$\mathbf{QM}_\infty$

1 Introduction

What I'll call theories of *ordinary quantum mechanics* (QM) concern systems of finitely many particles. There is a standard notion of what it takes to quantize such a system. One finds a *Hilbert space representation*, that is, a set of operators acting on a Hilbert space and obeying relations characteristic of the system at hand: *canonical commutation relations* (CCRs) for mechanical systems, or *canonical anticommutation relations* (CARs) for spin systems. Observables pertaining to the system are obtained by taking linear combinations of, and limits of linear combinations of, the representation-bearing operators. (This procedure yields what's known as the CCR/CAR *algebra*.) A state of the system is a well-behaved expectation value assignment to these observables. Thus a Hilbert space representation of the CCRs or CARs appropriate to a quantum system supplies a *kinematics*—an account of the possible states and the magnitudes in their scope—for that system.

Most interpreters of ordinary QM take quantum kinematics, in the form determined by a Hilbert space representation, as their point of departure. After that, they typically diverge. They disagree about whether to supplement these kinematics' bare quantum states with hidden variables. They disagree about whether quantum *dynamics* (that is, the time evolution of quantum states and observables) is collapse-ridden. They disagree about whether a quantum system can exhibit a determinate observable value its state cannot predict with certainty.[1]

Interpreters of ordinary QM have a lot to be anxious about. But they generally needn't worry that the structure framing interpretive disputes of the sort just catalogued might admit multiple, *physically inequivalent* instantiations. Suppose there were manifold realizations of the directive "quantize this system" which couldn't be understood as notational variants on one another. Anxiety, anxiety that there's no univocal quantum theory of the system for interpreters to disagree about to begin with, looms.

But, thanks to the Stone-von Neumann uniqueness theorem, not over interpreters of ordinary QM. The theorem states that all Hilbert space rep-

[1] Albert 1994 is an accessible overview of some of these disagreements and how they interact.

resentations of the CCRs for a particular classical Hamiltonian theory of finitely many particles stand to one another in a mathematical relation called *unitary equivalence*. Unitary equivalence is widely accepted as a standard of physical equivalence for Hilbert space representations. It follows that all these representations are simply and unalarmingly different ways of expressing the same quantum kinematics. For finitely many spin systems subject to CARs, Wigner's theorem likewise guarantees uniqueness. For systems of finitely many particles, the directive "quantize!" has a unique outcome.

These uniqueness theorems do not extend to quantum field theories (QFTs), which one obtains not by quantizing a system of finitely many particles, but by quantizing a field, an entity defined at every point in space(time). Neither do they apply to the thermodynamic limit of quantum statistical mechanics (QSM), where the number of systems one considers and the volume they occupy are taken to be infinite. I'll collect such theories under the heading "QM_∞". What's provoking about QM_∞ is this: According to very same criterion of physical equivalence by whose lights Hilbert space representations for ordinary quantum theories are reassuringly unique, a QM_∞ theory can admit infinitely many presumptively physically inequivalent Hilbert space representations. Interpreters of QM_∞ are prone to anxieties unknown to interpreters of ordinary QM.

This chapter chronicles those anxieties. It opens with brief accounts of quantization (§2) and its uniqueness for ordinary QM (§3). After illustrating the non-uniqueness of Hilbert space representations in QM_∞ (§4) and some ensuing interpretive questions (§5), it sketches the rudiments of the algebraic approach to quantum theories, which discloses a structure common even to unitarily inequivalent Hilbert space representations (§6). The final section articulates and evaluates different strategies for interpreting QM_∞ in the face of the non-uniqueness of Hilbert space representations. The chapter will be mathematically informal. It will also be limited in scope. Excellent correctives to both these failings are Halvorson and M*ü*ger (2007) for QFT,[2] and Sewell (2002) for QSM.

[2]See also Huggett (2000), Teller (1995), or Sklar (2002) for philosophical discussions of the important topic, neglected here, of the renormalization of interacting QFTs.

2 Quantizing

Representing the CCRs

In classical Hamiltonian mechanics, the state of a simple mechanical system is given by its position and momentum. The position and momentum variables q_i and p_j therefore serve as coordinates for the phase space M of possible states of the system. They're known as *canonical coordinates*. System observables are functions from M to $\mathbb{R}$. The position and momentum *observables*, for example, map points in phase space to their q_i and p_j coordinate values, respectively. All other observables can be expressed as functions of these observables. Of particular significance is the Hamiltonian observable H, which usually coincides with the sum of the system's kinetic and potential energies. The Hamiltonian helps identify dynamically possible trajectories $\mathbf{q}(t), \mathbf{p}(t)$ through phase space as those obedient to Hamilton's equations of motion, which are equivalent to Newton's second law:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{1}$$

Here, and for the sake of mathematical tractability, we'll confine our attention to linear dynamical systems, that is (more or less) those whose equations of motion are linear in the canonical coordinates.[3]

To prepare the ground for the quantization of a classical Hamiltonian thery, we ought to say a little more. We ought to say something about the *algebraic* structure of classical observables. As smooth functions on phase space, classical observables form a set that is also a vector space over the real numbers. To first approximation, an algebra is a linear vector space V endowed with a (not necessarily associative) multiplicative structure. The vector space of classical observables becomes a *Lie algebra* upon being equipped with a multiplicative structure supplied by the Poisson bracket. The Poisson bracket $\{f, g\}$ of classical observables $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ is

$$\{f, g\} := \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2}$$

[3] For a precise characterization, see Wald 1994, 14 ff. or Marsden and Raitu 1994, §2.7 ff.

It provides another way of formulating Hamilton's equations (1).

$$\frac{dq_i}{dt} = \{q_i, H\}, \ \frac{dp_i}{dt} = \{p_i, H\} \tag{3}$$

Notice that for observables p and q

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \ \{p_i, q_j\} = -\delta_{ij} \tag{4}$$

Considered both as coordinates and as observables, p_j and q_i are canonical *because* they satisfy (4).

Now, to quantize a classical theory cast in Hamiltonian form, one promotes its canonical observables to symmetric operators $\hat{q}_i, \hat{p}_i$ acting on a separable[4] Hilbert space $\mathcal{H}$ and obeying commutation relations corresponding to the Poisson brackets of the classical theory. For a mercifully simple classical theory with phase space $\mathbb{R}^{2n}$ and canonical observables q_i and p_i, these CCRS are (where $[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}$ and $\hat{I}$ is the identity operator) (cf. (4))

$$[\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0, [\hat{p}_i, \hat{q}_j] = -\frac{\hat{I}\delta_{ij}}{i} \tag{5}$$

Representing the CARs

To quantize a single spin system, one finds symmetric operators $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ acting on a Hilbert space $\mathcal{H}$ to satisfy the *Canonical Anticommutation Relations (CARs)*,

$$[\hat{\sigma}_x, \hat{\sigma}_y] = i\hat{\sigma}_z, \ \ [\hat{\sigma}_y, \hat{\sigma}_z] = i\hat{\sigma}_x, \ \ [\hat{\sigma}_z, \hat{\sigma}_x] = i\hat{\sigma}_y \tag{6}$$

$$\hat{\sigma} \cdot \hat{\sigma} = (\hat{\sigma}_x)^2 + (\hat{\sigma}_y)^2 + (\hat{\sigma}_z)^2 = 3I \tag{7}$$

Call these $\{\hat{\sigma}_i\}$ the Pauli spin observables. The generalization to n spin systems is straightforward. To quantize such a system, one finds for each spin system k a Pauli spin $\hat{\sigma}^k = (\hat{\sigma}^k_x, \hat{\sigma}^k_y, \hat{\sigma}^k_z)$ satisfying the CARs, expanded to include the requirement that spin observables for different systems commute:

$$[\hat{\sigma}^k_x, \hat{\sigma}^{k'}_y] = i\delta_{kk'}\hat{\sigma}^k_z, \ \text{etc.}, \tag{8}$$

[4] Recall that a separable Hilbert space is one with a countable basis.

Having crafted a Hilbert space representation of the CARs/CCRs, one prosecutes QM as usual. One enriches ones set of physical magnitudes by taking linear combinations, and limits of linear combinations, of the canonical magnitudes representing the CARs/CCRs. The result is the self-adjoint part of $\mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space carrying the representation.

3 Uniqueness in Ordinary QM

Werner and Irwin are instructed to quantize a system obeying CCRs and possessing n degrees of freedom. Paul and Wolfgang are asked to do the same for a CAR system of m degrees of freedom. No matter how idiosyncratically Werner and Irwin proceed, they produce the same quantization—or so the Stone-von Neumann theorem is generally understood to imply. No matter how perversely Paul and Wolfgang proceed, their quantizations coincide as well—or so Wigner's theorem is generally understood to imply. Getting a handle on how these implications work is a useful a prolegomenon to interpreting QM_∞.

Think of a physical theory as running a sort of modal toggle. The theory sorts logically possible worlds into two sorts: those that are (by its lights) physically possible, and those that are not. The content of a theory so construed consists of the set of worlds possible according to it. And two theories are physically equivalent – they have the same content – just in case they put the same worlds into the "physically possible" pile. This content coincidence criterion of physical equivalence, which descends from the modal toggle picture of a physical theory, is one part of the background against which the Stone-von Neumann and Wigner theorems emerge as a demonstrations of the physical equivalence of quantizations in their scope.

Another part of the background is a set of substantive assumptions about the particular shape of those quantizations, and in particular about the structures with respect to which they describe and individuate their possible worlds. A quantization is modelled as a pair $(\mathcal{M}, \mathcal{S})$, whose first entry denotes the set of physical magnitudes it recognizes, and whose second entry denotes its set of states. Given that a state is a map from magnitudes to their expectation values, an ideology about how such maps should behave could be sufficient to determine $\mathcal{S}$ given $\mathcal{M}$. For instance, a theory of ordinary QM equates $\mathcal{M}$ with $\mathfrak{B}(\mathcal{H})$ for some separable $\mathcal{H}$, and adopts the ideology that states should be normalized, linear, positive, and countably additive.

It follows from Gleason's theorem that ordinary QM must equate $\mathcal{S}$ with $\mathfrak{T}^+(\mathcal{H})$, the set of density operators on $\mathcal{H}$. Each $\hat{W} \in \mathfrak{T}^+(\mathcal{H})$ determines a state via the trace prescription, which assigns each self-adjoint $\hat{A} \in \mathfrak{B}(\mathcal{H})$ the expectation value $Tr(\hat{W}\hat{A})$. I persist in modelling a quantization as an ordered *pair* $(\mathcal{M}, \mathcal{S})$ because ideologies of state are discretionary, even optional. Imposing a *universal* rule for generating $\mathcal{S}$ given $\mathcal{M}$ would obliterate a certain sort of interpretational freedom.

The pair $(\mathfrak{B}(\mathcal{H}), \mathfrak{T}^+(\mathcal{H}))$ encapsulates what worlds are possible according to a quantization of the sort under discussion. The sets of worlds possible according to quantizations $(\mathfrak{B}(\mathcal{H}), \mathfrak{T}^+(\mathcal{H}))$ and $(\mathfrak{B}(\mathcal{H}'), \mathfrak{T}^+(\mathcal{H}'))$ coincide exactly when there are isomorphisms $i_{obs} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}')$ and $i_{state} : \mathfrak{T}^+(\mathcal{H}) \to \mathfrak{T}^+(\mathcal{H}')$ such that

$$Tr\left(i_{state}(\hat{W}) i_{obs}(\hat{A})\right) = Tr(\hat{W}\hat{A}) \tag{9}$$

for all $\hat{W} \in \mathfrak{T}^+(\mathcal{H})$ and all $\hat{A} \in \mathfrak{B}(\mathcal{H})$.[5] (9) ensures that to every state $\hat{W}$ in one quantization there stands a state $i_{state}(\hat{W})$ in the other that's the counterpart of the first in the sense that $i_{state}(\hat{W})$'s assignment of expectation values to observables $i_{obs}(\hat{A})$ perfectly mimics $\hat{W}$ assignment of expectation values to observables $\hat{A}$. We are a definition away from being able to announce a necessary and sufficient condition for the physical equivalence of quantizations.

> A Hilbert space $\mathcal{H}$, and a collection of operators $\{\hat{O}_i'\}$ is *unitarily equivalent* to $(\mathcal{H}', \{\hat{O}_i'\})$ if and only if there exists a one-to-one, linear, norm-preserving transformation ("unitary map") $U : \mathcal{H} \to \mathcal{H}'$ such that $U^{-1}\hat{O}_i' U = \hat{O}_i$ for all i.

Now for the announcement:

> Quantizations $(\mathfrak{B}(\mathcal{H}), \mathfrak{T}^+(\mathcal{H}))$ and $(\mathfrak{B}(\mathcal{H}'), \mathfrak{T}^+(\mathcal{H}'))$ are physically equivalent, if and only if $(\mathcal{H}, \mathfrak{B}(\mathcal{H}))$ and $(\mathcal{H}', \mathfrak{B}(\mathcal{H}'))$ are unitarily equivalent.

The U effecting their unitary equivalence furnishes both the bijection i_{state} from the first theory's state space to the second's and the bijection i_{obs} from the first theory's observable set to the second's. Unitary equivalence is the relation, the Stone-von Neumann theorem assures us, all representations of

[5]Clifton and Halvorson 2001 develop and deploy this criterion of physical equivalence.

the CCRs for n degrees of freedom (n finite) enjoy with one another.[6] It's the relation, Wigner's theorem assures us, all representations of the CARs for n degrees of freedom (n finite) enjoy with one another. According to the content coincidence criterion of physical equivalence, unitarily equivalent quantizations are physically equivalent.

4 Non-uniqueness in QM_∞

QM_∞ abounds with what the Stone-von Neumann and Wigner theorems declared unimaginable for ordinary QM: unitarily inequivalent representations of the CCR/CAR algebras. Anyone even minimally acquainted with philosophical treatments of QM has on hand the resources to describe a quantum system admitting unitarily inequivalent representations: a chain of infinitely many spin $\frac{1}{2}$ systems.[7]

As a warmup, consider a finite number of spin $\frac{1}{2}$ systems, arranged in a one-dimensional lattice. The CARs [(6)-(8)] specify what it takes to quantize this system. One way to build a Hilbert space representation fitting these specifications is to use a vector space $\mathcal{H}$ spanned by a basis consisting of sequences s_k, where k ranges from $-n$ to n, and each entry takes one of the values ± 1. (NB there are finitely many distinct such sequences–finitely many ways to map a set of finite cardinality into the set $\{+1, -1\}$.) Operators $\hat{\sigma}_z^m$ are introduced in such a way that sequences s_k whose m^{th} entry is ± 1 correspond to eigenvectors associated with the eigenvalue ± 1. Operators $\hat{\sigma}_x^k, \hat{\sigma}_y^k$ conspiring with these to satisfy the CARs are introduced by analogy to their single electron counterparts.

The *polarization* of a system is described by a vector whose magnitude ($\in [0,1]$) gives the strength and whose orientation gives the direction of the system's magnetization. On a single electron, it is represented by an observable $\hat{\mathbf{m}}$ whose three components correspond to three orthogonal components of quantum spin. For example, in the $+1$ eigenstate $|+\rangle$ of $\hat{\sigma}_z$ (understood as the z-component of spin), $\langle \hat{\mathbf{m}} \rangle = +1$ along the z axis. For the finite spin chain, the polarization observable has components $\hat{m}_i := \frac{1}{2n+1} \sum\limits_{k=-n}^{n} \hat{\sigma}_i^k$. $\hat{\mathbf{m}}$ belongs to $\mathfrak{B}(\mathcal{H})$ because its components are polynomials of Pauli spins.

[6]This needs to be qualified slightly. See Summers (2001) for details.

[7]Here I follow Sewell 2002, §2.3, to which I refer the reader for details.

In the basis sequence s_k, the polarization observable $\hat{\mathbf{m}}$ takes an expectation value of magnitude $\frac{1}{2n+1}\sum_{k=-n}^{n}[s_k]$ (where $[s_k]$ is the k^{th} entry of the sequence s_k) directed along the z axis. The polarization attains extreme values (of ± 1) for those sequences every term of which is the same. Let $\hat{W}$ be a state in this quantization assigning $\hat{m}_z$ the expectation value $+1$. Wigner's theorem ensures that any other representation of the CARs will be unitarily equivalent to this one. Any other representation of the CARs is thus guaranteed to contain a state $\hat{W}'$ (the image of $\hat{W}$ under the unitary map implementing the equivalence of the representations) and an observable $\hat{m}_z'$ (the image of $\hat{m}_z$ under that map) such that the expectation value of $\hat{m}_z'$ in the state $\hat{W}'$ is $+1$.

Now consider a doubly infinite chain, labelled by the positive and negative integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, of spin $\frac{1}{2}$ systems. As before, to quantize this system, we must associate with each site k a trio $\hat{\sigma}_k = (\hat{\sigma}_x^k, \hat{\sigma}_y^k, \hat{\sigma}_z^k)$ satisfying the CARs. But if we follow the strategy adopted for the finite spin chain, and attempt to construct our Hilbert space from a basis consisting all possible maps from $\mathbb{Z}$ to $\{\pm 1\}$, we are foiled. Because the set of such maps is non-denumerable, the Hilbert space we'd construct would be non-separable, breaking the tradition of using separable Hilbert spaces for physics.[8]

Here's one way to build a separable Hilbert space representation of the CARs for an infinite chain of spins. Start with the sequence $[s_k] = +1$ for $k \in \mathbb{Z}$, and add all sequences obtainable therefrom by finitely many local modifications. The resulting basis consists of all sequences for which all but a finite number of sites take the value $+1$. Continue to follow the model of the finite spin chain to introduce operators $\hat{\sigma}_k^+$ satisfying the CARs. I will call this representation $(\mathcal{H}^+, S^+)$.

Before considering how the polarization observable $\hat{\mathbf{m}}^+$ behaves on $(\mathcal{H}^+, S^+)$, let us assure ourselves that there is such an observable. To be an element of $\mathfrak{B}(\mathcal{H}^+)$, the polarization observable must be a polynomial of Pauli spins, or the limit (in the appropriate sense) of a sequence of such polynomials. The sequence $\hat{m}_i^N := \frac{1}{2N+1}\sum_{k=-N}^{N}\hat{\sigma}_i^k$ has a limit as $N \to \infty$ in $\mathcal{H}^+$'s weak topology, which provides the appropriate sense of convergence. A sequence $\hat{A}_i$ of operators on $\mathcal{H}$ converges to an operator $\hat{A}$ in $\mathcal{H}$'s weak operatory topology iff for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, $|\left\langle \psi|(\hat{A} - \hat{A}_i)|\phi \right\rangle|$ goes to 0 as i goes to ∞. Confining

[8]Halvorson 2004 defends just such iconoclasty.

attention to the sequence $\hat{m}_z^N$, observe that $\langle s_k|\hat{m}_z^N|s_k'\rangle = \frac{1}{2N+1}\sum_{k=-N}^{N}(\frac{[s_k]+[s_k']}{2})$ for basis sequences s_k and s_k'. Because -1 occurs only finitely many times in each basis sequence, $\frac{\langle s_k|\hat{m}_z^N|s_k'\rangle}{2N+1}$ will converge to 1 as $N \to \infty$, no matter what s_k and s_k' are. This shows that the sequence $\hat{m}_z^N$ converges weakly to the identity operator. We can imagine realizations of the CARs, modelled on $(\mathcal{H}^+, S^+)$, for which this is not so: for example, a representation whose "ground state" s_k is a sequence for which $\lim_{N\to\infty} \langle s_k|\, \hat{m}_i^N \,|s_k\rangle$ does not converge (e.g., the sequence $[s_k] = +1$ for $2^n < |k| \leq 2^{n+1}$ and n odd; $[s_k] = -1$ otherwise).

Other components of global polarization are also captured as weak limits of polynomials of Pauli spins. Each component of polarization is thus an element of $\mathfrak{B}(\mathcal{H}^+)$, and so an observable in ordinary QM's sense. For every state in the basis, $\langle \hat{\mathbf{m}}^+\rangle$ will be oriented along the z axis and take the value $\lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n}[s_k]$. Because for each basis element, all but a finite number of its entries take the value $+1$, this limit will be 1. Every state representable on S^+ will inherit this feature from the basis vectors in terms of which it is expressed: every state in the representation will have unit polarization in the positive z direction.

Because the chain is infinite, Wigner's theorem does not imply that the quantization just constructed is unique up to unitary equivalence. And it is not. Consider, for instance, a representation set in a Hilbert space whose basis elements correspond to the sequence $[s_k] = -1$ for $k \in \mathbb{Z}$, along with all sequences obtainable from this one by finitely many local modifications. The basis consists, then, of sequences for which all but a finite number of sites take the value -1. Operators $\hat{\sigma}_k^-$ satisfying the CARs are introduced in such a way that the n^{th} entry in the basis sequence s_k gives the expectation value of $\hat{\sigma}_z^n$. Call this representation $(\mathcal{H}^-, S^-)$. By parity of reasoning, $\hat{\mathbf{m}}^-$ is an observable in this quantization, one assigned the expectation value -1 by every state in S^-.

We can now see that $(\mathcal{H}^-, S^-)$ and $(\mathcal{H}^+, S^+)$ are not unitarily equivalent. Suppose, for contradiction, that they were. Then there'd be a unitary $U : \mathcal{H}^+ \to \mathcal{H}^-$ such that $U\hat{\sigma}_n^+U^{-1} = \hat{\sigma}_n^-$ for all n. Where $\hat{m}_N^{\pm} := \frac{1}{2N+1}\sum_{n=-N}^{N}\hat{\sigma}_n^{\pm}$, this implies that $\hat{m}_N^- = U\hat{m}_N^+U^{-1}$. For $|\psi^+\rangle$ and $|\psi^-\rangle$, unit vectors in $\mathcal{H}^+$ and $\mathcal{H}^-$related by $|\psi^-\rangle = U\ |\psi^+\rangle$, it follows that

$$\langle\psi^+|\hat{m}_N^+|\psi^+\rangle = \langle\psi^-|\hat{m}_N^-|\psi^-\rangle \tag{10}$$

But in the limit $N \to \infty$, (10) breaks down: the r.h.s. (which gives the expectation value the state $|\psi^-\rangle$ assigns the polarization observable $\hat{\mathbf{m}}^-$) and the l.h.s. (which gives the expectation value the state $|\psi^+\rangle$ assigns the polarization observable $\hat{\mathbf{m}}^+$) go to $+1$ and -1 respectively. The expectation values $|\psi^+\rangle$ assigns observables in $\mathfrak{B}(\mathcal{H}^+)$ do not replicate the expectation values $|\psi^-\rangle$ assigns the images of those observables (in $\mathfrak{B}(\mathcal{H}^-)$) under the map U. The quantizations $(\mathcal{H}^-, S^-)$ and $(\mathcal{H}^+, S^+)$ are not unitarily equivalent. We can generate continuously many unitarily inequivalent quantizations of the infinite spin chain by suitable modifications of the strategy used to obtain $(\mathcal{H}^-, S^-)$ and $(\mathcal{H}^+, S^+)$.[9] These quantizations are presumptively physically inequivalent as well.

The surfeit is not peculiar to representations of the CARs. The CCRs for infinitely many degrees of freedom also admit continuously many unitarily inequivalent representations. Indeed, the representation (q_i', p_i') obtained from a standard representation (q_i, p_i) by the apparently innocuous rescaling

$$p_i' = \frac{p_i}{a}, \quad q_i' = aq_i \tag{11}$$

is unitarily inequivalent to its progenitor (Segal 1967).

Unitarily inequivalent representations of the CARs contradict one another concerning the apparently physical matter of the magnitude and direction of the global polarization. There is also something evidently physical at stake between unitarily inequivalent representations of the CCRs. One way to see this is to consider the conventional representation for the CCRs arising from a classical field theory, such as Klein-Gordon theory for a fixed mass m (see Wald 1994, Ch. 3). The ensuing quantization tempts a particle interpretation. It incorporates a state ($|0\rangle$) whose energy and momentum suggests the absence of particles, as well as states ("single particle states")

[9] Here, roughly, is why: The strategy was, starting from a particular map s_0 from $\mathbb{Z}$ to $\{\pm 1\}$, to build a vector space from the collection of such maps differing from s_0 in only finitely many places. "Differing from one another in only finitely many places" is an equivalence relation on the set of maps from $\mathbb{Z}$ to $\{\pm 1\}$, which set has the cardinality of the continuum. Each equivalence class has the cardinality of the natural numbers and corresponds to a quantization of the infinite spin chain–different equivalence classes to unitarily inequivalent quantizations. So there must be continuously many inequivalent quantizations.

whose energies and momenta suggest the presence of one particle, and so on. Every state in the representation's Hilbert space can be obtained as a linear combination of states of this sort, or as a limit of a sequence of such linear combinations. The quantization moreover hosts an observable $\hat{N}$ that appears to count particles. $\hat{N}$'s expectation value in $|0\rangle$ is 0; $\hat{N}$'s expectation value in an n-particle state is n. Unconventional representations of the CCRs can also have structures inviting a particle interpretation: a vacuum state $|0'\rangle$, a total number operation $\hat{N}'$, and so on.

Let $(|0\rangle, \hat{N})$ and $(|0'\rangle, \hat{N}')$ stand for unitarily inequivalent, particle - interpretation-tempting quantizations. The vacuum state of one isn't even a possible state, according to the other; the total number operator of one just isn't a physical magnitude, according to the other. Applying to disjoint domains, the particle concepts circumscribed by inequivalent quantizations of the Klein-Gordon field are something like incommensurable. But the quantized Klein-Gordon field was supposed to describe just one uncomplicated kind of particle, the free boson of mass m. Continuously many unitarily —and so presumptively, physically—inequivalent quantizations of the Klein-Gordon field are an embarrassment of riches that demands further investigation[10]

Investigations of the formal structure of standard realizations of quantum field theoretic CCRs conducted in the 50s and 60s made possible an intriguing commentary on this embarrassment. Each concrete Hilbert space representation of the CCRs gives rise to an algebra which is representation-independent. The *Weyl algebra* for a system is constructed by taking linear combinations of operators satisfying the CCRs for that system, then closing in the *norm* topology, whose criterion of convergence is stricter than that of the weak topology we encountered earlier. This Weyl algebra is unique in the sense that no matter what Hilbert space realization of the CCRs for the system one starts with, one obtains the same Weyl algebra. Thus independent of its Hilbert space antecedents, the Weyl algebra for a system can be regarded as an *abstract* algebra, the algebraic structure shared by all its Hilbert space representations. For a system whose canonical observables obey CARs, one can likewise abstract from those CARs a single algebraic structure common to every Hilbert space representation of those CARs.

Haag and Kastler give voice to a suspicion prompted by this circumstance:

[10]For further discussion of inequivalent particle concepts in QFT, see Arageorgis et al 2003 and Clifton and Halvorson 2001.

> The relevant object is the abstract algebra and not the representation. The selection of a particular ... representation is a matter of convenience without physical implications. (Haag and Kastler 1964, 852))

Ordinary QM completes the kinematic template $(\mathcal{M}, \mathcal{S})$ in terms set by a particular concrete Hilbert space $\mathcal{H}$. The suspicion is that *these are the wrong terms* for QM_∞, that a theory of QM_∞ deploys states and observables that *aren't* states and observables in ordinary QM's sense. The balance of this chapter articulates and evaluates different strategies for making this suspicion precise.

5 Interpretting QM_∞: Some Questions

A question looms from the wreckage of the uniqueness theorems: how do theories of QM_∞ characterize and individuate physical possibilities? Understand this as follows: how on behalf of such theories should we complete the template $(\mathcal{M}, \mathcal{S})$? On the answer to this question hinges the answer to another: under what circumstances are theories of QM_∞ physically equivalent? To answer these questions is to begin to interpret QM_∞.

Before canvassing answers, we should consider criteria of adequacy to which we might hold them. These criteria are provisional; they're meant to help initiate, not terminate, the interrogation of interpretive options. The exigencies of interpreting QM_∞ may induce us to revise our sense of what it takes to comprehend the physical world by means of an empirical theory. To my mind, the capacity of interpretive projects to inspire such revision is what makes them philosophical.

In "The Theoretician's Dilemma," Hempel (1965) considers the case for taking the content of a scientific theory to be its "Craigified" part—that is, the connections it draws between observable phenomena, connections presented without the intermediary of (nonlogical) theoretical apparatus. Hempel resists the Craigification of theoretical content because he attributes theories a function which he considers their Craigified parts ill-suited to fulfill. That function is explanation. What promotes it, Hempel suggests, is the systematic unification between observable phenomena effected by the theoretical apparatus.

Without going deeper into the details of Hempel's position, we can extract from it one criterion of interpretive adequacy. An interpretation should recognize enough states and enough observables to enable the theory it interprets to discharge its explanatory duties. I don't need—I'm not sure there exists—a categorical imperative for identifying these duties. In applying the criterion, we can start from the explanatory aspirations of physicists using a theory, and aim for a reflective equilibrium between the assessment of interpretations for sustaining those aspirations, and the assessment of those aspirations as appropriate.

"Explanatory duties" is not only vague but also synechdochical. There are a host of things a theory ought to do, and interpretations of a theory are adequate only insofar as they enable the theory to do these things. Because we're approaching the matter of interpretation via the kinematic question of how to fill in the $(\mathcal{M}, \mathcal{S})$ template, it's worth emphasizing that one thing a physical theory ought to do is furnish *dynamics* for the systems it describes. An interpretation that completes the template in a way that hamstrings this dynamical project is on that count suspect.

I've been proceeding as though each theory were an island, complete unto itself. In this mode of address, adequacy of the sort under discussion is a purely internal affair: an interpretation ought to attribute a theory sets of observables and magnitudes rich enough to meet that theory's own aims. But externally oriented questions of sufficiency arise as well. We may, for instance, want to interpret a theory in a way that makes sense of its fit with environing theories and their interpretations. Thus in the case of the thermodynamic limit of QSM, we might hope for an interpretation that attributes the theory enough observables/magnitudes to bring it into a substantial explanatory relationship with thermodynamics. Again, existing theories are conceptual resources from which new theories are forged. Thus we might want an interpretation to recognize enough observables/magnitudes to enable an interpreted theory to serve as such a resource. Semiclassical quantum gravity gives a good example of how this desire plays out in the interpretation of QFT.

Semiclassical quantum gravity considers a quantum field on spacetime manifold subject to Einstein's field equations, with a quantum commodity (the expectation value $< T_{ab} >$ of the stress energy tensor T_{ab}) substituted for a classical one (plain old T_{ab}):

$$G_{ab} = 8\pi < T_{ab} > \tag{12}$$

One of the points of this hybridization is the clues it might hold for what a purebred quantum theory of gravity would look like. An interpretation of

QFT that withheld physical significance from a stress energy observable (or something like it) would blunt that point. It would fail to recognize enough observables to sustain QFT in this developmental task.

One vague and provisional criterion of adequacy for an interpretation of QM_∞ is that it recognize enough states and observables. Another is that it not recognize too many. Although this demand has undeniable intuitive pull, I find it difficult to motivate ecumenically. It is preaching to converted Ockhamists to declare parsimony desirable in itself. Perhaps it helps to observe that parsimony can serve recognized epistemological and metaphysical goals. For instance, there is a failure of parsimony that frustrates the venerable goal of *determinism*. For a theory to have any shot at determinism, its dynamics must specify how the values of some set of magnitudes at one time depend on the values of some set of magnitudes at another. Unparsimoniously recognizing more magnitudes than are dynamically tractable, or more states than can be put into one-to-one correspondence with valuations of tractable dynamical magnitudes, a completion of the kinematic template $(\mathcal{M}, \mathcal{S})$ threatens determinism. Of course, other components of an interpretation incorporating such a template could work to defuse this threat; see, for instance, defenses of substantivalism against the hole argument (Brighouse 1994).

6 Intermission: Algebraica

To better appreciate the suspicion raised at the close of §4, the algebraic framework that grounds it, and the interpretive options that framework makes available, let us undertake a brief review of the rudiments of algebraic approaches to quantum theories.

Intuitively, an *algebra* is just a collection of elements along with a way of taking their products and linear combinations. Insofar as it's the business of natural science to weave physical magnitudes into functional relationships, organizing physical magnitudes into an algebra underwrites that business. Officially, an algebra $\mathfrak{A}$ over the field $\mathbb{C}$ of complex numbers is a set of elements $(A, B, ...)$ with the following features:

> (i) First, $\mathfrak{A}$ is closed under a commutative, associative operation $+$ of binary addition, with respect to which $\mathfrak{A}$ forms a group.
>
> (ii) Second, $\mathfrak{A}$ is closed with respect to a binary multiplication

operation ·, which is associative and distributive with respect to addition, but not necessarily commutative.

(iii) Finally, $\mathfrak{A}$ is closed with respect to multiplication by complex numbers.

Features (i) and (iii) reveal an algebra to be something well-known to those acquainted with ordinary QM– a linear vector space – equipped by feature (ii) with a binary multiplication operation.

More specialized algebras augment this basic structure. A * *algebra*, is an algebra closed under an *involution* $^* : \mathfrak{A} \to \mathfrak{A}$.[11] $\mathfrak{B}(\mathcal{H})$ is a * algebra, with the adjoint operation † supplying the involution. In ordinary QM, an $\hat{A} \in \mathfrak{B}(\mathcal{H})$ such that $\hat{A}^\dagger = \hat{A}$ is *self-adjoint*, and represents an observable magnitude. This is generalized by algebraic QM, which deploys * algebras (without prejudice to whether they coincide with some $\mathfrak{B}(\mathcal{H})$ or not), and identifies their self-adjoint elements with observables.

C^*-algebras

One sort of interweaving of physical magnitudes constitutes one magnitude—the global polarization of an infinite spin chain, say—as the limit of a sequence of other magnitudes. In order to determine which sequences of elements of an algebra have limits, we need a criterion of convergence for such sequences. A *norm* underwrites such a criterion. A norm on $\mathfrak{A}$ is a function that assigns a non-negative real number to each element of $\mathfrak{A}$.[12] The *Hilbert space vector norm*, which maps each vector $|\psi\rangle$ in a Hilbert space to $\sqrt{\langle\psi|\psi\rangle}$, is an example. A space equipped with a norm is thereby equipped with the criterion of convergence (aka, speaking somewhat loosely, a *topology*) induced by that norm: A sequence of v_i of elements of the space V converges to the element v in the norm $\|\| : V \to \mathbb{R}$ if and only if the sequence of real numbers $\|v_n - v\|$ converges to 0 in the usual sense.

We've said enough to abstractly characterize an object of extraordinarily wide application. *A C^* algebra* $\mathfrak{A}$ is a * algebra over $\mathbb{C}$, equipped with a norm, satisfying $\|A^*A\| = \|A\|^2$ and $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in \mathfrak{A}$, with

[11] Satisfying: $(A^*)^* = A, (A+B)^* = A^* + B^*, (cA)^* = \bar{c}A^*(AB)^* = B^*A^*$ for all $A, B \in \mathfrak{A}$ and all complex c.

[12] In more detail, a norm on a vector space V is map $\|\|$ from V into the non-negative reals satisfying $\|cv\| = |c|\|v\|$, $\|v\|\|w\| = \|vw\|$, and $\|v\| + \|w\| > \|v + w\|$ for all $v, w \in V$ and all $c \in \mathbb{C}$.

respect to which norm it is complete.[13] A familiar example of a C^* algebra is the set $\mathfrak{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space $\mathcal{H}$. The Hilbert space operator norm,

$$\|A\| := \sup_{\sqrt{\langle\psi|\psi\rangle}=1} |A\,|\psi\rangle\,| \tag{13}$$

(where $|\psi\rangle$ ranges over $\mathcal{H}$, $|\ |$ is the vector norm, and $\sup_{x\in X} f(x)$ is the least upper bound of the values the function $f(x)$ assumes in the domain X) provides the C^*-algebraic norm. It is no accident that $\mathfrak{B}(\mathcal{H})$ is an example of a C^* algebra. Concretely characterized, *a C^* algebra is a norm-closed self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$ for some $\mathcal{H}$.*

While every $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra, the converse is not the case: a C^* -algebra need only be (upto isomorphism) a *sub*algebra of $\mathfrak{B}(\mathcal{H})$. To see how some elements of $\mathfrak{B}(\mathcal{H})$ could be left out of the norm closure of others, reflect that the norm topology is relatively strict. A Hilbert space sustains other operator topologies with looser admissions criteria. The weak operator topology, encountered in §4's discussion of whether there really was a polarization observable for the infinite spin chain, is one example. There are others: the strong topology, the σ-strong topology, and so on (see Bratelli and Robinson, 1987, §2.4.1 for a disentanglement). As the names imply, strong convergence implies weak convergence, but not vice-versa; norm convergence implies strong convergence, but not vice versa.

The infinite spin illustrates the asymmetry of these implications. As §4 indicates, the sequence $\hat{m}_z^N$ has a limit as $N \to \infty$ in $\mathcal{H}^+$'s weak topology. That limit is the identity operator. But the same sequence lacks a norm limit. In order for $\hat{m}_z^N$ to converge to $\hat{I}$ in $\mathcal{H}^+$'s norm topology, it must be the case that

$$\left\|\hat{m}_z^N - \hat{I}\right\| := \sup_{\sqrt{\langle\psi|\psi\rangle}=1} |(\hat{m}_z^N - \hat{I})\,|\psi\rangle\,| \tag{14}$$

approaches 0 as $N \to \infty$. But this won't happen. For each N, one can find a basis sequence s_k—one starting with N 0's—s.t. $\frac{1}{2n+1}\sum_{k=-N}^{N} \hat{m}_z^k\,|s_k\rangle = 0$. This implies that $\hat{m}_z^N\,|s_k\rangle = 0$, and so that $|(\hat{m}_z^N - \hat{I})\,|s_k\rangle\,| = 1$, and so that $\sup_{\sqrt{\langle\psi|\psi\rangle}=1} |(\hat{m}_z^N - \hat{I})\,|\psi\rangle\,| \geq 1$–which spoils norm convergence.

[13] That is, the limit of every norm-convergent sequence of elements of $\mathfrak{A}$ is itself an element of $\mathfrak{A}$.

A C^*-algebra that is conceived in abstraction admits a concrete Hilbert space representation. A *representation* of a C^*-algebra $\mathfrak{A}$ is an algebraic-structure-preserving map $\pi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$ from the C^*-algebra into (although not necessarily onto) the algebra $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. The notion of unitary equivalence extends straightforwardly to representations of C^*-algebras.

von Neumann algebras

The representation $\pi(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is closed in the norm topology (Bratteli and Robinson 1987, Prop. 2.3.1). Thus starting with $\pi(\mathfrak{A})$ and closing in the weaker topologies $\mathcal{H}$ makes available could eventuate in sets of observables richer than those contained in $\pi(\mathfrak{A})$ proper. Next we consider algebras obtained by following a recipe of this sort.

Concretely characterized, a *von Neumann algebra* $\mathfrak{R}$ is a *-algebra of bounded operators that is strong operator-closed in its action on some Hilbert space. If you want a von Neumann algebra, start with some self-adjoint algebra of bounded Hilbert space operators–$\pi(\mathfrak{A})$, say– and close in the strong operator topology.

Given an algebra $\mathfrak{D}$ of bounded operators on a Hilbert space $\mathcal{H}$, its *commutant* $\mathfrak{D}'$ is the set of all bounded operators on $\mathcal{H}$ that commute with every element of $\mathfrak{D}$. The commutant of $\mathfrak{B}(\mathcal{H})$, for example, consists of scalar multiples of the identity operator. Naturally enough, $\mathfrak{R}$'s *double commutant* $\mathfrak{D}''$ is $\mathfrak{D}'$'s commutant. Every element of $\mathfrak{B}(\mathcal{H})$ commutes with the identity operator, and so with every element of $\mathfrak{B}(\mathcal{H})'$. This makes $\mathfrak{B}(\mathcal{H})$ its own double commutant.

Here's why that's interesting: von Neumann's celebrated double commutant theorem links the "topological" characterization of a von Neumann algebra in terms of closure with an "algebraic" characterization in terms of commutants. Von Neumann showed that the strong and weak closures of a self-adjoint algebra $\mathfrak{D}$ of bounded Hilbert space operators coincide–and coincide as well with $\mathfrak{D}$'s double commutant. Thus a *von Neumann algebra* $\mathfrak{R}$ is a *-algebra of bounded operators such that $\mathfrak{R} = \mathfrak{R}''$. We've just seen a splendid example: $\mathfrak{B}(\mathcal{H})$. Where π is a representation of a C* algebra $\mathfrak{A}$, $\pi(\mathfrak{A})''$ – which according to the double commutant theorem is equivalent to $\pi(\mathfrak{A})$'s weak closure – is a von Neumann algebra, known as *the von Neumann algebra affiliated with the representation* π.

Isomorphic von Neumann algebras are called *quasi-equivalent*. The von

Neumann algebras $\pi(\mathfrak{A})''$ and $\phi(\mathfrak{A})''$ affiliated with the representations π and ϕ are quasi-equivalent if and only if there's a *isomorphism α from $\phi(\mathfrak{A})''$ to $\pi(\mathfrak{A})''$ such that $\alpha[\phi(A)] = \pi(A)$ for all $A \in \mathfrak{A}$–in which case the representations π and ϕ are said to be quasiequivalent as well. Unitarily equivalent representations are quasi-equivalent: the *isomorphism is implemented unitarily.

Algebraic States and GNS Representations

The algebraic approach to quantum theories associates observables pertaining to a system with the self-adjoint elements of a C^* algebra $\mathfrak{A}$ appropriate to the system. For instance, $\mathfrak{A}$ could be the algebra that results from the norm closure of a set of operators satisfying the CCRs/CARs for the system. The algebraic approach accommodates ordinary QM's picture of observables as a special case: $\mathfrak{A}$ is set equal to $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Ordinary QM conceives states on $\mathfrak{B}(\mathcal{H})$ as normed, positive, countably additive linear functionals over that algebra. Provided $dim(\mathcal{H}) > 2$, Gleason's theorem places states, so conceived, in one-to-one correspondence with density operators on $\mathcal{H}$. By contrast, the algebraic approach to QM defines states directly in terms of the observable algebra $\mathfrak{A}$, which it does not constrain to coincide with some $\mathfrak{B}(\mathcal{H})$. An algebraic *state* ω on $\mathfrak{A}$ is a linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ that is normed ($\omega(I) = 1$)) and positive ($\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$). $\omega(A)$ may be understood as the expectation value of (self-adjoint) $A \in \mathfrak{A}$. Like the states of ordinary QM, the set of states on a C^* algebra $\mathfrak{A}$ is convex. Its extremal elements–that is, states ω which cannot be expressed as non-trivial convex combinations of other states–are pure states; all other states are mixed.

Unlike ordinary quantum states, algebraic states are not required to be countably additive. In the most general case, the requirement would be otiose. The projection operators guaranteed to appear in $\mathfrak{A}$ are the additive and multiplicative identities 0 and I. With respect to that set of projection operators, countable additivity reduces to linearity and positivity.

Still, connections straightforward and surprising can be established between the Hilbert space and algebraic notions of state. Straightforwardly, a state $\hat{W}$ acting on a Hilbert space $\mathcal{H}$ carrying a representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ of an algebra $\mathfrak{A}$ defines an algebraic state ω on $\mathfrak{A}$. Simply set $\omega(A) = Tr(\hat{W}\pi(A))$ for all $A \in \mathfrak{A}$. Surprisingly, we can travel in the other direction, from an abstract algebraic state to its realization on a concrete Hilbert space. Let ω be a state on a C^* algebra $\mathfrak{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a faithful[14] representation $\pi_\omega : \mathfrak{A}. \rightarrow \mathfrak{B}(\mathcal{H}_\omega)$ of the algebra, and a cyclic[15] vector $|\Psi_\omega\rangle \in \mathcal{H}_\omega$ such that, for all $A \in \mathfrak{A}$, the expectation value the algebraic state ω assigns A is duplicated by the expectation value the Hilbert space state vector $|\Psi_\omega\rangle$ assigns the Hilbert space observable $\pi(A)$ (in symbols, $\omega(A) = \langle\Psi_\omega|\pi_\omega(A)|\Psi_\omega\rangle$ for all $A \in \mathfrak{A}$). The triple $(\mathcal{H}_\omega, \pi_\omega, |\Psi_\omega\rangle)$ is unique up to unitary equivalence.

Quasi-equivalence and Disjointness

There is a sort of kinship question we might want to raise about GNS representations. Obviously, a state ω on a C^* algebra $\mathfrak{A}$ can, by the agency of the cyclic vector $|\Psi_\omega\rangle$, be expressed in terms of ω's GNS representation. But what other states on $\mathfrak{A}$ can be expressed in terms of $(\mathcal{H}_\omega, \pi_\omega, |\Psi_\omega\rangle)$? In a clumsy and uninteresting sense, every state on $\mathfrak{A}$ can be so expressed, because every state on $\mathfrak{A}$ corresponds (via $\omega(\pi_\omega(A)) := \omega(A)$) to a normed, positive linear functional over $\pi_\omega(\mathfrak{A})$. So we'd better refine our question to make it interesting. Which algebraic states on $\mathfrak{A}$ (and thus on $\pi_\omega(\mathfrak{A})$) admit natural and well-behaved extensions to the von Neumann algebra $\pi_\omega(\mathfrak{A})''$ affiliated with ω's GNS representation? It is intuitively plausible that states on $\mathfrak{A}$ (and thus on $\pi_\omega(\mathfrak{A})$) whose extensions to $\pi_\omega(\mathfrak{A})''$ are unique will be states with (roughly speaking) nice continuity properties with respect to the topologies defined on that algebra.[16] States exhibiting these continuity properties with respect to a von Neumann algebra $\mathfrak{R}$ are called *normal* states of that algebra.

A state ω on a von Neumann algebra $\mathfrak{R}$ acting on a Hilbert space $\mathcal{H}$ is normal if and only if there is a density operator $\hat{W}$ such that $\omega(\hat{A}) = Tr(\hat{W}\hat{A})$ for all $\hat{A} \in \mathfrak{R}$ (Bratteli and Robinson 1987, Thm. 2.4.21). So states on $\mathfrak{A}$ interestingly akin to ω are states normal with respect to the von Neumann algebra affiliated with ω's GNS representation. This set of states – the set of algebraic states expressible as density matrices on ω's GNS representation– comprise what is known as a *folium*. In what follows, $\mathcal{F}_\omega$ will denote ω's folium.

We'll call states quasiequivalent when their GNS representations are. If

[14] π is faithful iff $\pi(A) = 0$ implies $A = 0$ for all $A \in \mathfrak{A}$.

[15] $|\Psi\rangle$ is cyclic for $\pi_\omega(\mathfrak{A})$ means $\{\pi_\omega(\mathfrak{A})\,|\Psi\rangle\}$ is *dense* in $\mathcal{H}$. That is, any vector $|\phi\rangle \in \mathcal{H}$ can be approximated to arbitrary precision by linear combinations of elements of $\{\pi_\omega(\mathfrak{A})\,|\Psi\rangle\}$.

[16] For an elaboration, see Bratteli and Robinson 1987, 75-79.

and only if states are quasiequivalent do their folia coincide. The coincidence of the folia of quasi-equivalent states has a simple explanation: von Neumann algebras affiliated with quasi-equivalent states are $*$-isomorphic; thus a state is normal on one such von Neumann algebra if and only if it's normal on the other.

Consider, by contrast, states ω and φ on $\mathfrak{A}$ whose folia have null intersection. No state on $\mathfrak{A}$ expressible as a density matrix on ω's GNS representation is so expressible on φ's GNS representation, and vice-versa. Such states are *disjoint*. "Disjointness," a relation that can obtain between algebraic states, radicalizes the relation of orthogonality that can obtain between vectors implementing pure Hilbert space states. Like orthogonal states, disjoint states assign one another a transition probability of 0.[17] Call states (of any stripe) *impossible relative to* one another just in case the transition probability between them is 0. Like orthogonal states, disjoint states are impossible relative to one another.

Disjointness radicalize orthogonality in the following sense. If pure states $|\phi\rangle$ and $|\psi\rangle$ on $\mathfrak{B}(\mathcal{H})$ are orthogonal, there exist a host of pure states possible relative to both. Normed superpositions of $|\phi\rangle$ and $|\psi\rangle$ are examples. By contrast, if ω and ρ on $\mathfrak{A}$ are disjoint, no pure state possible relative to one is possible relative to the other. This follows from the fact that pure states on $\mathfrak{A}$ are either quasi-equivalent or disjoint.

7 Interpreting QM_∞: Some answers

Hilbert space conservatism

The interpreter I'll label the *Hilbert space conservative* completes the kinematic template $(\mathcal{M}, \mathcal{S})$ on behalf of a theory of QM_∞ the same way it's standardly completed on behalf of a theory of ordinary QM. Our characterization of that way will be somewhat roundabout. Let $\mathfrak{A}$ be the C^* algebra obtained by taking the norm-closure of a Hilbert space representation of the CCRs/CARs for a system under consideration. Then the Hilbert space conservative maintains that the observables pertaining to the system are the self

[17]In algebraic QM, the transition probability between states ρ and ω on an observable algebra $\mathfrak{A}$ is given by $\frac{1}{4}\|\omega - \rho\|^2$, where $\| \circ \|$ is the norm on the space of linear functionals on $\mathfrak{A}$ (see Roberts and Roepstorff 1969 for a discussion). This probability is 0 when ω and ρ are disjoint.

adjoint elements of $\pi_\omega(\mathfrak{A})''$, the von Neumann algebra affiliated with the GNS representation of a pure state ω on $\mathfrak{A}$. It follows that $\pi_\omega(\mathfrak{A})''$ coincides with $\mathfrak{B}(\mathcal{H}_\omega)$, where $\mathcal{H}_\omega$ is the concrete separable Hilbert space carrying ω's GNS representation.[18] The Hilbert space conservative identifies possible states of the system with normal states on $\pi_\omega(\mathfrak{A})''$, that is, with ω's folium $\mathcal{F}_\omega$. Because these states coincide with $\mathfrak{T}^+(\mathcal{H}_\omega)$, the recapitulation of ordinary QM is complete. When the Hilbert space conservative offers $(\pi_\omega(\mathfrak{A})'', \mathcal{F}_\omega)$ as the kinematic pair for a theory of QM_∞, she's just dressing the kinematic pair $(\mathfrak{B}(\mathcal{H}), \mathfrak{T}^+(\mathcal{H}))$ typical of ordinary QM up in algebraic garb.

The Hilbert space conservative also transfers ordinary QM's content coincidence concept of physical equivalence to the setting of QM_∞. Theories specified upto kinematic pairs $(\pi_\omega(\mathfrak{A})'', \mathcal{F}_\omega)$ and $(\pi_{\omega'}(\mathfrak{A})'', \mathcal{F}_{\omega'})$ are physically equivalent, the Hilbert space conservative maintains, just in case $(\pi_\omega(\mathfrak{A})'', \mathcal{H}_\omega)$ and $(\pi_{\omega'}(\mathfrak{A})'', \mathcal{H}_{\omega'})$ are unitarily equivalent, from which it follows that $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$.

Of course, having said this much, the Hilbert space conservative is hardly done interpreting QM_∞. In ordinary QM, the uniqueness theorems assure us, once $\mathfrak{A}$ is fixed, all representations are unitarily equivalent. This is not so in QM_∞. The algebra $\mathfrak{A}$ that fixes a system's type admits continuously many unitarily inequivalent Hilbert space representations, and there are as many candidate completions of the kinematic template $(\pi_\omega(\mathfrak{A})'', \mathcal{F}_\omega)$ as there are disjoint pure states ω on $\mathfrak{A}$. Committed to unitary equivalence as a criterion of physical equivalence, the Hilbert space conservative must regard each of these as rival quantum theories. At most one of them can be the theory she's set out to interpret. Thus the burden of the Hilbert space conservative is to articulate and motivate a principle of privilege that enables her to enthrone a single unitary equivalence class of representations as physical, and to consign the rest to the dustbin of mere mathematical artifacts.

Promising principles of privilege apply to quantum field theories, and appeal to the symmetries of the spacetimes in which they're set. In a spacetime equipped with global timelike isometries (i.e. symmetries of the metric), a unitary equivalence class of representations of CCRs for the Klein-Gordon field exhibits energetics respecting those isometries. The respect is expressed by representations with the particle-interpretation-tempting struc-

[18] This follows via considerations so far suppressed: the GNS representation of a pure state ω is irreducible (Bratteli and Robinson 1987, Thm. 2.3.19 (57))—that is, the only subspaces of $\mathcal{H}_\omega$ invariant under π_ω are the zero subspace and $\mathcal{H}_\omega$ itself—and $\pi_\omega(\mathfrak{A})''$ is isormorphic to $\mathfrak{B}(\mathcal{H}_\omega)$ if π_ω is irreducible (Sakai 1971, Thm. 1.12.9).

ture sketched in §4, and takes roughly[19] the following form: a privileged Hamiltonian operator can be identified as the infinitesmal generator of time translations that leave the metric unchanged. A respectful representation has a vacuum state which is not only itself invariant under these timelike isometries, but also the lowest-energy eigenstate of the privileged Hamiltonian. What's more, each n-particle state of a respectful representation is an eigenstate of the privileged Hamiltonian, with an eigenvalue matching the energy of a corresponding classical state. Kay and Wald (1991) show that in suitably symmetric spacetimes, respectful representations of the Klein-Gordon CCRs are unique upto unitary equivalence, and have a particle-intepretation-tempting structure.

If it's reasonable to demand such respect, it's reasonable to distinguish invidiously between respectful and disrespectful representations. The invidious distinction makes available univocal and powerful particle talk. Particle notions incommensurable with, because circumscribed by representations unitarily inequivalent to, that of the respectful representation can be dismissed as unphysical. Each state in the privileged folium can be characterized in terms of its particle content; different states can be distinguished in terms of their particle content. If an adequate interpretation is obliged to understand QFT as particle physics—and that is how many of its practitioners do understand it—then the pursuit of Hilbert space conservatism in spacetime settings with enough symmetry to pick out a particle-friendly unitary equivalence class of representations is one route to adequacy.

But it's a route a that's blocked for QFTs set in spacetimes lacking the relevant symmetries, and a route that bypasses the thermodynamic limit of QSM, which is set in Euclidean three space. There, as we will see presently, Hilbert space conservatism is untenable.

Short of the thermodynamic limit, and in the setting of concrete Hilbert spaces, the *Gibbs state* equips QSM with a notion of equilibrium. The Gibbs state describing a system governed by Hamiltonian $\hat{H}$ in equilibrium at inverse temperature $\beta = \frac{1}{kT}$ is

$$\hat{\rho} = \frac{\exp(-\beta\hat{H})}{\mathrm{Tr}[\exp(-\beta\hat{H})]} \tag{15}$$

For realistic, finite quantum systems this Gibbs state is well-defined and unique (Ruelle 1969). For infinite quantum systems this needn't be so: the

[19] See Kay and Wald, 1991 for a rigorous exposition.

r.h.s. of (15) fails to be well defined.

Suppose that we aspire to construct a quantum statistical account of *phase structure.* Then we'd have reason to concoct a notion of equilibrium suited to infinite systems. For the apparent macroscopic explanandum is the existence, at certain temperatures, of multiple thermodynamic phases. If a statistical account of phase structure requires the existence, at these critical temperatures, of *multiple distinct* equilibrium states, answering to different thermodynamic phases, then, as long as we're dealing with finite systems, the very uniqueness of the Gibbs state frustrates our explanatory aspirations.

Explanatory hopes are revived in the thermodynamic limit by using the *KMS condition* to explicate a notion of equilibrium more general than that afforded by the Gibbs state.[20] The set of KMS equilibrium states with respect to a given dynamics has several striking features.[21] One is that a system of type $\mathfrak{A}$ in equilibrium at a temperature T occupies a state *disjoint* from the state of the same system in equilibrium at a different temperature (Bratteli and Robinson 1997, thm. 5.3.35). Provided neither temperature is 0, each equilibrium state is moreover – and mind-bogglingly – disjoint from every pure state on $\mathfrak{A}$.[22] And at critical temperatures where $\mathfrak{A}$ admits *multiple* KMS states, the extremal elements of that set —which correspond to the pure phases whose copresence constitutes the phase structure we aspire to explain (cf. Sewell 1984, §4.4)—are either equivalent or disjoint (Bratteli and Robinson 1997, Thm. 5.3.30).

Holding that physically possible states of a system of type $\mathfrak{A}$ lie in the folium of some pure state ω on $\mathfrak{A}$, Hilbert space conservatism paralyses the thermodynamic limit of QSM. Hilbert space conservatism implies that equilibrium at temperatures different from 0 is impossible – because no such equilbrium state lies in the folium of a pure state. Even if Hilbert space conservatism is relaxed to allow an *impure* state ϕ to pick out the privileged folium, it implies that at most *one* equilibrium state at a non-zero temperature – the one given by ϕ – is possible. This declaration hamstrings the attempt to explain phase structure in terms of the availability, at the critical temperatures at which they occur, of multiple distinct equilibrium states.

Even this preliminary review of Hilbert space conservatism suggests that

[20] For more on the KMS condition and why it suitably explicates equilibrium, see Bratteli and Robinson 1997, §5.3.

[21] The following statements suppress certain technical assumptions that generally hold at the thermodynamic limit; see the references for details.

[22] A circumstance Ruetsche 2004 discusses at some length.

its viability depends on *which* theory of QM_∞ — QFT in Minkowski spacetime? QFT in a less symmetric spacetime? The thermodynamic limit of QSM?? — one is interpreting. This sensitivity of viable interpretive options to particularities, even peculiarities, of the theory interpreted will be an ongoing theme.

Algebraic Imperialism

According to an interpretive strategy Arageorgis (1995) has dubbed "algebraic imperialism," the extra structure one obtains along with a concrete representation of (e.g.) the Weyl algebra is extraneous–the Weyl algebra, and not any of its concrete but disparate Hilbert space realizations, encapsulates the physical content of QFT . For Irving Segal, a pioneer of C^* algebraic approaches,

> [T]he important thing here is that the observables form some algebra, and not the representation Hilbert space on which they act. (Segal (1967),128))

Inspired by the representation independence of the Weyl algebra, algebraic imperialism construes a theory of QM_∞ not in terms of the Hilbert space setting of a particular concrete representation of the CARs/CCRs for that theory, but in terms of the abstract algebraic structure every such representation shares. Physical magnitudes pertaining to a system of type $\mathfrak{A}$ are given by the self-adjoint part of the abstract C^* algebra $\mathfrak{A}$; possible states are states in the algebraic sense on $\mathfrak{A}$, which I'll denote $\mathcal{S}_{\mathfrak{A}}$. $(\mathfrak{A}, \mathcal{S}_{\mathfrak{A}})$ thus gives the kinematic pair for a theory of QM_∞.

Algebraic imperialism and Hilbert space conservatism are rival accounts of the content of QM_∞. For there can arise algebraic states ω and ω' on an observable algebra $\mathfrak{A}$ of QM_∞ which are disjoint. Both are, according to the algebraic imperialist, possible states of the system associated with $\mathfrak{A}$. But they can't both lie in the folium of some pure state ω on $\mathfrak{A}$. Any ordinary QM kinematic pair admitting ω as a possible state is unitarily inequivalent to any ordinary QM kinematic pair admitting ω' as possible. Put another way, if unitary equivalence is criterial for physical equivalence, any theory admitting one state as possible will be physically inequivalent to a theory admitting the other as possible.

Apologetic Imperialism

This makes it clear that unitary equivalence is a criterion of physical equivalence ill-adapted to algebraic imperialism. Unitary equivalence distinguishes invidiously between disjoint folia; the algebraic approach does not. Recognizing a mismatch, early advocates of the algebraic approach (e.g., Haag and Kastler (1964), 851; cf. Robinson (1966), 488) offered instead of unitary equivalence *weak equivalence* as a criterion of physical equivalence for Hilbert space representations. It seem to me that these advocates regarded their recognition of unitarily inequivalent representations as requiring some sort of apology, an apology they couched in terms of weak equivalence. So I will call them *apologetic imperialists*.

Weak equivalence is a relation that holds (or not) between Hilbert space representations of a C* algebra $\mathfrak{A}$. Two representations are weakly equivalent exactly when no finite set of expectation values specified to finite accuracy can locate a state in one, rather than the other, representation. More precisely, $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are weakly equivalent if and only if, given any Hilbert space state on the first representation and the algebraic state ω it yields, and given any finite set of elements of the abstract algebra of observables and any finite set of non-zero margins of experimental error, there exists a Hilbert space state on the second representation yielding an algebraic state ω' that reproduces within those margins of errors ω's assignment of expectation values to those observables.

Now the apologizing imperialist "define[s] two representations to be *physically equivalent* if and only if they are weakly equivalent" (Robinson (1966), 488). A rough and ready operationism suggests a justification of the equation of physical and weak equivalence.

> We want the results of any finite set of measurements on a physical state to be equally well describable in terms of a density matrix on $\mathcal{H}_1$ or a density matrix on $\mathcal{H}_2$. As measurements are never totally accurate, "equally well" is to be understood as "to any desired degree of accuracy. (Kastler 1964, 180-181)

Any actual measurement performed on a system in the algebraic state ω will fix the expectation values of some finite collection of observables A_i, $i = 1, 2, ..., n$, within experimental error margins $\varepsilon_i > 0$. Accordingly, such a measurement cannot distinguish ω from any other state ω' satisfying $|\omega(A_i) - \omega'(A_i)| < \varepsilon_i$ for all i. Such states are, the apologetic imperialist contends, for all practical purposes, physically equivalent. And so, weakly equivalent representations are for all practical purposes physically equivalent as well.

A mathematical result known as Fell's theorem consolidates weak equivalence as a criterion of physical equivalence hospitable to algebraic approches. Fell's theorem implies that all faithful representations of a C^* algebra are weakly equivalent (see Wald (1994), Ch. 4.5). Defining physical equivalence as weak equivalence, the apologetic imperialist can conclude that, "All faithful representations of [the Weyl algebra] are physically equivalent" (Robinson (1966), 488))–and chastise the conservative for elevating surplus structure to physical significance:

> It is in this new notion of equivalence in field theory that the algebraic approach has its greatest justification. All the physical content of the theory is contained in the algebra itself; nothing of fundamental significance is added to a theory by its expression in a particular representation. (ibid.)

Notice that the observables with respect to which weakly equivalent representations are practically indistinguishable are elements of the abstract algebra $\mathfrak{A}$. Consider, for some concrete representation π_1 of $\mathfrak{A}$, an observable in the weak closure of that representation but without correlate in $\mathfrak{A}$. I will call such an observable *parochial* to the representation π_1. Examples of parochial observables include the total number operators (or more properly the projectors in their spectral resolution) encountered in suitable representations of the Weyl algebra for the free Klein-Gordon field. There is a π_1 normal state, the vacuum state, which assigns the total number operator expectation value 0. But there is no normal state in any representation π_2 unitarily inequivalent to π_1 that assigns this total number operator an expectation value within any finite ϵ of 0 (Clifton and Halvorson 2001, Prop. 11, 450). Every faithful representation of the Weyl algebra is weakly equivalent to every other. So apologetic imperialists would hail π_1 and π_2 as physically equivalent–even though all π_2-normal states assign an observable in the weak closure of π_1 –its total number operator–"expectation values"[23] arbitrarily *far* from that assigned by the π_1 vacuum state. Of course, this is a compelling criticism of weak equivalence as an explication of physical equivalence only

[23]The scare quotes because, strictly speaking, the observable is outside the domain of the state.

for those inclined—as imperialists are *not*—to regard parochial observables as physical.

The justifications apologetic imperialists offer for weak equivalence as ac criterion of physical equivalence for Hilbert space representations are, broadly speaking, operationalist. But they ultimately fail, *even by operationalist lights*. For, as Summers (2001) has observed, even operationalists ought to take states to be predictive instrumentalities. Operationalists should therefore expect the equivalence of states to extend to their predictions concerning future measurements. But Fell's theorem offers us no assurance that a π_1-normal state for all practical purposes indistinguishable from a π_2-normal state with respect to some set $\{A_i\}$ of algebraic observables will continue to mimic the first state's predictions with respect to an expanded or altered set of observables. The practical indistinguishability of states that secures the weak equivalence of the representations bearing them is only a backward-looking or static sort of indistinguishability. Before recognizing a normal state on one representation as a suitable counterpart to a normal state on another, an operationalist should demand not only backward-looking but forward-looking indistinguishability: the indistinguishability of putative counterpart states conceived as predictive instrumentalities. Thus the appeal to Fell's theorem should not convince even the operationalist that weak equivalence is an acceptable notion of physical equivalence.

Bold Imperialism

Apologies for algebraic liberality invoking weak equivalence fall flat. Insofar as weak equivalence is a criterion of equivalence for Hilbert space representations—which the imperialist *denies* to be repositories of physical content—these apologies were heading in the wrong direction to begin with. A bolder imperialism would define physical equivalence directly in terms of the imperialist kinematic pair $(\mathfrak{A}, \mathcal{S}_{\mathfrak{A}})$. Two such pairs $(\mathfrak{A}, \mathcal{S}_{\mathfrak{A}})$ and $(\mathfrak{A}', \mathcal{S}_{\mathfrak{A}'})$ are equivalent in the content coincidence sense (explicated for ordinary QM by (9)) only if their associated algebras are isomorphic. Once $(\mathfrak{A}, \mathcal{S}_{\mathfrak{A}})$ is identified as the appropriate kinematic pair for a theory of QM_∞ this criterion of physical equivalence follows straightforwardly. The challenge for the bold imperialist is to justify his choice of kinematic pair.

Segal, possibly the boldest imperialist of them all, rises to the challenge by an operationalist ploy of his own. All hands, Segal suggests, should grant that the canonical observables pertaining to a system are physically meaningful, and grant physical meaning as well to bounded functions of of finite sets of canonical observables. Segal's crucial move is to grant physical significance to the norm-closure but not (where they differ) the weak closure of this set of antecedently meaningful observables. He accepts that "[norm] approximation is operationally meaningful" because where f_n norm approximates f, f's "expectation value in any state is simply the limit of the expectation values of the approximating bounded functions" (1959, 347-348). To see the point of this observation, imagine a sequence of $A_i \in \mathfrak{A}$ that does not converge in $\mathfrak{A}$'s norm topology. Suppose further that in some representation π of $\mathfrak{A}$, the sequence $\pi(A_i)$ *does* weakly converge, to an operator A. Recall that π normal states on $\mathfrak{A}$ are those states ϕ whose extensions from $\pi(\mathfrak{A})$ to its weak closure $\pi(\mathfrak{A})''$ are unique. So a π-*abnormal* state ω on $\mathfrak{A}$ can be such that the sequence $\omega(A_i)$ fails to converge even though the sequence $\pi(A_i)$ converges weakly. Segal contends that A, and other parochial observables, lack operational significance because there are states whose expectation value assignments to observables conceded significance by all hands don't fix their expectation value assignments to A. Segal deems norm limits admissible because, if A_i norm converges to some limit A, $\omega(A_i)$ perforce converges as well.

The keystone difference, for Segal, between weak and norm limits, then, is that the world can get itself into a condition – the condition of state not normal with respect to the representation whose weak topology is used to take a weak limit–where the expectation value of weak limit floats free of any set, even an infinite one, of laboratory machinations, even perfectly accurate ones. With norm limits, this is not so. Thus Segal would vest norm, but not weak, limits with operational, and so physical, significance. It follows that he regards as significant the C*, but not (where they differ) the von Neumann algebras generated by elements satisfying the canonical relations defining a system, and regards as equivalent kinematic templates plying isomorphic C* algebras. Therein lies his bold imperialism.

Segal's derivation of imperialism from operationalism impresses me as deeper and less opportunistic than the apologetic imperialists' plea for weak equivalence. Segal rests his case on a principle like

> (SEGOP) Let et $\mathcal{P}$ be the set of bounded functions of canonical observables for a system S. X is an observable pertaining to S if and only if there is a sequence $X_i \in \mathcal{P}$ such that, for any state ω of S, $\omega(X_i)$ converges. When this condition is met, $\lim_{i\to\infty} \omega(X_i)$

is the expectation value of X in the state ω.

Reflecting the conviction that physical magnitudes are those for which there exist measurement protocols, SEGOP has operationalist roots. But the principle makes minimal appeal to the depressing actualities of laboratory life – its finitude, imperfect material conditions, or inaccuracy. Indeed, a confirmed metaphysician who foreswears operationalism might nevertheless accept (SEGOP) as an explication of the parsimony desideratum discussed near the close of §5. (SEGOP) is telling us which magnitudes we're committed to, once we've adoped a set of canonical magnitudes and space of possible states, and which magnitude-pretenders turn out to be free-riding fluff.

Of course, the catch is that to apply Segal's criterion, we need to know which states are possible. It follows that (SEGOP) won't help us settle the sort of interpretive dispute that consists in disagreeing, as the Hilbert space conservative and the algebraic imperialist disagree, about which states are possible to begin with. A Hilbert space conservative, confining possible states for a system of canonical type $\mathfrak{A}$ to those in the folium of a pure state ω on $\mathfrak{A}$ can appeal to SEGOP to earn observables parochial to $\pi_\omega(\mathfrak{A})$ physical significance. For she recognizes no π_ω-*ab*normal states that might interact with SEGOP to disqualify those observables.

SEGOP sets a tenet of interpretive good taste (if not a test of internal coherence) for an interpretation. But it can't adjudicate between competing interpretations that meet its scruples. We'll have to fall back on our vaguer criteria to call those contests.

Mixed strategies

By a "mixed" strategy for interpreting QM_∞, I mean one that mixes and matches elements of the kinematic pairs for the imperialist and conservative approaches. Two mixed strategies are discussed in Clifton and Halvorson (2001, 430 ff.). One offers $(\mathfrak{A}, \mathcal{F}_\omega)$ as the kinematic template for a system of canonical type $\mathfrak{A}$. The offers $(\pi_\omega(\mathfrak{A})'', \mathcal{S}_{\mathfrak{A}})$. Let's focus on the latter, which concedes to the conservative that observables reside in the von Neumann algebra generated by some concrete representation, but suspends the conservative's requirement that states be countably additive.

Our discussion of the Hilbert space conservative foreshadows one challenge for this mixed strategy. Suppose π_ω is the representation whose affiliated von Neumann algebra contains the physical magnitudes. If ω and ω'

are disjoint, some self-adjoint elements of the von Neumann algebra affiliated with $\pi_{\omega'}$ are disqualified from physical significance. The challenge is to justify this disqualification. In a way, the challenge is deeper than the Hilbert space conservative's problem of privilege. The conservative can at least observe a SEGOP-stype consonance between the states she recognizes and the observables she recognizes. The mixed strategy enjoys no such consonance. In $\mathcal{S}_{\mathfrak{A}}$ are states whose values on $\pi_\omega(\mathfrak{A})$ fix their values on $\pi_\omega(\mathfrak{A})''$, and states for which this is not so. So it seems our discussion of bold algebraic imperialism foreshadows a problem for the mixed strategy as well: it stands in violation of SEGOP. In a sense Segal has made precise, the strategy recognizes too many states for her observable set to bear.

Universalism

The last section found mixed strategies wanting not only in motivation but also in the sort of coherence articulated by SEGOP. This section sketches an interpretive stance I call "universalism." Universalism construes the content of a quantum theory of type $\mathfrak{A}$ by means of the extravagance of the *universal representation* $\mathfrak{U}$ of $\mathfrak{A}$.[24] The universal representation is the direct sum,[25] over states on $\mathfrak{A}$, of their GNS representations:

$$\mathfrak{U} = \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \pi_\omega(\mathfrak{A}) \tag{16}$$

$\mathfrak{U}$ acts on the direct sum Hilbert space $\mathcal{H}_u = \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \mathcal{H}_\omega$.

$\mathfrak{U}''$, the von Neumann algebra affiliated with this universal representation– aka the *universal enveloping von Neumann algebra*– will be a direct sum of von Neumann algebras affiliated with each π_ω:

$$\mathfrak{U}'' = \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \pi_\omega(\mathfrak{A})'' \tag{17}$$

[24] See Kronz and Lupher 2005 for one version of this proposal.

[25] Where $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces, their direct sum $\mathcal{H} \oplus \mathcal{H}'$ consists of all elements of the form $|\alpha\rangle \oplus |\beta\rangle$, $|\alpha\rangle \in \mathcal{H}$ and$|\beta\rangle \in \mathcal{H}'$. The rule for vector addition is $|\alpha_1\rangle \oplus |\beta_1\rangle + |\alpha_2\rangle \oplus |\beta_2\rangle = (|\alpha_1\rangle + |\alpha_2\rangle) \oplus (|\beta_1\rangle + |\beta_2\rangle)$. Inner products work like so: $(\langle\alpha_1| \oplus \langle\beta_1|)\,(|\alpha_2\rangle \oplus |\beta_1\rangle + |\beta_2\rangle) = \langle\alpha_1|\alpha_2\rangle + \langle\beta_1|\beta_2\rangle$. $\mathcal{H} \oplus \mathcal{H}'$ contains *only* those elements: the direct sum of two Hilbert spaces corresponds to their Cartesian, rather than their tensor, product.

(cf. Emch 1972, 120). An $\hat{A}$ that's parochial to a particular representation π_ϕ lacks a counterpart in the abstract algebra $\mathfrak{A}$ but has its avatar somewhere in $\mathfrak{U}''$: the direct sum of $\hat{A}$ acting on the Hilbert space $\mathcal{H}_\phi$ and the identity operator acting on every other Hilbert space in the expansion of $\mathcal{H}_u$. Thus there is no observable constructable from any Hilbert space representation of $\mathfrak{A}$ without a counterpart in the universal enveloping von Neumann algebra $\mathfrak{U}''$. And this abundance of physical magnitudes does not require the universalist to impoverish her state set. According to universalism, the states possible for a system of canonical type $\mathfrak{A}$ are the states normal with respect to the associated universal enveloping von Neumann algebra. And these are just states in the algebraic sense on $\mathfrak{A}$ (see Emch 1972, 120-121 for a proof). So the universalist is entitled to as rich a state set as the imperialist. What's more, because the states she recognizes are just those normal with respect to the observable algebra she recognizes, the universalist promises to survive the scrutiny of SEGOP.

The conservative commits invidious privilege. The universalist (like her religious counterpart) is adamantly non-commital. The imperialist haughtily ignores representation-specific observables that could be wage-earning members of physical theories. Ecumenically, the universalist welcomes all observables descended, however weakly, from canonical ones. Mixed strategies run afoul of SEGOP, a principle of interpretive good taste universalism abides. Notwithstanding this litany of accomplishments, Kastler is not converted. $\mathfrak{U}''$, he remarks, "contains all the operators of the standard theory — but it has no interesting algebraic structure" (1964, 184). Perhaps that is an aesthetic judgment. Here, tapping the ethos behind SEGOP, is a moral one: it is the task of an observable to discriminate between states, and universalism's observables shirk that task. Many observables it entertains, the parochial observables without correlate in the abstract algebra, are profoundly irrelevant to the vast majority of states it entertains. Where A is parochial to π_ω, it (or its representative in $\mathfrak{U}''$) takes expectation value 1 on every state disjoint from ω.[26] All states disjoint from ω look the same to an observable parochial to π_ω.

[26]Here's a sketch of why: Consider such a state ω'. Letting $\mathcal{H}_u = \mathcal{H}_\omega \oplus \mathcal{H}_{\omega'} \bigoplus_{\phi \neq \omega, \omega'} \mathcal{H}_\phi$ and $\mathfrak{U} = \pi_\omega \oplus \pi_{\omega'} \bigoplus_{\phi \neq \omega, \omega'} \pi_\phi$, ω' is implemented by the $\mathcal{H}_u$ vector $0 \oplus |\Psi_{\omega'}\rangle \oplus 0 \oplus \cdots$, and A is implemented by the operator $A \oplus I \oplus I \oplus \cdots \in \mathfrak{U}''$. Thus $\omega'(A) = \langle 0|A|0\rangle + \langle \Psi_{\omega'}|I|\Psi_{\omega'}\rangle + \langle 0|I|0\rangle \cdots = 1$.

Not only do universalism's parochial observables fail to discriminate between many of the states it recognizes, they also violate the spirit (if not the letter) of SEGOP. When π_ω-parochial A is assigned expectation value 1 by ω' disjoint from ω, that expectation value assignment is not a consequence of ω''s expectation value assignment to automatically significant canonical observables. It's made on a technicality, not on physical grounds.

Michael Redhead reports that Clark Glymour once branded contextualist responses to No Go results like the Kochen-Specker argument – responses asserting that as many physical magnitudes correspond to a self-adjoint degenerate Hilbert space operator as that operator has distinct eigenbases–the "deOckhamization of QM" (Redhead 1987, 135). Universalism threatens the deOckhamization of QM_∞. Where $\mathfrak{A}$ has continuously (or even countably) many disjoint representations $\{\pi_\omega\}$, an element $A \in \mathfrak{A}$ has continuously many representatives in the universal enveloping von Neumann algebra: where s is any finite subset of $\mathfrak{U}$-normal states, anything of the form $\bigoplus_{\omega \in s} \pi_\omega(A) \bigoplus_{\omega' \notin s} \pi_{\omega'}(A)$ does the trick. Each duplicates every other's functional relationships to canonical observables. Which is the real A? The answer $\bigoplus_{\omega} \pi_\omega(A)$, in effect a disjunction of individually parsimonious options, won't satisfy anyone inclined to ask the question to begin with.

Tempered Universalism

For a system of type $\mathfrak{A}$, the approach I call "tempered universalism" advances some criterion of privilege that identifies as physically reasonable a subset Φ of the full set $\mathcal{S}_{\mathfrak{A}}$ of algebraic states. The tempered universalist then constructs an algebra of physical magnitudes as the direct sum of the von Neumann algebras affiliated with states in the privileged set. The hope is that, with the appropriate principle of tempering, this strategy will prove less extravagant than universalism, but more generous than either conservatism or the imperialism.

Candidate tempering principles include

SYMMETRY MONGERING

> In the presense of spacetime symmetries Λ implemented by automorphisms α_λ of $\mathfrak{A}$, a state ω on $\mathfrak{A}$ is physical only if each symmetry α_λ can be implemented unitarily in π_ω. α can be im-

plemented unitarily in $(\pi, \mathcal{H})$ if and only if there exists a unitary $U \in \mathfrak{B}(\mathcal{H})$ such that

$$\pi(\alpha(A)) = U\pi(A)U^{-1} \text{for all } A \in \mathfrak{A} \tag{18}$$

For an early instance, Robinson (1966, 483) proposes a principle of this sort. Our discussion of promising criteria of privilege for the Hilbert space conservative shows how symmetry mongering might pay off. When α^t is a one-parameter family of symmetry-implementing automorphisms of $\mathfrak{A}$, the family of unitaries U_t demanded by the principle could have as an infinitesmal generator a magnitude that does physical work.

ACTUALISM

> A state ω on $\mathfrak{A}$ is physical only if it's the actual state of the system of interest, or appropriately related to that state.

Perhaps Primas suggests something of this sort when he writes,

> To select a particular representation means to concentrate the attention to a particular phenomenon and to forget effects of secondary importance. With such a choice (which is possible if we adopt an appropriate topology in our mathematical formalism) we lose irrelevant information so that the system becomes simpler. (1983, 174-5)

This principle of tempering threatens to fall victim to the complaint voiced about Hilbert space conservatism, that it excludes too many possible states. One version of that complaint pivoted on the fact that equilibrium states at different temperatures are disjoint. Universalism tempered by an actualism that failed to exploit the "appropriately related" clause would commit the modal solecism of declaring equilibria at temperatures other than the actual one impossible.

However, exploiting the appropriately related clause is a delicate business. A natural suggestion is that ω and ω' are appropriately related if they're dynamically accessible one from the other, that is, only if there's a "dynamical automorphism" α_t of $\mathfrak{A}$ s.t. $\omega = \omega' \circ \alpha_t$. But which automorphisms are dynamical? Given that the algebraic approach countenances non-unitarily implementable dynamics (documented in Arageorgis et al (2002)), one fraught question is whether these automorphisms must be unitarily implementable on the GNS representations of the states in question to qualify.

HADAMARDISM

> A state ω on $\mathfrak{A}$ is physical only if it satisfies the Hadamard condition, which means it supports a procedure for assigning an expectation value to a stress-energy observable

Wald (1994, §4.6) announces a principle of this sort. For reasons outlined in §5, this tempering principle makes the most sense for quantum field theories consorting with semi-classical quantum gravity. But there it can have dramatic consequences. In spacetimes with compact Cauchy surfaces (aka closed universes) all Hadamard vacuum states are unitarily equivalent (Wald 1994, 97). In open universes, this is not so.

DYNAMICISM

> A state ω on $\mathfrak{A}$ is physical only if it supports dynamics.

The cases to which this principle applies most directly come from the thermodynamic limit of quantum statistical mechanics. There a dynamics for subalgebras $\mathfrak{A}_V$ of the overall algebra $\mathfrak{A}$ – subalgebras associated with finite volumes V–may not have a *norm* limit as $V \to \infty$, but may have a weak limit in certain representations π_ω of $\mathfrak{A}$ (see Sewell 2002, §2.4.5. The idea behind this principle is that states whose GNS representations enable a well-defined global dynamics are more physical than those that frustrate such dynamics.

8 Conclusion: Tempering and Toggling

What all these tempering strategies have in common is that they complicate the modal toggle picture of a physical theory. On that picture, a physical theory sorts logically possible worlds into those that are (according to its kinematics) physically possible and those that are not. These physically possible worlds are, as it were, instantaneous; the theory's dynamical laws tell us which physically possible worlds are the time developments of which others. If the theory is lucky enough to be true, uninteresting contingency tells us which trajectory the actual world lies on. Different untempered interpretations offer different sorting mechanisms: a Hilbert space structure of observables, an abstract algebraic structure, a universal representation structure, and so on. But they deploy these sorting mechanisms, as it were, *a* *priori*. Conservatism, imperialism, mixed strategies, and universalism didn't need to peek at a theory's spacetime setting, its symmetries, its dynamics, or its relation to semi-classical quantum gravity to complete the kinematic template on its behalf. They aspire to a greater level of generality than that.

Tempered universalism does not. Tempered universalism peeks. Different tempering principles peek in different places, but they all peek. Peeking, each gives up on the idea that what is kinematically possible according to a quantum theory has everything to do with that theory's kinematical laws (that is, (anti)commutation relations) and nothing to do with its non-kinematic particulars. I take tempered universalism to suggest that to apply a theory of QM_∞ to a particular problem is not simply to select appropriate elements of a pre-configured set of worlds possible according to that theory. Rather, the configuration of that set can happen along with the application of the theory, and happen in different ways for different applications. The toggle is a myth that obscures the resourcefulness of theories of QM_∞.

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