NAME: ____________________________________

Professor/Section: ____________________________________

For each problem show **ALL** your steps and clearly **BOX** your final answer.

Problem	Points Possible	Points Earned
1	10	
2	10	
3	15	
4	5	
5	10	
6	10	
7	10	
8	15	
9	10	
10	5	
Total	100	

1. Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

a) Find a basis for the column space of A, and state *dim col* A.

b) Find a basis for the null space of A, and state *dim nul* A.

c) Find a basis for the row space of A, and state *dim row* A.

d) Find a vector perpendicular to *row* A.

2. Prove that, for an $n \times n$ matrix, A, $row\ A = (nul\ A)^{\perp}$.

   Hints: first show that if $\mathbf{x}$ is in $row\ A$, then $\mathbf{x}$ is orthogonal to every $\mathbf{u}$ in $nul\ A$ - what does this show? Then assume $rank\ A = r$ and examine the dimensions of $nul\ A$ and $(nul\ A)^{\perp}$ and finish the proof.

3. $A = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$

a) Find the determinant of A.

b) Find the eigenvalues and eigenvectors of A.

c) Is A diagonalizable? If yes,justify your answer and then find an invertable matrix P that diagonalizes A and check that it is correct. If no, justify your answer.

d) Is it possible to find an orthogonal matrix U which diagonalizes A? Justify your answer.

4. Let a matrix A have a characteristic polynomial

$$\lambda(\lambda + 5)(\lambda - 2)^2.$$

Is A invertible? Justifly your answer.

5. Consider the linearly independent set of vectors $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 5 \\ 6 \end{bmatrix} \right\}$, that define a subspace of $\mathbb{R}^4$, such that $W = \text{span}\{S\}$.

(a) Find an orthonormal basis, $\hat{S}$ for W.

(b) Find a vector $\mathbf{g}$ so that the four vectors $\hat{S}$ and $\mathbf{g}$ form an orthonormal basis for $\mathbb{R}^4$.

6. Let A and B be $n \times n$ matrices that commute with each other $(A\ B = B\ A)$. Prove that $A^k\ B = B\ A^k$.

7. **a)** Orthogonally diagonalize $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

b) Let $B = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$, find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & s \end{bmatrix}$ such that

$$B = P\ C\ P^{-1}.$$

8. Which of the following define an inner product on their respective vector space (justify your answer):

a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^2\, y_1 + x_2^2\, y_2 \quad \mathbf{x},\ \mathbf{y} \in \mathbb{R}^4$

b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\, y_1 + x_2\, y_2 \quad \mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$

c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\, y_3 + x_2\, y_2 + x_3\, y_1 \quad \mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$

d) $\langle \mathbf{x}, \mathbf{y} \rangle = 2\, x_1^2 + 2\, x_2^2 - y_1^2 - y_2^2 \quad \mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$

e) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\, y_1 - x_2\, y_2 - x_3\, y_3 \quad \mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$

9. Prove that $(AB)^T = B^T A^T$.

10. Let $W = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find the least squares approximation of

$b = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ in W.