NOTES ON BLACKWELL DOMINANCE WITH ONLY TWO STATES OF THE WORLD

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In these notes I discuss some properties of Blackwell dominance in the special case that there are only two possible states of the world. Let the set of possible states of the world be $\Omega = \{\omega_1, \omega_2\}$. Assume that each state is equally likely, but note that the concept of Blackwell dominance is independent of the prior.

A signal σ is a mapping that maps each state of the world into a probability distribution over some non-empty and finite set S: $\sigma^i : \Omega \to \Delta(S)$, where $\Delta(S)$ denotes the set of all probability distributions over S. We denote by $\sigma(s|\omega)$ the conditional probability of observing signal realization $s \in S$ conditional on the state being $\omega \in \Omega$. The probability that signal σ has realization $s \in S$ is:

$$p(s) = \frac{\sigma(s|\omega_1) + \sigma(s|\omega_2)}{2}.$$

If the decision maker observes realization $s \in S$ then her posterior probability that the state of the world is ω equals:

$$\mu(\omega|s) = \frac{\sigma(s|\omega)}{\sigma(s|\omega_1) + \sigma(s|\omega_2)}.$$

It suffices to keep track of the decision maker's probability of state 1. We introduce for this probability simplified notation:

$$\mu(\omega_1|s) \equiv \mu(s).$$

A decision problem D is a pair (A_D, u_D) consisting of a finite set of actions A_D and a utility function $u_D : A_D \times \Omega \to \mathbb{R}$. Let the set of all decision problems be $\mathcal{D}$. Suppose the decision maker's posterior probability of state ω_1 is $\mu \in [0,1]$. If she chooses an action $a \in A$, her expected utility is:

$$u_D(a, \mu) = \mu u_D(a, \omega_1) + (1 - \mu) u_D(a, \omega_2),$$

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where we use the same notation for the utility function that takes a belief about the state as its second argument as we use for the utility function that takes states of the world as its second argument. If the decision maker with belief μ chooses her action optimally from A_D, she achieves utility:

$$v_D(\mu) = \max_{a \in A_D} u_D(a, \mu).$$

If the decision maker first observes signal σ and then chooses an action from A_D that maximizes her expected utility given her posterior, then her ex ante expected utility is:

$$V_D^\sigma = \sum_{s \in S} p(s) v_D(\mu(s)).$$

A standard definition of Blackwell dominance is:

Definition 1. *Signal σ^1* Blackwell dominates *signal σ^2 if for all decision problems $D \in \mathcal{D}$:*

$$V_D^{\sigma^1} \geq V_D^{\sigma^2}.$$

We will now show in several steps that, when verifying Blackwell dominance, it is sufficient to consider decision problems in a small subset of $\mathcal{D}$. We use the following terminology:[1]

Definition 2. *A subset $\hat{\mathcal{D}}$ of the set $\mathcal{D}$ is* sufficient for Blackwell dominance *if a signal σ^1 Blackwell dominates signal σ^2 if and only if or all decision problems $D \in \hat{\mathcal{D}}$:*

$$V_D^{\sigma^1} \geq V_D^{\sigma^2}.$$

Let us call a decision problem "monotone" if for every action $a \in A$ we have: $u_D(a, \omega_1) \leq u_D(a, \omega_2)$.

Proposition 1. *The set of all monotone decision problems is sufficient for Blackwell dominance.*

Proof. Consider given signals σ^1 and σ^2, and suppose $V_D^{\sigma^1} \geq V_D^{\sigma 2}$ for every monotone decision problem. We have to prove that then $V_D^{\sigma^1} \geq V_D^{\sigma 2}$ for all decision problems. Consider any decision problem D that is not monotone. Suppose we modify D by

[1]The notion of a "sufficient" subset of decision problems is closely related to the notion of a "generator" of a stochastic order as defined in Chapter 2 of Alfred Müller and Dietrich Stoyan, *Comparison Methods for Stochastic Models and Risk*, Wiley, 2002.

subtracting the same constant $k > 0$ from $u_D(a, \omega_1)$ for all $a \in A$, leaving $u_D(a, \omega_2)$ unchanged. Call the new decision problem D'. We choose the constant k sufficiently large so that D' is monotone. Then we have by assumption:

$$V_{D'}^{\sigma^1} \geq V_{D'}^{\sigma^2}.$$

But note that for each signal $i \in \{1, 2\}$ we have:

$$V_{D'}^{\sigma^i} = V_D^{\sigma^i} - 0.5k,$$

where 0.5 is the prior probability of state 1. Therefore, we can conclude:

$$\begin{aligned} V_D^{\sigma^1} - 0.5k &\geq V_D^{\sigma^2} - 0.5k \Leftrightarrow \\ V_D^{\sigma^1} &\geq V_D^{\sigma^2}, \end{aligned}$$

which is what we wanted to prove. □

We call a decision problem "simple" if either A_D has only one element, or A_D has two elements, a and $\hat{a}$, and the utility function is of the form given by the table in Figure 1 where $k, \ell \in \mathbb{R}$.

	ω_1	ω_2
a	k	ℓ
$\hat{a}$	0	0

TABLE 1. Simple Decision Problems

Proposition 2. *The set of all monotone and simple decision problems is sufficient for Blackwell dominance.*

Proof. Consider given signals σ^1 and σ^2, and suppose $V_D^{\sigma^1} \geq V_D^{\sigma^2}$ for every monotone and simple decision problem D. We have to prove that then $V_D^{\sigma^1} \geq V_D^{\sigma^2}$ for all monotone decision problems. The following observation is crucial:

Claim 1. *For every monotone decision problem D there is a finite list of monotone and simple decision problems $D_1, D_2, \ldots, D_n$ such that the corresponding value functions satisfy:*

$$v_D(\mu) = \sum_{i=1}^{n} v_{D_i}(\mu) \text{ for all } \mu \in [0, 1].$$

Before we prove Claim 1 we show that Claim 1 implies Proposition 2. Claim 1 and the linearity of expected utility imply together:

$$V_D^{\sigma^1} = \sum_{i=1}^{n} V_{D_i}^{\sigma^1} \text{ and } V_D^{\sigma^2} = \sum_{i=1}^{n} V_{D_i}^{\sigma^2}.$$

By assumption: $V_{D_i}^{\sigma^1} \geq V_{D_i}^{\sigma^2}$ for all $i = 1, 2, \ldots, n$. Thus, $V_D^{\sigma^1} \geq V_D^{\sigma^2}$, follows.

It remains to prove Claim 1. Consider any decision problem D. It is easy to see that we can find n intervals of the form $[\alpha_i, \alpha_{i+1}] \subseteq [0, 1]$ and for each interval an action $a_i \in A$ such that:

(i) $\alpha_1 = 0$, $\alpha_i < \alpha_{i+1}$ for all $i \in \{1, 2, \ldots, n\}$, and $\alpha_{n+1} = 1$, and
(ii) for all $i \in \{1, 2, \ldots, n\}$ if $\mu \in [\alpha_i, \alpha_{i+1}]$ then $v_D(\mu) = u_D(a_i, \mu)$, i.e. action a_i maximizes expected utility for beliefs μ in the interval $[\alpha_i, \alpha_{i+1}]$.

If $n = 1$, then Claim 1 is trivial true because the value function v_D is the same as the value function for the monotone and simple decision problem in which only action a_1 is available, and the utility from action a_1 is the same as it is in D. Therefore, we assume from now onwards that $n \geq 2$.

The following observation will be key to the proof of Claim 1:

Claim 2. *If $\mu \in [\alpha_i, \alpha_{i+1}]$, then the decision maker's preferences over the actions $a_1, a_2, \ldots, a_n$ are "single-peaked," that is, if $j < i$ then:*

$$u_D(a_j, \mu) \leq u_D(a_{j+1}, \mu),$$

and if $j \geq i$ then:

$$u_D(a_j, \mu) \geq u_D(a_{j+1}, \mu).$$

Proof of Claim 2. To prove Claim 2 it is sufficient to show that:

$$u_D(a_j, \mu) - u_D(a_{j+1}, \mu) \text{ is weakly monotonically decreasing in } \mu.$$

To see that this is sufficient note that if $j < i$ then a_{j+1} is optimal for some $\mu \leq \alpha_i$, i.e. $u_D(a_j, \mu) - u_D(a_{j+1}, \mu) \leq 0$ for some $\mu \leq \alpha_i$. But then it follows from of the monotonicity in μ of utility differences that $u_D(a_j, \mu) - u_D(a_{j+1}, \mu) \leq 0$ for all $\mu \in [\alpha_i, \alpha_{i+1}]$, which is what we needed to show. Conversely, if $j \geq i$, then a_j is optimal for some $\mu \geq \alpha_{i+1}$, i.e. $u_D(a_j, \mu) - u_D(a_{j+1}, \mu) \geq 0$ for some $\mu \geq \alpha_i$. Then the monotonicity in μ of utility differences implies that $u_D(a_j, \mu) - u_D(a_{j+1}, \mu) \geq 0$ for all $\mu \in [\alpha_i, \alpha_{i+1}]$.

It thus suffices to prove that for all $i \in \{1, 2, \ldots, n-1\}$ the difference $u_D(a_i, \mu) - u_D(a_{i+1}, \mu)$ is weakly monotonically decreasing in μ. Writing out this difference we obtain:

$$\begin{aligned}
& u_D(a_i, \mu) - u_D(a_{i+1}, \mu) \\
= \; & (\mu u_D(a_i, \omega_1) + (1-\mu) u_D(a_i, \omega_2)) - (\mu u_D(a_{i+1}, \omega_1) + (1-\mu) u_D(a_{i+1}, \omega_2)) \\
= \; & u_D(a_i, \omega_2) - u_D(a_{i+1}, \omega_2) - ((u_D(a_i, \omega_2) - u_D(a_i, \omega_1)) - (u_D(a_{i+1}, \omega_2) - u_D(a_{i+1}, \omega_1)))\, \mu
\end{aligned}$$

which is weakly monotonically decreasing in μ if:

$$u_D(a_i, \omega_2) - u_D(a_i, \omega_1) \geq u_D(a_{i+1}, \omega_2) - u_D(a_{i+1}, \omega_1).$$

To prove this we note that there are μ, μ' such that $\mu' > \mu$, and:

$$\mu u_D(a_i, \omega_1) + (1-\mu) u_D(a_i, \omega_2) \geq \mu u_D(a_{i+1}, \omega_1) + (1-\mu) u_D(a_{i+1}, \omega_2)$$

and

$$\mu' u_D(a_i, \omega_1) + (1-\mu') u_D(a_i, \omega_2) \leq \mu' u_D(a_{i+1}, \omega_1) + (1-\mu') u_D(a_{i+1}, \omega_2)$$

Subtracting the second inequality from the first we obtain:

$$(u_D(a_i, \omega_2) - u_D(a_i, \omega_1))\,(\mu' - \mu) \geq (u_D(a_{i+1}, \omega_2) - u_D(a_{i+1}, \omega_1))\,(\mu' - \mu).$$

Dividing by $(\mu' - \mu)$, we get:

$$u_D(a_i, \omega_2) - u_D(a_i, \omega_1) \geq u_D(a_{i+1}, \omega_2) - u_D(a_{i+1}, \omega_1),$$

which is what we wanted to prove. $\square$

We now use Claim 2 to find an expression for the value functions v_D. By construction, if $\mu \in [\alpha_i, \alpha_{i+1}]$:

$$v_D(\mu) = u_D(a_i, \mu).$$

We can re-write this as:

$$v_D(\mu) = \sum_{j=i}^{n-1} (u_D(a_j, \mu) - u_D(a_{j+1}, \mu)) + u_D(a_n, \mu).^{2}$$

Using Claim 2, we can write this as:

$$v_D(\mu) = \sum_{j=i}^{n-1} \max\{u_D(a_j, \mu) - u_D(a_{j+1}, \mu), 0\} + u_D(a_n, \mu),$$

[2] We define summations where the index of the first term is smaller than the index of the last term to be equal to zero.

According to Claim 2, moreover, the first term on the right hand side of the following equation is zero, and therefore this equation follows, too:

$$v_D(\mu) = \sum_{j=1}^{i-1} \max\{u_D(a_j,\mu) - u_D(a_{j+1},\mu), 0\} + \sum_{j=i}^{n-1} \max\{u_D(a_j,\mu) - u_D(a_{j+1},\mu), 0\} + u_D(a_n,\mu).$$

We have now obtained:

$$v_D(\mu) = \sum_{j=1}^{n-1} \max\{(u_D(a_j,\mu) - u_D(a_{j+1},\mu)), 0\} + u_D(a_n,\mu),$$

which holds for all $\mu \in [0,1]$. Observe that each terms in the sum is the value function corresponding to a simple decision problem as shown in Table 1, where

$$k = u_D(a_j,\omega_1) - u_D(a_{j+1},\omega_1) \text{ and } \ell = u_D(a_j,\omega_2) - u_D(a_{j+1},\omega_2),$$

and the last term is the simple decision problem where only action a_n is available. We have thus written v as the sum of a value functions for a finite list of simple decision problems.

We still have to show that these simple decision problems are monotone. We have to show:

$$\begin{aligned} k_j &\leq \ell_j \Leftrightarrow \\ u_D(a_j,\omega_1) - u_D(a_{j+1},\omega_1) &\leq u_D(a_j,\omega_2) - u_D(a_{j+1},\omega_2) \Leftrightarrow \\ u_D(a_j,\omega_1) - u_D(a_j,\omega_2) &\leq u_D(a_{j+1},\omega_1) - u_D(a_{j+1},\omega_2), \end{aligned}$$

which was established at the end of the proof of Claim 2. Finally, the decision problem in which only action a_n is available is monotone because by assumption D is monotone. □

By normalizing utilities appropriately we can obtain an even smaller class of decision problems that are sufficient for Blackwell dominance. We define monotone and simple decision problems to be "canonical" if there is a $\kappa \in (0,1)$ such that utility is of the form shown in Table 2.

	ω_1	ω_2
a	$-(1-\kappa)$	κ
$\hat{a}$	0	0

TABLE 2. Canonical Decision Problems

Proposition 3. *The set of all canonical decision problems is sufficient for Blackwell dominance.*

Proof. Consider given signals σ^1 and σ^2, and suppose $V_D^{\sigma^1} \geq V_D^{\sigma^2}$ for every canonical decision problem. We have to prove that then $V_D^{\sigma^1} \geq V_D^{\sigma^2}$ for all monotone and simple decision problems. Consider any monotone and simple decision problem D that is not canonical. The inequality $V_D^{\sigma^1} \geq V_D^{\sigma^2}$ is trivially true if D has only one action, or if D has two actions but one of the actions is dominant: $k \leq \ell \leq 0$ or $0 \leq k \leq \ell$. The remaining case is that D has two actions and $k < 0 < \ell$. If we divide all payoffs by $\ell - k$, we obtain a canonical decision problem. Therefore, D can be obtained from a canonical problem by multiplying all utilities by a positive constant, say $\xi > 0$. But then for both signals $V_D^{\sigma^i} = \xi V_{D'}^{\sigma^i}$ where D' is canonical. By assumption $V_{D'}^{\sigma^1} \geq V_{D'}^{\sigma^2}$, and therefore $V_D^{\sigma^i} \geq V_D^{\sigma^2}$ follows. □

We have now obtained a one parameter family of decision problems that is sufficient for Blackwell dominance. Consider a signal σ^i. Let us calculate $V_D^{\sigma^i}$ where D is as in Figure 2. If the decision maker's probability of state ω_1 is μ, then the expected utility from a is: $-\mu(1-\kappa) + (1-\mu)\kappa = \kappa - \mu$. Thus, the decision maker will choose a whenever $\mu \leq \kappa$, and will then obtain expected utility $\kappa - \mu$. Otherwise the decision maker will chose $\hat{a}$ and will obtain expected utility zero. We therefore find:

$$V_D^{\sigma^i} = \sum_{\{s^i \in S^i | \mu^i(\omega_1|s^i) \leq \kappa\}} p^i(s^i)(\kappa - \mu^i(\omega_1|s^i)).$$

Let us define the right hand side of this equality as: $F^{\sigma^i}(\kappa)$. We obtain the corollary:

Corollary 1. *Signal σ^1 Blackwell dominates signal σ^2 if and only if for all $\kappa \in (0,1)$:*

$$F^{\sigma^1}(\kappa) \geq F^{\sigma^2}(\kappa).$$

Observe that $F^{\sigma^1}(\kappa)$ is equal to the integral from 0 to κ of the cumulative distribution function of posteriors when the decision maker observes the realization of signal σ^i. Thus, we have derived a familiar characterization of Blackwell dominance. In fact, the condition in Corollary 1 is well-known to be equivalent to the condition that the beliefs after observing signal σ^1 are a mean-preserving spread of the beliefs after observing signal σ^2.

Note that the functions F^{σ^1} and F^{σ^2} are both piecewise linear, that they have in common that $F^{\sigma^1}(0) = 0$ and $F^{\sigma^1}(1) = 0.5$, and that they are both convex. These observations together simply that the inequality in Corollary 1 needs to be checked only at the points in which F^{σ^1} has kinks, which are the points at which μ equals one of the posteriors that have positive probability under σ^1. Formally, we have:

Proposition 4. *Signal σ^1 Blackwell dominates signal σ^2 if and only if for all $s^1 \in S^1$:*

$$F^{\sigma^1}(\mu^1(\omega_1|s^1)) \geq F^{\sigma^2}(\mu^1(\omega_1|s^1)).$$

Proof. Assume that the inequality in Proposition 4 is satisfied. We need to prove that it follows that for all $\mu \in (0,1)$ we have:

$$F^{\sigma^1}(\mu) \geq F^{\sigma^2}(\mu).$$

But note that for every $\mu \in (0,1)$ there are $\mu' \leq \mu$ and $\mu'' \geq \mu$ such that:

$$F^{\sigma^1}(\mu') \geq F^{\sigma^2}(\mu') \text{ and } F^{\sigma^1}(\mu'') \geq F^{\sigma^2}(\mu'') \text{ and } F^{\sigma^1} \text{ is linear on } [\mu', \mu''.].$$

There is $\lambda \in [0,1]$ such that $\mu = \lambda\mu' + (1-\lambda)\mu''$. By the linearity of F^{σ^1} on $[\mu', \mu'']$ we have:

$$F^{\sigma^1}(\mu) = \lambda F^{\sigma^1}(\mu') + (1-\lambda)F^{\sigma^1}(\mu'').$$

Therefore:

$$F^{\sigma^1}(\mu) \geq \lambda F^{\sigma^2}(\mu') + (1-\lambda)F^{\sigma^2}(\mu'').$$

By the convexity of F^{σ^2}:

$$\lambda F^{\sigma^2}(\mu') + (1-\lambda)F^{\sigma^2}(\mu'') \geq F^{\sigma^2}(\lambda\mu' + (1-\lambda)\mu'') = F^{\sigma^2}(\mu).$$

We conclude:

$$F^{\sigma^1}(\mu) \geq F^{\sigma^2}(\mu),$$

which concludes the proof. □