Supplement to: "Efficient Compromising"

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This supplement proves Lemma 4 in the main paper. As explained in footnote 9 to the main paper we prove this lemma using an additional assumption. The additional assumption is that the decision rule f is *regular* in the sense of the following definition.

Definition: An incentive compatible decision rule f is called *regular* if $t_1 = 1 \Rightarrow f_A(t_1, t_2) = 0$ and $t_2 = 1 \Rightarrow f_C(t_1, t_2) = 0$.

Definition: Two decision rules f and $\tilde{f}$ are called "interim equivalent" if $U_i(t_i) = \tilde{U}_i(t_i)$ for every $i = 1, 2$ and every type $t_i \in [0, 1]$.

The next lemma provides the precise sense in which it is without loss of generality for us to restrict attention to regular decision rules.

Lemma: For every incentive compatible decision rule f there is a regular incentive compatible decision rule $\tilde{f}$ that is interim equivalent to f.

Proof: Let f be an incentive compatible decision rule. Define $\tilde{f}$ to be the decision rule that satisfies:

(i) If $t_i < 1$ for $i = 1, 2$ then:

$$\tilde{f}(t_1, t_2) = f(t_1, t_2)$$

(ii) If $t_1 = 1$ but $t_2 < 1$ then:

$$\begin{aligned}
\tilde{f}_A(t_1, t_2) &= 0 \\
\tilde{f}_B(t_1, t_2) &= f_A(t_1, t_2) + f_B(t_1, t_2) \\
\tilde{f}_C(t_1, t_2) &= f_C(t_1, t_2)
\end{aligned}$$

(iii) If $t_1 < 1$ but $t_2 = 1$ then:

$$\begin{aligned} \tilde{f}_A(t_1, t_2) &= f_A(t_1, t_2) \\ \tilde{f}_B(t_1, t_2) &= f_B(t_1, t_2) + f_C(t_1, t_2) \\ \tilde{f}_C(t_1, t_2) &= 0 \end{aligned}$$

(iv) If $t_1 = t_2 = 1$ then:

$$f_B(t_1, t_2) = 1$$

It is evident that the decision rule $\tilde{f}$ is interim equivalent to f. It remains to show that $\tilde{f}$ is incentive compatible. Note first that the move from f to $\tilde{f}$ does not affect agent i's incentives if agent i is of type $t_i = 1$. Such an agent's interim utilities under $\tilde{f}$ are the same as they are under f, whether the agent tells the truth or lies. Consider now an agent i with type $t_i < 1$. The agent's incentives to pretend to be some other type $t_i' < 1$ are by construction not affected. Finally, note that such an agent's incentives to pretend to be type $t_i' = 1$ have been negatively affected. Probability that was previously assigned to the agent's most preferred alternative has now been shifted to the compromise, which the agent by assumption values less than the compromise. Thus, the agent will find it optimal to report his type truthfully.

Q.E.D.

We now restate Lemma 4 of the main paper, incorporating the assumption that the decision rule is regular.

Lemma 4: Consider a function $\hat{f}_B : [0,1]^2 \to [0,1]$. For $i = 1, 2$ define the interim expected value of $\hat{f}_B$ to be: $\hat{q}_i(t_i) \equiv \int_0^1 \hat{f}_B(t_i, t_j) g(t_j) dt_j$ for all $t_i \in [0,1]$, where $j \neq i$. If there is a regular incentive compatible decision rule $f = (f_A, f_B, f_C)$ such that $f_B = \hat{f}_B$, then for $i = 1, 2$:

(i) $\hat{q}_i(t_i)$ is monotonically increasing in t_i;

(ii)

$$\int_0^1 \int_0^1 \hat{f}_B(t_1,t_2)\left(t_1 + \frac{G(t_1)}{g(t_1)} + t_2 + \frac{G(t_2)}{g(t_2)} - 1\right) g(t_1)g(t_2)dt_1dt_2$$

$$= \quad \hat{q}_1(1) + \hat{q}_2(1) - 1$$

Proof: The necessity of (i) was already shown in Lemma 1 of the main paper. To see why condition (ii) is necessary, suppose f is a regular incentive compatible decision rule as described in the Lemma 4, and note first that the fact that probabilities add up to one implies:

$$\int_0^1 p_1(t_1)g(t_1)dt_1 + \int_0^1 p_2(t_2)g(t_2)dt_2 + \int_0^1 \int_0^1 f_B(t_1,t_2)g(t_1)g(t_2)dt_1dt_2 = 1.$$

Next we observe that condition (ii) in Lemma 3 of the main paper implies for regular decision rules:

$$p_i(t_i) = q_i(1) - q_i(t_i)t_i - \int_{t_i}^1 q_i(s_i)ds_i.$$

Using this formula we can calculate the expected value of $p_i(t_i)$:

$$\begin{aligned}
&\int_0^1 p_i(t_i)g(t_i)dt_i \\
=\ &\int_0^1 \left(q_i(1) - q_i(t_i)t_i - \int_{t_i}^1 q_i(s_i)ds_i\right) g(t_i)dt_i \\
=\ &q_i(1) - \int_0^1 q_i(t_i)t_i g(t_i)dt_i - \int_0^1 \int_{t_i}^1 q_i(s_i)ds_i g(t_i)dt_i \\
=\ &q_i(1) - \int_0^1 q_i(t_i)t_i g(t_i)dt_i - \int_0^1 q_i(t_i)G(t_i)dt_i \\
=\ &q_i(1) - \int_0^1 q_i(t_i)\left(t_i + \frac{G(t_i)}{g(t_i)}\right) g(t_i)dt_i.
\end{aligned}$$

Now we can substitute these expressions into the equation with which we started:

$$\begin{aligned} q_1(1) - \int_0^1 q_1(t_1)\left(t_1 + \frac{G(t_1)}{g(t_1)}\right) g(t_1)dt_1 \\ q_2(1) - \int_0^1 q_2(t_2)\left(t_2 + \frac{G(t_2)}{g(t_2)}\right) g(t_2)dt_2 \\ + \int_0^1 \int_0^1 f_B(t_1, t_2) g(t_1) g(t_2) dt_1 dt_2 &= 1. \end{aligned}$$

Re-arranging terms, and recalling that $f_B = \hat{f}_B$ and that $q_i(t_i) = \hat{q}_i(t_i)$ yields the assertion.

Q.E.D.