Auyang and Sewell on Equilibrium and its Microanalysis

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1. Introduction

"Large composite systems are variegated and full of surprises," Sunny Auyang writes at the outset of her study of order and complexity in statistical physics, evolutionary biology, and economics (Auyang 1998)–a study which is itself variegated and full of surprises. "Perhaps the most wonderful" she continues, "is that despite their complexity on the small scale, sometimes they crystallize into large-scale patterns that can be conceptualized rather simply" (1). My discussion today of equilibrium and its microanalysis aims to crystallize some provocative features of Auyang's analysis of the conceptual structure of emergence in statistical physics. I'll try to suggest that Auyang's analysis begins to complicate a standard picture–a picture within the confines of which Auyang's book officially operates–of the conceptual structure of physical theories themselves.

In his recent *Quantum Mechanics and its Emergent Macrophysics*, the physicist Geoffry Sewell echoes Auyang's delight at the startling focus into which collective systems can fall. "The miracle of thermodynamics," he writes, "is that the equilibrium states of systems of enormously many particles can be classified in terms of just a few macroscopic variables" (2002, 144). Consider a gas, a system composed of enormously many (say, 10^{23}) gas molecules, where the classical characterization of *each one* of these molecules requires the specification of its (three component) position and its (three component) momentum. Thus, to specify the state of the gas by way of specifying the states of its microconstituents, one must supply on the order of 6×10^{23} numbers. Nevertheless, at equilibrium, the gas as a collective is characterized by a few bulk features–its energy, its pressure, its temperature, its volume, and so on–which remain more or less constant in time.

Equilibrium statistical mechanics seeks a *microphysical* underpinning for this (and other characteristic) bulk behavior. For example, from the vantage of statistical mechanics, the equilibrium state of a gas characterized by some small set of bulk parameters corresponds to a particular probability distribution over those microconfigurations of the gas's 10^{23} constituents that are consistent with those bulk parameters.

But how exactly is the micro-characterization related to the macro one? This is a special case of the focal question of Auyang's book: "What are the relations between theories for large composite systems and theories for their constituents, theoretical relations that give substance to the notion of composition?" (2). Auyang is most interested in the least transparent of such relations: *emergence*, whereby (roughly) a macrosystem manifests a character or property different in type from (and unreducible to) the characters and properties of its microconstituents. To better understand Auyang's account of emergence, I will discuss it in the context of equilibrium (macro)phenomena she identifies as archetypically emergent (183): phase transitions and spontaneous symmetry breaking.

2. What we can't have

I will start with a catalog of macrophenomena for which there is *no finite microanalysis*, in the following sense: A statistical physics that treats the systems manifesting these phenomena as composites of *finitely many* microconstituents is a statistical physics whose notion of equilibrium is too improverished to accommodate the phenomena.

2.1. Phase structure

At normal atmospheric pressure, water in equilibrium at 0^o centigrade can be solid or liquid; at 100^o centigrade it can be liquid or vapor. Iron is *ferromagnetic*: above 771^o C an iron bar at equilibrium exhibits no net magnetization (paramagnetic phase); below 771^o C, an iron bar at equilibrium exhibits spontaneous magnetization (ferromagnetic phase); at 771^o C, both phases, ferromagnetic and paramagnetic, are possible for it. These examples illustrate *phase structure*, the availability, to certain systems in equilibrium at certain temperatures, of a variety of different phases. (Phases are typically differentiated by appeal to an *order parameter*: the ferromagnetic phase, which requires the spins of most of the (outer shell) electrons in the iron to line up, is more highly ordered than the paramagnetic phase, in which the spins of constituent electrons, distributed every which way, tend to cancel one another out.) To accommodate phase structure, let us suppose, statistical physics must allow, for certain systems at certain temperatures, that a *multiplicity* of equilibrium states–and so a multiplicity of phases–are possible.

Here's the rub. The statistical physics of finite systems identifies equilibrium states with Gibbs states. And at finite temperatures, if the Gibbs state of a finite system exists, it is unique. So too then must be the phase available to a finite system in equilibrium at a finite temperature. Thus, finite microanalysis is incompatible with non-trivial phase structure, and incompatible as well with the multiplicity of equilibrium states in evidence at phase transitions.

2.2. Spontaneous symmetry breaking

The time evolution of physical systems is governed by their dynamical laws. A theory's dynamical laws serve to distinguish histories of a system which are possible, according to that theory, from those which are impossible. Consider, for instance, a pair of masses in otherwise empty space. Some sequences of configurations of this two body system–e.g. ones where one mass traverses an elliptical orbit with the other mass at one focus–obey Newtonian mechanics and Newton's law of universal gravitation; other sequences of configurations–e.g. ones where one mass traverses a rectangular orbit about the other–do not.

A noteworthy feature of the set of sequences of configurations satisfying the dynamical laws of Newtonian celestial mechanics is this: if you take any sequence in the set and 'rotate it' (e.g., in the elliptical orbit sequences, tip over the plane containing the major and minor axes of the ellipse), you'll wind up with another sequence in the set. This signals that the dynamical laws are *rotationally symmetric* (or isotropic); they favor no direction in space over any other. Rotation, an operation one can perform on configurations (or states) of the two body system, is a transformation implementing this symmetry. (Somewhat more [and possibly gratuitiously] technically, if $\mathfrak{h}$ is the set of histories allowed by some theory's dynamical laws, then θ is a dynamical symmetry of that theory iff $h \in \mathfrak{h}$ iff $\theta(h) \in \mathfrak{h}$. [Interpret $\theta(h)$ as follows: the history you obtain by performing the transformation implementing the symmetry–a rotation through a fixed angle say–on every state in the sequence of states comprising the history h.])

It is desirable that the *set* of equilibrium states for a given system at a given temperature be invariant under the dynamical symmetries of the theory governing that system. (Motivation?!: The idea is that, insofar as equilibrium is characterized *dynamically*, in terms of *stability* of bulk properties, you shouldn't be able to change something that is an equilibrium state into something that isn't by performing a transformation that makes no difference dynamically.) This does not, of course, require that *each* state in the set be unchanged by the symmetry transformation–think of a set of arrows pointing North, South, East and West on a flat map of Newfoundland, and the transformation "Rotate 90^o clockwise" in the plane of the surface of that map. Even though the transformation turns the East arrow into the South arrow, and the South arrow into the West arrow, and so on–even though, that is, no arrow in the set is invariant under the transformation–what you wind up with after you've applied the transformation to every arrow in the set is the set you started out with–the set itself is invariant under the transformation. (See also Auyangs's wonderful knitting needle illustration, p. 185.)

So simply requiring the set of equilibrium states at a given temperature to be invariant under the dynamical symmetries of the system for which they're equilibrium states does not preclude the existence of equilibrium states that *break* the symmetry. (A state s is said to break a symmetry implemented by a transformation θ iff applying the symmetry transformation to the state changes the state: $\theta(s) \neq s$.) If, however, the set of equilibrium states in question contains only one member, that set is invariant under dynamical symmetry transformations if and only if those symmetry transformations leave its sole member unchanged.

Now recall that according to the statistical physics of a finite system, if such a system has an equilibrium state at a finite temperature, then that equilibrium state is unique. This is exactly what inhibited a finite microanalysis of phase structure. We are now in a position to see that it inhibits as well a finite microanalysis of *symmetry breaking*, that is, of the existence of equilibrium states that break dynamical symmetries. For, according to the considerations above, the lone member of the set of equilibrium states of a finite system at a finite temperature must be invariant under that system's dynamical symmetry transformations. Finite microanalysis rules out broken symmetry. But nature appears to abound with equilibrium states that do break dynamical symmetries. Ferromagnetism furnishes one example. The dynamics of ferromagnetic substances are rotationally invariant (isotropic–no preferred direction in space). But ferromagnetic states aren't– spontaneously magnetized, an iron bar exerts a force in a *particular direction*; this directedness of a state of spontaneous magnetization breaks the rotational symmetry of the magnet's dynamics. (Crystals are another example–crystalline states generally break Euclidean symmetries but abide by those of a crystallographic subgroup of the Euclidean group.)

2.3. How to get it

To construct a statistical physics better suited to accommodate the macrophenomena of phase structure and spontaneous symmetry breaking, one must go to the thermodynamic limit. That is, one must allow the number of microconstituents one considers, and the volume they occupy, go to infinity, while keeping their energy density finite. Only in the thermodynamic limit can one introduce a notion of equilibrium that allows what the Gibbs notion of equilibrium for finite systems disallows: the multiplicity of equilibrium states at a finite temperature implicated both in phase structure and symmetry breaking. Auyang counts these phenomena as emergent (183). The next section tries to explain why.

3. Auyang on Emergence

For Auyang, *emergent* characters (and, she says, it is typically *characters* that emerge) are properties of composite systems that differ in type from the properties of their constituents. (Setting the standard for emergence high, Auyang additionally requires emergent characters of collective systems to defy reduction to the microcharacters of their constituents (177-179; cf. 45-46). What counts as reduction is a fraught question. For our purposes, it suffices to remark that Auyang takes it to be sufficient for reduction that a macrocharacter be defined as some sort of average or sum of microcharacters–that it be a *resultant* of microcharacters.)

To get a handle on what it is for properties or characters to differ in type, it is useful to consult Auyang's remarks on the role of state spaces in constituting the concepts of objects. "The state space of an individual contains all possible states that individuals of its kind can attain" (89). We've already encountered one example of a state space: a classical system of N particles is associated with a $6N$ dimensional state space $\mathbb{R}^{6N}$ called *phase space*; the state of such a system is specified by attributing a (three component) position and a (three component) momentum to each of its constituents; once its state is specified, so too are the values of every physical magnitude pertaining to the system, simply because classical magnitudes are functions of these $6N$ positions and momenta. A classical observable called a Hamiltonian determines, via Hamilton's equations, what dynamical evolutions are possible for the system. It thereby imposes trajectories on state space–paths traced out by states of systems as those states change in accord with dynamical laws.

On this picture, what's possible for a system–what sorts of configurations, and which time developments thereof–is encapsulated by a state space with dynamical trajectories imposed, i.e., *a state space type.* It is standard in the philosophical literature to identify the *content* of a theory with the state space types it assigns systems in its domain. The idea is that a theory takes on content by distinguishing states of affairs possible according to it from statesof affairs not possible according to it. State space types serve admirably as vessels of content, so construed. This is the *standard state space picture* of how theories take on contents that, I will try to suggest, Auyang begins to subvert.

But she also works perceptively within it. Consider another example of a state space, one which to which we will return presently. This state space is the two dimensional complex vector space $\mathbb{C}^2$ for a quantum spin $\frac{1}{2}$ system such as an electron. An important class of states of this system correspond to vecctors (of length 1); quantum observable magnitudes correspond to operators on this vector space. (An operator is a map that acts on a vector to produce a vector.) Roughly speaking, each observable corresponds to a pair of orthogonal ('perpindicular') (normed) vectors, called its *eigenvectors*, in the state space $\mathbb{C}^2$; all electron observables can be expressed in terms of a trio of electron observables, $\{\sigma_x, \sigma_y, \sigma_z\}$, called *Pauli spins.* Via Schrödinger's equation, a Hamiltonian operator imposes dynamical trajectories on this state space.

Now I would claim that *the* signal difference between quantum and classical mechanics is this: only if the electron's state vector serves also as an eigenvector of an observable is it straightforward to attribute the electron a determinate value for that observable. Unlike a classical particle, then, an electron can be assigned a state space state without thereby also being assigned a determinate value for every observable magnitude pertaining to it. This is because quantum state spaces and classical state spaces are differently configured. It aptly illustrates the perspicuity of Auyang's contention that "Since the state space is peculiar to a kind of individual, we can say it is part of the general concept of a kind" (89).

Different kinds–quantum systems and classical ones, say– dwell in different sorts of state spaces. Electrons are assigned states which are vectors in $\mathbb{C}^2$; systems of N classical particles are assigned states which are points in $\mathbb{R}^{6N}$. Accordingly, the characters (or properties or observables or magnitudes) of physical systems can differ in kind–and so be such that one might be appropriately said to emerge from the other–if they are constituted in terms of state spaces of different sorts: properties of systems of N classical particles are functions from $\mathbb{R}^{6N}$ to $\mathbb{R}$; electron observables are operators on $\mathbb{C}^2$. So one way properties or characters can differ in type–and so be candidates for emergence– is by pertaining to state spaces (and thereby to individuals) different in kind.

There is another way Auyang speaks about type-differences between systems. Consider the distinction drawn in the following passage–a distinction, Auyang suggest, profound enough that characters pertaining to systems of the second sort can be said to emerge from characters pertaining to systems of the first:

> Which one of the myriad possible states is actualized at any time is often determined by various dynamic or static rules subjected to constraints peculiar to the individual, such as boundary or initial conditions. *In other cases*, especially in more complex systems, accidents and contingent factors affect the history and hence the actual state of an individual. (90, italics mine)

We will see that Auyang takes systems involved in phase transitions and spontaneous symmetry breaking to be of the second sort, while mechanical aggregates of their microconstituents are of the first sort.

The distinction Auyang draws here appears to derive from the role of history and contingency in specifying the state of the system. But it is initially puzzling how any distinction so drawn could have the force Auyang desires of it. Consider her characterization of the typical progression by which a mechanical system is assigned, first, a state space type, and second, a state in that state space. First, equations of motion are specified (e.g., Newton's second law, or Schrödinger's equation). These are schemata for the time-development of a system. Next a Hamiltonian is specified; this suffices to convert the schema into the imposition of trajectories through elements of state space. Finally, "initial and boundary conditions for a particular system of the kind represented by the Hamiltoniann" are supplied (122-123). This serves to locate the state of the system in state space.

The first two steps in this progression accomplish on behalf of the theory a sort of nomic articulation, by identifying the laws pertaining to the system and configuring its state space accordingly. The last step identifies the contribution of contingency, in locating a system in one, rather than any other allowed, post in state space. As Auyang presents the progression, *the first two steps can be* *completed without taking the last.* Only after law has determined what is possible for a system does accident identify what is actual for it; accident contributes in no way to the configuration of state space types. This separation of questions of state space configuration (to which theoretical laws matter) from questions of state space occupation (to which contingency matters) reflects to the standard state space picture's identification of a theory's content with the set of states of affairs that are possible according to the theory (and corresponds as well to the allied idea that a theory constitutes the concept of a an object-kind by furnishing for that kind a state space type). For on the standard state space picture, the articulation of possibilities for systems precedes the specification of their actual states. And the possibilities articulated are *unimodal*; in particular, they are in no way conditioned or constrained by the particularities of any system for which they are possibilities.

Here then is the puzzle. The distinction Auyang draws in the passage cited above seems to amount to this, that in some systems, such as bulk ferromagnets, accident and history play a *pronounced* role in characterizing their states, and in other, simply specifying initial and boundary conditions suffices. This is perplexing, because 'history, contingency, particularity,' etc., seem after all to be just an evocative way to describe initial and boundary conditions. Hence the state specification of a composite system differs not at all from that of a micromechanical one, and Auyang's distinction comes to nothing.

To alleviate the perplexity, we must examine Auyang's account of phase transitions and spontaneous symmetry breaking more closely. Her point, it turns out, is not simply that accidents matter, but that they matter where (according to the usual state space picture) they shouldn't–to the very configuration of a system's state space. Our initial puzzlement owes to the fact that the distinction Auyang wants to draw is in tension with the idea that to constitute the concept of an object, one associates with that object a state space, with dynamical trajectories imposed. The distinction Auyang wants to draw is in tension with that idea because it is in tension with a presupposition (or at least constant concomitant) of the state space picture, that there's a sharp distinction between the configuration of a state space (that is, its construction, parameterization, and acquisition of dynamical trajectories, all of which proceed by appeal to dynamic and static *laws*) and the attribution to a system of a state in that state space (by specifying initial and/or boundary conditions, that is, contingencies).

4. Auyang on Phase Structure/SSB

Here is how I'll set things up: The composite system exhibiting the bulk paramagnetic/ferromagnetic phase transition–the iron sample, say–is the *ferromagnet*. This bulk system is microanalyzed as a collective of systems of half-integer spin–I'll call them electrons, and call the microanalyzing system the *n electron system*. Questions of emergence turn, in this terminology, on the relation between the ferromagnet and the n electron system.

Electron spin is a sort of intrinsic angular momentum. Moving charges generate magnetic fields, so with a spinning electron may be associated a magnetic moment (Auyang invites you picture it as a little arrow or compass needle) whose orientation depends on the details of that electron's spin. Owing to this magnetic moment, an electron behaves in an external magnetic field roughly as a compass needle would, experiencing a force tending to align it with the magnetic field lines. An electron also generates a tiny magnetic field of its own; the magnetic moment indicates its direction.

Models of the micro-physics of magnetism posit a collective containing large numbers of these electrons. Their spins are subject to two sorts of time-development. First, the spins of individual electrons rotate randomly, due mostly to thermal agitation. Second, spins of neighboring electrons interact in a way that tends to allign them. Both sorts of time-development are governed by *rotationally symmetric* (isotropic) dynamical laws. Intrinsically favoring no direction in space over any other, these laws are such that, if some process p satisfies them, so does every process $\theta_R(p)$ obtained by "rotating" p in space.

Aggregating these constituents, electrons conceived as tiny magnets, into a composite system, we can introduce a composite system macro-observable: the ferromagnet's *net magnetization*, intuitively the sum-effect of the magnetic moments of its myriad microconstituents. At high temperatures, a magnetic material exhibits no net magnetization, a fact typically explicated in terms of the micromodel as follows: at high temperatures "thermal agitation [tending to randomize the spins of constituent electrons] dominates. At any moment, there are as many spins pointing in one direction as in the opposite direction. Consquently the net magnetization vanishes" (188). In this case, the magnet occupies a *paramagnetic* state, which shares the rotational symmetry of the microdynamics–no net magnetization privileges a direction in space.

At lower temperatures, ferromagnets exhibit net magnetization. The microanalysis attributes this to the fact that at low temperatures, interactions between constituent spins dominate thermal effects. Spin-spin interactions tend to *align* the spins of neighboring electrons. The result, enhanced as the composite system tends toward a state in which all its spins are alligned, is a net spontaneous magnetization. Such spontaneous magnetization characterizes the *ferromagnetic* state of the magnet. The direction of this magnetization breaks the isotropy of the microdynamics: ferromagnetic states break dynamical symmetries.

The macroscopic transition from paramagnetic states to ferromagnetic ones–which in iron occurs at 771^o C–is an example of a phase transition. Magnetization serves as an *order parameter*: it is 0 in the highly disordered states wherein the spins of constituent electrons are randomized by thermal effects; it attains its maximum value in the highly ordered states in which the spins of all constituents are alligned. This change in the value of the order parameter as the macrosystem passes from a paramagnetic to a ferromagnetic state marks the phase transition. Insofar as paramagnetic states obey the dynamical symmetries of the systems at issue, and ferromagnetic states break them, this phase transition also illustrates spontaneous symmetry breaking.

But how does it illustrate *emergence*? Ferromagnetic and paramagnetic states alike seem to occupy state spaces of the same type as those occupied by their microconstituents (e.g., for an n electron system, a ferromagnetic state corresponds to an n electron state in which all n spins are alligned). They don't, it seems, assume chacters of different type by occupying a state space of a different type. Indeed, the macrocharacter of net magnetization whose change marks the phase transition appears merely to be the *resultant*, summed over the constituent electrons, of the microcharacter electron spin. What's more, the phase transition itself receives a lucid explanation in terms of the microdynamics. How is it that the macrocharacters manifested in phase transitions meet Auyang's stringent criteria of emergence, that they be different in kind from, and not reducible to, microcharacters?

For Auyang, the answer lies in a closer examination of the state space available to a ferromagnetic system, and how that space is configured. In light of the macroscopic phenomenology of a ferromagnet, in particular, its capacity for spontaneous magnetization, the statistical physics of n electron systems harbors quite a surprise: "According to the basic laws of statistical mechanics, to calculate the magnetization one averages over the microstates of the entire microstate space, that is, over all possible directions of spin alignments. The result is inevitably zero magnetization for ferromagnets" (188). If the ferromagnet has the same state space type as n electron systems–if the ferromagnet as a system is simply an aggregate of its constituents–then a ferromagnet is incapable of exhibiting ferromagnetism!

According to Auyang, physicists deal with this by "put[ting] [magnetism] in by hand" (188). The insertion procedure unfolds as follows: Physicists invoke an *imaginary* external field. This field renders the highly ordered spin configuration, in which all spins align with the external field, preferable to every other spin configuration. Next, they "take the infinite system limit to lock in the direction of spin alignment along the imagined field"(ibid.). (I'll have a bit more to say about this locking mechanism in the next paragraph, then again in Section 5.) Finally, they let the field go to 0, leaving the magnet in a state of "spontaneous" net magnetization. Auyang considers this theoretical maneuver highly significant. For it "underscores the history-dependence of the equilibrium state" (189), as well as its "accidental" (because no laws inform the choice of orientation of the fictitious external field) nature. It by appeal to the heightened contingency and historicity of its state specification that Auyang would differentiate the ferromagnet, and its characters, from an n electron system, and its characters–a differentiation required for the former to be properly said to emerge from the latter.

Still, it might seem that "putting magnetism in by hand" amounts simply to imposing very special initial/boundary conditions on the n electron system, so that the type distinction between ferromagnets (collective systems with emergent macrocharacters) and n electron systems (microsystems with microcharacters) I tried to run on behalf of Auyang in the last paragraph evaporates. Now evaporation is a suggestive specter to raise here, because it is a phase transition which does not also induce the breaking of a symmetry. The paramagnetic/ferromagnetic phase transition does, and Auyang's comments on symmetry breaking harbor further clues to the sense in which she takes phase transitions to manifest emergence.

"Broken symmetry," she writes, "obtains only if the system is somehow stuck to its particular state and prevented from going into other possible states" (186). (Notice how this "sticking" recalls the locking in of a ferromagnet to spin configuration favored by a fictitious external field.) Given that the *set* of equilibrium states is invariant under the symmetries in question, if the system, unstuck, were allowed to cycle through its equilibrium states, it would be aptly described by a convex combination of those states, a convex combination invariant under the symmetry transformation. But how to stick a system in a state of broken symmetry? "This is where the difference between *microscopic* and *macroscopic* systems becomes important" (186), Auyang writes.

If I follow her, the difference is important because the macroscopic corresponds to the thermodynamic limit of the microscopic. In the thermodynamic limit, Auyang claims, one finds not only broken symmetry, but also *broken ergodicity.* For an ergodic systems, almost all its microstates are such that the dynamical trajectories through them eventually reach almost all its other microstates. In the sort of "broken ergodicity" to which Auyang refers, a system stuck in a particular (e.g. symmetry breaking) macrostate can't evolve out of the set of microstates corresponding to that macrostate. The space of states available to the system has effectively shrunk to that set of microstates. Auyang comments, "With the breaking of ergodicity, *the microstate space of the system is radically different from its original microstate space.* The difference cannot be deduced from microscopic laws but has to be put in 'by hand,' bolstering the emergent nature of the macrostate" (187, italics mine).

We can now see that for Auyang, what distinguishes the bulk ferromagnet from its would-be microanalyst, the n electron system, is not simply that history/contingency affects the *state* occupied by the former–for surely, history and contingency, in the form of initial conditions affect the state occupied by the latter as well. What is significant for Auyang is that, for bulk ferromagnets, but not for their putatively microanalysing n electron systems, history and contingency have a role in shaping the *state spaces* associated with them–thus blurring the distinction that reigns in the micromechanical sphere between the wholly nomic matter of configuring state spaces and the wholly contingent one of which state a system occupies initially.

The "radical difference" in accessible state spaces renders the ferromagnet different in kind from the n electron systems constituting it; characters–broken symmetry, high order–supported by the constricted microstate space of the ferromagnet are thus characters different in type from those supported by the microstate spaces of n electron systems. This difference in type is signalled by an expansion in the set of concepts requisite to individuate macrosystems: "The concepts of broken symmetry and broken ergodicity signify that the categorical framework of macroscopic systems has been expanded to include a new general concept, the *particularity* of equilibrium states. Although the system's possibilities are still governed by universal laws, the particular state it realizes becomes preponderant, and demands understanding" (189). Such macro characters are thus, by Auyang's lights, emergent.

In the time that remains me, I propose to take one more look at ferromagnetism–

this time from the perspective of certain foundational questions about quantum theory that arise in the thermodynamic limit of quantum statistical mechanics, the theoretical setting hospitable to accounts of phase structure and broken symmetry. I will try to offer a redescription of what Auyang characterizes as the shrinking, and historicity, of the microstate space available a ferromagnet in a symmetry-breaking equilibrium state. On my redescription, in theoretical accounts accommodating broken symmetry and phase structure, the microstate space available a ferromagnet has not shrunk so much as the notion of the state space of a quantum system–indeed, the notion of what counts as a quantum theory–has expanded. This expansion is what makes the isolation of an asymmetric equilibrium state available as *an isolation.* On this view, what emerges at the thermodynamic limit are not properties of physical systems, but possibility structures and ways to think about them that cast in a different light what it is to be a property of a physical system.

5. Quantum Theories of Spin Chains

This section sketches very informally the quantum theory of chains of spin systems; the aim is to better appreciate Auyang's theses about emergence by rendering them in a different idiom.

The systems we'll consider are one-dimensional lattice systems, consisting of $2n+1$ spin systems (electrons) at a set of locations labelled by $k \in \{-n, -(n-1), ..., -1, 0, 1, ..., n\}$. We'll assume the dynamics of these systems to be rotationally invariant.

5.1. Single spin system

We'll need to know a bit more about spins. Where $\vec{v}$ is a vector picking out a direction in space, the observable $\sigma_{\vec{v}}$–spin in the $\vec{v}$ direction–is measured by passing electrons through a magnetic field whose strength varies as you move along the $\vec{v}$ axis. Electrons so passed invariably exit the magnet in one of two possible trajectories. Put another way, each spin observables has exactly two possible values: call them $\{+1, -1\}$. We can operationalize as many spin observables as there are spatial orientations along which to vary strengths of magnetic fields.

To construct a quantum theory of a single spin system, find a trio of spin observables $\{\sigma_{\vec{x}}, \sigma_{\vec{y}}, \sigma_{\vec{z}}\}$ manifesting a characteristic algebraic structure. This just means that there are ways to take sums, products, and squares of those observables, and that those sums, products, and squares obey certain relations, I'll call the *Pauli relations*. Here are some of them[1]:

$$\sigma_{\vec{x}}\sigma_{\vec{y}} - \sigma_{\vec{y}}\sigma_{\vec{x}} = 2i\sigma_{\vec{z}}$$
$$(\sigma_{\vec{x}})^2 + (\sigma_{\vec{y}})^2 + (\sigma_{\vec{z}})^2 = 3I$$

Every other observable pertaining to a single spin system may be defined in terms of these $\{\sigma_{\vec{x}}, \sigma_{\vec{y}}, \sigma_{\vec{z}}\}$. In quantum mechanics, the *state* of a system is a (normed, linear) functional that assigns every observable an *expectation value* (an expectation value of an observable in a state is the long run average you should obtain if you prepared an immense set of systems in that state and measured that observable on each of them). The complete set of states possible for a single spin system can be generated from a state in which the expectation value of $\sigma_{\vec{z}}$ is $+1$–call that state $|+>$–and a state in which the expectation value of $\sigma_{\vec{z}}$ is -1–call that state $|->$.[2]

The usual way to realize the quantum theory of a single spin system–to represent the full set of observables pertaining to that system, as well as the full set of states attainable by it–is on a two dimensional complex vector space. Observables correspond to bounded self-adjoint operators on this vector space (i.e. maps from vectors to vectors, which maps have certain special properties); trios of such observables satisfying the Pauli relations can be found; possible (pure) states of spin systems correspond to unit vectors in the vector space. Call the set of states $\mathcal{S}$ and the set of observables $\mathcal{O}$. Given an element of $\mathcal{S}$, there is a vector space operation one can perform (the trace operation) to precipitate from that element an expectation value assignment to every element of $\mathcal{O}$. Thus the vector space serves quite neatly as the setting of a quantum theory for a single spin system.

Suppose you and I each construct a quantum theory of a spin system. Mine unimaginatively features observables $\{\sigma_{\vec{x}}, \sigma_{\vec{y}}, \sigma_{\vec{z}}\}$ satisfying the Pauli relations and acting on a vector space *spanned* by the pair of states $|+>$ (in which the expectation value of $\sigma_{\vec{z}}$ is $+1$) and $|->$ (in which the expectation value of $\sigma_{\vec{z}}$ is -1). (This means that every vector in the space–and hence, every pure state

[1] In units where $\hbar/2 = 1$, and I is the identity observable.

[2] Although the details of the generation don't matter, here they are. The set of pure states of the system are given by $c_1|+> + c_2|->$ where c_1 and c_2 are complex numbers s.t. $|c_1|^2 + |c_2|^2 = 1$. Where $|\psi>$ and $|\phi>$ are pure states of the system, so is their convex combination–that is, the linear functional (map from observables to expectation value assignments) defined by $\lambda\rho_\psi + (1-\lambda)\rho_\phi$, where ρ_ψ and ρ_ϕ are the linear functionals defined by $|\psi>$ and $|\phi>$ respectively.

of the system–can be expressed as a linear combination of the states $|+>$ and $|->$. We also say that $|+>$ and $|->$ furnish a *basis* for the space.) You are more creative: you find operators $\{\sigma_{\vec{x}'}, \sigma_{\vec{y}'}, \sigma_{\vec{z}'}\}$ satisfying the Paul relations and acting on a vector space spanned by a pair of states $|v_+>$ (in which the expectation value of $\sigma_{\vec{v}}$ is $+1$) and $|v_->$ (in which the expectation value of $\sigma_{\vec{v}}$ is -1). Have you and I found *different* quantum theories of a spin system?

To answer this question requires a little thought about how to individuate physical theories. The idea (widespread and shared by Auyang) that a physical theory characterizes a system to which it applies in terms of a state space–a space of configurations possible for the system– suggests a reasonable working criterion for the physical equivalence of theories. Two theories of a system are equivalent, according to this criterion, if and only if the sets of states they declare possible for that system coincide. The quantum theories we're considering equate a state (or possibility) for a system with an expectation value assignment to the full set of observables pertaining to that system. Thus, according to our working criterion, two quantum theories of a system are physically equivalent if and only if every expectation value assignment offered by one has a counterpart in the other, and vice-versa.[3] For quantum theories realized, as your theory and mine of the spin system are, in vector spaces, the content coincidence criterion, so understood, turns out to be equivalent to the requirement that the vector space representations be *unitarily equivalent*. Under rather general circumstances, the quantizations of systems containing finitely many particles are unitarily equivalent (Wigner 1959). Thus you and I can be confident that, notational variances notwithstanding, our quantizations give the same theory.

Let us introduce a polarization observable m, meant to track the net magnetization, its direction as well as its magnitude, of the systems we're considering. For a single electron, m is just going to point where the spin of the system points. So $m = +1$ along z direction in $|+>$; $m = -1$ along z direction in $|->$; $m = 0$ when the state is an evenly weighted mixture of $|+>$ and $|->$; m has other directions and magnitudes for other states.

[3]This counterpart relation can be made more precise. Where (O, S) and (O', S') are the sets of observables and states for theories T and T' respectively, the counterpart relation requires that there be an isomorphism i_o from O to O' and an ismorphism i_s from S to S' s.t. $i_s(s)(i_o(o)) = s(o)$ for all $s \in S$ and all $o \in O$. For quantum theories realized in vector spaces, this isomorphism is implemented by a unitary operator.

5.2. Finite one-dimensional spin chain

Now suppose that we have a finite number of electrons, arranged in a one dimensional lattice. To find a quantum theory for this composite syste, we must equip each location k with a trio of spin observables $\{\sigma^k_{\vec{x}}, \sigma^k_{\vec{y}}, \sigma^k_{\vec{z}}\}$ satisfying Pauli relations–expanded to include the requirement that spin observables for different systems commute: for $k \neq k'$, $\sigma^k_{\vec{n}}\sigma^{k'}_{\vec{n}'} = \sigma^{k'}_{\vec{n}'}\sigma^k_{\vec{n}}$. Possible states of the chain of electrons are spanned by a basis consisting of sequences s_k, where k ranges from $-n$ to n, and each s_k takes one of the values □ 1 (understood as expectation values of $\sigma_{\vec{z}}$). So, for instance, the sequence assigning every s_k the value $+1$ corresponds to the state in which every spin in the chain assigns $\sigma_{\vec{z}}$ expectation value $+1$; the sequence assigning every s_k the value -1 corresponds to the state in which every spin in the chain assigns $\sigma_{\vec{z}}$ expectation value -1; a sequence alternating between $+1$ and -1 represents a chain in which the $\sigma_{\vec{z}}$ expectation values also alternate.

N.B. there are finitely many distinct such sequences–finitely many ways to map a set $\{k = -n \text{ to } n\}$ of finite cardinality into the set $\{+1, -1\}$. Alternately, each such sequence can be obtained from the sequence in which every $s_k = +1$ by flipping finitely many of the s_ks–a formal manipulation whose physical counterpart is performing finitely local modifications on the spin chain. A transformation that flips the k^{th} component of a spin chain while leaving the rest unchanged is a unitary transformation. For n finite, all representations of the quantum theory of a spin chain of length $2n+1$ are unitarily equivalent: their bases can be transformed into one another by sequences of (unitarily implementable) local flips and (unitarily implementable) local rotations.)

For such a spin chain, the polarization observable m takes, in a sequence s_k of the basis, an expectation value of magnitude $\frac{1}{2n+1}\sum\limits_{k=-n}^{n} s_k$ directed along the z axis. (The polarization of arbitrary states is more complicated; fortunately we don't need to go into it.) For "balanced" basis sequences containing an equal number of $+1$s and -1s, the magnetic moments of the constituent electrons cancel out, and the polarization is 0. Refusing to favor a direction in space, such sequences, as well as states expressed as linear combinations of such sequences, are rotationally symmetric. For 'unbalanced' sequences, the polarization m points up or down, depending on whether $+1$ or -1 predominates. m attains extreme values (of □ 1) for those sequences every term of which is the same. As n grows, so does the proportion of balanced states in the basis–eo ipso, the measure of states in the state space with no net polarization grows as well. Intuitively, a "typical" state in which the spins in the chain are randomly distributed is a state in which the magnetic moment of those spins cancel one another out, yielding no net polarization. Atypical states, where the values of spin are not random, but positively correlated across significant lengths of the chain, are those yielding a net polarization and so violating the rotational symmetry of the spin dynamics.

One might suspect that, as $n \to \infty$, the measure of unbalanced states in the basis set approaches 0–rendering symmetry-breaking states rare indeed (this underlies the result Auyang cites, that at equilibrium the magnetization of a large spin chain is 0). The next section reveals that in the thermodynamic limit, as the number of spins on the chain approaches ∞, the structure of the quantum theory of the system changes dramatically.

5.3. Infinite one-dimensional chain

Now our system is a doubly infinite chain of electrons, labelled by the positive and negative integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. Once again, to construct a quantum theory of this system is to associate with each electron k a trio of observables $\{\sigma^k_{\vec{x}}, \sigma^k_{\vec{y}}, \sigma^k_{\vec{z}}\}$ satisfying Pauli relations, subject to the constraint that observables associated with different electrons commute. The vectors spaces on which these observables act must have a denumerable basis. (N.B. the set of all possible maps from $\mathbb{Z}$ to {□ 1} –corresponding in our conventions to all possible sequences s_k–is non-denumerable.) One way to construct such a basis is to start with the sequence $s_k = +1$ for $k \in \mathbb{Z}$, and add all sequences obtainable from this one by finitely many local modifications. The resulting basis consists of all sequences for which all but a finite number of sites s_k take the value $+1$ (again understood as the state in which the expectation value of $\sigma^k_{\vec{z}} = +1$ for all k). Operators satisfying the Pauli relations for the infinite chain of electrons can be defined on the vector space spanned by these states (Sewell 2002, §2.3 has details). Let us call the ensuing representation of the quantum theory of an infinite chain of spins S^+.

Next consider how the polarization observable m behaves on S^+. For every state in the basis, m will be oriented along the z axis and take the value $m = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} s_k$. Because for each basis element, all but a finite number of the s_ks take the value $+1$, the limit of this sequence of partial sums is $+1$, m's maximum value, and the polarization of the basis element is a unit vector in the positive

z direction. Every state representable on S^+ will inherit this feature from the basis vectors in terms of which it is expressed: every state in the representation will have a net magnetic moment in the positive z direction, in bold defiance of the rotational symmetry of system's dynamics! Where (oh where) have states with *other* polarizations gone?

The answer is that they've gone to *other* representations of the algebra of observables for infinite spin chains. 'Other' here is to be taken quite seriously: unlike alternative representations of the algebras of observables for finite spin chains discussed above, these representations *fail to be unitarily equivalent* to the one with which we started (S^+). According to our working criterion of physical equivalence for quantum theories, then, these representations constitute theories of the infinite spin chain distinct from that afforded by S^+. (There's something awfully fishy here. The spin systems are supposed to have rotational symmetric dynamics, but–it will appear–the only way to build rotationally symmetric states is to combine states from (according to our working criterion) distinct theories, issuing a weird sort of modal chimera. What gives? More on this presently.)

Here's a way to build one such representation: start with the sequence $s_k = -1$ for $k \in \mathbb{Z}$, and add all sequences obtainable from this one by finitely many local modifications. The resulting basis consists of all sequences for which all but a finite number of sites s_k take the value -1 (again understood as the state in which the expectation value of $\sigma^k_{\vec{z}} = -1$ for all k). Operators satisfying the Pauli relations for the infinite chain of electrons can be defined on the vector space spanned by these states (Sewell 2002, §2.3 has details). Call the resulting representation S^-. Some noteworthy features of S^-: there's no way to reach any state in S^+ by performing finitely many local modifications to S^-, and vice-versa. This is a symptom of the unitary *in*equivalence of those representations; it is also an exposition of what Auyang describes as "locking in" a magnetization by taking an infinite volume limit. Another symptom is the behavior in S^- of the polarization observable $\boldsymbol{m}$. By parity of reasoning, because in every basis element of S^- all but a finite number of the the s_ks take the value -1, on every state in S^-, $\boldsymbol{m}$ will be given by a unit vector in the *negative* z direction.[4] That is, every state in S^- has a maximum magnetic moment in the negative z direction, boldly breaking the rotational symmetry of the spin dynamics.

And we could keep going: the infinite chain of spin systems admits infinitely many unitarily inequivalent representations. Instead I will pause to remark

[4]See Sewell 2002 (17-18) for a proof that the unitary inequivalence of S^+ and S^- follows from this.

what else becomes available along with this curious abundance of physically inequivalent vector space quantum theories at the thermodynamic limit: an expanded notion of what it is to be an equilibrium state–a KMS state–that includes Gibb states–equilibrium states of finite systems–as special cases. Moreover, and very suggestively, KMS equilibrium states at a finite temperatures *needn't* be unique–opening the possibility, closed by the Gibbs notion of equilibrium for finite systems, of accommodating phase structure and spontaneous symmetry breaking.

Different KMS equilibrium states of a system at a given finite temperature, corresponding to different phases, as well as symmetry breaking KMS states, resemble the representations S^+ and S^- in the following respect: KMS equilibrium states corresponding to different phases of a bulk system correspond as well to unitarily inequivalent vector space quantum theories of that system's microconstituents. To underscore this last point, let ω and ω' be KMS equilibrium states representing different phases of the same system at the same temperature. ω and ω' are also each states in a vector space realization of a quantum theory for the system in equilibrium. No state in ω's vector space quantum theory is a state in ω''s, and vice-versa. That is to say, ω and ω' are *disjoint* states.[5]

6. Re: Emergence

This concluding section will view Auyang's analysis of emergence, as manifested in phase structure and broken symmetry, through the lens of the foregoing sketch of quantum theories of spin chains. The theoretical treatment of the ferromagnet, the bulk system exhibiting emergent behavior on Auyang's analysis, is furnished by the thermodynamic limit of quantum statistical mechanics. The thermodynamic limit makes available what the finite microanalysis of the ferromagnet provided by a vector space quantum theory for n electron systems lacked: the multiplicity of (KMS) equilibrium states for a system at finite temperatures requisite to accommodate phase structure and spontaneous symmetry breaking.

To muster this multiplicity in an account of those phenomena, however, we must revisit entrenched assumptions about what a quantum theory is (a vector space theory), and about how any physical theory takes on content (on the standard state space picture, by distinguishing the worlds possible according

[5]Many claims in this paragraph are made a little less imprecisely in Ruetsche (2003).

to it from the worlds impossible according to it). Where ω and ω' are KMS states suited to represent different phases of a system–ferromagnetic and paramagnetic say–these assumptions imply that if the ω phase is a possible state of the system, the ω' phase is impossible (because the states ω and ω' as disjoint belong to physically inequivalent theories of the system). But–this is the whole point of going to the thermodynamic limit to craft a notion of equilibrium admitting multiplicities of equilibrium states–two different phases of a system at equilibrium are most decidedly *both* possible states of that system. Either our conception of what it is to be a quantum theory, or our picture of how theories carve out spaces of physical possibility, must change. I would suggest that both should.

The order parameter by which the (highly ordered) ferromagnetic phase is distinguished from the (less ordered) paramagnetic phase is (presumably) an archetypically emergent character for Auyang. One way an order parameter differs in type from characters featured in the ferromagnet's microanaylsis–observables appearing in the vector space quantum theories of n electron systems–is that its domain is much broader than the domain of any vector space observable. For the order parameter is defined on both the disjoint states ω and ω' representing the ferromagnetic and the paramagnetic phase. But there is no single vector space quantum theory in which both those states appear as states. Ergo there is no single vector space quantum observable corresponding to the order parameter. This reflects something of the texture of Auyang's remarks that emergent characters differ in type from their micro-bases.

The foregoing presents a sense in which the thermodynamic limit of quantum statistical mechanics affords the ferromagnet an expanded state space, compared to those made available by its finite microanalysis. Now, as Auyang emphasizes, measured against expectations developed in the context of finite microanalysis, there is something peculiar about how the ferromagent occupies this expanded state space. Often, at the thermodynamic limit, equilibrium dynamics for systems are state-dependent (see Sewell, 31[6]), and impose dynamical trajectories only on the particular vector space quantum theory in which the equilibrium state in question resides. That is, it can happen that the dynamics arising for the equilibrium state ω isn't defined on the vector space housing the equilibrium state ω', and vice-versa. Thus, although the equilibrium state ω' is not impossi-

[6] I'm running roughshod over a lot of distinctions (e.g., discrete vs. continuous systems, and models people actually have in hand vs. models they have some ideas about) that might be important here.

ble for a system whose actual state happens to be ω–although it's not impossible that iron in its ferromagnet phase at 771^o be in its paramagnetic phase instead–the ω' possibility is distinctly remote, compared to other possibilities for the system.

This remoteness is not simply that ω' is not among the states to which ω-based dynamics connect ω. It is, moreover and more significantly, that the 'ω-dynamics' *don't apply to* ω'–a ω' world differs from a ω world not only with respect to what is actual but also with respect to details of dynamical law. (Compare a classical state space with dynamical trajectories imposed: although there may be no trajectory connecting states s and s', the same dynamical laws–Hamilton's equation, with a specific form for the Hamiltonian supplied–impose trajectories through both states.) Partaking in a nomic structure more similar to ω's, states which admit the 'ω-dynamics' (these will be states representable in the same vector space quantum theory as ω) might be said to be *more possible*, with respect to ω, than states, such as ω', that don't. The standard state space picture of theoretical content, assuming as it does that physical possibility is unimodal, discourages talk of this sort.

But, as Auyang has brought it to our attention, talk of this sort arises when physicist's approaches erode the sharp distinction the standard state space picture would draw between the nomic business of configuring a system's state space, and the contingent matter of which particular initial state a particular system happens to occupy. Insofar as dynamical laws can be state dependent, issues of state space configuration become entangled with issues of which state a system occupies, and contingency's footfalls impress allowed dynamical trajectories upon state space. I am not prepared to say whether this subversion of the order of law and contingency, this complication of the notion of physical modality, is a generic feature of phenomena rightly termed *emergent* . But it is a striking feature on which Auyang's discussion of phase structure and broken symmetry casts intriguing light.

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