Mirror Symmetry Through Polytopes

Physical and Mathematical Dualities

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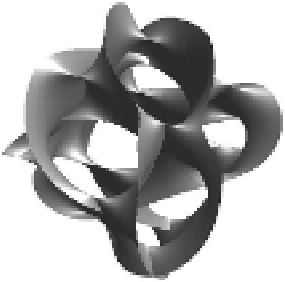


Outline

String Theory and Mirror Symmetry

Some Complex Geometry

Reflexive Polytopes

From Polytopes to Spaces

Where's the Theory of Everything?

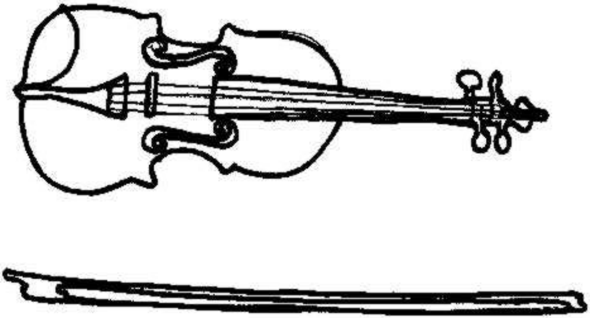
- We understand gravity on a large spatial scale (planets, stars, galaxies).

Figure: S. Bush et al.

- We understand quantum physics on a small spatial scale (electrons, photons, quarks).

Are Strings the Answer?

- "Fundamental" particles are strings vibrating at different frequencies.



- Strings wrap other dimensions!

T-Duality

Pairs of Universes

An extra dimension shaped like a circle of radius R and an extra dimension shaped like a circle of radius α'/R yield indistinguishable physics! (The slope parameter α' has units of length squared.)

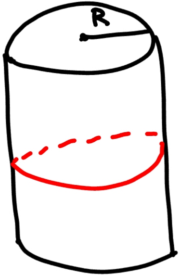


Figure: Large radius, few windings

Figure: Small radius, many windings

Building a Model

Locally, space-time should look like

$$M_{3,1} \times V.$$

- $M_{3,1}$ is four-dimensional space-time
- V is a d-dimensional complex manifold
- Physicists require $d = 3$ (6 real dimensions)
- V is a Calabi-Yau manifold

Mirror Symmetry

Physicists say . . .

- Calabi-Yau manifolds appear in pairs (V, V°).
- The universes described by $M_{3,1} \times V$ and $M_{3,1} \times V^{\circ}$ have the same observable physics.

Mathematicians say . . .

- Calabi-Yau manifolds appear in paired families $(V_{\alpha}, V_{\alpha}^{\circ})$.
- The families V_{α} and V_{α}° have dual geometric properties.

Realizing Mirror Symmetry Geometrically

We need:

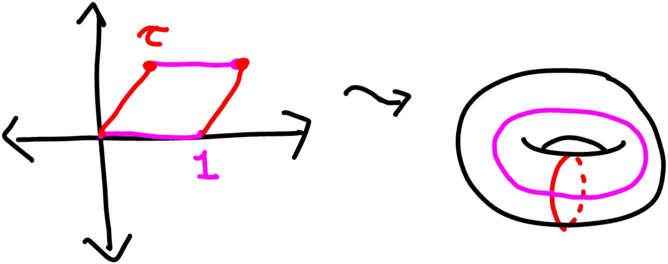
- Complex manifolds
- which are Calabi-Yau
- and arise in paired or “mirror” families
- with dual geometric properties.

Varying complex structure in one family should correspond to varying Kähler structure in the other family.

Complex Structure

An n-dimensional complex manifold is a geometric space which looks locally like $\mathbb{C}^n$.

Example: Elliptic Curves



We can think of varying the parameter τ as either changing the complex manifold, or changing the complex structure on an underlying topological 2-torus.

Kähler Structure

Standard Product

We can pair vectors v and w with their tails at a point z in $\mathbb{C}$ using the product for complex numbers:

$$\langle v, w\rangle = v\overline{w}$$

Note that $\langle v, v\rangle = ||v||^2$.

More generally, if $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ are vectors with their tails at a point $\mathbf{z} = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$, their standard Hermitian product is given by

$$\begin{aligned} \langle \vec{v}, \vec{w}\rangle &= \sum v_i \overline{w_i} \\ &= \vec{v}^{\mathrm{T}} \overline{\vec{w}} \end{aligned}$$

Kähler Structure

Hermitian metrics

A Hermitian metric H tells us how to pair tangent vectors at any point of a complex manifold and obtain a complex number.

$$H(\vec{v}, \vec{w}) = \overline{H(\vec{w}, \vec{v})}$$

Kähler Structure

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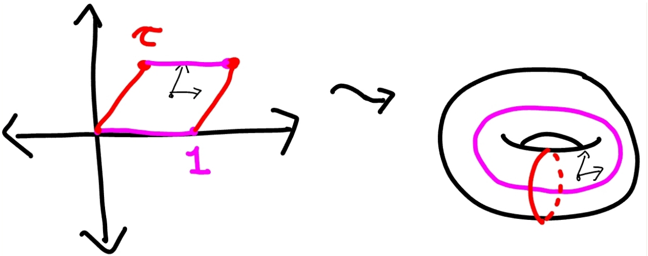
Elliptic Curve Example

We can use the standard Hermitian product on $\mathbb{C}$ to describe a Hermitian metric for tangent vectors to an elliptic curve.



A Kähler metric is a special type of Hermitian metric which can be written in local coordinates as follows. If $\vec{v}$ and $\vec{w}$ are vectors with their tails at a point $\mathbf{z}_0$ in $\mathbb{C}^n$,

$$\kappa(\vec{v}, \vec{w}) = \vec{v}^{\mathrm{T}} \left(\mathrm{I} + \mathrm{G}(\mathbf{z})\right) \overline{\vec{w}}.$$

Here I is the identity matrix and

$$\mathrm{G}(\mathbf{z}) = \begin{pmatrix} g_{11}(\mathbf{z}) & \dots & g_{1n}(\mathbf{z}) \\ \vdots & \ddots & \vdots \\ g_{n1}(\mathbf{z}) & \dots & g_{nn}(\mathbf{z}) \end{pmatrix}$$

vanishes up to order 2 at $\mathbf{z}_0$.

The Geometric Ingredients of Mirror Symmetry

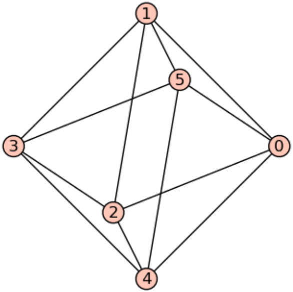
We need:

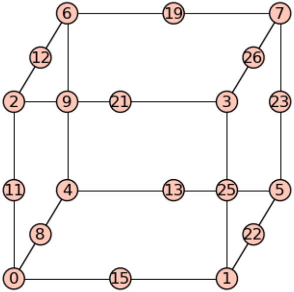
- Complex manifolds V
- which are Calabi-Yau
- and arise in paired or "mirror" families
- with dual geometric properties.

Varying complex structure in one family should correspond to varying Kähler structure in the other family.

Batyrev's Insight

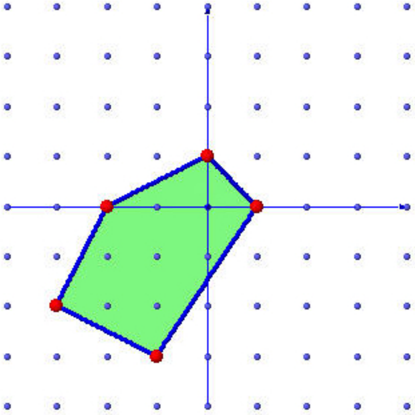
We can describe mirror families of Calabi-Yau manifolds using combinatorial objects called reflexive polytopes.





Lattice Polygons

The points in the plane with integer coordinates form a lattice N. A lattice polygon is a polygon in the plane which has vertices in the lattice.



Reflexive Polygons

We say a lattice polygon is reflexive if it has only one lattice point, the origin, in its interior.

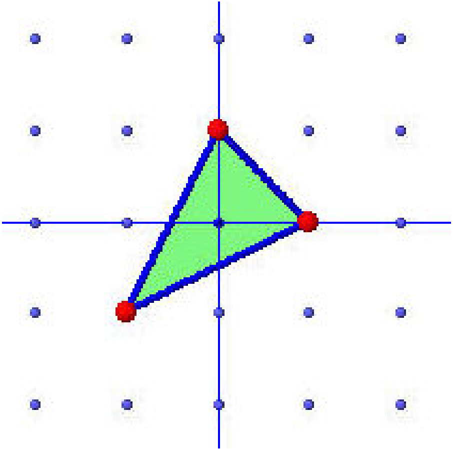
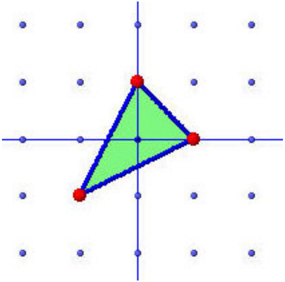


Figure: A reflexive triangle

Describing a Reflexive Polygon



- List the vertices

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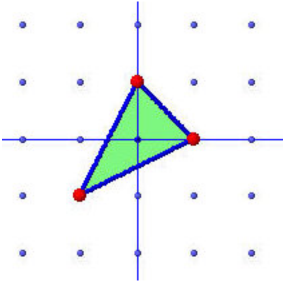

- List the vertices

$$\{(0,1),(1,0),(-1,-1)\}$$

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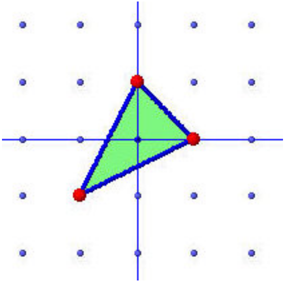

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- List the equations of the edges

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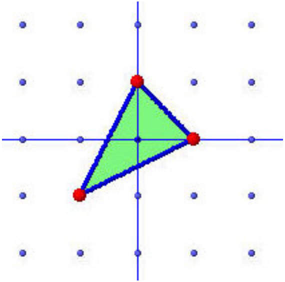

- List the vertices

$$\{(0,1),(1,0),(-1,-1)\}$$

- List the equations of the edges

$$\begin{aligned} -x - y &= -1 \\ 2x - y &= -1 \\ -x + 2y &= -1 \end{aligned}$$

A Dual Lattice

Let M be another copy of the points in the plane with integer coordinates.
The dot product lets us pair points in N with points in M:

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2$$

Polar Polygons

Edge equations define new polygons

Let Δ be a lattice polygon in N which contains $(0,0)$. The polar polygon Δ° is the polygon in M given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

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$$\begin{aligned}(x, y) \cdot (-1, -1) &= -1 \\ (x, y) \cdot (2, -1) &= -1 \\ (x, y) \cdot (-1, 2) &= -1\end{aligned}$$

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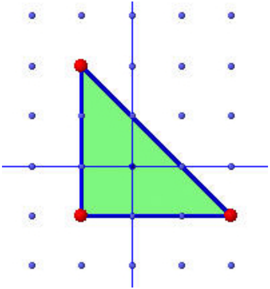


Figure: Our triangle's polar polygon

Mirror Pairs

If Δ is a reflexive polygon, then:

- Δ° is also a reflexive polygon
- $(\Delta^\circ)^\circ = \Delta$.

Δ and Δ° are a mirror pair.

A Polygon Duality

Mirror pair of triangles

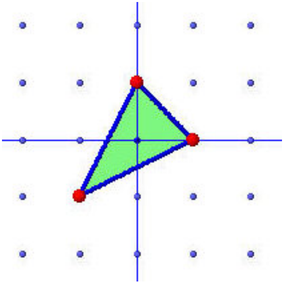


Figure: 3 boundary lattice points

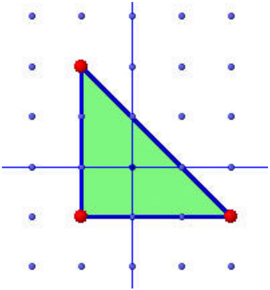


Figure: 9 boundary lattice points

$$3 + 9 = 12$$

Mirror Pairs of Polygons

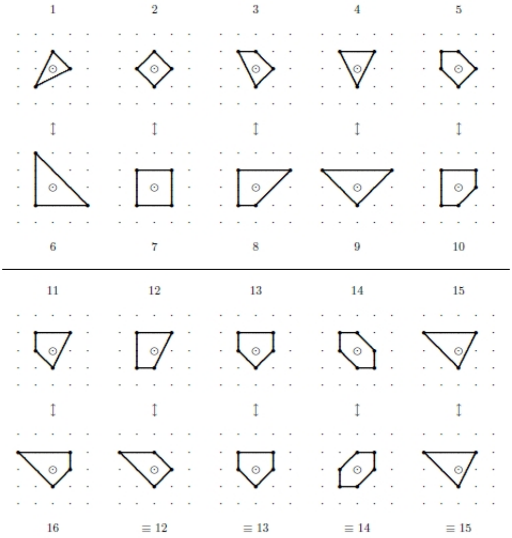
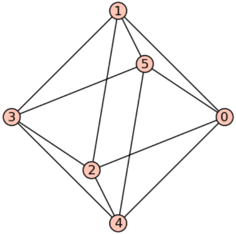


Figure: F. Rohsiepe, "Elliptic Toric K3 Surfaces and Gauge Algebras"

Other Dimensions

Definition

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ be a set of points in $\mathbb{R}^k$. The polytope with vertices $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ is the convex hull of these points.



Polar Polytopes

Let N be the lattice of points with integer coordinates in $\mathbb{R}^k$. A lattice polytope has vertices in N.
As before, we have a dual lattice M and a dot product

$$(n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) = n_1 m_1 + \cdots + n_k m_k$$

Definition

Let Δ be a lattice polygon in N which contains $(0, 0)$. The polar polytope Δ° is the polytope in M given by:

$$\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

Reflexive Polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

- If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- Δ and Δ° are a mirror pair.

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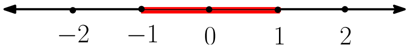

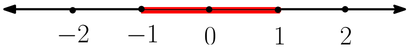

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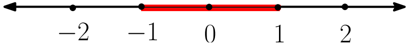

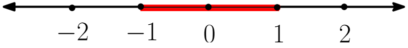

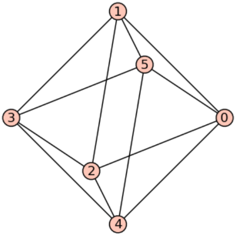

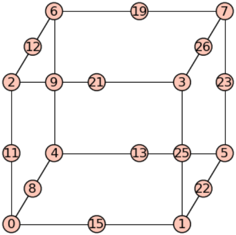

Mirror Polytopes Yield Mirror Spaces

$$\begin{array}{ccc}
\text{polytope} & \longleftrightarrow & \text{polar polytope} \\
\downarrow & & \downarrow \\
\text{Laurent polynomial} & \longleftrightarrow & \text{mirror Laurent polynomial} \\
\downarrow & & \downarrow \\
\text{space} & \longleftrightarrow & \text{mirror space}
\end{array}$$

From Polytopes to Polynomials

- Standard basis vectors in $N \leftrightarrow$ variables z_i

$$\begin{aligned}(1,0,\dots,0) &\leftrightarrow z_1\\ (0,1,\dots,0) &\leftrightarrow z_2\\ &\dots\\ (0,0,\dots,1) &\leftrightarrow z_n\end{aligned}$$

- Lattice points in $\Delta^\circ \leftrightarrow$ monomials defined on $(\mathbb{C}^*)^n$

$$(m_1,\dots,m_k) \leftrightarrow z_1^{(1,0,\dots,0)\cdot(m_1,\dots,m_k)} z_2^{(0,1,\dots,0)\cdot(m_1,\dots,m_k)} \dots z_k^{(0,0,\dots,1)\cdot(m_1,\dots,m_k)}$$

- $\Delta^\circ \leftrightarrow$ Laurent polynomials p_α defined on $(\mathbb{C}^*)^n$

Example

The One-Dimensional Reflexive Polytope

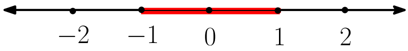

Figure: Δ

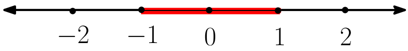

Figure: Δ°

- Standard basis vectors in $N \leftrightarrow$ variables z_i

$$(1) \leftrightarrow z_1$$

Example

The One-Dimensional Reflexive Polytope

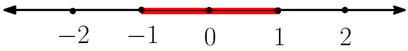

Figure: Δ

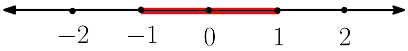

Figure: Δ°

- Standard basis vectors in $N \leftrightarrow$ variables z_i

$$(1) \leftrightarrow z_1$$

- Lattice points in $\Delta^\circ \leftrightarrow$ monomials defined on $(\mathbb{C}^*)^n$

$$(-1) \leftrightarrow z_1^{(1)\cdot(-1)} = z_1^{-1}$$
$$(0) \leftrightarrow z_1^{(1)\cdot(0)} = 1$$
$$(1) \leftrightarrow z_1^{(1)\cdot(1)} = z_1$$

Example

Continued

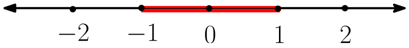

Figure: Δ

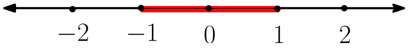

Figure: Δ°

- $\Delta^\circ \leftrightarrow$ Laurent polynomials p_α defined on $(\mathbb{C}^*)^n$

$$\Delta^\circ \leftrightarrow p_\alpha = \alpha_{(-1)} z_1^{-1} + \alpha_{(0)} + \alpha_{(1)} z_1^{1}$$

Each choice of parameters $(\alpha_{(-1)}, \alpha_{(0)}, \alpha_{(1)})$ defines a Laurent polynomial.

From Polynomials to Spaces

The solutions to the Laurent polynomials p_α describe geometric spaces.

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Example: One Dimensional Polytope

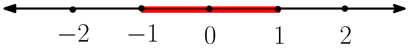


Figure: Δ

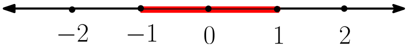
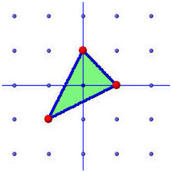


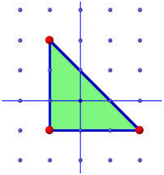
Figure: Δ°

Solutions to $\alpha_{(-1)}z_1^{-1} + \alpha_{(0)} + \alpha_{(1)}z_1^{1} = 0$ define pairs of nonzero points in the complex plane.

- $-z_1^{-1} + z_1 = 0$
  $z_1 = \pm 1$
- $z_1^{-1} + z_1 = 0$
  $z_1 = \pm i$

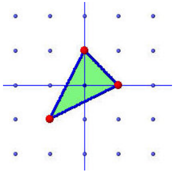
Example: Two-Dimensional Polytopes

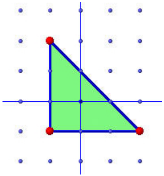




$$\alpha_{(-1,2)} z_1^{-1} z_2^2 + \cdots + \alpha_{(2,-1)} z_1^2 z_2^{-1} = 0$$

Example: Two-Dimensional Polytopes





$$\alpha_{(-1,2)} z_1^{-1} z_2^{2} + \cdots + \alpha_{(2,-1)} z_1^{2} z_2^{-1} = 0$$

Figure: Real part of a curve

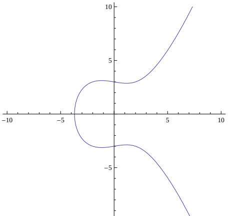
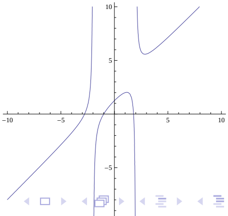


Figure: Another real curve



Example: Four-Dimensional Polytopes

Let Δ be the four-dimensional polytope with vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.

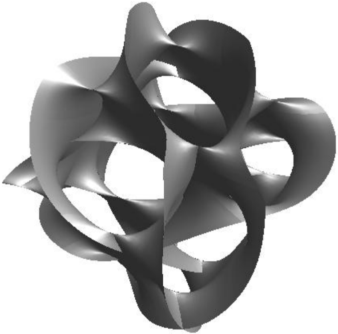


Figure: Slice of a Calabi-Yau threefold

Compactifying

Our Laurent polynomials p_α define spaces which are not compact: $||z_i||$ can be infinitely large. We can solve this problem by adding in some "points at infinity" using a standard procedure from algebraic geometry.

The resulting compact spaces V_α are Calabi-Yau varieties of dimension $d = k - 1$.

- When $k = 2$, for generic choice of α, the V_α are elliptic curves.
- When $k = 4$, for generic choice of α, the V_α are smooth 3-dimensional Calabi-Yau manifolds.

Mirror Symmetry

$$
\begin{array}{ccc}
\text{polytope} & \longleftrightarrow & \text{polar polytope} \\
\downarrow & & \downarrow \\
\text{Laurent polynomials } p_\alpha & \longleftrightarrow & \text{mirror Laurent polynomials } p_\alpha^\circ \\
\downarrow & & \downarrow \\
\text{spaces } V_\alpha & \longleftrightarrow & \text{mirror spaces } V_\alpha^\circ
\end{array}
$$

Counting Complex Moduli

The possible deformations of complex structure of V_α form a complex vector space of dimension $h^{d-1,1}(V_\alpha)$.
For $k \geq 4$,

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

- $\ell() =$ number of lattice points
- $\ell^*() =$ number of lattice points in the relative interior of a polytope or face
- The Γ° are codimension 1 faces of Δ°
- The Θ° are codimension 2 faces of Θ°
- $\hat{\Theta}^\circ$ is the face of Δ dual to Θ°

Counting Kähler Moduli

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

- $\ell()$ = number of lattice points
- $\ell^*()$ = number of lattice points in the relative interior of a polytope or face
- The Γ are codimension 1 faces of Δ
- The Θ are codimension 2 faces of Θ
- $\hat{\Theta}$ is the face of Δ dual to Θ

Comparing V and V°

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

Comparing V and V°

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

$$h^{1,1}(V_\alpha^\circ) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

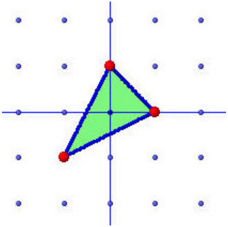
$$h^{d-1,1}(V_\alpha^\circ) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

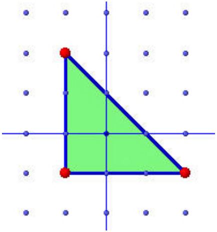
Mirror Symmetry from Mirror Polytopes

We have mirror families of Calabi-Yau varieties V_α and V_α° of dimension $d = k - 1$.

$$h^{1,1}(V_\alpha) = h^{d-1,1}(V_\alpha^\circ)$$
$$h^{d-1,1}(V_\alpha) = h^{1,1}(V_\alpha^\circ)$$

An Example

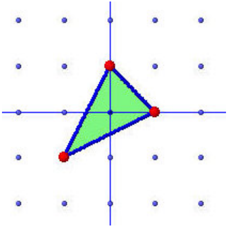


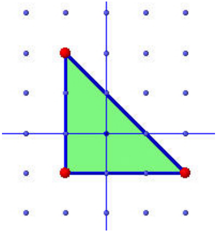


Four-dimensional analogue:

- Δ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- Δ° has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

An Example





Four-dimensional analogue:

- Δ has vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$.
- Δ° has vertices $(-1,-1,-1,-1)$, $(4,-1,-1,-1)$, $(-1,4,-1,-1)$, $(-1,-1,4,-1)$, and $(-1,-1,-1,4)$.

$$\begin{aligned} h^{1,1}(V_\alpha) &= \ell(\Delta) - n - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta}) \\ &= 6 - 4 - 1 - 0 - 0 = 1. \end{aligned}$$

Example (Continued)

- Δ has vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$.
- Δ° has vertices $(-1,-1,-1,-1)$, $(4,-1,-1,-1)$, $(-1,4,-1,-1)$, $(-1,-1,4,-1)$, and $(-1,-1,-1,4)$.

$$h^{1,1}(V_\alpha) = 1$$

$$h^{3-1,1}(V_\alpha) = \ell(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$
$$= 126 - 4 - 1 - 20 - 0 = 101.$$

The Hodge Diamond

Calabi-Yau Threefolds

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(V) & & 0 & \\ 1 & & h^{2,1}(V) & & h^{2,1}(V) & & 1 \\ & 0 & & h^{1,1}(V) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The Hodge Diamond

Calabi-Yau Threefolds

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(V) & & 0 & \\
1 & & h^{2,1}(V) & & h^{2,1}(V) & & 1 \\
 & 0 & & h^{1,1}(V) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

V_α

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

V_α°

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$