Robustness of Bertrand's Equilibrium in a General Model of Product-Differentiation *)

by

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Abstract

We introduce a model of oligopolistic product-differentiation which allows for somewhat greater generality than more common models do. We then address two questions. The first concerns the continuity properties of Nash-equilibria in prices with respect to exogeneously given product-specifications. In particular we investigate in how far the Bertrand-equilibrium is robust under perturbations of the firms' product-specifications. The second question concerns the existence of subgame-perfect equilibria of two stage games in which the firms first choose product-specifications and then prices.

1.Introduction

This paper tries to make three contributions to the theory of product-differentiation. The first is that we propose a model of product-differentiation which seems to allow for somewhat greater generality than more common models do.

We then assume that there is a given set of firms, and that these firms play a two stage game. In the first stage each firm chooses the product-variant which it wants to produce. In the second stage each firm chooses a price.

Within this model we then address two problems which appear to us to be important for the theory of product-differentiation and which do not seem to have been treated systematically before.

The first problem concerns the Nash-equilibria of the second stage subgames, i.e. the price-equilibria. We ask in how far these equilibria will change continuously as the first stage outcome changes. In particular we shall ask whether the Bertrand-equilibrium which results if several firms produce totally identical products is robust under perturbations of the firms' product types.

The second question concerns the existence of subgame-perfect equilibria [1)] of the whole two stage game.

We now explain these points in some more detail.

We first turn to the model. The model is constructed in analogy to well-known examples such as Hotelling-type models (e.g. Osborne and Pitchik (1987)) or the model of vertical product-differentiation discussed in Shaked and Sutton (1983) but we have tried to achieve somewhat greater generality.

We take the space of all product-variants to be some abstract metric space. We then introduce a demand- and a cost-fucnction. The demand-function is assumed to be continuous, and to satisfy some further basic requirements, but no specific functional form is imposed. As regards costs we assume that all product-variants can be produced at constant marginal costs whereby marginal costs are allowed to vary continuously with product-variants.

As was explained above our first question about the model then concerns Nash-equilibria in prices whereby the firms' product-variants are taken as exogeneously given. Our model will be such that pure strategy equilibria in prices don't necessarily exist. Instead we shall work with mixed strategy equilibria the existence of which can easily be proved.[2)]

Our main interest is then in showing that the correspondence mapping product-variants into price-equilibria is upper hemicontinuous if the space of mixed strategies is endowed with the topology of weak convergence.

Upper hemicontinuity is unproblematic as long as the firms' profit-functions are continuous. This however will not always be the case. If several firms produce identical products their profit-functions will exhibit the familiar Bertrand-discontinuities. In such points the upper hemicontinuity of the correspondence of price-equilibria is nontrivial.

Now if several firms produce identical products then the Bertrand-argument implies that at least two of these firms choose their price equal to marginal costs with probability one. To prove upper hemicontinuity in such points one has to show the robustness of this result under small perturbations of the firms' product-specifications, i.e. one has to prove that if several firms produce very similar but not identical products at least two of them will choose their price close to marginal costs with a probability of almost one.

We think that the robustness of the Bertrand-equilibrium is an economically interesting problem because the Bertrand-equilibrium result is the simplest result providing non-cooperative foundations for the notion of perfect competition. However the underlying assumption that the firms' product-variants are totally identical seems rather extreme. Hence it is useful to know that the result is robust under perturbations of the product-variants.

Mathematically the problem of the robustness of the Bertrand-equilibrium is related to a problem which Dasgupta and Maskin (1986) and Simon (1987) studied while proving the existence of mixed strategy equilibria in games with discontinuous payoffs. Our mathematical arguments will be extensions of ideas from that literature.

After our analysis of price-equilibria for exogeneously given product-variants we shall also consider the full two stage game in which firms choose in a first stage which product-variants they wish to produce and in the second stage they choose prices. Our interest is in the existence of subgame-perfect equilibria in such a game.

Since the existence of price-equilibria is unproblematic we focus on the first stage game assuming that in this game the firms anticipate how the second stage equilibria depend on the first stage outcome. The question is then whether the continuity-property obtained in the first part of our analysis suffices to guarantee the existence of an equilibrium of this game.

The answer depends on the precise specification of the first stage game. If the decisions in this game are made simultaneously, then we need further restrictive assumptions, in particular, that the second stage equilibrium profits are unique. Moreover we obtain only the existence of a mixed strategy equilibrium. If on the other hand the decisions in the first stage are made sequentially, as e.g. in Prescott and Visscher (1977), then we obtain the existence of a pure strategy equilibrium of the first stage game without having to assume uniqueness of the second stage equilibrium profits.

It might appear obvious that in the sequential move case the first stage game has a subgame-perfect equilibrium in pure strategies since it is then a game of perfect information. It can however be seen from papers such as Harris (1985) that this is a wrong impression. As soon as players' choice sets are infinite the existence of subgame-perfect equilibria in pure strategies is nontrivial even in games of perfect information.Our existence proof will in fact be an application of Harris' results.

The paper is organised as follows: In section 2 we present our framework. Section 3 studies equilibria in prices. In section 4 the full two stage game with endogeneous choice of product-variants is considered. Section 5 concludes. Proofs are in an appendix.

2.The Model

In this section we introduce the three basic elements of our model: the set of all product-variants, the cost-function, and the demand-function.

We begin with the set of all product-variants. We denote this set by V. We assume that V is nonempty, and that it is endowed with some metric. Intuitively we shall think of this metric as representing some notion of similarity of product-variants. Hence if the distance between two product-variants is very small then we shall think of these products as being very similar to each other.

We turn next to the cost-function. We assume that all product-variants in V can be produced at constant marginal costs. However marginal costs may be different for different variants. The function $c : V \longrightarrow \mathbb{R}_+$ assigns to each product-variant v in V the marginal costs $c(v)$ of producing v. We assume:

(A.1) c is continuous.

Hence if two product-variants are rather similar to each other the costs of producing them don't differ too much either.

The third element of our model is the demand-function. It describes the consumers' buying behaviour for every given set of offers made by the firms.

To give a formal definition of the demand-function we hence first have to define what we understand by an "offer". An offer ω is an element (v, p) of $V \times \mathbb{R}_+$ where v is the product-variant which is being offered, and p is the price at which it is offered.

Offers will be made by firms. In our model there will be $F (\in \mathbb{N})$ firms. So there will be at most F offers. However there may be less because different firms may make the same offer. Therefore we define a "set of offers" to be a subset Ω of $V \times \mathbb{R}_+$ with at least one and at most F elements. By $\hat{\Omega}$ we shall denote the set of all such sets. $\hat{\Omega}$ is thus a subset of the power set of $V \times \mathbb{R}_+$.

The demand-function d will now describe for each set Ω of offers which quantities of the various offers in Ω are bought by consumers. The domain of d will thus be $\hat{\Omega}$. Now for technical purposes (especially the formulation of assumption (A.2) below) it will be convenient to take as the range of d the set of all finite Borel-measures [3] on $V \times \mathbb{R}_+$ which we denote by $M(V \times \mathbb{R}_+)$. Thus d is of the form $d : \hat{\Omega} \longrightarrow M(V \times \mathbb{R}_+)$. We assume also that for each Ω in $\hat{\Omega}$ the support of $d(\Omega)$ is contained in Ω.

The interpretation of d is now as follows: Let $\Omega \in \hat{\Omega}$ and $\omega \in \Omega$. Then the quantity of the offer ω which consumers want to buy if they can choose from Ω is just what the measure $d(\Omega)$ assigns to $\{\omega\}$, i.e. $d(\Omega)(\{\omega\})$. In the following we shall use a simplified notation and write $d(\omega, \Omega)$ for this quantity.

Our first and most important assumption for d is now a continuity assumption. To be able to formulate this assumption we endow the domain $\hat{\Omega}$ of d with the Hausdorff-metric, and we endow the range $M(V \times \mathbb{R}_+)$ of d with the topology of weak convergence [4]. Our assumption is then:

(A.2) d is continuous.

To illustrate the meaning of (A.2) we consider two special cases. The first is that the number F of firms equals one. Then $\hat{\Omega}$ can be identified with $V \times \mathbb{R}_+$, and (A.2) just means that the demand for an offer ω is a continuous function of ω.

Suppose next that $F = 2$. Then every element Ω of $\hat{\Omega}$ is a set of the form $\{\omega_1, \omega_2\}$ (identity of ω_1, ω_2 not excluded). (A.2) then requires that whenever a sequence $(\Omega^n)_{n \in \mathbb{N}} = (\{\omega_1^n, \omega_2^n\})_{n \in \mathbb{N}}$ converges in Hausdorff-distance to $\Omega^o = \{\omega_1^o, \omega_2^o\}$, then $(d(\Omega^n))_{n \in \mathbb{N}}$ converges weakly to $d(\Omega^o)$.

Now suppose first that $\omega_1^o = \omega_2^o$. Then $(\Omega^n)_{n \in \mathbb{N}}$ converges to Ω^o if and only if both $(\omega_1^n)_{n \in \mathbb{N}}$ and $(\omega_2^n)_{n \in \mathbb{N}}$ converge to $\omega_1^o (= \omega_2^o)$. Moreover $(d(\Omega^n))_{n \in \mathbb{N}}$ converges weakly to $d(\Omega^o)$ if and only if the sum of the demand for all offers in Ω^n, i.e. $(\sum_{\omega \in \Omega^n} d(\omega, \Omega^n))_{n \in \mathbb{N}}$ converges to the demand for the offer in Ω^o, i.e. to $d(\omega_1^o, \Omega^o)$ $(= d(\omega_2^o, \Omega^o))$.

Suppose next that $\omega_1^o \neq \omega_2^o$. In this case $(\Omega^n)_{n \in \mathbb{N}}$ converges to Ω^o if and only if for all $n \in \mathbb{N}$ ω_1^n and ω_2^n can be defined such that $(\omega_1^n)_{n \in \mathbb{N}}$ converges to ω_1^o and $(\omega_2^n)_{n \in \mathbb{N}}$

converges to ω_2^o. Moreover $(d(\Omega^n))_{n\in\mathbb{N}}$ converges to $d(\Omega^o)$ if and only if $(d(\omega_1^n, \Omega^n))_{n\in\mathbb{N}}$ converges to $d(\omega_1^o, \Omega^o)$ and $(d(\omega_2^n, \Omega^n))_{n\in\mathbb{N}}$ converges to $d(\omega_2^o, \Omega^o)$.

In all cases the continuity requirement (A.2) seems to be a natural assumption provided that the metric on V represents the similarity of product-variants correctly.

Clearly the continuity-assumptions (A.1) and (A.2) will be the crucial assumptions for our robustness result in the next section. However we shall need four additional assumptions for the demand-function d.

The first assumption is a simple implication of consumers' rationality. It says that the consumers' behaviour doesn't change if an offer the demand for which is zero is replaced by an offer involving the same product-variant as the original offer but at a higher price.

(A.3) Let $\Omega \in \hat{\Omega}$, $(v,p) \in \Omega$, $d((v,p),\Omega) = 0$ and $\tilde{p} > p$.

Then $d((\Omega \setminus \{(v,p)\}) \cup \{(v,\tilde{p})\}) = d(\Omega)$.

Observe that (A.3) means of course also that the demand for the new offer is zero: $d((v,\tilde{p}),(\Omega \setminus \{(v,p)\}) \cup \{(v,\tilde{p})\}) = d((v,\tilde{p}),\Omega) = 0$.

The next assumption is in the same spirit as (A.3) and is as (A.3) an implication of consumers' rationality. It says that the consumers' behaviour doesn't change if an offer the demand for which is zero is taken out of the market.

(A.4) Let $\Omega \in \hat{\Omega}$, $\omega \in \Omega$, and $d(\omega,\Omega) = 0$.

Then $d(\Omega) = d(\Omega \setminus \{\omega\})$ provided that $\Omega \setminus \{\omega\} \neq \emptyset$.

Next we assume that whenever one product-variant is offered at different prices then consumers won't buy this product-variant at any price above the lowest available price.

(A.5) Let $\Omega \in \hat{\Omega}$, $(v,p) \in \Omega$, $(\tilde{v},\tilde{p}) \in \Omega$, and suppose that $v = \tilde{v}$ while $p > \tilde{p}$.

Then $d((v,p),\Omega) = 0$.

Our final assumption is somewhat more problematic than the preceding three assumptions. We assume that there is some maximum price $\bar{p} > 0$ such that an offer can attract strictly positive demand only if it involves a price below $\bar{p}$.

(A.6) There is some $\bar{p} > 0$ such that for all $\Omega \in \hat{\Omega}$ and $(v,p) \in \Omega$

$$p \geq \bar{p} \text{ implies } d((v,p),\Omega) = 0.$$

The existence of some kind of upper boundary for the consumers' readiness to buy is certainly required for the results in this paper. (A.6) is an admittedly crude way of introducing such a boundary. It appears to be possible that less crude ways of introducing such a boundary exist. But we have not pursued this point.

3.Equilibria in Prices

In this section we introduce a finite set of firms and then define for any given choice of product-variants by these firms a game which models price competition among the firms. Our interest is then in proving the upper hemicontinuity of the correspondence mapping product-variants into Nash-equilibria in prices. In this context we prove in particular the robustness of the Bertrand-equilibrium.

a) The Game

We assume that there is a finite set $F = \{1, 2, ..., F\}$ $(F \geq 2)$ of firms. Each firm $i \in F$ produces some product-variant $v_i \in V$. We write $\underline{v}$ for the F-tuple $(v_1, ..., v_F)$. In this section $\underline{v}$ will be treated as exogeneous.

For any given F-tuple $\underline{v} \in V^F$ we introduce a game played by the firms in F. We shall refer to this game as "the game corresponding to $\underline{v}$". It is as follows: Every firm $i \in F$ chooses a price $p_i \in I\!R_+$. Choices are made simultaneously. We write $\underline{p}$ for the F-tuple $(p_1, ..., p_F)$. Firms seek to maximise their profits. For every $i \in F$ and $\underline{p} \in I\!R_+^F$ firm i's profit is:

$$Pr_i(\underline{v}, \underline{p}) = (p_i - c(v_i))S_i(\underline{v}, \underline{p})$$

where $S_i(\underline{v}, \underline{p})$ denotes firm i's sales. They are defined by:

$$S_i(\underline{v}, \underline{p}) = \frac{d((v_i, p_i), \{(v_j, p_j) \mid j \in F\})}{\#\{j \in F \mid (v_j, p_j) = (v_i, p_i)\}}$$

Hence $S_i(\underline{v}, \underline{p})$ is the demand for firm i's offer (v_i, p_i) divided by the number of firms making this offer.

Observe now that if in the above game a firm charges a price above $\bar{p}$ it might as well charge $\bar{p}$ itself since by (A.3) both this firm's own profit (which of course equals zero) and all other firms' profits remain unchanged. We shall therefore in the following take the firms' choice sets to be $[0; \bar{p}]$ rather than $I\!R_+$. Oviously this does not affect the substance of the game.

This completes the definition of the game corresponding to $\underline{v}$.

b) Mixed Strategy Equilibria

Since we have not made any concavity assumptions the games defined above need not have pure strategy equilibria. We shall study instead mixed strategy equilibria. The existence of such equilibria is easily shown (see below).

Let any fixed $\underline{v} \in V^F$ be given. In the game corresponding to $\underline{v}$ a mixed strategy for a firm $i \in F$ will be a Borel probability measure μ_i on $[0;\bar{p}]$. We denote the set of all such measures by M. We write $\underline{\mu}$ for the F-tuple $(\mu_1, ..., \mu_F)$.

In the following we shall identify (also notationally) a pure strategy $p \in [0;\bar{p}]$ with the probability measure $\mu \in M$ the mass of which is concentrated on p, i.e. that satisfies $\mu(\{p\}) = 1$.

Given $\underline{\mu} \in M^F$ and $i \in F$ the expected profit of firm i is:

$$Pr_i(\underline{v}, \underline{\mu}) = \int ... \int Pr_i(\underline{v}, \underline{p}) d\underline{\mu}$$

$\underline{\hat{\mu}} \in M^F$ is called a "Nash-equilibrium of the game corresponding to $\underline{v}$" iff

$$Pr_i(\underline{v}, \underline{\hat{\mu}}) \geq Pr_i(\underline{v}, (\mu_i, \hat{\mu}_{-i}))$$

for all $i \in F$ and $\mu_i \in M$. Here $(\mu_i, \hat{\mu}_{-i})$ denotes that element of M^F that one obtains if one replaces the i-th component of $\underline{\hat{\mu}}$ by μ_i.

A first result is then:

Proposition 1: For every $\underline{v} \in V^F$ at least one Nash-equilibrium of the game corresponding to $\underline{v}$ exists.

Proposition 1 is a simple consequence of the fixed point theorem due to Ky Fan and Glicksberg (Glicksberg (1952)). It is proved in the appendix.

We can now define a correspondence E which maps every F-tuple $\underline{v}$ of product-variants into the set $E(\underline{v})$ of all Nash-equilibria of the game corresponding to $\underline{v}$. We also define a correspondence Pr^E which maps every F-tuple $\underline{v}$ into the set $Pr^E(\underline{v})$ of all equilibrium profit combinations, i.e. $Pr^E(\underline{v}) = \{(Pr_1(\underline{v},\underline{\hat{\mu}}),...,Pr_F(\underline{v},\underline{\hat{\mu}})) \mid \underline{\hat{\mu}} \in E(\underline{v})\}$. In the remainder of this section our interest will be in the upper hemicontinuity of these two correspondences.

c) The Bertrand-Equilibrium and its Robustness

The upper hemicontinuity of the correspondences E and Pr^E is problematic only in points in which some firms' profit-functions are discontinuous. Given our continuity assumptions (A.1) and (A.2) this can only be the case if several firms produce identical product-variants. In this subsection we therefore concentrate on such situations.

As a first step we consider a special but representative case, namely the case in which all firms produce the same product-variant. We shall return to the more general case at the end of this subsection.

Now if all firms produce the same product-variant then our price competition game is just a standard Bertrand game with a continuous demand-function which cuts the price axis at a unique point[5]. (Note that for this statement to be true we need all assumptions (A.2)-(A.6).)

We shall now again restrict attention to a special case, namely to that in which demand at marginal costs is strictly positive. This is the case to which the usual Bertrand-result applies. Again we shall come back to the more general case at the end of this subsection.

We now begin by stating formally the Bertrand-result for mixed strategies:

Proposition 2: Suppose that $\underline{v}^o \in V^F$ satisfies $v_1^o = ... = v_F^o = v^o \in V$.

Suppose also that $d((v^o, c(v^o)), \{(v^o, c(v^o))\}) > 0$.

Then $\underline{\hat{\mu}} \in M^F$ is a Nash-equilibrium of the game corresponding to $\underline{v}^o$ if and only if

(i) $\hat{\mu}_i = c(v^o)$ for at least two $i \in F$, and

(ii) $\text{supp}(\hat{\mu}_i) \subseteq [c(v^o); \bar{p}]$ for all $i \in F$.

This mixed strategy version of the Bertrand-equilibrium result has been proved by Harrington (1988, theorem 1).

We add to proposition 2 an (obvious) corollary which deals with profits:

Corollary 1: Let $\underline{v}^o$ be as in proposition 2, and let $\underline{\hat{\mu}}$ be a Nash-equilibrium of the game corresponding to $\underline{v}^o$. Then $Pr_i(\underline{v}^o, \underline{\hat{\mu}}) = 0$ for all $i \in F$.

To prove the upper hemicontinuity of the correspondence E in points $\underline{v}^o$ of the type described above now means to show that the Bertrand-equilibrium result is robust against perturbations of the firms' product-variants.

For this we endow the space of all mixed strategies with the topology of weak convergence. Given this topology robustness of the Bertrand-equilibrium result means the following: Suppose that the combination $\underline{v}$ of the firms' product-variants is close to a point $\underline{v}^o$ of the type described above, but that not all firms' product-variants are totally identical. Then in any Nash-equilibrium $\underline{\hat{\mu}}$ of the game corresponding to $\underline{v}$ at least two firms must assign a probability of almost one to prices close to marginal costs, and all firms must assign a probability of almost one to prices not below marginal costs.

Formally it is convenient to state this result in terms of sequences:

Proposition 3: Let $\underline{v}^o$ be as in proposition 2.

For every $n \in I\!N$ let $\underline{v}^n \in V^F$. Suppose $\underline{v}^n \longrightarrow \underline{v}^o$.

For every $n \in I\!N$ let $\underline{\hat{\mu}}^n \in M^F$ be a Nash-equilibrium of the game corresponding to $\underline{v}^n$. Suppose $\underline{\hat{\mu}}^n \longrightarrow \underline{\hat{\mu}}^o$.

Then $\underline{\hat{\mu}}^o$ is a Nash-equilibrium of the game corresponding to $\underline{v}^o$.

Proposition 3 is nontrivial because in the Bertrand-game the firms' profit-functions are discontinuous. Mathematically the problem with which proposition 3 deals is in fact quite similar to a problem which Dasgupta and Maskin (1986) and Simon (1987) studied while proving the existence of mixed strategy equilibria in games with discontinuous payoffs.

Their approach was to look at increasingly fine discretizations of the given game. For these discretizations mixed strategy equilibria were known to exist. The question then became under which conditions the limit of mixed strategy equilibria of the discretizations would be an equilibreum of the original game.

At a more abstract level one is hence seeking conditions under which the limit of mixed strategy Nash-equilibria of games which in some sense approximate a game with discontinuous payoffs will be a Nash-equilibrium of the discontinuous game itself.

The problem with which proposition 3 deals is clearly of the same type. Since moreover the Bertrand-game satisfies the assumptions in Dasgupta and Maskin (1986) and Simon (1987) it is not surprising that our proof of proposition 3 is an adaptation of arguments from that literature.

The proof is contained in the appendix.

To conclude our treatment of the Bertrand-situation we add to proposition 3 a straightforward corollary that deals with equilibrium profits:

Corollary 2: Under the assumptions of proposition 3: $Pr_i(\underline{v}, \underline{\hat{\mu}}^n) \longrightarrow 0$ for all $i \in F$.

Corollary 2 is proved in the appendix.

We have now concluded our treatment of the Bertrand-situation. As was explained above the Bertrand-situation is just one special case of those situations in which firms' profit-functions may be discontinuous. We shall now explain briefly how the above results have to be modified in more general cases.

So suppose first that not all but some firms produce the same product-variant v. Assume that all other firms play any arbitrary mixed strategies. We can then calculate some "expected demand-function" for the product-variant v. This function has the same properties as the deterministic demand-function in the Bertrand-game discussed above.

If now moreover expected demand at marginal costs is strictly positive, then we obtain in analogy to proposition 2 that the firms producing v don't have an incentive to deviate if and only if at least two of them set their price equal to marginal costs with probability one, and all of these firms choose a price not below marginal costs with probability one. Of course one also obtains an analog of corollary 1.

Moreover one can then show that this result is robust against perturbations of the product-variants produced by all firms and of the strategies chosen by the firms not producing v in the limit. The robustness result can be formulated in the same way in which proposition 3 was formulated above. Moreover the proof of the result is virtually identical with the proof of proposition 3. Of course one also obtains the analog of corollary 2.

It remains to discuss what happens if demand resp. expected demand at marginal costs is zero. In this case it is trivial to show that all firms producing the same product-variant have to choose with probability one a price at which demand resp. expected demand equals zero. Hence they will all make zero profits. Also these results are robust. This can again be shown using the same arguments which prove proposition 3 and corollary 2.

d) The Equilibrium Correspondences

We now return to the equilibrium correspondences E and Pr^E. Using the results of the preceding subsection it is now straightforward to prove that:

Proposition 4: E and Pr^E are closed-valued and upper hemicontinuous on V^F.

Proposition 4 is nontrivial only in points in which several firms produce identical products. For these points however proposition 4 was proved in the preceding subsection.[6)] Therefore the proof of proposition 4 is omitted.

4.Equilibria in Product-Variants

We now turn to the two stage game in which in the first stage the firms choose the product-variants which they want to produce and in the second stage the firms choose prices. Our interest is in proving the existence of a subgame-perfect equilibrium in such a game.

Now the correspondence E which was defined in the previous section describes for every outcome of the first stage the set of all mixed strategy Nash-equilibria of the ensuing second stage subgame, and the correspondence Pr^E describes for every outcome of the first stage the set of all equilibrium profits of the ensuing second stage subgame.

From the definition of subgame-perfect equilibria (Selten (1975)) it is hence clear that a subgame-perfect equilibrium of the two stage game exists if and only if one can find a selection of the correspondence Pr^E such that the first stage game has a subgame-perfect equilibrium provided that the payoffs in this game are those described by the given selection.

The results of the previous section enable us to derive two existence results. They relate to two different specifications of the first stage game. The first specification is that the firms make their choices in the first stage simultaneously. The second specification is that they make these choices sequentially, i.e. firm 1 comes first, then firm 2 follows, etc. until finally firm F makes its choice.

So consider first the case of simultaneous decisions. As there are no concavity properties in our model there cannot be any result on the existence of pure strategy equilibria in this case. Instead we turn to mixed strategy equilibria.

Suppose that Pr^E is singleton-valued, i.e. that the second stage equilibrium profits are uniquely determined. By proposition 4 these profits are then a continuous function of the decisions in the first stage. If we then assume in addition that V is compact then we can apply the fixed point theorem due to Ky Fan and Glicksberg (Glicksberg (1952)) to the first stage game and we obtain that the first stage game has a mixed strategy equilibrium. Therefore the following proposition holds:

Proposition 5: If V is compact and if Pr^E is singleton-valued then the game with simultaneous decisions in the first stage has at least one subgame-perfect equilibrium (which may involve mixed strategies in both stages).

We observe that the condition that V is compact is not without economic substance. It implies that if the number F of firms becomes large at least some firms in F must produce product-variants which are very similar to each other.

The second condition in proposition 5 however appears to be more problematic. But it is not clear whether it can be relaxed. If Pr^E is not singleton-valued, then we only know from proposition 4 that it is closed-valued and upper hemicontinuous. But then it might not have a continuous selection, in which case the Ky Fan-Glicksberg theorem cannot be applied.

We now turn to the case of sequential decisions in the first stage. For special examples of product-differentiation this version of the first stage game has previously been considered e.g. by Prescott and Visscher (1977), Lane (1980) and Neven (1984). Economic motivations for this specification of the first stage game are given in these references.

The main interest of the authors quoted above was in the numerical investigation of subgame-perfect equilibria in their models. None of them however gave an analytical proof of the existence of such equilibria. The following result which can be proved using an existence theorem due to Harris (1985) closes this gap:

Proposition 6: If V is compact then the game with sequential decisions in the first stage has at least one subgame-perfect equilibrium (which may involve mixed strategies in the second stage but which involves pure strategies only in the first stage).

Observe that in contrast to proposition 5 proposition 6 does not require Pr^E to be singleton-valued. Moreover in proposition 6 we obtain the existence of a pure strategy equilibrium of the first stage game whereas in proposition 5 we obtained only the existence of a mixed strategy equilibrium of this game.[7)]

5.Conclusion

In this section we want to discuss one assumption which although important for our results is only implicit in our model and has not yet been emphasised explicitly. It is the assumption that market demand is described by a continuous function rather than an upper hemicontinuous correspondence.[8)]

In more special models of product-differentiation one frequently finds that for some constellations of offers the individual consumers may be indifferent between several options and that hence demand behaviour at the individual level must be described by an upper hemicontinuous correspondence rather than a continuous function.

Now in some models this problem disappears in the process of aggregation and at the market level demand is a continuous function. In such models normally also the other assumptions made in this paper are satisfied, and hence the results of this paper apply. An example of such a model is the model in Shaked and Sutton (1983).

In other models however also market demand is an upper hemicontinuous correspondence, not a function. In such models in order to be able to study the firms' choice of product-variants and prices one has to choose a selection of the demand-correspondence.

If this selection is continuous then one is back in the framework of this paper. However it need not be. Indeed sometimes no continuous selection exists. An example of such a case is Hotelling's model (Osborne and Pitchik (1987)).

A natural question is hence in how far the results of this paper can be extended to the case in which demand is described by an upper hemicontinuous correspondence rather than by a function.

It is quite easy to see that such an extension is not possible without further assumptions. With discontinuous demand-behaviour even the Bertrand-equilibrium result need not hold. The problem is that the Bertrand-type of undercutting need not be profitable at points at which demand is discontinuous. At such points undercutting although increasing a firm's share in market demand discontinuously may at the same time discontinuously reduce the size of total demand.

On the other hand Osborne and Pitchik's (1987) results show that in Hotelling's model at least for some selection of the demand-correspondence the above problem does not occur and the Bertrand-equilibrium result does hold (their proposition 1) and is also robust under perturbations of the firms' locations (their proposition 2). One would need to investigate whether a more general principle is here at work.

Appendix

Proof of Proposition 1

Let $\underline{v} \in V^F$. Consider the following restricted version of the game corresponding to $\underline{v}$: Firms $i \in F$ with $v_i \neq v_j$ for all $j \in F, j \neq i$, still choose their price from $[0;\bar{p}]$. But firms $i \in F$ with $v_i = v_j$ for some $j \in F, j \neq i$ have just a single option: $\min\{c(v_i), \bar{p}\}$. Profits are unchanged.

The restricted game then satisfies the assumptions of the Ky Fan-Glicksberg theorem (Glicksberg (1952)) and therefore it has a Nash-equilibrium in mixed strategies. Furthermore every Nash-equilibrium of the restricted game is also a Nash-equilibrium of the unrestricted game. Therefore also this game has a Nash-equilibrium.

Proof of Proposition 3

The basic idea for our proof is taken from Simon (1987). It is as follows: Define for every $i \in F, \underline{v} \in V^F$ and $\mu_{-i} \in M^{F-1}$:

$$Pr_i^{sup}(\underline{v}, \mu_{-i}) = \sup_{p_i \in [0;\bar{p}]} Pr_i(\underline{v}, (p_i, \mu_{-i}))$$

i.e. $Pr_i^{sup}(\underline{v}, \mu_{-i})$ is the supremum of firm i's profits if the product-variants $\underline{v}$ are given, and if the other firms choose the strategies μ_{-i}. Two lemmas will now be used to prove proposition 3:

Lemma 1: $\sum_{i \in F} Pr_i$ is continuous in $(\underline{v}^o, \underline{\hat{\mu}}^o)$.

Lemma 2: For all $i \in F$ Pr_i^{sup} is lower semicontinuous in $(\underline{v}^o, \hat{\mu}^o_{-i})$.

To show that lemmas 1 and 2 imply proposition 3 we note first that the assumption that for all $n \in I\!N$ $\underline{\hat{\mu}}^n$ is a Nash-equilibrium of the game corresponding to $\underline{v}^n$ can also be written as:

$$Pr_i(\underline{v}^n, \underline{\hat{\mu}}^n) = Pr_i^{sup}(\underline{v}^n, \hat{\mu}^n_{-i})$$

for all $n \in I\!N$ and $i \in F$. Hence

$$\sum_{i \in F} Pr_i(\underline{v}^n, \underline{\hat{\mu}}^n) = \sum_{i \in F} Pr_i^{sup}(\underline{v}^n, \hat{\mu}^n_{-i})$$

for all $n \in I\!N$. Taking if necessary convergent subsequences we then obtain:

$$\lim_{n\to\infty} \sum_{i\in F} Pr_i(\underline{v}^n, \underline{\hat{\mu}}^n) = \lim_{n\to\infty} \sum_{i\in F} Pr_i^{sup}(\underline{v}^n, \hat{\mu}^n_{-i})$$

Applying lemmas 1 resp. 2 to the left hand side resp. right hand side of this equality we can deduce:

$$\sum_{i\in F} Pr_i(\underline{v}^o, \underline{\hat{\mu}}^o) \geq \sum_{i\in F} Pr_i^{sup}(\underline{v}^o, \hat{\mu}^o_{-i})$$

On the other hand it is a tautology that for all $i \in F$:

$$Pr_i(\underline{v}^o, \underline{\hat{\mu}}^o) \leq Pr_i^{sup}(\underline{v}^o, \hat{\mu}^o_{-i})$$

The last two inequalities together imply for all $i \in F$:

$$Pr_i(\underline{v}^o, \underline{\hat{\mu}}^o) = Pr_i^{sup}(\underline{v}^o, \hat{\mu}^o_{-i})$$

which means that $\underline{\hat{\mu}}^o$ is a Nash-equilibrium of the game corresponding to $\underline{v}$, and hence is just what we wanted to prove.

It remains to prove lemmas 1 and 2. Now lemma 1 is trivial given our continuity assumptions (A.1) and (A.2). We only prove lemma 2. Lemma 2 will be deduced from the following results which describes some basic properties of the firms' profit-functions in the point $\underline{v}^o$:

Lemma 3: For all $i \in F$ and $\underline{p} \in [0;\bar{p}]^F$ Pr_i is continuous in $(\underline{v}^o, \underline{p})$ provided that $p_i \neq p_j$ for all $j \in F, j \neq i$.

Lemma 4: For all $i \in F$, $p_i \in [0;\bar{p}]$ and $\epsilon > 0$ there is some nondegenerate interval $[\pi^o, \pi^1] \subseteq [0;\bar{p}]$ such that for all $\tilde{p}_i \in [\pi^o;\pi^1]$ and for all $p_{-i} \in [0;\bar{p}]^{F-1}$ $Pr_i(\underline{v}^o, (\tilde{p}_i, p_{-i})) \geq Pr_i(\underline{v}^o, (p_i, p_{-i})) - \epsilon$.

We do not give a formal proof of these results but we briefly comment on them.

As regards lemma 3 it should first be noted that "continuity" here means continuity in both product-variants and prices. This is important for what follows. The fact now that lemma 3 holds is an immediate consequence of our continuity-assumptions.

Whereas lemma 3 concerns the continuity of firms' profits also under perturbations of the product-variants in lemma 4 the product-variants $\underline{v}^o$ are taken as given, and a simple property of the payoffs in the game corresponding to $\underline{v}^o$, i.e. of the Bertrand-game, is described. It is that for every price p_i which a firm can choose there is a nondegenerate interval $[\pi^o; \pi^1]$ of other prices which yield (at least) almost the same profit as the given price p_i. This is in fact quite easy to prove. In most cases the interval $[\pi^o; \pi^1]$ can be chosen to consist of prices slightly below p_i.

It is interesting to note that the properties of the firms' profits which are described by lemmas 3 and 4 are in fact quite similar to conditions which Dasgupta and Maskin (1986) use. In particular lemma 3 corresponds to Dasgupta and Maskin's requirement that discontinuities of payoff-functions occur only on low-dimensional continuous manifolds, and lemma 4 corresponds to Dasgupta and Maskin's condition that payoffs be weakly lower-semicontinuous.

Given lemmas 3 and 4 we can now easily prove lemma 2. For every $m \in I\!N$ let $\underline{v}^m \in V^F$ and $\hat{\mu}^m_{-i} \in M^{F-1}$. Suppose $\underline{v}^m \longrightarrow \underline{v}^o$ and $\hat{\mu}^m_{-i} \longrightarrow \hat{\mu}^o_{-i}$. We have to prove:

$$\liminf_{m \to \infty} Pr_i^{sup}(\underline{v}^m, \hat{\mu}^m_{-i}) \geq Pr_i^{sup}(\underline{v}^o, \hat{\mu}^o_{-i})$$

This follows if for every $p_i \in [0; \bar{p}]$ and $\epsilon > 0$

$$\liminf_{m \to \infty} Pr_i^{sup}(\underline{v}^m, \hat{\mu}^m_{-i}) \geq Pr_i(\underline{v}^o, (p_i, \hat{\mu}^o_{-i})) - \epsilon$$

This in turn follows if for every $p_i \in [0; \bar{p}]$ and $\epsilon > 0$ there is some $\tilde{p}_i \in [0; \bar{p}]$ such that

$$\lim_{m \to \infty} Pr_i(\underline{v}^m, (\tilde{p}_i, \hat{\mu}^m_{-i})) \geq Pr_i(\underline{v}^o, (p_i, \hat{\mu}^o_{-i})) - \epsilon$$

Taking p_i and ϵ as given and fixed we shall construct such a $\tilde{p}_i$. For this we apply lemma 4 to p_i and ϵ and hence obtain some nondegenerate interval $[\pi^o; \pi^1]$ with the properties asserted in lemma 4. If we can now find some $\tilde{p}_i \in [\pi^o;\pi^1]$ such that

$$\lim_{m\to\infty} Pr_i(\underline{v}^m,(\tilde{p}_i,\hat{\mu}^m_{-i})) = Pr_i(\underline{v}^o,(\tilde{p}_i,\hat{\mu}^o_{-i}))$$

then the claim follows from lemma 4. Now take any $\tilde{p}_i \in [\pi^o;\pi^1]$ and denote for every $m \in I\!N_o$ by ν^m that probability-measure on $V^F \times [0;\bar{p}]^F$ that assigns probability one to $\underline{v}^m$ and that conditional on $\underline{v}^m$ agrees with $(\tilde{p}_i,\hat{\mu}^m_{-i})$. The property which we want $\tilde{p}_i$ to have can then also be written as:

$$\int Pr_i(\underline{v},\underline{p})\, d\nu^m \longrightarrow \int Pr_i(\underline{v},\underline{p})\, d\nu^o$$

Now clearly the sequence $(\nu^m)_{m\in I\!N}$ converges weakly to ν^o for all $\tilde{p}_i$. Hence by theorem 5.2 in Billingsley (1968,p.31) the claim holds if ν^o assigns probability zero to points of discontinuity of Pr_i. According to lemma 3 this will be so if

$$\hat{\mu}^o_j(\{\tilde{p}_i\}) = 0$$

for all $j \in F$, $j \neq i$. Now the measures $\hat{\mu}^o_j (j \in F,\ j \neq i)$ can have only countably many atoms. But there are uncountably many elements of $[\pi^o;\pi^1]$. Hence some $\tilde{p}_i$ in $[\pi^o;\pi^1]$ must have the required property.

Proof of Corollary 2

By lemma 1:

$$\sum_{i\in F} Pr_i(\underline{v}^n,\underline{\hat{\mu}}^n) \longrightarrow \sum_{i\in F} Pr_i(\underline{v}^o,\underline{\hat{\mu}}^o)$$

By corollary 1 the right hand side equals zero. On the other hand we must have $Pr_i(\underline{v}^n,\underline{\hat{\mu}}^n) \geq 0$ for all $i \in F$ and $n \in I\!N$ because firms can always avoid losses by choosing $\bar{p}$. But then the claim of corollary 2 follows immediately.

Proof of Proposition 6[9)]

To prove proposition 6 we have to prove the existence of a selection of the correspondence Pr^E and of a combination of pure strategies for the first stage game such that the strategy-combination constitutes a subgame-perfect equilibrium of the first stage game provided that all firms' payoffs in this game are described by the given selection.

For our proof of this we shall use theorem 1 in Harris (1985). A special version of this result which is still sufficient for our purposes is as follows: Consider a game played by a finite number of players in which the players move sequentially and in which each players' set of available choices is independent of all other players' choices and equal to some compact metric space. Suppose that every player's payoff is a continuous function of all players' choices. Then this game has a subgame-perfect equilibrium in pure strategies.

Proposition 6 would be an immediate consequence of this result if we could be sure that Pr^E has a continuous selcetion. However we only know that Pr^E is closed-valued and upper hemicontinuous, and these properties do not suffice for the existence of a continuous selection. Nevertheless we shall derive proposition 6 from Harris' theorem. For this we proceed as follows: We first introduce an artifical new game which satisfies the conditions of Harris' theorem and hence has a subgame-perfect equilibrium in pure strategies. Using the equilibrium of the artificial game we shall then construct the equilibrium and the selection needed to prove proposition 6.

The artificial game is as follows: There are $F+1$ players. The first F players are just the F firms. The $(F+1)$-th player is an artifical player.

These players make sequential choices. First the F firms choose in the order of their indices, then the $(F+1)$-th player chooses. Each of the F firms chooses some product-variant $v \in V$. The $(F+1)$-th player chooses an F-tuple $\underline{\pi} = (\pi_1, ..., \pi_F)$ from the range of Pr^E, i.e. from $\bigcup_{\underline{v} \in V^F} Pr^E(\underline{v})$.

For all $i \in F$ player i's payoff is π_i, i.e. the i-th component of player $F+1$'s choice. Player $F+1$ himself seeks to minimise the distance between the pair $(\underline{v}, \underline{p})$ and the graph of Pr^E. Observe that this distance is well defined because by proposition 4 the latter set is compact.

Now one can check that this game satisfies the conditions of Harris' theorem. Hence it has a subgame-perfect equilibrium in pure strategies. Moreover it is clear that in such an equilibrium for all choices $\underline{v}$ of the first F players player $F+1$ will choose some $\underline{\pi} \in Pr^E(\underline{v})$. Such a choice is always possible. By the definition of player $F+1$'s payoff it is moreover also optimal.

Hence player $F+1$'s strategy is a selection of Pr^E. Now this selection together with the first F players' equilibrium strategies clearly satisfies the statment in the first paragraph of this proof. Thus the proof is complete.

Footnotes

1) In the sense of Selten (1975).

2) Of course there are intuitive problems with mixed strategies. However here we shall not go into these.

3) A Borel-measure ν on $V \times I\!R_+$ is called "finite" iff it has finite total mass, i.e. iff $\nu(V \times I\!R_+) < \infty$.

4) The Hausdorff-metric is defined in Hildenbrand (1974, p.16). The notion of weak convergence is usually defined only for probability measures (see Billingsley (1968)). However this definition can immediatley be extended to general finite measures. Hence a sequence $(\nu^n)_{n \in I\!N}$ of finite measures converges weakly to a finite measure ν^o iff for all bounded and continuous real valued functions f we have: $\int f \, d\nu^n \longrightarrow \int f \, d\nu^o$.

5) More precisely: There is a unique $\hat{p} \in [0; \bar{p}]$ such that the demand for the product-variant is zero iff it is offered at the price $\hat{p}$ or at any price above this price.

6) See theorem 1 in Hildenbrand (1974, p.24) which demonstrates how results about sequences translate into results about closed-valuedness and upper hemicontinuity.

7) A result which is related to our proposition 6 has been obtained by Salant (1986). He considers a more special model of product-differentiation with sequential choice of product-variants in a first stage and simultaneous choice of quantities (rather than prices) in a second stage. He proves the existence of a subgame-perfect equilibrium in pure strategies. Since he assumes quantitiy rather than price competition in the second stage and moreover obtains unique second stage equilibrium profits his result is rather different from ours.

8) Upper hemicontinuity appears to be natural provided that consumers' preferences are in some sense continuous with respect to the metric on the space of all product-variants.

9) Although formally our proposition 6 is not a completely immediate implication of the results in Harris (1985) it is nontheless quite obvious from the proofs in that paper

that proposition 6 holds. It is therefore only to make this paper self-contained that we include an explicit proof of proposition 6.

References

Billingsley, P., 1968, Convergence of Probability Measures, New York: John Wiley and Sons.

Dasgupta, P. and E. Maskin, 1986, "The Existence of Equilibrium in Discontinuous Economic Games, I.: Theory", Review of Economic Studies, 53, 1-26.

Glicksberg, I., 1952, "A Further Generalisation of Kakutani's Fixed Point Theorem with Application to Nash Equilibrium Points", Proceedings of the American Mathematical Society, 38, 170-174.

Harrington, J.E., 1988, "A Re-Evaluation of Perfect Competition as the Solution to the Bertrand Price Game", mimeo., The Johns Hopkins University, Baltimore.

Harris, C., 1985, "Existence and Characterisation of Perfect Equilibrium in Games of Perfect Information", Econometrica, 53, 613-628.

Hildenbrand, W., 1974, Core and Equilibria of a Large Economy, Princeton: Princeton University Press.

Lane, W.J., 1980, "Product Differentiation in a Model with Endogeneous Sequential Entry", The Bell Journal of Economics, 11, 273-280.

Neven, D., 1984, "Sequential Equilibrium with Foresight in Hotelling's Model", mimeo., Nuffield College, Oxford.

Osborne, M.J. and C. Pitchik, 1987, "Equilibrium in Hotelling's Model of Spatial Competition", Econometrica, 55, 911-922.

Prescott, E.C. and M. Visscher, 1977, "Sequential Location Among Firms with Foresight", The Bell Journal of Economics, 8, 378-393.

Salant, D.J., 1986, "Equilibrium in a Spatial Model of Imperfect Competition with Sequential Choice of Location and Quantities", Canadian Journal of Economics, 19, 685-715.

Selten, R., 1975, "Re-Examination of the Perfectness Concept for Equilibrium Points in Extensive Games", International Journal of Game Theory, 4, 25-55.

Shaked, A. and J. Sutton, 1983, "Natural Oligopolies", Econometrica, 51, 1469-1483.

Simon, L.K., 1987, "Games with Discontinuous Payoffs", Review of Economic Studies, 54, 569-597.