale R-algebra S. We shall check that S comes from an almost finite étale R^+-algebra by first doing so locally on $X := \mathrm{Spa}(R, R^+)$, and then patching the local ones together using Proposition 10.0.6, using Lemma 10.0.8 to ensure that the patched R-algebra coincides with S.

More precisely, we first check that there exists a cover $\{V_i\}$ of X by rational subsets such that the finite étale $\mathcal{O}_X(V_i)$-algebra $S \otimes_R \mathcal{O}_X(V_i)$ lifts (necessarily uniquely) to an almost finite étale $\mathcal{O}_X^+(V_i)$-algebra S_i^+. Fix a point $x \in X$. Then, varying through rational subsets $U \subset X$ containing x, we obtain

$$\operatorname*{colim}_{x \in U} \mathcal{O}_X(U)_{fet} \simeq \widehat{k(x)}_{fet}$$

by Corollary 10.0.5 as

$$\operatorname*{colim}_{x \in U} (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \simeq (\widehat{k(x)}, \widehat{k(x)^+})$$

in the category of complete uniform affinoid Tate rings. Applying this formula in the tilted setting, using the almost purity theorem, and untilting, we get

$$\operatorname*{colim}_{x \in U} \mathcal{O}_X^+(U)_{afet} \simeq \widehat{k(x)^+}_{afet}.$$

Now the almost purity theorem for perfectoid fields shows that the canonical map

$$\widehat{k(x)^+}_{afet} \to \widehat{k(x)}_{fet}$$

is an equivalence. It follows that the canonical map

$$\operatorname*{colim}_{x \in U} \mathcal{O}_X^+(U)_{afet} \to \operatorname*{colim}_{x \in U} \mathcal{O}_X(U)_{fet}$$

is an an equivalence. In particular, there exists some rational subset $V \subset X$ containing x such that the finite étale $\mathcal{O}_X(V)$-algebra $S \otimes_R \mathcal{O}_X(V)$ lifts to (necessarily uniquely) to an almost finite étale $\mathcal{O}_X^+(V)$-algebra. As x varies, this gives the desired cover $\{V_i\}$ of X. We shall write $V_{ij} = V_i \cap V_j$.

Thus, for each i, we have a strongly finite étale map $U_i := \mathrm{Spa}(S_i, S_i^+) \to V_i$ of affinoid perfectoid spaces, where $S_i = S_i^+[\frac{1}{\pi}]$, whose underlying finite étale map corresponds to the finite étale $\mathcal{O}_X(V)$-algebra $S \otimes_R \mathcal{O}_X(V)$. By full faithfulness in Corollary 10.0.7, there is a canonical isomorphism $U_i \times_{V_i} V_{ij} \simeq U_j \times_{V_j} V_{ij}$: both sides are strongly finite étale perfectoid spaces over V_{ij} whose underlying finite étale $\mathcal{O}_X(V_{ij})$-algebras are canonically identified with $S \otimes_R \mathcal{O}_X(V_{ij})$. These isomorphisms satisfy the cocycle condition, so the U_i's can be glued together to give a perfectoid space Y with a map $Y \to X$ that is strongly finite étale over each V_i. But then $Y \to X$ is strongly finite étale (by definition), so $Y = \mathrm{Spa}(T, T^+)$ for a strongly finite étale perfectoid affinoid (R, R^+)-algebra (T, T^+) by Proposition 10.0.6.

It remains to check that $T \simeq S$. Both T and S are finite étale R-algebras, and hence finite projective as R-modules. Write $\widetilde{T}$ and $\widetilde{S}$ for the corresponding sheaves of $\mathcal{O}_X$-modules, as described in Lemma 10.0.8. Then $\widetilde{T}$ and $\widetilde{S}$ are isomorphic: they are identified with each other over each V_i in a manner that is compatible over the overlaps V_{ij}. It follows from Lemma 10.0.8 that there is a unique isomorphism $\phi : T \simeq S$ patching together the local ones over the V_i's. As the latter are R-algebra isomorphisms, the same is true for ϕ, so the claim follows. □

In Theorem 10.0.1, one would expect that finite étale *covers* of R lift to almost finite étale *covers* of R^+. While this is not formal from the conclusion of Theorem 10.0.1, it is true.

versChar0 **Proposition 10.0.9** (Finite étale covers give almost split almost finite étale covers)**.** *If $(R, R^+) \to (S, S^+)$ is finite étale map of perfectoid affinoid K-algebras with $R \to S$ being injective (and thus faithfully flat), then $R^+ \to S^+$ is almost faithfully flat. In this case, the map $R^+ \to S^+$ is almost split as an R^+-module map.*

Proof. When K has characteristic p, this follows from Remark 4.3.5. In general, $R^+ \to S^+$ is almost flat by almost purity. For almost faithful flatness, we must check that if $M \otimes_{R^+} S^+ \overset{a}{\simeq} 0$, then $M \overset{a}{\simeq} 0$. By filtering M, we may assume $M = R^+/I$ is cyclic. In this case, we must check: given an ideal $I \subset R^+$, if $\mathfrak{m}S^+ \subset IS^+$, then $\mathfrak{m}R^+ \subset I$.

We first note that $\mathrm{Spec}(R^+/I) \subset \mathrm{Spec}(R^+/\mathfrak{m}R^+)$ inside $\mathrm{Spec}(R^+)$. Indeed, we have the analogous containments inside $\mathrm{Spec}(S^+)$, so it suffices to show $\mathrm{Spec}(S^+/IS^+) \to \mathrm{Spec}(R^+/I)$ is surjective. But this is a consequence of the surjectivity of $\mathrm{Spec}(S^+) \to \mathrm{Spec}(R^+)$, and the latter holds true as its image is closed under specializations (as $R^+ \to S^+$ is integral) and contains generic points (as $R^+ \to S^+$) is injective. In particular, I contains π^N for some $N \gg 0$.

Thanks the previous reduction, for the first part of the proposition, it now suffices to show that $R^+/\pi^N \to S^+/\pi^N$ is almost faithfully flat. By devissage, this reduces to the case $N = 1$. But the latter follows via tilting from the characteristic p case settled in Remark 4.3.5, so we are done.

To get the almost splitting, one could argue directly by showing that the trace map is almost surjective: this assertion can be checked modulo π, and thus follows from tilting and Remark 4.3.5 granting the compatibility of the trace map with reduction modulo π in the almost world. Alternately, using the almost finite projectivity of S^+ as an R^+-module, it is easy to see that

$$\mathrm{Ext}^1_{R^+}(Q, R^+) \otimes_{R^+} S^+ \to \mathrm{Ext}^1_{R^+}(Q, S^+)$$

is an almost isomorphism for any R^+-module Q. We apply this to $Q = S^+/R^+$ and the tautological class $\alpha \in \mathrm{Ext}^1_{R^+}(Q, R^+)$ measuring the failure to split $R^+ \to S^+$. We must show that α is almost zero. By the almost faithful flatness of $R^+ \to S^+$ and the preceding almost isomorphism, it is enough to show that the image of $\alpha \otimes 1$ in $\mathrm{Ext}^1_{R^+}(Q, S^+)$ along the displayed map above is almost zero. But the latter corresponds to the obstruction to splitting the base change $S^+ \to S^+ \otimes_{R^+} S^+$ of $R^+ \to S^+$ along itself, and is thus 0 (as the multiplication map yields a splitting). $\square$

Chapter 11

The direct summand conjecture

In this chapter, we explain an application of the preceding theory (due to André) in resolving Hochster's direct summand conjecture:

thm:DSC

Theorem 11.0.1 (Direct summand conjecture)**.** *Let A_0 be a regular noetherian ring, and let $f_0 : A_0 \to B_0$ is a finite injective map. Then f_0 admits an A_0-module splitting.*

Remark 11.0.2. Discuss historical relevance of DSC

An elementary but useful observation that shall be used repeatedly is the following: given maps $A_0 \to B_0 \to C_0$ of commutative rings, if the composite $A_0 \to C_0$ admits a splitting, so does $A_0 \to B_0$. We discuss some special cases next.

Example 11.0.3 (Characteristic 0)**.** If A_0 is any normal domain containing $\mathbf{Q}$, then any finite injective map $f_0 : A_0 \to B_0$ admits a splitting. To see this, we may replace B_0 by a suitable quotient to assume that B_0 is also a domain. In fact, the trace map $\mathrm{Tr} : B_0 \to A_0$ is surjective since the composition $A_0 \xrightarrow{f_0} B_0 \xrightarrow{\mathrm{Tr}} A_0$ is multiplication by the nonzero integer $d := [\mathrm{Frac}(B_0) : \mathrm{Frac}(A_0)]$. Thus, choosing any element $b \in B_0$ with $\mathrm{Tr}(b) = 1$ (such as $b = \frac{1}{d}$) provides a splitting $s_0 : B_0 \to A_0$ by the formula $s_0(c) := \mathrm{Tr}(bc)$.

Example 11.0.4 (Finite étale maps)**.** If $f_0 : A_0 \to B_0$ is an injective finite étale map, then it is easy to see that f_0 is split. Indeed, as above, it is enough to show that the trace map Tr is surjective. But this assertion is local for the étale topology on A_0, so we can assume that B_0 is just a finite product of copies of A_0, where everything is clear.

Example 11.0.5 (Small dimensions)**.** If $f_0 : A_0 \to B_0$ is an injective map of noetherian rings that realizes B_0 as a finite projective A_0-module, then f_0 is split. In particular, this applies if A_0 is regular of dimension ≤ 2, and B_0 is the normalization of A_0 in a finite extension of $\mathrm{Frac}(A_0)$ (which is always possible to arrange by refining by B_0); this settles Theorem 11.0.1 in dimension ≤ 2. To see this claim, note that f_0 is split if and only if the extension

$$0 \to A_0 \to B_0 \to Q_0 \to 0$$

(which defines Q_0) splits as A_0-modules. If one writes $\alpha \in \mathrm{Ext}^1_{A_0}(Q_0, A_0)$ for the extension class, then it is easy to see that the formation of α is compatible with base change. In particular, as the formation of Ext-groups commutes with faithfully flat base change (as Q_0 is a finitely presented A_0-module), we reduce to the case where A_0 is a local ring. But then B_0 is a finite free A_0-module. Working modulo the maximal ideal shows that $A_0 \to B_0$ is the inclusion of a free summand as an A_0-module, so f_0 is split.

Remark 11.0.6. Heitmann's proved Theorem 11.0.1 in dimension 3.

The strategy for proving Theorem 11.0.1 is roughly as follows:

1. Choose $g \in A_0$ such that $A_0[\frac{1}{g}] \to B_0[\frac{1}{g}]$ is finite étale.

2. Construct a huge extension $A_0 \to A_\infty$ such that A_∞ is the ring of integral elements in a perfectoid K-algebra (where K is some auxiliary perfectoid field), the map $A_0 \to A_\infty$ is almost faithfully flat, and the element $g \in A_0$ acquires a compatible system of p-power roots in A_∞. This step will be accomplished using Theorem 9.4.3.

3. Show that the base change map $A_\infty \to B_0 \otimes_{A_0} A_\infty$ is almost split. If $g \mid p^N$, then this is deduced from the almost purity theorem. In general, for each $n > 0$, we may apply the preceding argument to get a splitting over the rational subsets $U_n := X\left(\frac{p^n}{g}\right)$ of $X := \mathrm{Spa}(A_\infty[\frac{1}{p}], A_\infty)$ using the same argument. Note that $U_n \subset U_{n+1}$, and $\mathrm{colim}_n U_n = U$ is the Zariski open subset of X defined by $g \neq 0$. We then prove (see Theorem 11.2.1 and Corollary 11.2.2) a quantitative form of the perfectoid Riemann extension theorem that (roughly) says that A_∞ is isomorphic to $\lim_n \mathcal{O}^+_X(U_n)$, and that the same holds true for $\mathrm{Ext}^i_{A_\infty}(-, A_\infty)$ and $\lim_n \mathrm{Ext}^i_{A_\infty}(-, \mathcal{O}^+_X(U_n))$; this allows us to deduce that the vanishing of the relevant obstruction class on X from the corresponding statement on each U_n. (More precisely, the previous statement is only almost true, where almost mathematics is measured with respect to $(pg)^{\frac{1}{p^\infty}}$.)

4. Descend the almost splitting over A_∞ to an honest splitting over A_0. This step relies on the almost faithful flatness of the map $A_0 \to A_\infty$ and Krull's intersection theorem.

Remark 11.0.7. Hochster has shown that, in order to prove Theorem 11.0.1, it is sufficient to do so for regular rings A_0 of the form $W[\![x_1, ..., x_d]\!]$, where W is the ring of Witt vectors of a perfect field k of characteristic p. In the language of commutative algebra, it suffices to prove Theorem 11.0.1 in the *unramified* case. We shall use this reduction in the proof given below, although it is not criticial to the arguments.

11.1 DSC for maps which are unramified in characteristic 0

In this section, we fix a perfect field k of characteristic p, and let $W = W(k)$ be the ring of Witt vectors. Our goal is to explain how to prove Theorem 11.0.1 in the special case where f_0 is unramified after inverting p. In terms of strategy outlined above, this leads to the following simplifications: (a) the construction of the relevant extension $A_0 \to A_\infty$ is straightforward, and (b) the Riemann extension theorem is not needed.

nramified **Theorem 11.1.1.** *Set $A_0 = W[\![x_1, ..., x_d]\!]$. Let $f_0 : A_0 \to B_0$ be a finite injective map such that $f_0[\frac{1}{p}]$ is finite étale. Then f_0 admits an A_0-module splitting.*

Proof. Write $Q_0 := B_0/A_0$, so there is a tautological obstruction class $\alpha \in \mathrm{Ext}^1_{A_0}(Q_0, A_0)$ whose vanishing is equivalent to the theorem. We shall check that this class vanishes after a faithfully flat cover of A_0.

Set A_∞ to be the p-adic completion of $A_0[p^{\frac{1}{p^\infty}}, x_i^{\frac{1}{p^\infty}}]$. Then $(A_\infty[\frac{1}{p}], A_\infty)$ is a perfectoid affinoid K-algebra for $K = \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$. The natural map $A_0 \to A_\infty$ is faithfully flat (INSERT PROOF). Thus, it is enough to check that $A_\infty \to B_0 \otimes_{A_0} A_\infty$ is split. By the flatness of $A_0 \to A_\infty$, we have a base change isomorphism

$$\mathrm{Ext}^1_{A_0}(Q_0, A_0) \otimes_{A_0} A_\infty \simeq \mathrm{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty). \tag{11.1}$$ eq:DSCPro

and thus

$$\mathrm{Ann}_{\mathrm{Ext}^1_{A_0}(Q_0,A_0)}(\alpha) A_\infty = \mathrm{Ann}_{\mathrm{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty)}(\alpha \otimes 1).$$

As $A_0 \to A_\infty$ is faithfully flat, this also gives

$$\mathrm{Ann}_{\mathrm{Ext}^1_{A_0}(Q_0,A_0)}(\alpha) = A_0 \cap \mathrm{Ann}_{\mathrm{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty)}(\alpha \otimes 1),$$

and the same holds true for powers of the ideal. By Krull's intersection theorem, it is thus enough to show that

$$p \in \left(\mathrm{Ann}_{\mathrm{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty)}(\alpha \otimes 1)\right)^{p^n}$$

for all $n \geq 0$. We shall check the stronger statement that

$$p^{\frac{1}{p^n}} \in \mathrm{Ann}_{\mathrm{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty)}(\alpha \otimes 1)$$

for all $n \geq 0$. Now under the isomorphism (11.1), the class $\alpha \otimes 1$ is the obstruction to splitting $A_\infty \to B_0 \otimes_{A_0} A_\infty$. Thus, we must check that $A_\infty \to B_0 \otimes_{A_0} A_\infty$ is almost split. We shall do so by constructing an extension $B_0 \to B_\infty$ such that $A_\infty \to B_\infty$ is almost split.

Let $(A_\infty[\frac{1}{p}], A_\infty) \to (B_\infty[\frac{1}{p}], B_\infty)$ be the finite étale map obtained by setting $B_\infty[\frac{1}{p}] := B_0 \otimes_{A_0} A_\infty[\frac{1}{p}]$, so B_∞ is the integral closure of A_∞ in $B_\infty[\frac{1}{p}]$. By the almost purity theorem 10.0.1, the map $A_\infty \to B_\infty$ is almost finite étale. Proposition 10.0.9 implies that this map is almost split. But, since B_0 is integral over A_0, there is an obvious A_∞-algebra map $A_\infty \to B_0 \otimes_{A_0} A_\infty \to B_\infty$. As the composite is almost split, we are done. □

Remark 11.1.2 (DSC in characteristic p)**.** The argument given above to prove Theorem 11.1.1 readily adapts to prove the general case of Theorem 11.0.1 when A_0 has characteristic p (which is Hochster's theorem). The role of p is played by any element $g \in A_0$ such that $A_0[\frac{1}{g}] \to B_0[\frac{1}{g}]$ is the composition of a purely inseparable map followed by a finite étale map; such elements always exist by field theory. One then replaces the map $A_0 \to A_\infty$ used above with the map $A_0 \to A_{0,perf}$ to the perfection (which is faithfully flat by the regularity of A_0), while the role of B_∞ is played by the perfection $B_{0,perf}$. In this case, the relevant consequence of almost purity can be proven in an elementary fashion. Indeed, the map $A_{0,perf} \to B_{0,perf}$ is an injective integral map of perfect rings that is finite étale after inverting $g \in A$, and is thus almost split by the argument in Remark 4.3.5.

11.2 Quantitative Riemann Hebbarkeitssatz

sec:RH

In complex geometry, Riemann's extension theorem shows that bounded functions on a Zariski open subset of a complex manifold extend uniquely across the boundary to give bounded functions on the manifold itself. This result admits an analog in perfectoid geometry (provided "bounded" is interpreted as sections of the sheaf $\mathcal{O}_X^+$). The goal of this section is prove a finer version of this result adapted to algebraic applications.

Let K be perfectoid field, and let (R, R^+) be a perfectoid affinoid K-algebra. Fix some $g \in R^+$ that lifts to $R^{+\flat}$, i.e., g admits a compatible system $\{g^{\frac{1}{p^n}}\}$ of p-power roots. Let $t \in K^\circ$ be any pseudouniformizer that lifts to $K^{\circ\flat}$. We write $U_n := X\left(\frac{t^n}{g}\right) \subset X$ for the rational subset where $|t^n| \leq |g(x)|$. As $|t^n| \to 0$, we have a tower of rational open subsets

$$U_0 \subset U_1 \subset U_2 \subset ... \subset U_n \subset ..$$

of rational subsets of X, and the union $U_\infty := \cup_n U_n$ is the Zariski open subset where $|g(x)| \neq 0$. We write

$$A^+\langle \frac{t^n}{g} \rangle := \mathcal{O}_X^+(U_n)$$

of the ring of functions on U_n bounded above by 1. We shall show that the projective system $\{A^+\langle \frac{t^n}{g} \rangle\}$ approximates the ring A^+ is a rather strong sense:

m:QuantRH **Theorem 11.2.1.** *Fix some integer $m \geq 0$, and assume g is a nonzerodivisor in A^+/t^m. Then the projective system of natural maps*

$$\left\{ f_n : A^+/t^m \to A^+\langle \frac{t^n}{g} \rangle / t^m \right\}$$

is an almost-pro-isomorphism in the sense of Definition 11.3.1. In fact, we have the following more precise pair of assertions:

1. *The kernels $\ker(f_n)$ are almost 0 for each $n \geq 0$.*
2. *For each $k \geq 0$ and $c \geq p^k m$, the transition map $\operatorname{coker}(f_{n+c}) \to \operatorname{coker}(f_n)$ has image almost annihilated by $g^{\frac{1}{p^k}}$.*

The assumption on g imposed above is not necessary for the almost-pro-isomorphy conclusion, but it simplifies the argument and is harmless in applications.

Proof. As both t and g admit a compatible system of p-power roots, using Lemma 9.2.5 (2) and (3), it is sufficient to prove the analogous assertions for the map

$$\Big\{f_n : A^+/t^m \to M_n := A^+[X_n^{\frac{1}{p^\infty}}]/(t^m, \forall \ell : g^{\frac{1}{p^\ell}} X_n^{\frac{1}{p^\ell}} - t^{\frac{n}{p^\ell}})\Big\}.$$

As g is a nonzerodivisor modulo t^m, the same is true for $g^{\frac{1}{p^k}}$ for all $k \geq 0$. Using this, it is easy to see that the kernels are almost zero for each $n \geq 0$. For the assertion about cokernels, fix some $k \geq 0$. We must check that $g^{\frac{1}{p^k}} \cdot X_{n+c}^{\frac{i}{p^\ell}} \in M_{n+c}$ maps into $A^+/t^m \subset M_n$ under the transition map $M_{n+c} \to M_n$ for $c \geq p^k m$ and all $i, \ell \geq 0$. Note that the transition map carries $X_{n+c}^{\frac{i}{p^\ell}}$ to $X_n^{\frac{i}{p^\ell}} \cdot t^{\frac{ci}{p^\ell}}$. Thus, if $p^\ell \leq ip^k$, then $t^m \mid t^{\frac{ci}{p^\ell}}$ as $c \geq p^k m$, so $X_{n+c}^{\frac{i}{p^\ell}}$ maps to 0 in M_n, and there is nothing to show. On the other hand, if $p^\ell \geq ip^k$, then $g^{\frac{i}{p^\ell}} \mid g^{\frac{1}{p^k}}$ in A^+, so we can write

$$g^{\frac{1}{p^k}} \cdot X_{n+c}^{\frac{i}{p^\ell}} = g^{\frac{1}{p^k} - \frac{i}{p^\ell}} \cdot g^{\frac{i}{p^\ell}} \cdot X_{n+c}^{\frac{i}{p^\ell}} = g^{\frac{1}{p^k} - \frac{i}{p^\ell}} \cdot t^{\frac{(n+c)i}{p^\ell}} \in M_{n+c},$$

which obviously maps into $A^+/t^m \subset M_n$. □

On taking limits over n and m, the preceding theorem implies that

$$A^+ \to \lim_n A^+\langle \frac{t^n}{g} \rangle$$

is injective with cokernel annihilated by $(tg)^{\frac{1}{p^\infty}}$. Reinterpreting these rings in terms of the adic spectrum, this says that the natural map

$$\mathcal{O}_X^+(X) \to \mathcal{O}_X^+(U_\infty) := \lim_n \mathcal{O}_X^+(U_n) \tag{11.2}$$ eq:RH

is injective cokernel annihilated by $(tg)^{\frac{1}{p^\infty}}$. Thus, in the world of almost mathematics with respect to the ideal $(tg)^{\frac{1}{p^\infty}}$, the above map is an isomorphism, i.e., every function on the Zariski open set $U_\infty \subset X$ that is bounded above by 1 extends uniquely to a function on X that is bounded above by 1. The finer statement of Theorem 11.2.1 can then be interpreted as saying that the almost isomorphism (11.2) holds true for essentially "diagrammatic" reasons, and thus one may apply A^+-linear functors to both sides of (11.2), commute them past the limit on the right, and still obtain a similar almost isomorphism. It is this consequence (where the functor is $\mathrm{Ext}^1_{A^+}(Q, -)$ for some A^+-module Q) that is most relevant for Theorem 11.0.1, so we isolate it:

uantRHExt **Corollary 11.2.2.** *For any A^+-complex Q, any integer $m \geq 0$, and any integer i, the natural map*

$$g : \mathrm{Ext}^i_{A^+}(Q, A^+/t^m) \to \lim_n \mathrm{Ext}^i_{A^+}(Q, A^+\langle \frac{t^n}{g} \rangle / t^m)$$

has kernel and cokernel annihilated by $(tg)^{\frac{1}{p^\infty}}$.

The proof below uses the formalism of almost-pro-isomorphisms introduced in the next section.

Proof. The natural map $A^+/p^m \to R\lim_n A^+\langle \frac{t^n}{g}\rangle/p^m$ has cone annihilated by $(tg)^{\frac{1}{p^\infty}}$. Hence, the map

$$\mathrm{RHom}_{A^+}(Q, A^+/t^m) \to \mathrm{RHom}_{A^+}(Q, R\lim_n A^+\langle \frac{t^n}{g}\rangle/t^m)$$

also has cone annihilated by $(tg)^{\frac{1}{p^\infty}}$. It thus suffices to work with the right side above. As right derived functors commute with right derived functors, we have

$$\mathrm{RHom}_{A^+}(Q, R\lim_n A^+\langle \frac{t^n}{g}\rangle/t^m) \simeq R\lim_n \mathrm{RHom}_{A^+}(Q, A^+\langle \frac{t^n}{g}\rangle/t^m)$$

Using the Milnor sequence for derived functors of inverse limits, we get a short exact sequence

$$0 \to R^1 \lim \mathrm{Ext}^{i-1}_{A^+}(Q, A^+\langle \frac{t^n}{g}\rangle/t^m) \to \mathrm{Ext}^i_{A^+}(Q, R\lim_n A^+\langle \frac{t^n}{g}\rangle/t^m) \to \lim_n \mathrm{Ext}^i_{A^+}(Q, A^+\langle \frac{t^n}{g}\rangle/t^m) \to 0$$

It is thus enough to show that the term on the left is annihilated by $(tg)^{\frac{1}{p^\infty}}$. But this follows from the last clause in Corollary 11.3.5 using Theorem 11.2.1. ☐

11.3 Almost-pro-isomorphisms

To prove Corollary 11.2.2 above, it is convenient to extend the notion of almost mathematics to the world of pro-objects. As this discussion is independent of the perfectoid theory, we revert to the basic setup of almost mathematics. Thus, fix a ring R equipped with a flat ideal $\mathfrak{m}$ satisfying $\mathfrak{m} = \mathfrak{m}^2$, so there is a good notion of almost mathematics in this context. Recall that there is the classical notion of the associated abelian category $\mathrm{Pro}(\mathrm{Mod}_R)$ of pro-R-modules: its objects are diagrams $\{M_i\}_{i\in I}$ of R-modules indexed by a cofiltered category I[1], and maps[2] are defined by

$$\mathrm{Hom}(\{M_i\}_{i\in I}, \{N_j\}_{j\in J}) := \lim_j \operatorname*{colim}_i \mathrm{Hom}(M_i, N_j).$$

It is tempting to mimic the same definition in the almost world, i.e., to contemplate the category $\mathrm{Pro}(\mathrm{Mod}_R^a)$. However, as with the notion of almost finite generation, this naive guess turns out to be too strong: the resulting notion of almost-pro-isomorphism would not accommodate Theorem 11.2.1. Instead, the following variant works well:

stProIsom **Definition 11.3.1.** A pro-R-module $\{M_i\}_{i\in I}$ is R-modules is *almost-pro-zero* if for each $\epsilon \in \mathfrak{m}$ and $i \in I$, there exists some $j \geq i$ such that the transition map $M_j \to M_i$ has image killed by ϵ; a map $\{M_i\}_{i\in I} \to \{N_j\}_{j\in J}$ of pro-R-modules is called an *almost-pro-isomorphism* if the kernel and cokernel pro-objects are almost-pro-zero.

[1] In the application of Theorem 11.2.1, it is enough to restrict to the case where I is the poset of natural numbers, i.e., with projective systems of modules indexed by the positive integers.

[2] Given a map $f : \{M_i\}_{i\in I} \to \{N_j\}_{j\in J}$ in the pro-category, we can always replace the source and target by isomorphic objects to assume that the map is represented by an actual map $\{f_k : M_k \to N_k\}_{k\in K}$ of projective systems indexed by the same indexing poset K.

One checks that the collection of almost-pro-zero R-modules forms a thick Serre subcategory Σ of $\mathrm{Pro}(\mathrm{Mod}_R)$, and that almost-pro-isomorphisms are exactly the maps that become isomorphisms in the quotient $\mathrm{Pro}(\mathrm{Mod}_R)^a := \mathrm{Pro}(\mathrm{Mod}_R)/\Sigma$ (see [SP, Tag 02MN] for more on Serre quotients). In particular, almost-pro-isomorphisms are closed under compositions and satisfy the 2-out-of-3 property. The following lemma gives alternate characterizations of almost-pro-zero objects:

erization **Lemma 11.3.2.** *Let $M := \{M_i\}_{i\in I}$ be a pro-R-module. The following are equivalent:*

1. *M is almost-pro-zero.*
2. *For each $\epsilon \in \mathfrak{m}$, the inclusion $\{M_i[\epsilon]\}_{i\in I} \to \{M_i\}_{i\in I}$ is a pro-isomorphism.*
3. *For each $\epsilon \in \mathfrak{m}$, the pro-$R$-module $\{M_i\}_{i\in I}$ is pro-isomorphic to a pro-R-module $\{N_j\}_{j\in J}$ with N_j killed by ϵ.*

In this case, all homology groups of $R\lim_i M_i$ are almost zero.

Proof. It is clear that $1 \Rightarrow 2 \Rightarrow 3$. The implication $3 \Rightarrow 1$ follows from the definition of morphisms of pro-R-modules. For the last part, we use (2) to conclude that $R\lim_i M_i \simeq R\lim_i M_i[\epsilon]$ for each $\epsilon \in \mathfrak{m}$. Thus, the homology groups of $R\lim_n M_n$ are killed by ϵ for each $\epsilon \in \mathfrak{m}$, as wanted. ☐

There is also a direct characterization of almost-pro-isomorhisms:

Isomorphy **Lemma 11.3.3.** *Let $\{f_i : M_i \to N_i\}_{i\in I}$ be a map projective systems of R-modules indexed by cofiltered category I, and write f for the associated map of pro-R-modules. Then f is an almost-pro-isomorphism if and only if for each $\epsilon \in \mathfrak{m}$, we can find a map $b_\epsilon : \{N_i\}_{i\in I} \to \{M_n\}_{i\in I}$ of pro-R-modules such that both compositions $f \circ b_\epsilon$ and $b_\epsilon \circ f$ are multiplication by ϵ.*

Proof. The existence of the b_ϵ's immediately shows that $\{\ker(f_i)\}_{i\in I}$ and $\{\mathrm{coker}(f_i)\}_{i\in I}$ are almost-pro-zero, so one direction is clear. Conversely, assume that f is an almost-pro-isomorphism. By factoring f through the image, it suffices to show the claim separately when each f_i is either surjective or injective. Assume each f_i is injective, so we can identify M_i with a submodule of N_i. Our assumption on f then says $\{N_i/M_i\}_{i\in I}$ is almost-pro-zero. Unwinding definitions, this means the following: if $n_{ji} : N_j \to N_i$ is the transition map, then the map $\epsilon \cdot n_{ji}$ has image contained in M_i for $j \gg i$. This exactly says that multiplication by ϵ on the pro-R-module $\{N_i\}$ factors (necessarily uniquely) over the inclusion $\{M_i\}_{i\in I} \subset \{N_i\}_{i\in I}$, giving the desired map. The dual case where each f_i is surjective is handled similarly. ☐

Using the previous characterization, we obtain:

roRlinear **Lemma 11.3.4.** *Let $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be an R-linear functor. Then F preserves almost-pro-isomorphisms and almost-pro-zero modules.*

Proof. As almost-pro-zero pro-R-modules are exactly those that are almost-pro-isomorphic to the 0 pro-R-module, it is enough to check F preserves almost-pro-isomorphisms. But this follows from the characterization in Lemma 11.3.3. ☐

Putting these things together, we obtain:

oConstant **Corollary 11.3.5.** *Let $\{M_i\}_{i\in I} \to \{N_j\}_{j\in J}$ be a almost-pro-isomorphism, and let $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be an R-linear functor. Then $R\lim_i F(M_i) \to R\lim_j F(N_j)$ is an almost isomorphism on each cohomology group. If particular, if $\{M_i\}_{i\in I}$ is almost-pro-isomorphic to a constant pro-system, then $R^k \lim_i F(M_i)$ is almost zero for $k \neq 0$.*

Proof. By Lemma 11.3.4, it is enough to show that if $f : \{M_i\}_{i\in I} \to \{N_j\}_{j\in J}$ is an almost-pro-isomorphism, then $R\lim_i M_i \to R\lim_j M_j$ is an almost isomorphism. By changing our pro-R-modules by pro-isomorphic ones, we may assume that $I = J$ and that f is represented by a projective system of maps $f_i : M_i \to N_i$. The characterization in Lemma 11.3.2 then shows that $R\lim_i f_i$ has cone annihilated by ϵ for all $\epsilon \in \mathfrak{m}$, which clearly implies the claim. □

11.4 Proof of DSC

We shall need the following consequence of the Artin-Rees lemma

vanishing **Lemma 11.4.1.** *Let R be a noetherian ring equipped with an ideal I. Fix finitely generated R-modules M and N with N being I-adically complete. Then $\mathrm{Ext}^1_R(M,N) \simeq \lim_n \mathrm{Ext}^1_R(M, N/I^nN)$.*

Proof. Using a finite free resolution for M, it is easy to see that

$$\mathrm{RHom}_R(M,N) \simeq R\lim_n \mathrm{RHom}_R(M, N/I^nN).$$

Thus, there is a Milnor short exact sequence

$$0 \to R^1 \lim \mathrm{Hom}_R(M, N/I^n) \to \mathrm{Ext}^1_R(M,N) \to \lim \mathrm{Ext}^1_R(M, N/I^nN) \to 0.$$

We must show that the term on the left is 0. We shall verify the stronger assertion that the pro-R-modules $\{\mathrm{Hom}_R(M,N)/I^n\}_{n\geq 1}$ and $\{\mathrm{Hom}_R(M, N/I^nN)\}_{n\geq 1}$ are pro-isomorphic via the natural map. For this, we use the following formulation of the Artin-Rees lemma: the functor $P \mapsto \{P/I^n\}$ is an exact functor from finitely generated R-modules P to pro-R-modules. To apply this, pick a presentation

$$F_1 \to F_0 \to M \to 0$$

with F_i being finite free. Applying $\mathrm{Hom}_R(-,N)$ gives an exact sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(F_0,N) \to \mathrm{Hom}_R(F_1,N).$$

The previously mentioned form of the Artin-Rees lemma then yields an exact sequence of pro-R-modules of the form

$$0 \to \{\mathrm{Hom}_R(M,N)/I^n\}_{n\geq 1} \to \{\mathrm{Hom}_R(F_0,N)/I^n\}_{n\geq 1} \to \{\mathrm{Hom}_R(F_1,N)/I^n\}_{n\geq 1}.$$

Repeating this analysis using the functor $\mathrm{Hom}_R(-, N/I^nN)$ instead gives an exact sequence of pro-R-modules

$$0 \to \{\mathrm{Hom}_R(M, N/I^nN)\}_{n\geq 1} \to \{\mathrm{Hom}_R(F_0, N/I^nN)\}_{n\geq 1} \to \{\mathrm{Hom}_R(F_1, N/I^nN)\}_{n\geq 1}.$$

Comparing the sequences yields the lemma as $\mathrm{Hom}_R(F_i, N/I^nN) \simeq \mathrm{Hom}_R(F_i,N)/I^n$ since F_i is finite free. □

Proof of Theorem 11.0.1. We break the proof into the following steps:

1. *Reformulation in terms of obstruction classes.* We may assume $A_0 = W[\![x_1, ..., x_d]\!]$ by Hochster's theorem. Let $Q_0 = B_0/A_0$, and let $\alpha_{A_0} \in \mathrm{Ext}^1_{A_0}(Q_0, A_0)$ be the obstruction to splitting f_0. We shall check that $\alpha_{A_0} = 0$. For any A_0-algebra R, we write $\alpha_R \in \mathrm{Ext}^1_R(Q_0 \otimes^L_{A_0} R, R)$ for the corresponding obstruction to splitting $f_0 \otimes^L_{A_0} R$; this is also just the image of α_{A_0} under the map

$$\mathrm{Ext}^1_{A_0}(Q_0, A_0) \to \mathrm{Ext}^1_{A_0}(Q_0, R) \simeq \mathrm{Ext}^1_R(Q_0 \otimes^L_{A_0} R, R).$$

Note that $\alpha_R = 0$ if the (non-derived) base change $f_0 \otimes_{A_0} R$ admits an R-linear splitting. Using Lemma 11.4.1, it is enough to check that $\alpha_{A_0/p^m} = 0$ for all $m \gg 0$. Choose any $g \in A_0$ coprime to p such that $A_0[\frac{1}{pg}] \to B_0[\frac{1}{pg}]$ is finite étale. Using the perfectoid theory, for each $m \gg 0$, we shall construct in (2) a faithfully flat A_0/p^m-algebra R_m such that $p^2 g \in \mathrm{Ann}(\alpha_{R_m})^{p^n}$ for all $n \geq 0$. Granting this, by the flatness of $A_0/p^m \to R_m$, we have

$$\mathrm{Ext}^1_{A_0/p^m}(Q_0 \otimes^L_{A_0} A_0/p^m, A_0/p^m) \otimes_{A_0/p^m} R/p^m \simeq \mathrm{Ext}^1_{R/p^m}(Q_0 \otimes^L_{A_0} R/p^m, R/p^m).$$

The faithful flatness of $A_0/p^m \to R/p^m$ then shows that

$$\mathrm{Ann}(\alpha_{A_0/p^m}) = \mathrm{Ann}(\alpha_{R_m}) \cap A_0/p^m \subset R_m$$

and similarly for powers, so $p^2 g \in \mathrm{Ann}(\alpha_{A_0/p^m})^{p^n}$ for all n. Krull's intersection theorem then implies that either $p^2 g = 0$ in A_0/p^m or that $\mathrm{Ann}(\alpha_{A_0/p^m}) = (1)$. As $g \neq 0$, the first possibility cannot happen for $m \gg 0$, so $\alpha_{A_0/p^m} = 0$ for $m \gg 0$, as wanted.

2. *Constructing faithfully flat extensions.* Let C^+ be the p-adic completion of $A_0[p^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, ..., x_d^{\frac{1}{p^\infty}}]$, set $C = C^+[\frac{1}{p}]$, and let $K = \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$. Then (C, C^+) is a perfectoid affinoid K-algebra, and the natural map $A_0 \to C^+$ is faithfully flat modulo[3] any power of p. Applying Theorem 9.4.3 to (C, C^+) equipped with the element $g \in C^+$ gives a map $(C, C^+) \to (A, A^+)$ of perfectoid affinoid K-algebras such that $g \in A^+$ admits a compatible system of p-power roots, and that $C^+/p^m \to A^+/p^m$ is almost faithfully flat for all m. For each $m \geq 0$, we shall check in (3) that α_{A^+/p^m} is annihilated by $(pg)^{\frac{1}{p^\infty}}$. Granting this, we get $pg \in \mathrm{Ann}(\alpha_{A^+/p^m})^{p^n}$ for all n. Almost faithful flatness of $C^+/p^m \to A^+/p^m$ then shows that $p^2 g \in \mathrm{Ann}(\alpha_{C^+/p^m})^{p^n}$ for all n (where the extra power of p handles passage from the almost world to the real world). Thus, taking $R_m = C^+/p^m$ will present the construction required in (1).

3. *Almost vanishing over a faithfully flat extension.* Using the notation of §11.2 with $t = p$, we obtain a projective system of A^+-algebras $\{A^+\langle \frac{p^n}{g} \rangle\}$. The base change of the map $A_0 \to B_0$ along $A_0 \to A^+\langle \frac{p^n}{g} \rangle$ is finite étale after inverting p (as $g \mid p$ in this ring). Set $B(n)^+$ to be the integral closure of $A^+\langle \frac{p^n}{g} \rangle$ inside $A^+\langle \frac{p^n}{g} \rangle \otimes_{A_0} B_0[\frac{1}{p}]$, so $B(n)^+$ refines $B_0 \otimes_{A_0} A^+\langle \frac{p^n}{g} \rangle$. By the almost purity theorem and Proposition 10.0.9, the map $A^+\langle \frac{p^n}{g} \rangle \to B(n)^+$ is almost

[3] In fact, as A_0 is noetherian, this forces to the map be faithfully flat on the nose, but we do not need that here.

split, and hence the same holds modulo p^m. In particular, $\alpha_{A^+\langle \frac{p^n}{g} \rangle / p^m}$ is almost zero for all n and m. Corollary 11.2.2 then shows that α_{A^+/p^m} is annihilated by $(pg)^{\frac{1}{p^\infty}}$ for all m, as wanted in (2).

□

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