Bertrand-Competition without Demand-Rationing

1. Introduction

It is well known that Bertrand's model of price competition has the following unsatisfactory feature: If there are at least two firms and if these firms have identical cost functions with constant marginal costs there is a unique equilibrium and in this equilibrium the price equals marginal costs and the firms have zero profits. Thus in the case of constant marginal costs Bertrand's price competition leads to the competitive outcome even if the number of firms is small.

This paper analyzes the case that firms have identical cost functions with increasing marginal costs. In this case the crucial question is whether the firms have the possibility to ration demand or not, i.e. whether for a given price-system firms are able to restrict their sales to an upper limit which is smaller than their demand or whether they have to satisfy their entire demand. This question is important because one can easily think of situations in which a firm might wish to undercut its competitors without satisfying the entire demand at the lower price in order to avoid too high increases in marginal costs.

Empirically it seems to be plausible that firms have some possibility to ration demand. But gametheoretic models which include this possibility are somewhat problematic since equilibria in pure strategies frequently do not exist. This paper considers the empirically less plausible case in which firms have no possibility to ration demand. This case has the advantage that pure-strategy-equilibria exist under rather general conditions. It is shown that in this case some "pathologies" emerge which are different from the "pathologies" of the Bertrand-model in the case of constant marginal costs: There is a continuum of pure-strategy-equilibria of the Bertrand-game. The competitive price is one of the possible equilibrium prices, but there are also equilibrium prices which are higher or smaller than the competitive price. Furthermore the set of Bertrand-prices does not shrink to the competitive price as the number of firms tends to infinity.

These results are presented in the following sections. All proofs are straightforward and therefore omitted.

2. The Model

I consider a market for a homogeneous good. Market demand is described by a demand function

$$D: R_+ \longrightarrow R_+$$
$$p \longmapsto D(p)$$

which satisfies the following assumptions:

There is a $\hat{p} > 0$ such that:

(A.1) $D(p) > 0$ if $0 \leq p < \hat{p}$
(A.2) D is differentiable on $]0 ; \hat{p}[$
(A.3) $D'(p) < 0$ if $0 < p < \hat{p}$
(A.4) D is continuous at $\hat{p}$
(A.5) $D(p) = 0$ if $p \geq \hat{p}$

There are n (≥ 2) identical firms. The firms have no technological capacity constraints. Their production costs are described by a cost function

$$C: R_+ \longrightarrow R_+$$
$$x \longmapsto C(x)$$

Assumptions for C are:

(A.6) C is continuously differentiable on R_+
(A.7) $C(0) = 0$
(A.8) $C'(x) > 0$ if $x > 0$
(A.9) C is strictly convex on R_+

These assumptions imply that the average costs $C(x)/x$ are increasing. Finally I assume that

$$\text{(A.10)} \quad C'(0) < \hat{p} \ .$$

This excludes an obviously uninteresting case.

Firms play the following game: Each firm i chooses a price $p_i \in R_+$. Firms seek to maximize their profits. Sales are determined as follows: Let p be the lowest price and let m be the number of firms charging the lowest price. Then the firms with the lowest price sell $D(p)/m$, all the other firms don't sell anything.

The Nash-equilibria of this game will be called "Bertrand-equilibria", the corresponding minimum-prices will be called "Bertrand-prices". It will be interesting to compare the outcomes of the Bertrand-game with the competitive outcome. Obviously there is a unique competitive price. In the following this price is denoted by p_c.

3. Results

For the analysis of the Bertrand-equilibria it is useful to consider the following class of functions:

$$\Pi_m : R_+ \longrightarrow R$$
$$p \longmapsto \Pi_m(p) := p \cdot \frac{D(p)}{m} - C\left(\frac{D(p)}{m}\right)$$

where $m \in \mathbb{N}$. $\Pi_m(p)$ is the profit of a firm choosing the price p if p is the lowest price and if there are m-1 other firms which also choose p. Obviously for a given number n of firms the set of Bertrand-equilibria depends only on the functions Π_m with $1 \leq m \leq n$. The following lemma contains a list of some properties of these functions:

Lemma: For every $m \in \mathbb{N}$:

(i) $\Pi_m(p) = 0$ if $p \geq \hat{p}$

(ii) There is a $\bar{p}_m \in \,]C'(0)\,;\,\hat{p}[$ such that

$\Pi_m(p) < 0$ if $0 \leq p < \bar{p}_m$

$\Pi_m(p) = 0$ if $p = \bar{p}_m$

$\Pi_m(p) > 0$ if $\bar{p}_m < p < \hat{p}$

(iii) $\bar{p}_m > \bar{p}_{m+1}$

For every $m \in \mathbb{N}$ with $m \geq 2$:

(iv) There is a $\bar{\bar{p}}_m \in \,]\bar{p}_1\,;\,p[$ such that

$\Pi_m(p) > \Pi_1(p)$ if $0 \leq p < \bar{\bar{p}}_m$

$\Pi_m(p) = \Pi_1(p)$ if $p = \bar{\bar{p}}_m$

$\Pi_m(p) < \Pi_1(p)$ if $\bar{\bar{p}}_m < p < \hat{p}$

(v) $\bar{\bar{p}}_m > \bar{\bar{p}}_{m+1}$

(vi) Π_1 is increasing on $[0\,;\,\bar{\bar{p}}_m]$

From this lemma one can easily derive the following characterization of Bertrand-equilibria:

Proposition 1: A price-vector $(p_1, p_2, \ldots, p_n)$ is a Bertrand-equilibrium if and only if

$$p_1 = p_2 = \ldots = p_n \in [\,\bar{p}_n\,;\,\bar{\bar{p}}_n\,]$$

holds.

The next proposition concerns the relation between the possible outcoms of the Bertrand-game and the competitive outcome:

Proposition 2: The competitive price p_c is an interior point of the interval $[\,\bar{p}_n\,;\,\bar{\bar{p}}_n\,]$.

So the Bertrand-game may lead to the competitive price p_c, but it may also lead to prices which are higher or smaller than p_c.

I now turn to the question how the set of Bertrand-prices and the competitive price p_c change if the number n of firms increases. I assume that the functions D and C are given and fixed. From the parts (iii) and (v) of the lemma we know already that the boundary points of the interval of Bertrand-prices decrease if n increases. The following proposition describes the behaviour of these points as n tends to infinity:

Proposition 3: As n tends to infinity $\bar{p}_n$ converges to C'(0) and $\bar{\bar{p}}_n$ converges to $\bar{p}_1$.

This proposition shows that the set of Bertrand-prices converges to the closed interval whose lower boundary are the marginal costs at zero and whose upper boundary is the "break-even-price" of a monopolist, i.e. the lowest price which a monopolist can choose without making losses.
For the competitive price it is obvious that it decreases as n increases. The following proposition concerns the convergence of p_c:

Proposition 4: As n tends to infinity the competitive price p_c converges to C'(0).

From propositions 3 and 4 it is obvious that the set of Bertrand-prices does not shrink to the competitive price as the number of firms becomes large.

4. Why $\bar{p}_1$ is a Bertrand-Price for any given Number of Firms

In this section a simple graphic figure is used to illustrate the nature of the Bertrand-equilibria. The following figure shows the market-demand-curve (D), the split demand-curve (D/n) and the average-cost-curve (AC):

(Insert figure 1)

From the lemma and proposition 1 follows that $\bar{p}_1$ is a Bertrand-price for any given number of firms. This is iluustrated in the figure.

Assume that all firms choose $\bar{p}_1$. Then each firm has a profit which is equal to the rectangle A. Does it pay for a firm to deviate? If it charges a price which is higher than $\bar{p}_1$ then its profit will be zero because it doesn't sell anything. If it undercuts $\bar{p}_1$ then it will make losses because its sales will be so high that the average costs exceed the price. In the figure this is shown for the price p. A firm which undercuts $\bar{p}_1$ and chooses the price p will make losses which are equal to the rectangle B. So no firm has an incentive to deviate and p is a Bertrand-price.

The figure shows clearly that this argument is only possible because we have assumed that firms are not able to ration demand. Similarly all the results contained in this paper depend crucially on this assumption.

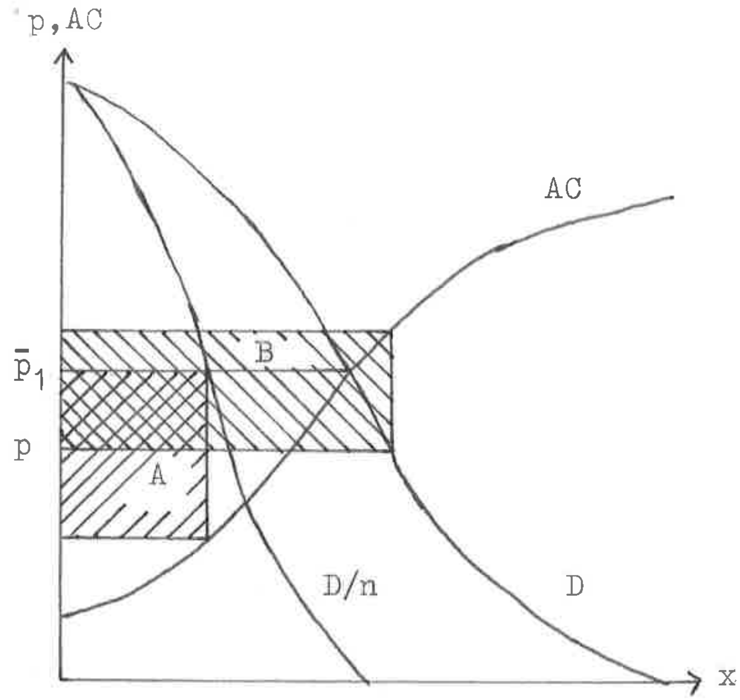

figure 1

Proofs

Proof of the lemma:

(i)

Part (i) is trivial.

(ii)

If p lies in the interval $[0 ; \hat{p}[$ we can write for $\Pi_m(p)$:

$$\Pi_m(p) = F_m(p) \cdot \frac{D(p)}{m}$$

where

$$F_m(p) = p - \frac{C\left(\frac{D(p)}{m}\right)}{\frac{D(p)}{m}}$$

Since $p \in [0 ; \hat{p}[$ implies $D(p) > 0$ it is sufficient for the proof of part (ii) of the lemma to consider the function F_m. Four properties of F_m are relevant:

(1) F_m is continuous.

This follows immediately from the continuity of D and C.

(2) $F_m(C'(0)) < 0$

Proof:

$$F_m(C'(0)) < 0 \qquad \Longleftrightarrow$$

$$C'(0) - \frac{C\left(\frac{D(C'(0))}{m}\right)}{\frac{D(C'(0))}{m}} < 0 \qquad \Longleftrightarrow$$

$$C'(0) \cdot \frac{D(C'(0))}{m} < C\left(\frac{D(C'(0))}{m}\right)$$

The last inequality is an implication of the convexity of C and the assumption that $C(0) = 0$ holds.

(3) $\lim_{p \to \hat{p}} F_m(p) > 0$

Proof:

$$\lim_{p \to \hat{p}} F_m(p)$$

$$= \lim_{p \to \hat{p}} \left(p - \frac{C\left(\frac{D(p)}{m} \right)}{\frac{D(p)}{m}} \right)$$

$$= p - \lim_{x \to 0} \frac{C(x)}{x}$$

From L'Hospital's rule follows: $\lim_{x \to 0} C(x)/x = C'(0)$. Therefore the last term equals:

$$p - C'(0)$$

This is by assumption positive.

From (1), (2) and (3) follows that there is a $\bar{p}_m \in]C'(0) ; \hat{p}[$ with $F_m(\bar{p}_m) = 0$. This implies of course $\Pi_m(\bar{p}_m) = 0$. For the proof of the remaining claims of part (ii) of the lemma it is now sufficient to show that

(4) F_m is increasing von $[0 ; \hat{p}[$.

This is rather evident from the definition of F_m if one remembers that the demand decreases as p increases and that our assumptions for the cost function C imply that the average costs increase.

(iii)

From (ii) follows that $\bar{p}_m > \bar{p}_{m+1}$ is equivalent to:

$$\Pi_{m+1}(\bar{p}_m) > 0$$

$$\bar{p}_m \cdot \frac{D(\bar{p}_m)}{m+1} - C\left(\frac{D(\bar{p}_m)}{m+1} \right) > 0$$

$$\bar{p}_m \cdot \frac{D(\bar{p}_m)}{m} - \frac{m+1}{m} C\left(\frac{D(\bar{p}_m)}{m+1}\right) > 0$$

The convexity of C implies together with the assumption $C(0) = 0$ that

$$\frac{m+1}{m} C\left(\frac{D(\bar{p}_m)}{m+1}\right) < C\left(\frac{m+1}{m} \frac{D(\bar{p}_m)}{m+1}\right) = C\left(\frac{D(\bar{p}_m)}{m}\right)$$

holds. Therefore it is sufficient to show that

$$\bar{p}_m \frac{D(\bar{p}_m)}{m} - C\left(\frac{D(\bar{p}_m)}{m}\right) \geq 0$$

$$\Pi_m(\bar{p}_m) \geq 0$$

holds. But according to the definition of $\bar{p}_m$ $\Pi_m(\bar{p}_m)$ is zero.

(iv)

Consider the difference $\Pi_1(p) - \Pi_m(p)$. If p lies in the interval $[0 ; \hat{p}[$ we can write for this difference:

$$\Pi_1(p) - \Pi_m(p) = G(p) \cdot \frac{D(p)}{m}$$

where

$$G_m(p) = m F_1(p) - F_m(p)$$

(I adopt the definition of the functions F_1 and F_m from the proof of part (ii).) Since $p \in [0 ; \hat{p}[$ implies $D(p) > 0$ it is sufficient for the proof of part (iv) of the lemma to consider the function G_m. Four properties of G_m are relevant:

(1) G_m is continuous.

This follows immediately from the continuity of D and C.

(2) $G_m(\bar{p}_1) < 0$

Proof:

$$G_m(\bar{p}_1) < 0 \qquad \Longleftrightarrow$$

$$m\ F_1(\bar{p}_1) - F_m(\bar{p}_1) < 0$$

Since $F_1(\bar{p}_1)$ is zero this is equivalent to

$$F_m(\bar{p}_1) > 0$$

This follows immediately from part (iii) of the lemma.

(3) $\lim\limits_{p \to \hat{p}} G_m(p) > 0$

Proof:

$$\lim_{p \to \hat{p}} G_m(p)$$

$$= \lim_{p \to \hat{p}} (m\ F_1(p) - F_m(p))$$

$$= m \cdot \lim_{p \to \hat{p}} F_1(p) - \lim_{p \to \hat{p}} F_m(p)$$

In the proof of part (ii) it was shown that

$$\lim_{p \to \hat{p}} F_1(p) = \lim_{p \to \hat{p}} F_m(p) = \hat{p} - C'(0)$$

holds. Therefore we have

$$\lim_{p \to \hat{p}} G_m(p)$$

$$= (m-1)\ (\hat{p} - C'(0))$$

This is by assumption positive.

From (1), (2) and (3) follows that there is a $\bar{\bar{p}}_m \in \,]\bar{p}_m ; p[$ with $G_m(\bar{\bar{p}}_m) = 0$. This implies $\Pi_m(\bar{\bar{p}}_m) = \Pi_1(\bar{\bar{p}}_m)$. For the proof of the remaining claims of part (iv) of the lemma it is now sufficient to show that

(4) G_m is increasing on $[0 ; \hat{p}[$.

Proof:

Since G_m is differentiable it is sufficient to show that for every $p \in]0 ; \hat{p}[$ we have:

$$G_m'(p) > 0 \quad \Leftrightarrow$$

$$m\ F_1'(p) - F_m'(p) > 0 \quad \Leftrightarrow$$

$$m\ \left(1 - \frac{D(p)\ C'(D(p))\ D'(p) - C(D(p))\ D'(p)}{(D(p))^2}\right)$$

$$- \left(1 - \frac{\frac{D(p)}{m}\ C'(\frac{D(p)}{m})\ \frac{D'(p)}{m} - C(\frac{D(p)}{m})\ \frac{D'(p)}{m}}{(\frac{D(p)}{m})^2}\right) > 0 \quad \Leftrightarrow$$

$$m\ (D(p))^2 - m\ D(p)\ C'(D(p))\ D'(p) - m\ C(D(p))\ D'(p)$$

$$> (D(p))^2 - D(p)\ C'(\frac{D(p)}{m})\ D'(p) - m\ C(\frac{D(p)}{m})\ D'(p)$$

This follows from:

$$m\ D(p)\ C'(D(p))\ D'(p) + m\ C(D(p))\ D'(p)$$

$$< D(p)\ C'(\frac{D(p)}{m})\ D'(p) + m\ C(\frac{D(p)}{m})\ D'(p) \quad \Leftrightarrow$$

$$m\ D(p)\ C'(D(p)) + m\ C(D(p))$$

$$> D(p)\ C'(\frac{D(p)}{m}) + m\ C(\frac{D(p)}{m})$$

and this is implied by

$$C'(D(p)) > C'(\frac{D(p)}{m}) \quad \wedge \quad C(D(p)) > C(\frac{D(p)}{m})$$

This is true because both C and C' are increasing.

(v)

From (iv) follows that $\bar{\bar{p}}_m > \bar{\bar{p}}_{m+1}$ is equivalent to

$$\pi_1(\bar{\bar{p}}_m) > \pi_{m+1}(\bar{\bar{p}}_m) \qquad \Longleftrightarrow$$

$$\bar{\bar{p}}_m\, D(\bar{\bar{p}}_m) - C(D(\bar{\bar{p}}_m)) > \bar{\bar{p}}_m \frac{D(\bar{\bar{p}}_m)}{m+1} - C\left(\frac{D(\bar{\bar{p}}_m)}{m+1}\right) \qquad \Longleftrightarrow$$

$$\bar{\bar{p}}_m > \frac{C(D(\bar{\bar{p}}_m)) - C\left(\frac{D(\bar{\bar{p}}_m)}{m+1}\right)}{D(\bar{\bar{p}}_m) - \frac{D(\bar{\bar{p}}_m)}{m+1}}$$

By definition $\bar{\bar{p}}_m$ satisfies:

$$\pi_1(\bar{\bar{p}}_m) = \pi_m(\bar{\bar{p}}_m) \qquad \Longleftrightarrow$$

$$\bar{\bar{p}}_m\, D(\bar{\bar{p}}_m) - C(D(\bar{\bar{p}}_m)) = \bar{\bar{p}}_m \frac{D(\bar{\bar{p}}_m)}{m} - C\left(\frac{D(\bar{\bar{p}}_m)}{m}\right) \qquad \Longleftrightarrow$$

$$\bar{\bar{p}}_m = \frac{C(D(\bar{\bar{p}}_m)) - C\left(\frac{D(\bar{\bar{p}}_m)}{m}\right)}{D(\bar{\bar{p}}_m) - \frac{D(\bar{\bar{p}}_m)}{m}}$$

Therefore it is sufficient to prove:

$$\frac{C(D(\bar{\bar{p}}_m)) - C\left(\frac{D(\bar{\bar{p}}_m)}{m+1}\right)}{D(\bar{\bar{p}}_m) - \frac{D(\bar{\bar{p}}_m)}{m+1}} < \frac{C(D(\bar{\bar{p}}_m)) - C\left(\frac{D(\bar{\bar{p}}_m)}{m}\right)}{D(\bar{\bar{p}}_m) - \frac{D(\bar{\bar{p}}_m)}{m}}$$

This is obviously proved if it can be shown that

$$\frac{C(x) - C(y)}{x - y} \quad \text{where } y < x$$

is increasing in y. The first derivative of this term with respect to y is:

$$\frac{(x - y)(-C'(y)) - (C(x) - C(y))(-1)}{(x - y)^2}$$

This is positive if and only if:

$$C'(y) < \frac{C(x) - C(y)}{x - y}$$

As one can easily verify this is true because C is convex.

(vi)

Since Π_1 is differentiable it is sufficient to show that

$$\Pi_1'(p) > 0$$

holds for every $p \in]0 ; \bar{\bar{p}}_m[$. This is equivalent to

$$p\, D'(p) + D(p) - C'(D(p))\, D'(p) > 0$$

Since D(p) is positive and D'(p) is negative this follows from:

$$p < C'(D(p))$$

In the remaining parts of this proof it is shown that this inequality is true for every $p \in]0 ; \bar{\bar{p}}_m[$. From part (iv) follows that for every p in this interval the following inequality holds:

$$\Pi_m(p) > \Pi_1(p) \qquad \Leftrightarrow$$

$$p\,\frac{D(p)}{m} - C\left(\frac{D(p)}{m}\right) > p\, D(p) - C(D(p)) \qquad \Leftrightarrow$$

$$p < \frac{C(D(p)) - C\left(\frac{D(p)}{m}\right)}{D(p) - \frac{D(p)}{m}}$$

It is easy to show that the convexity of C implies that the ratio on the right hand side of this inequality is smaller than C'(D(p)). Therefore

$$p < C'(D(p))$$

follows.

Proof of proposition 1:

The proof consists of two parts:

(i)

In the first part of the proof it is shown that for every price-vector $(p_1, p_2, \ldots, p_n)$ which is a Bertrand-equilibrium $p_1 = p_2 = \ldots = p_n$ holds. This is proved indirectly. Assume that in a Bertrand-equilibrium prices are not identical. Let p be the lowest price and let m be the number of firms which choose p. I now distinguish between three cases:

(1) $p < \hat{p}$ and $\Pi_m(p) < 0$

Then consider a firm which chooses the price p. This firm makes losses. Therefore it would be profitable for this firm to deviate from $(p_1, p_2, \ldots, p_n)$ and to choose a price which is higher than p. Thus it could avoid the losses. So $(p_1, p_2, \ldots, p_n)$ is not a Bertrand-equilibrium.

(2) $p < \hat{p}$ and $\Pi_m(p) \geq 0$

Then consider a firm which chooses a price which is higher than p. This firm doesn't sell anything and therefore its profit is zero. For this firm it would be profitable to deviate from $(p_1, p_2, \ldots, p_n)$ and to choose the price p too because then its profit would be $\Pi_{m+1}(p)$ and the parts (ii) and (iii) of the lemma show that $\Pi_m(p) \geq 0$ implies that $\Pi_{m+1}(p)$ is positive. Therefore the vector $(p_1, p_2, \ldots, p_n)$ is not a Bertrand-equilibrium.

(3) $p \geq \hat{p}$

Then the profit of every firm is zero. If a firm deviates from $(p_1, p_2, \ldots, p_n)$ and chooses a price p' which is smaller than p its profit will be $\Pi_1(p')$. Part (ii) of the lemma shows that there is a price p' which is smaller than p and for which $\Pi_1(p')$ is positive. Therefore each firm has an incentive to deviate from $(p_1, p_2, \ldots, p_n)$. So this pricevector is not a Bertrand-equilibrium.

(ii)

In the second part of this proof it is shown that a pricevector $(p_1, p_2, \dots, p_n)$ with $p_1 = p_2 = \dots = p_n = p$ is a Bertrand-equilibrium if and only if p is an element of the interval $[\bar{p}_n ; \bar{\bar{p}}_n]$. The definition of Bertrand-equilibria implies immediately that $(p_1, p_2, \dots, p_n)$ is a Bertrand-equilibrium if and only if the following two conditions are satisfied:

$$(*) \quad \Pi_n(p) \geq 0$$

$$(**) \quad \Pi_n(p) \geq \sup_{p' \in [0;p[} \Pi_1(p')$$

I show first that these two conditions can't be satisfied if the price p is not contained in the interval $[\bar{p}_n ; \bar{\bar{p}}_n]$. For all p with $p < \bar{p}_n$ the first condition is violated. This shows part (ii) of the lemma. For all p with $\bar{\bar{p}}_n < p < \bar{p}_n$ we have according to part (iv) of the lemma: $\Pi_1(p) > \Pi_n(p)$. For continuity reasons this implies that there is a p' which is very close to p but smaller than p for which $\Pi_1(p') > \Pi_n(p')$ holds. Therefore p doesn't satisfy (**). For $p \geq \hat{p}$ we have: $\Pi_n(p) = 0$. But part (ii) of the lemma implies that there is a $p' < p$ with $\Pi_1(p') > 0$. So again (**) is violated. So it is proved that for every price which is not contained in the interval $[\bar{p}_n ; \bar{\bar{p}}_n]$ either condition (*) or condition (**) is violated.
I show next that every price p with $p \in [\bar{p}_n ; \bar{\bar{p}}_n]$ satisfies both conditions. For the first condition this is an immediate consequence of the lemma. So consider condition (**). In part (vi) of the lemma it was stated that Π_1 is increasing on $[0 ; \bar{\bar{p}}_n]$. Therefore $p \leq \bar{\bar{p}}_n$ implies:

$$\sup_{p' \in [0;p[} \Pi_1(p') = \Pi_1(p)$$

So (**) is equivalent to:

$$\Pi_n(p) > \Pi_1(p)$$

Part (iv) of the lemma shows that this is satisfied for every $p \in [\bar{p}_1 ; \bar{\bar{p}}_n]$.

Proof of proposition 2:

The lemma shows that a price p is an interior point of the interval $[\bar{p}_n ; \bar{\bar{p}}_n]$ if and only if $\Pi_n(p) > 0$ and $\Pi_n(p) > \Pi_1(p)$ holds. So it has to be proved that the competitive price p_c satisfies these two conditions.

$\Pi_n(p_c) > 0$ can be proved as follows: First it is easy to see that p_c satisfies the following condition:

$$p_c = C'\left(\frac{D(p_c)}{n}\right)$$

Since the strict convexity of C and the assumption $C(0) = 0$ imply that for positive quantities average costs are smaller than marginal costs and since $D(p_c)$ is positive we can conclude:

$$p_c > \frac{C\left(\frac{D(p_c)}{n}\right)}{\frac{D(p_c)}{n}} \qquad \Leftrightarrow$$

$$p_c \frac{D(p_c)}{n} - C\left(\frac{D(p_c)}{n}\right) > 0 \qquad \Leftrightarrow$$

$$\Pi_n(p_c) > 0$$

The second condition: $\Pi_n(p_c) > \Pi_1(p_c)$ can be proved as follows: By definition p_c satisfies:

$$\Pi_n(p_c) \geq p_c\, x - C(x)$$

for every nonnegative x with $x \neq D(p_c)/n$. Actually one can easily show that as a consequence of the strict convexity of C the inequality is strict if x is positive and $x \neq D(p_c)/n$ holds. Since $D(p_c)$ is positive substituting $D(p_c)$ for x yields:

$$\Pi_n(p_c) > p_c\, D(p_c) - C(D(p_c)) = \Pi_1(p_c)$$

This completes the proof.

Proof of proposition 3:

The two statements of the proposition are proved seperately:

(i) $\lim_{n \to \infty} \bar{p}_n = C'(0)$

The parts (ii) and (iii) of the lemma show that the sequence $(\bar{p}_n)$ is decreasing and bounded below by C'(0). This implies that it is convergent and that

$$\lim_{n \to \infty} \bar{p}_n \geq C'(0)$$

holds. Therfore it is sufficient to show that

$$\lim_{n \to \infty} \bar{p}_n \leq C'(0)$$

holds. Now $\bar{p}_n$ is defined by the following equation:

$$\Pi_1(\bar{p}_n) = 0 \qquad \Leftrightarrow$$

$$\bar{p}_n \frac{D(\bar{p}_n)}{n} - C\left(\frac{D(\bar{p}_n)}{n}\right) = 0 \qquad \Leftrightarrow$$

$$\bar{p}_n = \frac{C\left(\frac{D(\bar{p}_n)}{n}\right)}{\frac{D(\bar{p}_n)}{n}}$$

Since the average costs increase this implies:

$$\bar{p}_n < \frac{C\left(\frac{D(0)}{n}\right)}{\frac{D(0)}{n}}$$

Taking limits on both sides yields:

$$\lim_{n \to \infty} \bar{p}_n \leq \lim_{n \to \infty} \frac{C\left(\frac{D(0)}{n}\right)}{\frac{D(0)}{n}} = \lim_{x \to 0} \frac{C(x)}{x}$$

According to L'Hospital's rule this is equivalent to:

$$\lim_{n \to \infty} \bar{p}_n \leq C'(0)$$

This had to be shown.

(ii) $\lim_{n \to \infty} \bar{\bar{p}}_n = \bar{p}_1$

The parts (iii) and (iv) of the lemma show that the sequence $(\bar{\bar{p}}_n)$ is decreasing and bounded below by $\bar{p}_1$. This implies that it is convergent and that

$$\lim_{n \to \infty} \bar{\bar{p}}_n \geq \bar{p}_1$$

holds. Therefore it is sufficient to show that

$$\lim_{n \to \infty} \bar{\bar{p}}_n \leq \bar{\bar{p}}_1$$

holds. For the proof of this inequality consider any p with $\bar{p}_1 < p < \hat{p}$. If n is sufficiently high $\Pi_1(p) > \Pi_n(p)$ holds as the following inequalities show:

$$\Pi_1(p) > \Pi_n(p) \qquad \Leftrightarrow$$

$$p\,D(p) - C(D(p)) > p\,\frac{D(p)}{n} - C\left(\frac{D(p)}{n}\right) \qquad \Leftarrow$$

$$p\,D(p) - C(D(p)) > p\,\frac{D(p)}{n} \qquad \Leftrightarrow$$

$$n > \frac{p\,D(p)}{p\,D(p) - C(D(p))}$$

But the lemma and proposition 1 show that $\Pi_1(p) > \Pi_n(p)$ implies:

$$\bar{\bar{p}}_n < p$$

So this inequality holds for every p with $\bar{p}_1 < p < \hat{p}$ if n exceeds a certain value (which depends on p). This implies:

$$\lim_{n\to\infty} \bar{\bar{p}}_n \leq \bar{p}_1$$

which had to be shown.

Proof of proposition 4:

In this proof I denote by $p_{c,n}$ the price which is the competitive price if the number of firms is n. I have to prove that

$$\lim_{n\to\infty} p_{c,n} = C'(0)$$

holds. The previous results show that the sequence $(p_{c,n})$ is decreasing and bounded below by C'(0). This implies that it is convergent and that

$$\lim_{n\to\infty} p_{c,n} \geq C'(0)$$

holds. Therfore it is sufficient to show that

$$\lim_{n\to\infty} p_{c,n} \leq C'(0)$$

holds. It is rather obvious that for every $n \in \mathbb{N}$

$$p_{c,n} \leq C'\left(\frac{D'(0)}{n}\right)$$

holds because for prices which are higher than C'(D(0)/n) the total supply exceeds D(0) which is the highest possible demand. Taking limits on both sides of the inequality yields:

$$\lim_{n\to\infty} p_{c,n} \leq \lim_{n\to\infty} C'\left(\frac{D(0)}{n}\right) = C'\left(\lim_{n\to\infty}\frac{D(0)}{n}\right) = C'(0)$$

This completes the proof.