What's it take to interpret a physical theory?

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1 Introduction

What's it take to interpret a physical theory? The question is ambiguous. In one sense it asks what it takes *to be* an interpretation of a physical theory; that is, what *qualifies* an intellectual offering as an interpretation of a theory. In another sense, it asks what it takes *to construct* an interpretation of a physical theory; that is, what resources are fair game for articulating an offering satisfying the qualifying criteria. I believe that a fairly widespread (although often implicit) answer to the question in its first sense is:

> (i) An interpretation of a physical theory is a characterization of the worlds possible according to that theory.

And I believe that a fairly widespread (although even more often implicit) answer to the question in its second sense is:

> (ii) To construct an interpretation of a physical theory, one may draw upon that theory's own formal apparatus, as well as ones own metaphysical scruples.

An example of drawing on a theory's formal apparatus in the service of its interpretation is setting up a correspondence between worlds possible according to the General Theory of Relativity and triples (M, g_{ab}, T_{ab}) satisfying the field equations, or between worlds possible according to the General Theory of Relativity and equivalence classes under diffeomorphism of such triples. An example of drawing on ones metaphysical scruples in the service of interpretation is opting on the basis of Leibniz shift arguments for the correspondence between possible worlds and equivalence classes.

My motivating concern is with the adequacy of the foregoing picture to the rough and tumble of how physical theories do what they ought to do, which is (among other things) unify and explain phenomena. But my particular agenda here is quite narrow. It's to explicate one expression of the picture framed by the tenets (i) and (ii), and to chronicle the difficulties it faces. The expression is the *maximal beable approach* to interpreting quantum theories, devised for the setting, familiar from philosophical expositions such as Redhead (1987) or Hughes (1989), of what I'll call *ordinary quantum mechanics*, wherein the observables pertaining to a quantum system are the self-adjoint elements of $\mathfrak{B}(\mathcal{H})$, the collection of bounded operators acting on a separable Hilbert space $\mathcal{H}$, aka a Type I von Neumann algebra. The maximal beable approach struggles when it encounters quantum theories of a sort less precedented in the philosophical literature, but typical of a setting I'll call QM_∞, encompassing quantum field theories and the thermodynamic limit of quantum statistical mechanics. In general, the observables pertaining to a QM_∞ system belong to von Neumann algebras of types more general, and more surprising, than the Type I algebra $\mathfrak{B}(\mathcal{H})$. I aim here to indicate why this is so and how it spells trouble for the maximal beable approach.

The paper is organized as follows. §2 presents a series of prominent interpretations of ordinary QM as instances of the maximal beable approach. §3 begins with a rudimentary review of lattice theory, then deploys the terms introduced to recharacterize the maximal beable approach in ordinary QM. The crux of the recharacterization is that the maximal beable approach hinges on the presence in $\mathfrak{B}(\mathcal{H})$ of minimal projection operators, that is, atoms in its projection lattice. §4 documents the absence of minimal projection operators from a variety of observable algebras encountered in QM_∞. There I contend that this absence leaves the maximal beable approach unable to say either what quantum probabilities are probabilities for or what the values of those probabilities are. A closing section takes stock.

2 Ordinary QM and its interpretation

2.1 Ordinary QM

What I'll call a theory of ordinary QM associates with a quantum system the collection $\mathfrak{B}(\mathcal{H})$ of bounded operators acting on a separable Hilbert space $\mathcal{H}$; identifies observables pertaining to the system with self-adjoint elements of $\mathfrak{B}(\mathcal{H})$; and identifies possible states of the system with normed, positive, countably additive linear functionals $\omega : \mathfrak{B}(\mathcal{H}) \to \mathbb{C}$. Taking the real number $\omega(A)$ as the *expectation value*—that is, long run experimental average—of the observable associated with the self-adjoint element $A \in \mathfrak{B}(\mathcal{H})$ endows the theory with empirical content. For Hilbert spaces of dimension greater than 2 , Gleason's theorem sets states on $\mathfrak{B}(\mathcal{H})$ in one-to-one correspondence with density operators (trace class operators of trace one) in $\mathfrak{B}(\mathcal{H})$. Where W is the density operator implementing the state ω, the prescription $\omega(A) = Tr(WA)$ for all $A \in \mathfrak{B}(\mathcal{H})$ establishes the correspondence. Schrödinger's equation equips the kinematical structure just specified with dynamics. An isolated system in initial state $W(0)$, with Hamiltonian (energy) operator H, evolves in a time t to the state $W(t) = e^{-iHt}W(0)e^{iHt}$.[1] The evolution operators e^{-iHt} are unitary and form a strongly continuous family.

The associations and identifications just laid out are in the first instance.

[1]Setting factors of Planck's constant to 1.

Interpretations of ordinary QM can and do complicate or revise them. An interpretation of ordinary QM undertakes to say what the world would have to be like, in order for a theory of the structure just laid out to be empirically adequate. Thus an interpretation characterizes a space of possible worlds as worlds of which ordinary QM is empirically adequate.[2]

2.2 The Maximal Beable Approach

An interpretation of a quantum theory should answer three related questions about the probabilities that theory assigns. First,

> Q1. What are quantum probabilities probabilities for?

The worlds possible according to a quantum theory are the recipients of its probability assignments. Thus, to answer Q1 is to characterize the worlds possible according to a quantum theory, that is, to begin to interpret it.

Second,

> Q2. What is the nature of these probabilities?

If an answer to Q1 identifies the worlds possible according to a quantum theory, an answer to Q2 tells us how to understand the probabilities that quantum theory assigns those worlds.

An interpretation of a quantum theory should also disclose that theory as *empirical*, that is, as committed to specific testable predictions about the course of nature. Hence the third question

> Q3. What values do these probabilities have?

An answer to Q3—which typically will either invoke the Born Rule directly [textbook, modal, hidden variable] or reconstitute it from other principles [Many Worlds]—equips the quantum theory with articulate empirical content, in the form of explicit statistical predictions. To be sure, it is the job of the theory itself to generate statistical predictions, but it is the job of the theory's interpretation to explicate those predictions as predictions about the possibilities the interpretation allies with the theory.

A natural way for an interpretation of a theory of ordinary QM to proceed is to take each state W on $\mathfrak{B}(\mathcal{H})$ to correspond to a set of possible configurations of the system associated with $\mathfrak{B}(\mathcal{H})$, that is, worlds (consisting of that system) possible according to the theory. (The W/world correspondence is allowed to be one-many rather than required to be one-one to accommodate approaches which interpret quantum probabilities in terms of statistical distributions.) A noble interpretation will attempt to characterize possible worlds without appeal to problematic and ambiguous notions such as 'measurement,'

[2] Why not 'true', a more common formulation? To make rooms for positions, recognized as interpretive, that alter the substance of the physics just presented, for instance by revising Schrödinger dynamics or by enriching the observable set with magnitudes, such as position and momentum in Bohmian mechanics, without correlate in $\mathfrak{B}(\mathcal{H})$.

'subjectivity,' 'classicality,' and so on. Familiar with such appeals from various contemporary explications of Copenhagen orthodoxy (for a sample, see §I of Wheeler and Zurek (1983)), John Bell conjures an interpretive strategy in the form of an alternate future for quantum mechanics:

> ...it is interesting to speculate on the possibility that a future theory will not be *intrinsically* ambiguous and approximate. Such a theory could not be fundamentally about 'measurements,' for that would again imply incompleteness of the system and unanalyzed interventions from outside. Rather it should again become possible to say of a system not that such and such may be *observed* to be so, but that such and such *be* so. The theory would be not be about '*observ*ables' but about '*be*ables'. (Bell 1987 (1973), 41)

Drawing upon the interpretive resource of the theoretical apparatus, suppose that the properties in whose terms a quantum system associated with $\mathfrak{B}(\mathcal{H})$ is characterized correspond to self-adjoint elements of $\mathfrak{B}(\mathcal{H})$, or more precisely to the assignment of eigenvalues to those elements. The option of taking W to describe an ensemble of systems, each of which exhibits a determinate eigenvalue for each observable pertaining to it, is exercised at (what most commentators' metaphysical scruples reckon to be[3]) considerable cost, if the dimension of $\mathcal{H}$ is greater than 2. Bell imagines another option:

> Could not one just promote *some* of the observables of the present quantum theory to the status of beables? The beables would then be represented by linear operators in the state space. The values which they are allowed to *be* would be eigenvalues of those operators. For the general state the probability of a beable *being* a particular value would be calculated just as was formerly calculated the probability of *observing* that value. (Bell 1987 (1973), 41)

The project Bell envisions is the *maximal beable approach* to interpreting quantum theories: to first approximation, the approach of identifying the largest set of quantum observables pertaining to a system that can (subject to constraints arising from metaphysical scruples) enjoy determinate values simultaneously. Determinate value assignments to this set indicate what worlds are possible according to the quantum theory; that is, they answer Q1.

Bell has the maximal beable approach aspire to *explicate* quantum probabilities. Successfully executing the approach, we come to understand not only what quantum probabilities are probabilities for (specific patterns of beable instantiation) but also how to calculate the values of those probabilities, and so answer Q3. This sort of explication is intregral to understanding QM as an empirical theory — a theory whose empirical commitments take the form of *characterized* possible worlds assigned *explicit* probabilities.

[3]Famously, Bell's metaphysical scruples are exceptional; see, for instance, Bell 1987 (1966), 8-9.

Clifton and Halvorson translate the interpretive approach Bell conjures into the formal apparatus of operator theory. Their articulation casts $\mathfrak{B}(\mathcal{H})$ as a * *algebra*, so we will begin with an introduction to that notion.

Definition 1 *An* ***algebra*** $\mathfrak{A}$ *over the field* $\mathbb{C}$ *of complex numbers is a set of elements* $(A, B, ...)$ *that is (Ai) closed under a commutative, associative operation* $+$ *of binary addition; (Aii) closed with respect to a binary multiplication operation* $\cdot$*, which is associative and distributive with respect to addition, but not necessarily commutative; and (Aiii) closed with respect to multiplication by complex numbers. A* **-algebra is an algebra closed under an involution* $^* : \mathfrak{A} \to \mathfrak{A}$.[4]

Simply put, an algebra is a closed linear vector space (Ai and Aii) equipped with a multiplication operation (Aiii). $\mathfrak{B}(\mathcal{H})$ is an example. A subset of $\mathfrak{B}(\mathcal{H})$ satisfying (Ai)-(Aiii)—the linear closure of the spectral resolution of a non-degenerate self adjoint element of $\mathfrak{B}(\mathcal{H})$ for example—is a *subalgebra* of $\mathfrak{B}(\mathcal{H})$. Like $\mathfrak{B}(\mathcal{H})$ all the algebras considered here are * algebras.

Three more definitions preface Clifton and Halvorson's articulation of the maximal beable approach.

Definition 2 *A* ***state*** ω *on a C** *algebra* $\mathfrak{A}$ *is a normed, positive, linear functional* $\omega : \mathfrak{A} \to \mathbb{C}$.

This is just a state in the sense familiar from ordinary quantum mechanics, generalized to algebras that needn't take the form $\mathfrak{B}(\mathcal{H})$, and relieved of the (seemingly natural!) requirement of countable additivity. The next definition unpacks the intuitive notion of possessing a determinate value:

Definition 3 *A state* ω *on a subalgebra* $\mathfrak{D}$ *of a C** *algebra* $\mathfrak{A}$ *is* ***dispersion free*** *iff* $\omega(A^2) = (\omega(A))^2$ *for each self adjoint* $A \in \mathfrak{D}$.

Thus a system in a dispersion free state ω on $\mathfrak{D}$ possesses a determinate value for every self-adjoint element of $\mathfrak{D}$. These elements are beables for a system in ω. (NB a dispersion-free state needn't be countably additive, even if $\mathfrak{D}$ contains enough projection operators for a requirement of countable additivity to make sense.)

When ω is dispersion-free on $\mathfrak{D}$, every observable in $\mathfrak{D}$ possesses a determinate value, moreover one which ω predicts with certainty. The next definition is introduced to accommodate the possibility that a state corresponds to a straightforward statistical mixture over possible patterns of beable instantiation.

[4]satisfying:

$$(A^*)^* = A, \quad (A+B)^* = A^* + B^*, \quad (cA)^* = \bar{c}A^*, \quad (AB)^* = B^*A^*$$

for all $A, B \in \mathfrak{A}$ and all complex c (where the overbar denotes complex conjugation).

Definition 4 *A state* ω *on a subalgebra* $\mathfrak{D}$ *of a* C^* *algebra* $\mathfrak{A}$ *is a* ***mixture of dispersion-free states*** *on* $\mathfrak{D}$ *iff there exists a probability measure* μ_ω *on the space* Λ *of dispersion free states* ω_λ *on* $\mathfrak{D}$ *such that for all* $A \in \mathfrak{D}$

$$\omega(A) = \int_\Lambda \omega_\lambda(A) d\mu_\omega(\lambda) \tag{1}$$

Now, for Clifton and Halvorson, to take the maximal beable approach to interpreting QM is to proceed as follows

> Given a state on an algebra of observables, characterize those subalgebras of 'beables,' that are maximal with respect to the property that the state's restriction to the subalgebra is a mixture of dispersion-free states. Such *maximal beable subalgebras* could then represent maximal sets of observables with simultaneously determinate values distributed in accordance with the state's expectation values. (Halvorson and Clifton 1999, 2442; cf. Bub 1997, 117, 119)

Underlying the maximal beable approach are metaphysical scruples about the nature of properties (to be instantiated is to take a value that is not disjunctive or interval-valued or otherwise fuzzy, but dispersion-free) and the desirability of plenitude (the imperative to find *maximal* beable algebras: if an interpretation can add to the list of properties instantiated in a quantum world, it should).

Maximal beable subalgebras, so defined, exist for any state (Halvorson and Clifton 1999, 2447). In the case of *faithful states*, maximal beable subalgebras are readily characterized.

Definition 5 *A state* ω *on* $\mathfrak{A}$ *is* ***faithful*** *iff* $\omega(A) = 0$ *implies* $A = 0$ *for all* $A \in \mathfrak{A}$.

Fact 1 *If* ω *is a faithful state on* $\mathfrak{A}$ *then it can be represented as a mixture of dispersion free states on a subalgebra* $\mathfrak{D}$ *of* $\mathfrak{A}$ *iff* $\mathfrak{D}$ *is abelian (that is, pairwise commuting). (Cf. Clifton 2000, 172)*

The maximal beable approach to interpretting a theory of ordinary QM sets the generic C* algebra $\mathfrak{A}$ in the foregoing characterizations equal to $\mathfrak{B}(\mathcal{H})$. It follows from Fact 1 that maximal beable subalgebra of $\mathfrak{B}(\mathcal{H})$ for a faithful state is a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$, that is, an abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ not properly contained in any other abelian subalgebra of $\mathfrak{B}(\mathcal{H})$. There is a simple recipe for generating a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$: start with a complete set $\{E_i\}$ of orthogonal one-dimensional projection operators in $\mathfrak{B}(\mathcal{H})$ and close in the weak topology. The result will be an algebra consisting of every element of $\mathfrak{B}(\mathcal{H})$ that has $\{E_i\}$ as a spectral resolution.[5] Call this the maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ *generated by* $\{E_i\}$.

Most physically interesting states in QM_∞ are faithful, so what follows will deal exclusively with faithful states. Bub 1997 gives a comprehensive account of the maximal beable approach in the presence of non-faithful states.

[5]See Beltrametti and Cassinelli 1981, §3.2 for an argument.

2.3 Variations on the maximal beable approach

The maximal beable approach is a template for interpreting QM wherein a possible world is a 'maximal set of co-obtaining properties' (Bub 1997, 18), with a property understood as the assignment of an eigenvalue to a self-adjoint operator. Different extant variations on the maximal beable approach can be individuated by how they complete that template—that is, by how they identify maximal beable subalgebras—and by the commentary on quantum probability they append to it.

Collapse

Collapse interpretations conjoin the *eigenstate-eigenvector link*, according to which a system in a state $|\psi\rangle$ has a determinate value for A if and only if $|\psi\rangle$ is an eigenstate of A, in which case the value is the associated eigenvalue, to the ignoble *postulate of measurement collapse*, according to which an A measurement performed on a system in a state $|\psi\rangle$ instantaneously and discontinuously collapses $|\psi\rangle$ to the A eigenstate corresponding to the outcome observed, with the Born rule probabilities prescribed by $|\psi\rangle$ furnishing a probability distribution over candidate collapses.

Now, assume non-degenerate A, with eigenstates $\{|\alpha_i\rangle\}$, is measured on an object in initial state $|\psi\rangle = \sum_i c_i|\alpha_i\rangle$. According to the postulate of measurement collapse, the density operator $W = \sum_i |c_i|^2 E_{\alpha_i}$ (where E_{α_i} is the projection operator for the subspace spanned by $|\alpha_i\rangle$) describes the post-measurement object state. W is faithful if none of the c_is is 0. The complete orthogonal set of projections E_{α_i}, as well as every $A \in \mathfrak{B}(\mathcal{H})$ having that set as a spectral resolution, is a maximal abelian subalgebra, call it $\mathfrak{D}_A$, of $\mathfrak{B}(\mathcal{H})$. It's also the set of observables that have determinate values after measurement, according to the eigenstate-eigenvalue link.[6] Thus a collapse interpretation instantiates the maximal beable approach. Its commentary on quantum probability answers (Q1)-(Q3) as follows:

(Q1): Each projection operator E_{α_i} encodes a possible condition of the system described by W via the recipe that the value of A in $\mathfrak{D}_A$ is the eigenvalue associated with E_{α_i}. (NB This is the eigenvector/eigenvalue link again.)

(Q2) Quantum probabilities are epistemic; they reflect our ignorance of the endpoint of collapse.

(Q3) The probability of a collapse to the condition encoded by E_{α_i} is $Tr(WE_{\alpha_i})$. Notice that because the E_{α_i} are orthogonal, these probabilites are non-interfering.

"Bohm"ian interpretations

A "Bohmian" interpretation dictates a prefered determinate observable. For simplicity, suppose that it's discrete and non-degenerate and call it R. (Bohm,

[6] Or, to be more careful, the intersection, over the collection of possible collapses, of the sets that have determinate values.

of course, supposes R to be position, an operator with a continuous spectrum. Technical difficulties attendant upon this complication—difficulties not with articulating Bohmian mechanics but with shoe-horning it into a framework where properties correspond to self-adjoint operators on a separable Hilbert space—motivate the shudder quotes around "Bohm" in the heading.) R's eigenprojections E_{r_i} correspond to pairwise orthogonal one-dimensional subspaces of $\mathcal{H}$; they generate a maximal abelian subalgebra $\mathfrak{D}_R$ of $\mathfrak{B}(\mathcal{H})$. This is the beable subalgebra, according to "Bohm." Notably, it is independent of W if W is faithful. Possible conditions of a system described by W are coded by R's eigenprojections E_{r_i}; as before, the eigenstate-eigenvalue link is the decoder (Q1). The probability that the system is in the condition coded by E_{r_i} is the Born Rule probability $Tr(WE_{r_i})$ (Q3); this probability is epistemic (Q2).

Modal interpretations

Same song, slightly different verse. Suppose W is faithful and non-degenerate. Then $\mathfrak{D}_W$, the maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ generated by W's spectral projections E_i, is the beable algebra; possible conditions of the system ("value states") stand in one-to-one correspondence with W's spectral projections, with the eigenstate-eigenvalue link dictating what's true of a system in value state E_i (Q1); the probability that the system is in the condition coded by E_i is the Born Rule probability $Tr(WE_i)$ (Q3); this probability is epistemic (Q2).

Relative state formulations

Same song, different register. For the sake of exposition, suppose that the universe admits a preferred decomposition into subsystems, and that the Hilbert space for each subsystem has a prefered basis (dictating the character of the 'worlds' corresponding to different branches of the universal wave function). Let W be the reduced state of some prefered subsystem induced by the universal wave function, and let $\{E_i\}$ be the complete set of eigenprojections onto the preferred basis for that subsystem. Then proceed as with the modal interpretation, only with this epicycle: the E_i's keep track not of value states (mutually exclusive possible conditions of the system) but of *worlds*, of which there are many, and to none of which the system is confined simpliciter. In the world kept track of by E_i, what's true of the system is given by applying the eigenstate-eigenvector link to E_i (Q1). The profligate metaphysics precludes a straightforward epistemic interpretation of quantum probabilities; there are other options, for instance, that Born rule probabilities encapsulate the degrees of belief of a rational observer in a universe truly described by a relative state formulation (Wallace 2003, Dorr 2007; for another approach to relative state formulation probabilities, see Barrett 1999).

3 von Neumann algebras and their projection lattices

Here I introduce some apparatus for re-describing what I've just reviewed, in order to notice a possible problem: in the case that the von Neumann algebra pertaining to a system fails to be a Type I factor, the formal pivot on which all the foregoing interpretations hinged—the encoding one-dimensional projections E_i (and their analogs), decoded by the eigenstate-eigenvalue link to answer Q1 and plugged into the trace prescription to answer Q3—can go missing.

3.1 von Neumann algebras

Given an algebra $\mathfrak{D}$ of bounded operators on a Hilbert space $\mathcal{H}$, its *commutant* $\mathfrak{D}'$ is the set of all bounded operators on $\mathcal{H}$ that commute with every element of $\mathfrak{D}$. So, for example, the commutant of $\mathfrak{B}(\mathcal{H})$ consists of scalar multiples of the identity operator. $\mathfrak{D}$'s *double commutant* $\mathfrak{D}''$ is $\mathfrak{D}'$'s commutant. Every element of $\mathfrak{B}(\mathcal{H})$ commutes with the identity operator, and so with every element of $\mathfrak{B}(\mathcal{H})'$. This makes $\mathfrak{B}(\mathcal{H})$ its own double commutant. It also qualifies $\mathfrak{B}(\mathcal{H})$ as a von Neumann algebra.[7]

Definition 6 *A* ***von Neumann algebra*** $\mathfrak{M}$ *is a* **-algebra of bounded operators such that* $\mathfrak{M} = \mathfrak{M}''$.

$\mathfrak{B}(\mathcal{H})$ is moreover a von Neumann factor:

Definition 7 *A von Neumann algebra* $\mathfrak{M}$ *is a* ***factor*** *if and only if* $\mathfrak{M} \cap \mathfrak{M}'$ *contains only multiplies of the identity.*

For the sake of simplicity, we will confine our attention here to factor algebras.

Every von Neumann algebra $\mathfrak{M}$ is a subalgebra of $\mathfrak{B}(\mathcal{H})$ for some $\mathcal{H}$, but it doesn't follow that every $\mathfrak{M}$ has projection operators corresponding to every subspace of the Hilbert space on which it acts. Let $\mathcal{P}(\mathfrak{M})$ be the set of projection operators in the von Neumann algebra $\mathfrak{M}$. It is a consequence of von Neumann's double commutant theorem that $\mathfrak{M} = \mathcal{P}(\mathfrak{M})''$. A typology of von Neumann algebras originating with Murray and von Neumann (1936) is based on the character of the projections $\mathcal{P}(\mathfrak{M})$ does contains. The characters that interest us most are infinite projections and minimal projections.

[7] Topologically characterized, a *von Neumann algebra* $\mathfrak{M}$ is a *-algebra of bounded operators that is strong operator-closed in its action on some Hilbert space. Von Neumann showed that the strong and weak closures of a self-adjoint algebra $\mathfrak{D}$ of bounded Hilbert space operators coincide—and coincide as well with $\mathfrak{D}$'s double commutant. For more on von Neumann algebras and operator topologies, consult Kadison and Ringrose (1997a, Ch. 5 and 1997b, Ch. 6).

Infinite, finite, and minimal projections

The *range* of a projection E in von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$ is the linear span of $\{|\psi\rangle \in \mathcal{H} : E\,|\psi\rangle = |\psi\rangle\}$. Thus the range of E is a closed subspace of $\mathcal{H}$ (cf. Kadison and Ringrose 1997a, Prop. 2.5.1). Two projections E and F in $\mathfrak{M}$ are *equivalent* (written $E \sim F$) just in case their ranges are isometrically embeddable into one another, *by an isometry that is an element of* $\mathfrak{M}$. Equivalence so construed is manifestly relative to $\mathfrak{M}$. When E's range is a subspace of F's range (written $E \leq F$), E is a *subprojection* of F. Equivalent criteria are that $FE = EF = E$ and that $|E|\psi\rangle| \leq |F|\psi\rangle|$ for all $|\psi\rangle \in \mathcal{H}$. We use the subprojection relation to define the relation *weaker than* (written $\preceq$), which imposes a partial order on projections in a von Neumann algebra: E is *weaker than* F if and only if E is equivalent to a subprojection of F. Because $\preceq$ is a partial order, $E \preceq F$ and $F \preceq E$ together imply that $E \sim F$.

A projection $E \in \mathfrak{M}$ is *infinite* if and only if there's some projection $E_0 \in \mathfrak{M}$ such that $E_0 < E$ and $E \sim E_0$. In this case, E_0's range is both a proper subset of, and isometrically embeddable, into E's range. $E \in \mathfrak{M}$ is *finite* if and only if it is not infinite.

A non-zero projection $E \in \mathfrak{M}$ is *minimal* if and only if E's only subprojections are 0 and E itself. It follows that minimal projections are finite. The minimal projections of the factor $\mathfrak{B}(\mathcal{H})$ are the one-dimensional ones.

A Classification of von Neumann Algebras

The Murray-von Neumann classification applies in the first instance to von Neumann algebras which are factors; on such algebras, the weaker than relation $\preceq$ imposes a total order (see Kadison and Ringrose 1997b, Prop. 6.2.6).

> Type I: Type I factors contain minimal projections, which are therefore also finite.

The algebras $\mathfrak{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space—that is, the observable algebras of ordinary QM—are Type I factors, and each Type I factor is isomorphic to some $\mathfrak{B}(\mathcal{H})$. Type II and III factors may be less familiar.

> Type II: Type II factors contain no minimal projections, but do contain (non-zero) finite projections.

Indeed, in a sense that can be made precise (Sunder 1987, §1.3), Type II factors have projections whose ranges are subspaces of *fractional* dimension.

> Type III: Type III factors have no (non-zero) finite projections and so no minimal projections. All their projections are infinite and therefore equivalent (cf. Kadison and Ringrose 1997b, Corr. 6.3.5).

For concrete examples of factors of Types II and III, see Sunder (1987).

3.2 The lattice of projections

The set $\mathcal{P}(\mathfrak{M})$ is partially ordered by the relation $\leq$ of subspace inclusion, defined in terms of the Hilbert space on which $\mathfrak{M}$ acts. Thus $E \leq F$ if E's range is a (not necessarily proper) subspace of F's range. (The following will employ expressions like "E" to refer both to projections and to the closed subspaces that are their ranges.) This partial order enables us to define for each pair of elements $E, F \in \mathcal{P}(\mathfrak{M})$, their *greatest lower bound* (aka *meet*) $E \cap F$ as the projection whose range is largest closed subspace of $\mathcal{H}$ that is contained in both E and F; and their *least upper bound* (aka *join*) $E \cup F$ as the projection whose range is smallest closed subspace of $\mathcal{H}$ that contains both E and F. Thus $E \cap F$ is just the projection whose range is the intersection of E's range and F's, and $E \cup F$ is just the projection whose range is the linear span of E's range and F's. A *lattice* is a partially ordered set every pair of elements of which has both a least upper bound and a greatest lower bound. Thus the foregoing definitions render $\mathcal{P}(\mathfrak{M})$ a lattice. Indeed, $\mathcal{P}(\mathfrak{M})$ is an orthocomplemented lattice.

Definition 8 (orthocomplemented lattice) *A lattice S has a zero element 0 s.t. $0 \leq a$ for all $a \in S$ and a unit element 1 s.t. $a \leq 1$ for all $a \in S$. The zero operator (the projection operator for the null subspace) is the zero element of $\mathcal{P}(\mathfrak{M})$ and the identity operator I is the unit element. The* ***complement*** *of an element a of a lattice S is an element $a' \in S$ such that $a' \cup a = 1$. A lattice is complemented if each of its elements has a complement. It's* ***orthocomplemented*** *if these complements obey*

$$a'' = a \qquad a \leq b' \text{ if and only if } b' \leq a' \tag{2}$$

The complement $E^{\perp}$ of $E \in \mathcal{P}(\mathfrak{M})$ supplied by the projection $I - E$, whose range is the orthogonal complement of E's range.

$\mathcal{P}(\mathfrak{M})$ is not necessarily a distributive lattice.

Definition 9 *All a, b, c in a* ***distributive lattice*** *S satisfy the distributive law*

$$\begin{aligned} a \cup (b \cap c) &= (a \cup b) \cap (a \cup c) \\ a \cap (b \cup c) &= (a \cap b) \cup (a \cap c) \end{aligned} \tag{3}$$

Let $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$, and let A, B, and C be projections whose ranges are subspaces spanned by the vectors $\mathcal{H} \ni |\alpha>$, $|\beta>$ and $|\gamma>= |\alpha> + |\beta>$, where $|\alpha>$ and $|\beta>$ are orthogonal. It is easy to verify that (3) breaks down. However, in the special case that $\mathfrak{M}$ is abelian, $\mathcal{P}(\mathfrak{M})$ is distributive. Indeed, it's a *Boolean lattice* (aka a *Boolean algebra*), that is, is a distributive complemented lattice. The simplest Boolean lattice is the set $\{0, 1\}$, where each element is the other's complement and meet and join correspond to set-theoretic intersection and union respectively. Call this lattice B_2. Notice that B_2's elements can be put into one-one correspondence with the truth-values *false* (0) and *true* (1).

A *Boolean (or two-valued) homomorphism* between Boolean lattices B and B_2 is a map $h : B \to B_2$ preserving Boolean operations.[8] Construing B as a lattice of propositions, we can construe lattice operations — join ($\cup$), meet ($\cap$), and complement ($'$)—as logical operations — disjunction (v), conjunction (&), and negation ($\sim$), respectively. Given this construal, a two-valued homomorphism $h : B \to B_2$ on a Boolean lattice B is a *truth valuation on B respecting the classical truth tables* for disjunction, conjunction, and negation.

When a Boolean lattice B is finite— that is, it has finitely many elements— there's a simple recipe for obtaining its two-valued homomorphisms. An element a of a lattice S is an *atom* if and only if S contains no non-zero elements "smaller" than a.[9] We encountered the notion of "atom" in a different guise earlier in this section: the minimal projections in von Neumann algebra $\mathfrak{M}$ (if it has any) are the atoms of the lattice $\mathcal{P}(\mathfrak{M})$. An atom a in a Boolean lattice B generates a two-valued homomorphism h_a on B as follows:

$$\begin{aligned} h_a(b) &= 1 \text{ if } a \leq b \\ h_a(b) &= 0 \text{ otherwise} \end{aligned} \tag{5}$$

If B is finite *all* its two-valued homomorphisms are determined in this way (Bell and Machover 1977, Cor. 5.3).

3.3 The maximal beable approach revisited

We can use the notions just reviewed to recharacterize ordinary QM instantiations of the maximal beable approach. Given an ordinary QM system in a faithful state ω on $\mathfrak{B}(\mathcal{H})$, familiar interpretations use a maximal beable subalgebra $\mathfrak{D}$ of $\mathfrak{B}(\mathcal{H})$ to identify a maximal abelian subalgebra $\mathcal{P}(\mathfrak{D})$ of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. As we have seen, different interpretations identify $\mathcal{P}(\mathfrak{D})$ in different ways. But they all share the strategy of using atoms in $\mathcal{P}(\mathfrak{D})$—minimal projection operators in the Type I von Neumann algebra $\mathfrak{B}(\mathcal{H})$—to code worlds possible for the system in state ω. Each world possible corresponds to a two-valued homomorphism on $\mathcal{P}(\mathfrak{D})$ determined by its coding atom. The classical semantic structure of such worlds is an attractive feature of the maximal beable approach. The state ω defines a probability distribution over these homomorphisms: where W is the density matrix in $\mathfrak{B}(\mathcal{H})$ implementing ω, the probability distribution is determined by plugging the atoms generating the

[8] to wit,

$$\begin{aligned} h(a \cup b) &= h(a) \cup h(b) \\ h(a \cap b) &= h(a) \cap h(b) \\ h(a') &= h(a)' \\ h(0) &= 0 \\ h(1) &= 1 \end{aligned} \tag{4}$$

(NB the second and third (equivalently the second and fourth) are sufficient to define a Boolean homomorphism; the remaining properties are consequences.)

[9] That is, for all $b \in S$, $b \leq a$ implies $b = a$ or $b = 0$.

homomorphisms into the trace prescription. Another attractive feature of the maximal beable approach is that the probabilities obtained by restricting ω to $\mathcal{P}(\mathfrak{D})$ will be non-interfering, because $\mathcal{P}(\mathfrak{D})$ is abelian. They have the character of classical probabilities, and can bear either epistemic or subjective interpretations just as well as those probabilities can.

Of course, there is nothing to stop us from extending the maximal beable approach beyond ordinary QM by replacing $\mathfrak{B}(\mathcal{H})$ with an *arbitary* von Neumann algebra $\mathfrak{M}$. The maximal beable recipe becomes

> **Maximal beable recipe** *Given a system in a faithful state ω on a von Neuman algebra $\mathfrak{M}$, identify a maximal abelian subalgebra $\mathcal{P}(\mathfrak{D})$ of $\mathcal{P}(\mathfrak{M})$. Two-valued homomorphisms on $\mathcal{P}(\mathfrak{D})$ correspond to worlds possible for the system; ω defines a probability distribution over those.*

The next section sets out inducements to pursue such an extension, in the form of physically significant algebras of quantum observables not isomorphic to $\mathfrak{B}(\mathcal{H})$.

4 QM$_\infty$

4.1 Physical setting in which Type III factors arise

Von Neumann algebras not isomorphic to $\mathfrak{B}(\mathcal{H})$ for some separable $\mathcal{H}$ are commonplace in QM$_\infty$. Consider, for instance, axiomatic approaches to local relativistic Quantum Field Theory, which associate with each open bounded region O of Minkowski spacetime a von Neumann algebra $\mathfrak{M}(O)$ of observables pertaining (in some sense) to that region. These local algebras are typically Type III. For example, in the Minkowski vacuum state for the mass $m \geq 0$ Klein-Gordon field, if O is a region with non-empty spacelike complement, the standard axioms imply that $\mathfrak{M}(O)$ is a Type III factor (Araki 1964).

For another example, consider the thermodynamic limit of quantum statistical mechanics (reached by letting the number of systems one considers and the volume they occupy go to ∞ while keeping their density finite). Equilibrium states at finite temperatures correspond to Type III factors for a wide variety of physically interesting systems: Bose and Fermi gases, the Einstein crystal, the BCS model (see Emch 1972, 139-140; Bratelli and Robinson 1997, Corr. 5.3.36). At temperatures at which phase transitions occur (if there are any for the systems in question), equilibrium states correspond to direct sums/integrals of Type III factors.

Type II factor states also abound in QM$_\infty$, as do non-factor algebras of Types II and III. But the examples of Type III factors just provided are enough to provoke the question: can strategies and techniques for interpreting QM evolved in the environment of Type I factors be adapted to the environment of QM$_\infty$?

4.2 Atomlessness

There is an impediment to adapting familiar variations on the maximal beable approach to the Type III factor algebras typical of QM_∞. The impediment is that if $\mathfrak{M}$ is a type III factor, its maximal abelian subalgebras—the very things the maximal beable approach would cast as maximal beable algebras—lack minimal projections—the very things familiar variations on the maximal beable approach use to characterize the worlds possible according to a quantum theory, and to assign probabilities to those worlds.

Let's approach this impediment by way of some examples. First, an atomless Boolean algebra. Consider the collection of subsets of $\mathbb{R}$ of the form $[x, \infty)$, partially ordered by set theoretic inclusion, with set theoretic union, intersection, and complementation supplying the Boolean operations of $\cup$, $\cap$, and $\sim$. This Boolean algebra is atomless: for any element $X = [x, \infty)$, there's an element $Y = [y, \infty)$, $y > x$ such that $Y < X$.

Next, an atomless maximal abelian von Neumann algebra: Let $\mathcal{H}$ be the separable Hilbert space L_2 of square integrable functions on the unit interval $[0, 1]$ equipped with the Lebesgue measure. Where S is a borel subset of $[0, 1]$, let χ_S be the operator on L_2 corresponding to multiplication by the characteristic function for f. Notice that if S is a set of measure 0, then χ_S is the zero operator on L_2, and that if T and V differ by a set of measure 0, χ_T and χ_V are the same operator on L_2. The collection $\{\chi_S\} = \mathfrak{D}_Q$ (with addition and multiplication defined pointwise) is a maximal abelian von Neumann algebra acting on $\mathcal{H}$ (Kadison and Ringrose 1997b, Example 5.1.6; see also Halvorson 2001).[10] The following consideration suggest (truly (Kadison and Ringrose 1997b, Lemma 8.6.8) that $\mathfrak{D}_Q$ lacks atoms. For each measurable subset X of $[0, 1]$, the characteristic function χ_X is a projection in $\mathfrak{D}_Q$. $\chi_Y < \chi_X$ iff Y is a measurable subset of X. But every measurable subset of X itself has a measurable proper subset. Thus no projection in $\mathfrak{D}_Q$ is minimal. It turns out that every atomless maximal abelian von Neumann algebra is isomorphic to $\mathfrak{D}_Q$ (Kadison and Ringrose 1997b, 665 ff.).

To see that such atomlessness afflicts any maximal abelian subalgebra of a Type III factor algebra, suppose that $\mathfrak{M}$ is such an algebra, and that $\mathfrak{D}$ is a maximal abelian subalgebra of $\mathfrak{M}$. And suppose, for reductio, that E is an atom in $\mathcal{P}(\mathfrak{D})$. Because $\mathfrak{M}$ contains no minimal projections, there exists $F \in \mathfrak{M}$ such that $F < E$. Because E is an atom in $\mathcal{P}(\mathfrak{D})$, $F \notin \mathcal{P}(\mathfrak{D})$. But then $\mathcal{P}(\mathfrak{D})$ is not a maximal abelian subalgebra of $\mathcal{P}(\mathfrak{M})$, which is our contradiction.

> "Proof." $\mathcal{P}(\mathfrak{D})$ is non-maximal because there is an element of $\mathcal{P}(\mathfrak{M})$ in $\mathcal{P}(\mathfrak{D})$'s commutant but not in $\mathcal{P}(\mathfrak{D})$. That element is F. Because $\mathcal{P}(\mathfrak{D})$, its hypothesized atom E is either orthogonal to or a subprojection of every other element G of $\mathcal{P}(\mathfrak{D})$. That is, for any

[10] To be more precise, $\mathfrak{D}_Q$ consists of equivalence classes of characteristic functions for Borel subsets of $[0, 1]$, where χ_T and χ_V belong to the same equivalence class if and only if T and V differ by a set of measure 0.

$G \in \mathcal{P}(\mathfrak{D})$, either (i) $EG = E$, or (ii) $EG = 0$. In case (i), $F < G$, because $F < E$ by hypothesis, and subspace inclusion is transitive. $F < G$ implies $FG = GF = F$. So F commutes with G. In case (ii), $FG = FEG = 0$, and F commutes with G. Thus F lies in $\mathcal{P}(\mathfrak{D})$'s commutant but not in $\mathcal{P}(\mathfrak{D})$.

We conclude that any maximal abelian subalgebra of the projection lattice of a Type III factor algebra is atomless.

Familiar variations on the maximal beable approach appeal to atoms in the projection lattice of a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ to answer Q1-Q3. When, as the generalization to QM_∞ demands, a Type III factor $\mathfrak{M}$ is substituted for $\mathfrak{B}(\mathcal{H})$ as the observable algebra of interest, the atomlessness of $\mathfrak{M}$'s maximal abelian subalgebras and their projection lattices impedes the adaptation of these variations to the more general environment.

4.3 Ultrafilters to the Rescue?

Atomless though they may be, maximal abelian subalgebras of $\mathcal{P}(\mathfrak{M})$ for arbitrary $\mathfrak{M}$ admit *ultrafilters*. Here I'll indicate why this is so, explain how it's being so might lend hope to the maximal beable approach to QM_∞, then temper that hope.

Ultrafilters on maximal abelian subalgebras of $\mathcal{P}(\mathfrak{M})$ for arbitrary $\mathfrak{M}$

Definition 10 (Ultrafilter) *A **filter** on a lattice S is a non-empty proper subset F of S such that for all $a, b \in S$*

1. *if a and b are both elements of F, then so is their meet $a \cap b$;*
2. *if a is an element of F and $a \leq b$, then b is an element of F;*
3. $0 \notin F$.

*As subsets of S, filters can be ordered by inclusion. An **ultrafilter** on S is a filter on S that's not a proper subset of any other filter on S.*

An example of an ultrafilter is the subset F_a of B mapped to 1 by the homomorphism h_a defined in (5):

$$F_a = \{b \in B \text{ such that } a \leq b\} \tag{6}$$

F_a's distinctions do not end there. For F_a is also a *principal ultrafilter*, which is just an ultrafilter on a lattice generated (in the manner of (6)) by an atom of that lattice.

We care about ultrafilters because an ultrafilter on a Boolean lattice amounts to a truth valuation:

Fact 2 *Each ultrafilter F of a Boolean lattice B generates a two-valued homomorphism $h : B \to B_2$ via*

$$\begin{aligned} h(a) &= 1 \textit{ if } a \in F \\ h(a) &= 0 \textit{ if } a \notin F \end{aligned} \tag{7}$$

One way to demonstrate the existence of an ultrafilter is

Fact 3 (Ultrafilter Extension Theorem) *Any subset S of a Boolean lattice possessing the finite meet property*[11] *is contained in some ultrafilter (Bell and Machover 1977, Cor. 3.8).*

The proof of the Ultrafilter Extension Theorem invokes Zorn's lemma, which is equivalent to the axiom of choice. Because a general Boolean lattice needn't have atoms, its ultrafilters needn't be principal. The Ultrafilter Extension Theorem may tell us that they exist, but not what they look like. This reticence will be treated in more detail presently.

No matter what the type of a von Neumann algebra $\mathfrak{M}$, if $\mathfrak{M}$ admits a faithful state, $\mathcal{P}(\mathfrak{M})$ will have maximal abelian subalgebras $\mathcal{P}(\mathfrak{D})$ (where $\mathfrak{D}$ is a maximal abelian subalgebra of $\mathfrak{M}$), because maximal beable subalgebras always exist, and coincide with maximal abelian subalgebras in the presence of faithful states. In virtue of Fact 1, a faithful state on $\mathfrak{M}$ is a dispersion-free state on the abelian algebra $\mathfrak{D}$. The projection lattice of $\mathcal{P}(\mathfrak{D})$ is also a Boolean lattice admitting ultrafilters, because each dispersion-free state on $\mathfrak{D}$ induces an ultrafilter (and so a two-valued homomorphism) on $\mathcal{P}(\mathfrak{D})$. (This follows from the fact that if ω is dispersion free on $\mathfrak{D}$, $\omega(A)$ lies in A's spectrum for each $A \in \mathfrak{D}$. The elements of $\mathcal{P}(\mathfrak{D})$ mapped to 1 by dispersion-free ω thus constitute an ultrafilter for $\mathcal{P}(\mathfrak{D})$.)

Hope?

These ultrafilters won't be principle (i.e. generated by atoms), but so what? Recall the recipe for the maximal beable approach spelled out in §3.3. Atoms are invoked nowhere in that recipe. Although familiar variations on the maximal beable approach code facts and mediate probability assignments by atoms, the master recipe explicitly calls only for maximal abelian subalgebras (the beables) and two-valued homomorphisms on their projection lattices (the possible worlds, obtained as consistent eigenvaluations on those beables). And these ingredients, we've just seen, *will* be available in the more general setting. So even a general von Neumann algebra harbors possible worlds in the maximal beable approach's favored sense of maximal sets of co-obtaining properties.

What's more, for a state ω on a general von Neumann algebra $\mathfrak{M}$, we can still express ω (a la equation (1)) as a mixture of dispersion free states on a maximal abelian subalgebra $\mathfrak{D}$ of $\mathfrak{M}$. That is, for all $A \in \mathfrak{D}$, there will be a

[11] That if $x_1, ..., x_n \in S$ then $x_1 \cap \ldots \cap x_n \neq 0$.

probability measure μ_ω on the space Λ of dispersion free states ω_λ on $\mathfrak{D}$ such that

$$\omega(A) = \int_\Lambda \omega_\lambda(A) d\mu_\omega(\lambda)$$

And this is all the maximal beable approach requires for a quantum state to define a probability distribution over possible worlds in its favored sense. Regarding the maximal beable approach in these general terms suggests that, despite the proclivities of variations developed for ordinary QM, the maximal beable approach does not presuppose that $\mathcal{P}(\mathfrak{M})$ contains atoms.

Tempered

But upon closer examination, the promise of non-principle ultrafilters rings hollow. Consider the states ω_λ in the expression above, and the ultrafilters they define. What is a possible world coded by such an ultrafilter *like*? This is a fair question; in fact, it's the question Q1 a maximal beable approach to a system in a state ω on $\mathfrak{M}$ must answer to lend content to the theory interpreted. It is also a thorny question, as Hans Halvorson makes clear: "Although we 'know' that there are ultrafilters (i.e. pure states[12]) on [atomless $\mathfrak{D}$)], we do not know this because someone has constructed an example of such an ultrafilter... We are told that there is some pure state ω on $\mathfrak{D}$, but we are not given a recipe for determining the value $\omega(A)$ for an arbitrary element $A \in \mathfrak{D}$" (Halvorson 2001, 41). The rub is that the ultrafilters in question are non-constructable. Their existence is demonstrated by appeal to the ultrafilter extension theorem, which by way of presupposing Zorn's Lemma, presupposes the axiom of choice. To show that these ultrafilters exist, one finds a family of elements of $\mathcal{P}(\mathfrak{D})$ satisfying the finite meet property, then invokes the ultrafilter extension theorem to conclude that this family belongs to an an ultrafilter on $\mathcal{P}(\mathfrak{D})$.[13]

The ineffability of ultrafilters on non-atomic $\mathcal{P}(\mathfrak{D})$ suggests that we have no handle, analogous to the one supplied in the Type I case by applying the eigenstate-eigenvalue link to the atom generating a principle ulatrafilter, on how to decode the facts these ultrafilters encode. It also suggests that we have no handle, analogous to the one supplied in the Type I case by applying the trace prescription to the system state and the atom generating a principle ultrafilter, on how to assign those facts probabilities. Confronted with the non-atomic von Neumann algebras of QM_∞, then, the maximal beable approach shirks two key interpretive tasks. First, it fails to explicitly characterize the worlds possible according to the theory. It tells us that there exist homomorphisms defined by pure states ω_λ corresponding to these possible worlds, but it

[12]Cf. [Kadison and Ringrose 1997a, Prop. 4.4.1] If ω is dispersion free on abelian $\mathfrak{D}$, then ω is a pure state of $\mathfrak{D}$.

[13]For the sake of space, I am setting to one side the alarming fact that if $\mathcal{P}(\mathfrak{D})$ lacks atoms, then pure states on $\mathfrak{D}$ fail to be countably additive. For a more complete discussion of the non-constructability of ultrafilters on nonatomic algebras, see Halvorson (2001), which also investigates ways to avoid the difficulties this non-constructability creates.

doesn't identify those states or lend content to those possible worlds. Second, it fails to explicate the probabilities the theory assigns to these worlds. Again, we know that a probability distribution μ_λ exists, but ignorant of the identities of states ω_λ, we're ignorant as well of what probabilities ω assigns them. Thus the maximal beable approach fails to equip QM_∞ with empirical content, in the form of specific probabilities assignments to explicitly characterized worlds possible for systems described by non-atomic von Neumann algebras.

5 What now?

Taking the maximal beable approach to QM_∞, I have suggested, will leave us in difficulty. But why? Here is a non-comprehensive list of possible culprits.

At its most general, the maximal beable approach makes a resource of the formal apparatus of von Neumann algebras and their self-adjoint elements. In particular, the maximal beable approach assimilates a physical property to a determinate eigenvalue assignment to a self-adjoint element of the von Neumann algebra of observables for the system whose property it is. Perhaps this is the wrong formal apparatus. Perhaps the salient algebra has been misidentified. Perhaps C^* algebras (cf. Segal 1959) or universal enveloping von Neumann algebras (cf. Primas 1983) will prove more fruitful frameworks. Or perhaps the aspiration to explicate the metaphysical commodity *property* so directly in terms of formal apparatus is misguided. (I think that the discipline inherent in the aspiration respects the contribution of physics to philosophy of physics. I don't think such respect is a *mandatory* component of philosophical reflection on physics. But I think it's nice that it's a component of some reflections.)

The maximal beable approach also rests on a set of metaphysical scruples, for instance, that a possible world is a *maximal* set of co-obtaining properties. Perhaps these scruples are to blame for the approach's inadequacy to QM_∞. Rob Clifton's (2000) strategy for interpretting the exotic von Neumann algebras encountered in QFT dispenses with the maximality scruple. To oversimplify, given a state ω on a von Neumann algebra $\mathfrak{M}$, Clifton identifies a beable algebra $\mathfrak{D}$ that isn't a maximal abelian subalgebra of $\mathfrak{M}$ but *is* the largest abelian subalgebra of $\mathfrak{M}$ characterizable, by means Clifton deems admissible, in terms of ω and $\mathfrak{M}$. In other words, $\mathfrak{D}$ can be embedded in larger abelian subalgebras of $\mathfrak{M}$, but ω doesn't tell us which ones. The merits of refraining from plenitude deserve further debate. Some demerits of Clifton's particular strategy for refraining have been discussed elsewhere (Clifton 2000, Earman and Ruetsche 2005): many observable algebras admit a dense set of states for which Clifton's beable algebra is trivial, in the sense that it contains only multiples of the identity operator!

The suggestion I find interesting is that the maximal beable approach goes wrong by allowing itself only the resources of formal apparatus and metaphysical scruples. However, the strategy of modifying the sets of *those* resources exploited by the maximal beable approach opens up many promising avenues for developing interpretations of QM_∞. Thus the considerations aired in this essay aren't even the shadow of an argument that those resources are inadequate to the task of interpreting QM_∞.

About that task, this discussion holds one lesson at least. Minimal projections, and interpretive strategies they underwrite, are not a perfectly general feature of the sorts of von Neumann algebras that arise in physical applications. It follows that assumptions widespread in the semantics of ordinary QM— assumptions that pure states are both fact encoders and probability bearers — are upset by QM_∞. The maximal beable approach can be freed of these assumptions and extended to QM_∞. But due to the non-constructability of ultrafilters on non-atomic abelian von Neumann algebras, the extension shirks the main tasks of quantum semantics: the *characterization* of worlds possible according to a quantum theory, and the *explication* of the probabilities that theory assigns those worlds. It appears that we can't interpret QM_∞ by *simply* extending or even adapting our favorite semantics for ordinary QM to infinite quantum systems.

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