Polytopes, Polynomials, and String Theory

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August 2010

Outline

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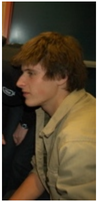
  - Dhruv Ranganathan ’12
  - Paul Riggins ’12
- Ursula Whitcher
  - Daniel Moore ’11
  - Dmitri Skjorshammer ’11

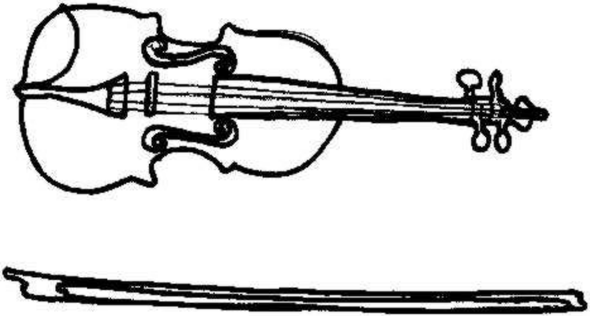




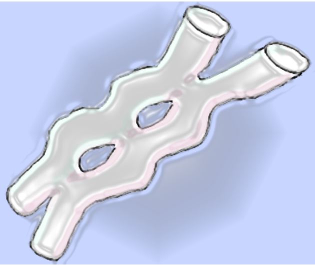
String Theory

- “Fundamental” particles are strings vibrating at different frequencies.
- Strings wrap extra, compact dimensions.



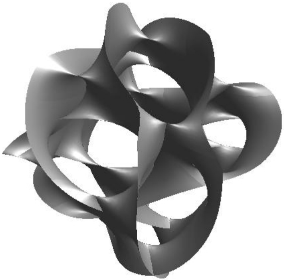
Gromov-Witten Theory

- Worldsheets of strings
- Moduli spaces of maps of curves
- Strong enumerative results



Mirror Symmetry

- Extra, compact dimensions are Calabi-Yau varieties.
- Mirror symmetry predicts that Calabi-Yau varieties should occur in paired or mirror families.
- Varying the complex structure of one family corresponds to varying the Kähler structure of the other family.



Lattices

Let N be a lattice isomorphic to $\mathbb{Z}^n$. The dual lattice M of N is given by $\mathrm{Hom}(N, \mathbb{Z})$; it is also isomorphic to $\mathbb{Z}^n$. We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w \rangle$.

Cones

A cone in N is a subset of the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a set of vectors $\{v_1, \ldots, v_m\} \subset N$. We assume that cones are strongly convex, that is, they contain no line through the origin.

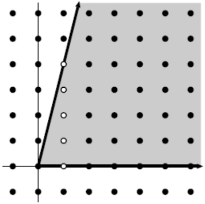


Figure: Cox, Little, and Schenk

Fans

A fan Σ consists of a finite collection of cones such that:

- Each face of a cone in the fan is also in the fan
- Any pair of cones in the fan intersects in a common face.

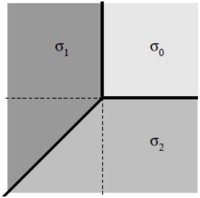
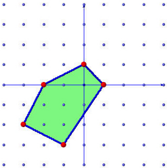


Figure: Cox, Little, and Schenk

Simplicial Fans

We say a fan Σ is simplicial if the generators of each cone in Σ are linearly independent over $\mathbb{R}$.

Lattice Polytopes



A lattice polytope $\diamond$ is the convex hull of a finite set of points in a lattice. We assume that our lattice polytopes contain the origin.

Definition

Let Δ be a lattice polytope in N which contains $(0,0)$. The polar polytope Δ° is the polytope in M given by:

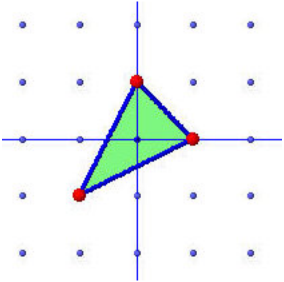
$$\{(m_1,\ldots,m_k) : (n_1,\ldots,n_k)\cdot(m_1,\ldots,m_k) \geq -1 \text{ for all } (n_1,n_2) \in \Delta\}$$

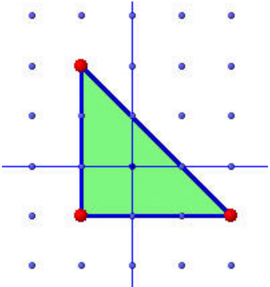
Reflexive Polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.





Fans from Polytopes

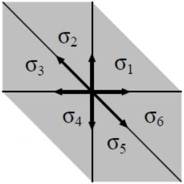


Figure: Cox, Little, and Schenk

We may define a fan using a polytope in several ways:

1. Take the fan R over the faces of $\diamond \subset N$.
2. Refine R by using other lattice points in $\diamond$ as generators of one-dimensional cones.
3. Take the fan S over the faces of $\diamond^\circ \subset M$.

Toric Varieties as Quotients

- Let Σ be a fan in $\mathbb{R}^n$.
- Let $\{v_1, \ldots, v_q\}$ be generators for the one-dimensional cones of Σ.
- Σ defines an n-dimensional toric variety V_Σ.
- V_Σ is the quotient of a subset $\mathbb{C}^q - Z(\Sigma)$ of $\mathbb{C}^q$ by a subgroup of $(\mathbb{C}^*)^q$.

Example

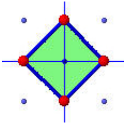


Figure: Polygon $\diamond$

Let R be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0, 0, z_3, z_4)$ or $(z_1, z_2, 0, 0)$.

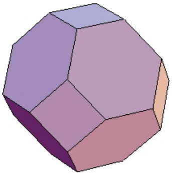
$$V_R = (\mathbb{C}^4 - Z(\Sigma))/\sim$$

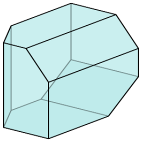
$$(z_1, z_2, z_3, z_4) \sim (\lambda_1 z_1, \lambda_1 z_2, z_3, z_4)$$
$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, \lambda_2 z_3, \lambda_2 z_4)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1 \times \mathbb{P}^1$.

Blowing Up

- Adding cones to a fan Σ corresponds to blowing up subvarieties of V_Σ
- We can use blow-ups to resolve singularities or create new varieties of interest
- Dhruv Ranganathan and Paul Riggins classified the symmetries of all toric blowups of $\mathbb{P}^3$, including varieties corresponding to the associahedron, the cyclohedron, and the graph associahedra.





Calabi-Yau Hypersurfaces

- Let $\diamond$ be a reflexive polytope, with polar polytope $\diamond^\circ$.
- Let R be the fan over the faces of $\diamond$
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- Let $\{v_k\} \subset \diamond \cap N$ generate the one-dimensional cones of Σ

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The following polynomial defines a Calabi-Yau hypersurface in V_Σ:

$$f = \sum_{x \in \diamond^\circ \cap M} c_x \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1}$$

If $n = 3$, f defines a K3 surface.

Quasismooth Hypersurfaces

Let Σ be a simplicial fan, and let X be a hypersurface in V_Σ. Suppose that X is described by a polynomial f in homogeneous coordinates.

Definition

If the products $\partial f/\partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on X, we say X is quasismooth.

Picard-Fuchs Equations

- A period is the integral of a differential form with respect to a specified homology class.
- Periods of holomorphic forms encode the complex structure of varieties.
- The Picard-Fuchs differential equation of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family.

The Residue Map in $\mathbb{P}^n$

Let X be a smooth hypersurface in in $\mathbb{P}^n$ described by a homogeneous polynomial f. Then there exists a residue map

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Let Ω_0 be a homogeneous holomorphic n-form on $\mathbb{P}^n$. We may represent elements of $H^n(\mathbb{P}^n - X)$ by forms $\frac{P\Omega_0}{f^k}$,where P is a homogeneous polynomial.

Let $J(f) =< \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}} >$. We have an induced residue map

$$\mathrm{Res} : \mathbb{C}[z_1, \ldots, z_{n+1}]/J \to H^{n-1}(X).$$

The Griffiths-Dwork Technique in $\mathbb{P}^n$

We want to compute the Picard-Fuchs equation for a one-parameter family of Calabi-Yau hypersurfaces X_t in $\mathbb{P}^n$.

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- Thus, there must be $\mathbb{C}(t)$-linear relationships between derivatives of periods of the holomorphic form
- Use Res to convert to a polynomial algebra problem in $\mathbb{C}(t)[z_1, \ldots, z_{n+1}]/J$

The Griffiths-Dwork Technique

Advantages and Disadvantages

Advantages

We can work with arbitrary polynomial parametrizations of hypersurfaces.

Disadvantages

We need powerful computer algebra systems to work with $\mathbb{C}(t)[z_1, \ldots, z_{n+1}]/J$.

Extending the Griffiths-Dwork Technique

Cox and Katz discuss the Griffiths-Dwork technique for:

- Weighted projective spaces
- Ample hypersurfaces in varieties obtained from simplicial fans
- Discrete group quotients of these spaces

Problem

What if the fan over the faces of a polytope is not simplicial?

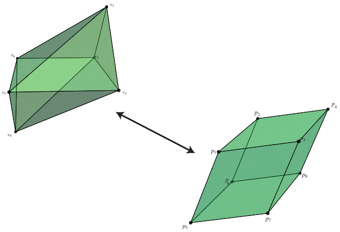
Extending the Residue Map

- Anvar Mavlyutov showed that Res can be defined for semiample, quasismooth hypersurfaces in simplicial toric varieties.

$$\mathrm{Res} : \mathbb{C}[z_1, \ldots, z_q]/J \to H^{n-1}(X).$$

- In this case, Res may not be injective.

The Skew Octahedron



- Let $\diamond$ be the reflexive octahedron with vertices $v_1 = (1, 0, 0)$, $v_2 = (1, 2, 0)$, $v_3 = (1, 0, 2)$, $v_4 = (-1, 0, 0)$, $v_5 = (-1, -2, 0)$, and $v_6 = (-1, 0, -2)$.
- $\diamond$ contains 19 lattice points.
- Let R be the fan obtained by taking cones over the faces of $\diamond$. Then R defines a toric variety V_R which is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

A Symmetric Hypersurface Family

$$
\begin{aligned}
f(t) = t \cdot\ & z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 z_9 z_{10} z_{11} z_{12} z_{13} z_{14} z_{15} z_{16} z_{17} \\
& + z_1^2 z_2^2 z_5^2 z_6^2 z_7^2 z_8^2 z_9 z_{10} z_{11} z_{12} z_{13} z_{14} z_{15} \\
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& + z_1^2 z_3^2 z_5^2 z_6 z_7^2 z_8 z_9 z_{11}^2 z_{12} z_{14}^2 z_{15} z_{16} \\
& + z_3^2 z_4^2 z_5^2 z_6 z_7 z_9^2 z_{10} z_{11}^2 z_{12} z_{14} z_{16}^2 z_{17} \\
& + z_0^2 z_1^2 z_3^2 z_7 z_8 z_{11} z_{12} z_{13} z_{14}^2 z_{15}^2 z_{16} z_{17} \\
& + z_0^2 z_3^2 z_4^2 z_9 z_{10} z_{11} z_{12} z_{13} z_{14} z_{15} z_{16}^2 z_{17}^2
\end{aligned}
$$

The Picard-Fuchs Equation

Theorem (MSW)

Let $\omega = \int \mathrm{Res}\left(\frac{\Omega_0}{f}\right)$. *Then* ω *is the period of a holomorphic form, and satisfies the Picard-Fuchs equation*

$$\frac{\partial^3\omega}{\partial t^3} + \frac{6(t^2-32)}{t(t^2-64)}\frac{\partial^2\omega}{\partial t^2} + \frac{7t^2-64}{t^2(t^2-64)}\frac{\partial\omega}{\partial t} + \frac{1}{t(t^2-64)}\omega = 0$$

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- As expected, the differential equation is third-order
- The differential equation is a symmetric square

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