K3 Surfaces with $\mathcal{S}_4$ Symmetry

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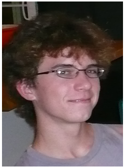
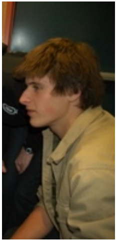


Outline

K3 surfaces

K3 surfaces are named after Kummer, Kähler, Kodaira . . .

and the mountain K2.

Examples of K3 surfaces

All K3 surfaces are diffeomorphic.

- Smooth quartics in $\mathbb{P}^3$
- Double covers of $\mathbb{P}^2$ branched over a smooth sextic

$$w^2 = f_6(x, y, z)$$

- Hypersurfaces in certain 3-dimensional toric varieties

K3 surfaces from elliptic curves

Let E_1 and E_2 be elliptic curves, and let $A = E_1 \times E_2$.

- The Kummer surface $Km(A)$ is the minimal resolution of $A/\{\pm 1\}$.
- The Shioda-Inose surface $SI(A)$ is the minimal resolution of $Km(A)/\beta$, where β is an appropriately chosen involution.

The Hodge diamond of a K3 surface

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Any K3 surface X admits a nowhere-vanishing holomorphic two-form ω which is unique up to scalar multiples.

The Picard group

$$\mathrm{Pic}(X) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$$

$$0 \leq \mathrm{rank}\,\mathrm{Pic}(X) \leq 20$$

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- We may identify $\mathrm{Pic}(X)$ with the Néron-Severi group of algebraic curves using Poincaré duality.
- $\mathrm{Pic}(X) \subset \omega^{\perp}$
- $\mathrm{rank}\ \mathrm{Pic}(X)$ can jump within a family of K3 surfaces

Varying complex structure for K3 surfaces

Let X_α be a family of K3 surfaces, and let M be a free abelian group. Suppose

$$M \hookrightarrow \mathrm{Pic}(X_\alpha).$$

Then:

- $\omega \perp M$ for each X_α
- If M has rank 19, then the variation of complex structure has 1 degree of freedom.

Picard-Fuchs equations

- A period is the integral of a differential form with respect to a specified homology class.
- Periods of holomorphic forms encode the complex structure of varieties.
- The Picard-Fuchs differential equation of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family.
- Solutions to Picard-Fuchs equations for holomorphic forms on Calabi-Yau varieties define the mirror map.

Picard-Fuchs equations for rank 19 families

Let M be a free abelian group of rank 19, and suppose $M \hookrightarrow \mathrm{Pic}(X_t)$.

- The Picard-Fuchs equation is Fuchsian.
- The Picard-Fuchs equation is a rank 3 ordinary differential equation.

Symmetric Squares

The symmetric square of the differential equation

$$a_2 \frac{\partial^2 A}{\partial t^2} + a_1 \frac{\partial A}{\partial t} + a_0 A = 0$$

is

$$a_2^2 \frac{\partial^3 A}{\partial t^3} + 3a_1 a_2 \frac{\partial^2 A}{\partial t^2} + (4a_0 a_2 + 2a_1^2 + a_2 a_1' - a_1 a_2') \frac{\partial A}{\partial t} + (4a_0 a_1 + 2a_0' a_2 - 2a_0 a_2') A = 0$$

where primes denote derivatives with respect to t.

Picard-Fuchs equations and symmetric squares

Theorem

*[D00, Theorem 5] The Picard-Fuchs equation of a family of rank-*19 *lattice-polarized K3 surfaces can be written as the symmetric square of a second-order homogeneous linear Fuchsian differential equation.*

Some Picard rank 19 families

- Hosono, Lian, Oguiso, Yau:

$$x + 1/x + y + 1/y + z + 1/z - \Psi = 0$$

- Verrill:

$$(1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)(xyz)$$

- Narumiya-Shiga:

$$\begin{aligned} Y_0 + Y_1 + Y_2 + Y_3 - 4tY_4 \\ Y_0 Y_1 Y_2 Y_3 - Y_4^4 \end{aligned}$$

Lattices

Let N be a lattice isomorphic to $\mathbb{Z}^n$. The dual lattice M of N is given by $\mathrm{Hom}(N, \mathbb{Z})$; it is also isomorphic to $\mathbb{Z}^n$. We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w \rangle$.

Cones

A cone in N is a subset of the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a set of vectors $\{v_1, \ldots, v_m\} \subset N$. We assume that cones are strongly convex, that is, they contain no line through the origin.

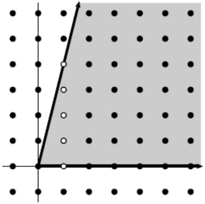


Figure: Cox, Little, and Schenk

Fans

A fan Σ consists of a finite collection of cones such that:

- Each face of a cone in the fan is also in the fan
- Any pair of cones in the fan intersects in a common face.

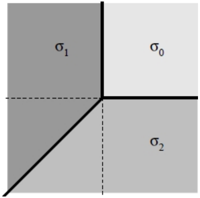
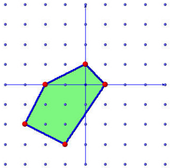


Figure: Cox, Little, and Schenk

Simplicial fans

We say a fan Σ is simplicial if the generators of each cone in Σ are linearly independent over $\mathbb{R}$.

Lattice polytopes



A lattice polytope $\diamond$ is the convex hull of a finite set of points in a lattice. We assume that our lattice polytopes contain the origin.

Definition

Let Δ be a lattice polytope in N which contains $(0,0)$. The polar polytope Δ° is the polytope in M given by:

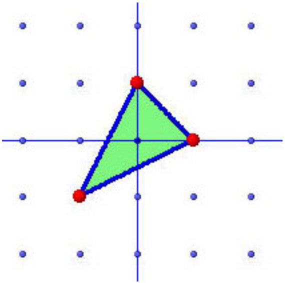
$$\{(m_1,\ldots,m_k) : (n_1,\ldots,n_k)\cdot(m_1,\ldots,m_k) \geq -1 \text{ for all } (n_1,n_2) \in \Delta\}$$

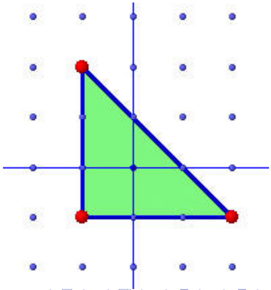
Reflexive polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.





Fans from polytopes

We may define a fan using a polytope in several ways:

1. Take the fan R over the faces of $\diamond \subset N$.
2. Refine R by using other lattice points in $\diamond$ as generators of one-dimensional cones.
3. Take the normal fan S to $\diamond^\circ \subset M$.

Toric varieties as quotients

- Let Σ be a fan in $\mathbb{R}^n$.
- Let $\{v_1, \ldots, v_q\}$ be generators for the one-dimensional cones of Σ.
- Σ defines an n-dimensional toric variety V_Σ.
- V_Σ is the quotient of a subset $\mathbb{C}^q - Z(\Sigma)$ of $\mathbb{C}^q$ by a subgroup of $(\mathbb{C}^*)^q$.
- Each one-dimensional cone corresponds to a coordinate z_i on V_Σ.

Example

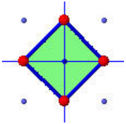


Figure: Polygon $\diamond$

Let R be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0,0,z_3,z_4)$ or $(z_1,z_2,0,0)$.

$$V_R = (\mathbb{C}^4 - Z(\Sigma))/\sim$$

$$(z_1,z_2,z_3,z_4) \sim (\lambda_1 z_1, \lambda_1 z_2, z_3, z_4)$$
$$(z_1,z_2,z_3,z_4) \sim (z_1, z_2, \lambda_2 z_3, \lambda_2 z_4)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1 \times \mathbb{P}^1$.

K3 hypersurfaces

- Let $\diamond$ be a 3-dimensional reflexive polytope, with polar polytope $\diamond^\circ$.
- Let R be the fan over the faces of $\diamond$
- Let Σ be a simplicial refinement of R
- Let $\{v_k\} \subset \diamond \cap N$ generate the one-dimensional cones of Σ
- Let c_x be complex numbers

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- Let c_x be complex numbers

The following polynomial defines a K3 surface X in V_Σ:

$$f = \sum_{x \in \diamond^\circ \cap M} c_x \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1}$$

Quasismooth and regular hypersurfaces

Let Σ be a simplicial fan, and let X be a hypersurface in V_Σ. Suppose that X is described by a polynomial f in homogeneous coordinates.

Definition

If the derivatives $\partial f / \partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on X, we say X is quasismooth.

Quasismooth and regular hypersurfaces

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Definition

If the products $z_i \partial f / \partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on X, we say X is regular and f is nondegenerate.

Semiample hypersurfaces

- Let R be a fan over the faces of a reflexive polytope
- Let Σ be a refinement of R
- We have a proper birational morphism $\pi : V_\Sigma \to V_R$
- Let Y be an ample divisor in V_R, and suppose $X = \pi^*(Y)$

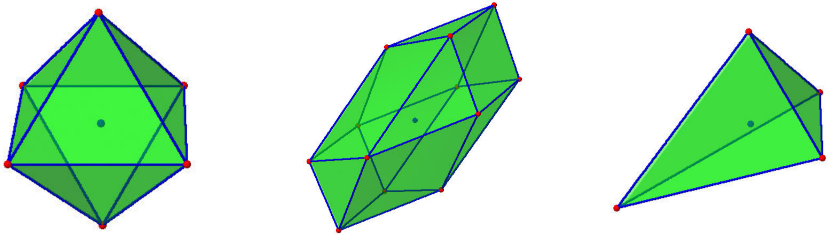
Then X is semiample:

Definition

We say that a Cartier divisor D is *semiample* if D is generated by global sections and the intersection number $D^n > 0$.

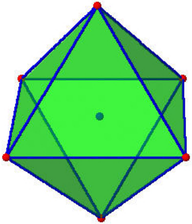
Toric realizations of the rank 19 families

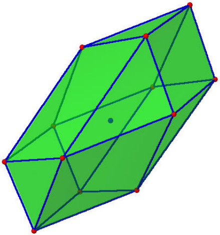
The polar polytopes $\diamond^\circ$ for [HLOY04], [V96], and [NS01].

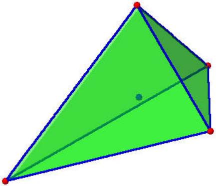


$$f(t) = \left(\sum_{x \,\in\, \mathrm{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x\rangle + 1} \right) + t \prod_{k=1}^{q} z_k .$$

What do these polytopes have in common?







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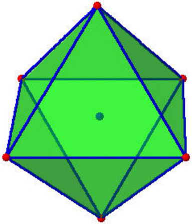
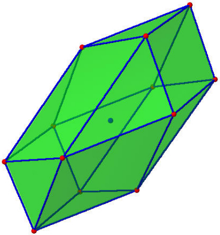
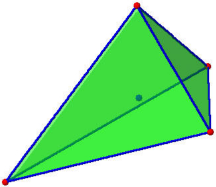

- The only lattice points of these polytopes are the vertices and the origin.

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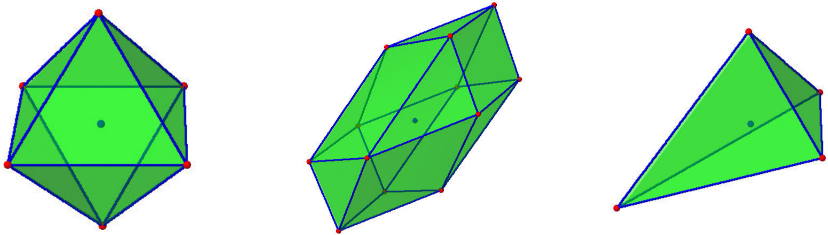

- The only lattice points of these polytopes are the vertices and the origin.
- The group G of orientation-preserving symmetries of the polytope acts transitively on the vertices.

Another symmetric polytope

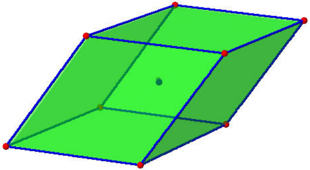

Figure: The skew cube

$$f(t) = \left(\sum_{x \in \text{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_k .$$

Dual rotations

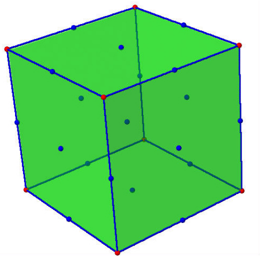

Figure: $\diamond$

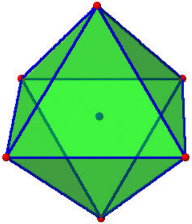

Figure: $\diamond^\circ$

We may view a rotation as acting either on $\diamond$ (inducing automorphisms on X_t) or on $\diamond^\circ$ (permuting the monomials of $f(t)$).

Symplectic Group Actions

Let G be a finite group of automorphisms of a K3 surface. For $g \in G$,

$$g^*(\omega) = \rho\omega$$

where ρ is a root of unity.

Definition

We say G acts *symplectically* if

$$g^*(\omega) = \omega$$

for all $g \in G$.

A subgroup of the Picard group

Definition

$$S_G = ((H^2(X, \mathbb{Z})^G)^\perp$$

Theorem ([N80a])

S_G is a primitive, negative definite sublattice of $\mathrm{Pic}(X)$.

The rank of S_G

Lemma

- *If X admits a symplectic action by the permutation group $G = \mathcal{S}_4$, then $\mathrm{Pic}(X)$ admits a primitive sublattice S_G which has rank 17.*
- *If X admits a symplectic action by the alternating group $G = \mathcal{A}_4$, then $\mathrm{Pic}(X)$ admits a primitive sublattice S_G which has rank 16.*

Why is the Picard rank 19?

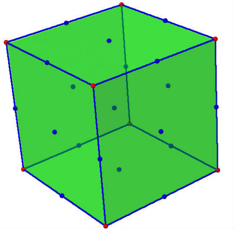

Figure: $\diamond$

We can use the orbits of G on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

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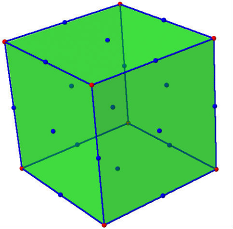

Figure: $\diamond$

We can use the orbits of G on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

- For the families of [HLOY04] and [V96], and the family defined by the skew cube, we conclude that $17 + 2 = 19$.
- For the family of [NS01], we conclude that $16 + 3 = 19$.

The residue map

We will use a residue map to describe the cohomology of a K3 hypersurface X:

$$\mathrm{Res} : H^3(V_\Sigma - X) \to H^2(X).$$

Anvar Mavlyutov showed that Res is well-defined for quasismooth, semiample hypersurfaces in simplicial toric varieties.

Two ideals

Definition

The Jacobian ideal $J(f)$ is the ideal of $\mathbb{C}[z_1, \ldots, z_q]$ generated by the partial derivatives $\partial f/\partial z_i$, $i = 1 \ldots q$.

Definition

[BC94] The ideal $J_1(f)$ is the ideal quotient

$$\langle z_1 \partial f/\partial z_1, \ldots, z_q \partial f/\partial z_q \rangle : z_1 \cdots z_q.$$

The induced residue map

Let Ω_0 be a holomorphic 3-form on V_Σ. We may represent elements of $H^3(V_\Sigma - X)$ by forms $\frac{P\Omega_0}{f^k}$, where P is a polynomial in $\mathbb{C}[z_1, \ldots, z_q]$.

Mavlyutov described two induced residue maps on semiample hypersurfaces:

- $\mathrm{Res}_J : \mathbb{C}[z_1, \ldots, z_q]/J \to H^2(X)$ is well-defined for quasismooth hypersurfaces
- $\mathrm{Res}_{J_1} : \mathbb{C}[z_1, \ldots, z_q]/J_1 \to H^2(X)$ is well-defined for regular hypersurfaces.

Whither injectivity?

Res_J is injective for smooth hypersurfaces in $\mathbb{P}^3$, but this does not hold in general.

Theorem

[M00] If X is a regular, semiample hypersurface, then the residue map Res_{J_1} is injective.

The Griffiths-Dwork technique

Plan

We want to compute the Picard-Fuchs equation for a one-parameter family of K3 hypersurfaces X_t.

- Look for $\mathbb{C}(t)$-linear relationships between derivatives of periods of the holomorphic form
- Use Res_J to convert to a polynomial algebra problem in $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$

The Griffiths-Dwork technique

Procedure

1. $$\frac{d}{dt}\int \mathrm{Res}\left(\frac{P\Omega}{f^k(t)}\right) = \int \mathrm{Res}\left(\frac{d}{dt}\left(\frac{P\Omega}{f^k(t)}\right)\right)$$
$$= -k\int \mathrm{Res}\left(\frac{f'(t)P\Omega}{f^{k+1}(t)}\right)$$

The Griffiths-Dwork technique

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$$= -k\int \mathrm{Res}\left(\frac{f'(t)P\Omega}{f^{k+1}(t)}\right)$$

2. Since $H^*(X_t, \mathbb{C})$ is a finite-dimensional vector space, only finitely many of the classes $\mathrm{Res}\left(\frac{d^j}{dt^j}\left(\frac{\Omega}{f^k(t)}\right)\right)$ can be linearly independent

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3. Use the reduction of pole order formula to compare classes of the form $\mathrm{Res}\left(\frac{P\Omega}{f^{k+1}(t)}\right)$ to classes of the form $\mathrm{Res}\left(\frac{Q\Omega}{f^k(t)}\right)$

The Griffiths-Dwork technique

Implementation

Reduction of pole order

$$\frac{\Omega_0}{f^{k+1}} \sum_i P_i \frac{\partial f}{\partial x_i} = \frac{1}{k} \frac{\Omega_0}{f^k} \sum_i \frac{\partial P_i}{\partial x_i} + \text{exact terms}$$

We use Groebner basis techniques to rewrite polynomials in terms of $J(f)$.

The Griffiths-Dwork technique

Advantages and disadvantages

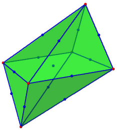
Advantages

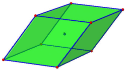
We can work with arbitrary polynomial parametrizations of hypersurfaces.

Disadvantages

We need powerful computer algebra systems to work with $J(f)$ and $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$.

The Skew Octahedron





- Let $\diamond$ be the reflexive octahedron shown above.
- $\diamond$ contains 19 lattice points.
- Let R be the fan obtained by taking cones over the faces of $\diamond$. Then R defines a toric variety $V_R \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.
- Consider the family of K3 surfaces X_t defined by $f(t) = \left(\sum_{x \,\in\, \mathrm{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x\rangle + 1}\right) + t \prod_{k=1}^{q} z_k$.
- X_t are generally quasismooth but not regular.

The Picard-Fuchs equation

Theorem ([KLMSW10])

Let $A = \int \mathrm{Res}\left(\frac{\Omega_0}{f}\right)$. Then A is the period of a holomorphic form on X_t, and A satisfies the Picard-Fuchs equation

$$\frac{\partial^3 A}{\partial t^3} + \frac{6(t^2-32)}{t(t^2-64)}\frac{\partial^2 A}{\partial t^2} + \frac{7t^2-64}{t^2(t^2-64)}\frac{\partial A}{\partial t} + \frac{1}{t(t^2-64)}A = 0.$$

As expected, the differential equation is third-order and Fuchsian.

Symmetric square root

The symmetric square root of our Picard-Fuchs equation is:

$$\frac{\partial^2 A}{\partial t^2} + \frac{(2t^2 - 64)}{t(t^2 - 64)} \frac{\partial A}{\partial t} + \frac{1}{4(t^2 - 64)} A = 0.$$

Mirror Moonshine

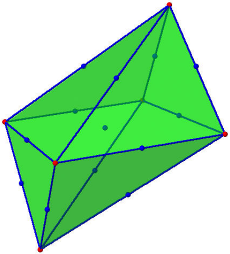
Mirror Moonshine for a one-parameter family of K3 surfaces arises when there exists a genus 0 modular group $\Gamma \subset PSL_2(\mathbb{R})$ such that . . .

- The Picard-Fuchs equation gives the base of the family the structure of a (pull-back of a) modular curve $\overline{\mathbb{H}/\Gamma}$.
- The mirror map is commensurable with a hauptmodul for Γ.
- The holomorphic solution to the Picard-Fuchs equation is a Γ-modular form of weight 2.

Mirror Moonshine from geometry

Example	[HLOY04]	[V96]
Shioda-Inose structure	$SI(E_1 \times E_2)$ E_1, E_2 are 6-isogenous	$SI(E_1 \times E_2)$ E_1, E_2 are 3-isogenous
$\mathrm{Pic}(X)^\perp$	$H \oplus \langle 12 \rangle$	$H \oplus \langle 6 \rangle$
Γ	$\Gamma_0(6) + 6$	$\Gamma_0(6) + 3 \subset \Gamma_0(3) + 3$

Geometry of the skew octahedron family



- X_t is a family of Kummer surfaces
- Each surface can be realized as $Km(E_t \times E_t)$
- The generic transcendental lattice is $2H \oplus \langle 4 \rangle$

The modular group

We use our symmetric square root and the table of [LW06] to show that:

$$\begin{aligned}\Gamma &= \Gamma_0(4|2)\\ &= \left\{ \begin{pmatrix} a & b/2 \\ 4c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \;\middle|\; a, b, c, d \in \mathbb{Z} \right\}\end{aligned}$$

$\Gamma_0(4|2)$ is conjugate in $PSL_2(\mathbb{R})$ to $\Gamma_0(2) \subset PSL_2(\mathbb{Z}) = \Gamma_0(1) + 1$.

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The next big polytope . . .