Convex Bodies with Minimal Mean Width

A.A. Giannopoulos[1], V.D. Milman[2], and M. Rudelson[3]

[1] Department of Mathematics, University of Crete, Iraklion, Greece
[2] School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
[3] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

1 Introduction

Let K be a convex body in $\mathbb{R}^n$, and $\{TK \mid T \in SL(n)\}$ be the family of its *positions*. In [GM] it was shown that for many natural functionals of the form

$$T \mapsto f(TK), \qquad T \in SL(n),$$

the solution T_0 of the problem

$$\min\{f(TK) \mid T \in SL(n)\}$$

is *isotropic* with respect to an appropriate measure depending on f. The purpose of this note is to provide applications of this point of view in the case of the *mean width functional* $T \mapsto w(TK)$ under various constraints.

Recall that the *width* of K in the direction of $u \in S^{n-1}$ is defined by $w(K,u) = h_K(u) + h_K(-u)$, where $h_K(y) = \max_{x\in K}\langle x, y\rangle$ is the *support function* of K. The width function $w(K,\cdot)$ is translation invariant, therefore we may assume that $o \in \mathrm{int}(K)$. The *mean width* of K is given by

$$w(K) = \int_{S^{n-1}} w(K,u)\sigma(du) = 2\int_{S^{n-1}} h_K(u)\sigma(du),$$

where σ is the rotationally invariant probability measure on the unit sphere S^{n-1}.

We say that K has *minimal mean width* if $w(TK) \geq w(K)$ for every $T \in SL(n)$. The following isotropic characterization of the minimal mean width position was proved in [GM]:

Fact. *A convex body K in $\mathbb{R}^n$ has minimal mean width if and only if*

$$\int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du) = \frac{w(K)}{2n}$$

for every $\theta \in S^{n-1}$. Moreover, if $U \in SL(n)$ and UK has minimal mean width, we must have $U \in O(n)$. □

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Our first result is an application of this fact to a "reverse Urysohn inequality" problem: The classical Urysohn inequality states that $w(K) \geq (|K|/\omega_n)^{1/n}$ where ω_n is the volume of the Euclidean unit ball D_n, with equality if and only if K is a ball. A natural question is to ask for which bodies K

$$a_n := \max_{|K|=1} \min_{T\in SL(n)} w(TK)$$

is attained, and what is the precise order of growth of a_n as $n \to \infty$. Examples such as the regular simplex or the cross-polytope show that $a_n \geq c\sqrt{n}\sqrt{\log(n+1)}$. On the other hand, it is known that every symmetric convex body K in $\mathbb{R}^n$ has an image TK with $|TK| = 1$ for which

$$w(TK) \leq c_1\sqrt{n}\log[d(X_K, \ell_2^n) + 1],$$

where $X_K = (\mathbb{R}^n, \|\cdot\|_K)$ and d denotes the Banach-Mazur distance. This statement follows from an inequality of Pisier [Pi], combined with work of Lewis [L], Figiel and Tomczak-Jaegermann [FT]. John's theorem [J] implies that

$$\min_{T\in SL(n)} w(TK) \leq c_2\sqrt{n}\log(n+1),$$

for every symmetric convex body K with $|K| = 1$, and a simple argument based on the difference body and the Rogers-Shephard inequality [RS] shows that the same holds true without the symmetry assumption. Therefore,

$$c\sqrt{n}\sqrt{\log(n+1)} \leq a_n \leq c_3\sqrt{n}\log(n+1).$$

Here, we shall give a precise estimate for the minimal mean width of zonoids (this is the class of symmetric convex bodies which can be approximated by Minkowski sums of line segments in the Hausdorff sense):

Theorem A. *Let Z be a zonoid in $\mathbb{R}^n$ with volume $|Z| = 1$. Then,*

$$\min_{T\in SL(n)} w(TZ) \leq w(Q_n) = \frac{2\omega_{n-1}}{\omega_n},$$

where $Q_n = [-1/2, 1/2]^n$.

For our second application, we consider the class of origin symmetric convex bodies in $\mathbb{R}^n$. Every symmetric body K induces a norm $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$ on $\mathbb{R}^n$, and we write X_K for the normed space $(\mathbb{R}^n, \|\cdot\|_K)$. The polar body of K is defined by $\|x\|_{K^\circ} = \max_{y\in K} |\langle x, y\rangle| = h_K(x)$, and will be denoted by K°. Whenever we write $(1/a)|x| \leq \|x\|_K \leq b|x|$, we assume that a, b are the smallest positive numbers for which this inequality holds true for every $x \in \mathbb{R}^n$.

We consider the average

$$M(K) = \int_{S^{n-1}} \|x\|_K \sigma(dx)$$

of the norm $\|\cdot\|_K$ on S^{n-1}, and define $M^*(K) = M(K^\circ)$. Thus, $M^*(K)$ is half the mean width of K. We will say that K has *minimal* M if $M(K) \leq M(TK)$ for every $T \in SL(n)$. Equivalently, if K° has minimal mean width.

Our purpose is to show that if K has minimal M, then the volume radius of K is bounded by a function of b and M. Actually, it is of the order of b/M. The precise formulation is as follows:

Theorem B. *Let K be a symmetric convex body in $\mathbb{R}^n$ with minimal M, such that $(1/a)|x| \leq \|x\|_K \leq b|x|$, $x \in \mathbb{R}^n$. Then,*

$$\frac{b}{M} \leq \left(\frac{|K|}{|(1/b)D_n|}\right)^{1/n} \leq c\frac{b}{M}\log\left(\frac{2b}{M}\right),$$

where $c > 0$ is an absolute constant.

Our last result concerns optimization of the width functional under a different condition. We say that an n-dimensional symmetric convex body K is in the *Gauss-John position* if the minimum of the functional

$$\mathbb{E}\|g\|_{TK}$$

under the constraint $TK \subseteq D_n$ is attained for $T = I$. That is, K° has minimal mean width under the condition $TK \subseteq D_n$ (it minimizes M under the condition $a(TK) \leq 1$).

We can consider this optimization problem only for positive self-adjoint operators T. Since the norm of T should be bounded to guarantee that $TK \subseteq D_n$ and the norm of T^{-1} should be bounded as well, there exists T for which the minimum is attained. Denote by γ the standard Gaussian measure in $\mathbb{R}^n$. Then, we have the following decomposition.

Theorem C. *Let K be in the Gauss-John position. Then there exist: $m \leq n(n+1)/2$, contact points $x_1, \ldots, x_m \in \partial K \cap S^{n-1}$ and numbers $c_1, \ldots, c_m > 0$ such that $\sum_{i=1}^m c_i = 1$ and*

$$\int_{\mathbb{R}^n} (x \otimes x - I)\|x\|_K d\gamma(x) = \int_{\mathbb{R}^n} \|x\|_K d\gamma(x) \cdot \left(\sum_{i=1}^m c_i x_i \otimes x_i\right).$$

The Gauss-John position is not equivalent to the classical John position. Examples show that, when K is in the Gauss-John position, the distance between D_n and the John ellipsoid may be of order $\sqrt{n/\log n}$.

2 Reverse Urysohn Inequality for Zonoids

The proof of Theorem A will make use of a characterization of the minimal surface position, which was given by Petty [Pe] (see also [GP]): Recall that the area measure σ_K of a convex body K is defined on S^{n-1} and corresponds to the usual surface measure on K via the Gauss map: For every Borel $V \subseteq S^{n-1}$, we have

$$\sigma_K(V) = \nu\left(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ is in } V\}\right),$$

where ν is the $(n-1)$-dimensional surface measure on K. If $A(K)$ is the surface area of K, we obviously have $A(K) = \sigma_K(S^{n-1})$. We say that K has *minimal surface area* if $A(K) \leq A(TK)$ for every $T \in SL(n)$. With these definitions, we have:

2.1 Theorem. *A convex body K in $\mathbb{R}^n$ has minimal surface area if and only if*

$$\int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) = \frac{A(K)}{n}$$

for every $\theta \in S^{n-1}$. Moreover, if $U \in SL(n)$ and UK has minimal surface area, we must have $U \in O(n)$. □

Recall also the definition of the projection body ΠK of K: it is the symmetric convex body whose support function is defined by $h_{\Pi K}(\theta) = |P_\theta(K)|$ where $P_\theta(K)$ is the orthogonal projection of K onto $\theta^\perp$, $\theta \in S^{n-1}$. It is known that Z is a zonoid in $\mathbb{R}^n$ if and only if there exists a convex body K in $\mathbb{R}^n$ such that $Z = \Pi K$. By the formula for the area of projections, this can be written in the form

$$h_Z(x) = \frac{1}{2}\int_{S^{n-1}} |\langle x, u\rangle| \sigma_K(du).$$

Then, the characterization of the minimal mean width position and Theorem 2.1 imply the following:

2.2 Lemma. *Let $Z = \Pi K$ be a zonoid. Then, Z has minimal mean width if and only if K has minimal surface area.*

Proof. The proof (modulo the characterization of the minimal mean width position) may be found in [Pe]: By Cauchy's surface area formula,

$$A(K) = \frac{n\omega_n}{\omega_{n-1}} \int_{S^{n-1}} h_Z(\theta)\sigma(d\theta).$$

If f_2 is a spherical harmonic of degree 2, the Funk-Hecke formula shows that

$$\int_{S^{n-1}} f_2(u)|\langle u, \tau\rangle| \sigma(du) = c_n f_2(\tau)$$

for all $u, \tau \in S^{n-1}$, where c_n is a constant depending only on the dimension. Therefore,

$$\begin{aligned}\int_{S^{n-1}} f_2(u) h_Z(u) \sigma(du) &= \frac{1}{2}\int_{S^{n-1}}\int_{S^{n-1}} f_2(u)|\langle u, \tau\rangle| \sigma(du)\sigma_K(d\tau)\\ &= \frac{c_n}{2}\int_{S^{n-1}} f_2(\tau)\sigma_K(d\tau).\end{aligned}$$

Since $u \mapsto \langle u, \theta \rangle^2$ is homogeneous of degree 2, this implies

$$\int_{S^{n-1}} h_Z(u)\langle u, \theta\rangle^2 \sigma(du) = \frac{c_n}{2} \int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du)$$

for every $\theta \in S^{n-1}$. The characterizations of the minimal mean width and the minimal surface area positions make it clear that Z has minimal mean width if and only if K has minimal surface area. □

Our next lemma is a well-known fact, proved by K. Ball [B]:

2.3 Lemma *Let $\{u_j\}_{j\le m}$ be unit vectors in $\mathbb{R}^n$ and $\{c_j\}_{j\le m}$ be positive numbers satisfying*

$$I = \sum_{j=1}^{m} c_j u_j \otimes u_j.$$

If $Z = \sum_{j=1}^{m} \alpha_j [-u_j, u_j]$ for some $\alpha_j > 0$, then

$$|Z| \ge 2^n \prod_{j=1}^{m} \left(\frac{\alpha_j}{c_j}\right)^{c_j}. \qquad \square$$

We apply this result to the projection body of a convex body with minimal surface area.

2.4 Lemma *If K has minimal surface area, then*

$$A(K) \le n|\Pi K|^{1/n}.$$

Proof. We may assume that K is a polytope with facets F_j and normals u_j, $j = 1, \dots, m$. Then, Theorem 2.1 is equivalent to the statement

$$I = \sum_{j=1}^{m} c_j u_j \otimes u_j$$

where $c_j = n|F_j|/A(K)$ (see [GP]). On the other hand,

$$\Pi K = \frac{A(K)}{2n} \sum_{j=1}^{m} c_j [-u_j, u_j].$$

We now apply Lemma 2.3 for $Z = \Pi K$, with $\alpha_j = \frac{A(K)}{2n} c_j$:

$$|\Pi K| \ge 2^n \prod_{j=1}^{m} \left(\frac{A(K)}{2n}\right)^{n}. \qquad \square$$

Remark. In the previous argument, equality can hold only if $(u_j)_{j\le m}$ is an orthonormal basis of $\mathbb{R}^n$ (see [Ba]). This means that if K is a polytope then equality in Lemma 2.3 can hold only if K is a cube.

Proof of Theorem A. Let Z be a zonoid with minimal mean width and volume $|Z| = 1$. By Lemma 2.2, Z is the projection body ΠK of some convex body K with minimal surface area. We have

$$w(Z) = 2\int_{S^{n-1}} h_Z(u)\sigma(du) = 2\int_{S^{n-1}} |P_u(K)|\sigma(du) = \frac{2\omega_{n-1}}{n\omega_n}A(K).$$

By Lemma 2.4, the area of K is bounded by $n|Z|^{1/n} = n$. We have equality when K is a cube, and this corresponds to the case $Z = Q_n$. Therefore,

$$w(Z) \leq w(Q_n) = \frac{2\omega_{n-1}}{\omega_n}. \qquad \Box$$

Remark. Urysohn's inequality and Theorem A show that if Z is a zonoid with $|Z| = 1$, then

$$\alpha_n \sqrt{\frac{2}{\pi e}}\sqrt{n} \leq \min_{T\in SL(n)} w(TZ) \leq \beta_n \sqrt{\frac{2}{\pi}}\sqrt{n},$$

where $\alpha_n, \beta_n \to 1$ as $n \to \infty$.

3 Volume Ratio of Symmetric Convex Bodies with Minimal M

For the proof of Theorem B we will need the following fact which was proved in [GM]:

3.1 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ with minimal M. Then, for every $\lambda \in (0,1)$ there exists a $[(1-\lambda)n]$-dimensional subspace E of $\mathbb{R}^n$ such that*

$$\frac{b}{r(\lambda)}|x| \leq \|x\|_K \leq b|x| \quad , \quad x \in E, \tag{3.1}$$

where $r(\lambda) \leq c\frac{b}{M\lambda^{1/2}}\log(\frac{2b}{M\lambda})$, and $c > 0$ is an absolute constant. $\Box$

Actually, the proof of Theorem 3.1 shows that the statement holds true for a random $[(1-\lambda)n]$-dimensional subspace E of $\mathbb{R}^n$. One can assume that for every $k \leq n - \frac{n}{c\log^2 n}$ we have the result with probability greater than $1 - \frac{1}{n}$ (this formulation is correct when $n \geq n_0$, where $n_0 \in \mathbb{N}$ is absolute). This assumption on the measure of subspaces satisfying (3.1) implies that there is an increasing sequence of subspaces $E_1 \subset E_2 \subset \ldots \subset E_{k_0}$, where $k_0 = [n - \frac{n}{c\log^2 n}]$ and $\dim E_k = k$, so that (3.1) holds for each E_k with $r = r(k/n)$.

We will also need the following

3.2 Lemma. *Let K be a symmetric convex body in $\mathbb{R}^n$, such that $(1/a)|x| \leq \|x\|_K \leq b|x|$. If E is a k-dimensional subspace of $\mathbb{R}^n$, then*

$$\frac{|K|}{|D_n|} \leq \left(Ca(\frac{n}{n-k})^{1/2} \right)^{n-k} \frac{|K \cap E|}{|D_k|},$$

where $C > 0$ is an absolute constant.

Proof. Let E be a k-dimensional subspace of $\mathbb{R}^n$. Replacing K by $(1/a)K$, we may assume that $a = 1$, so $K \subset D_n$. Using Brunn's theorem we see that

$$\begin{aligned} |K| = \int_{P_{E^\perp}(K)} |K \cap (E+y)| dy &\leq |P_{E^\perp}(K)||K \cap E| \\ &\leq |K \cap E||D_{n-k}|. \end{aligned}$$

This shows that

$$\frac{|K|}{|D_n|} \leq \frac{|D_k||D_{n-k}|}{|D_n|} \frac{|K \cap E|}{|D_k|} \leq \left(C\frac{n}{n-k} \right)^{(n-k)/2} \frac{|K \cap E|}{|D_k|}. \qquad \square$$

Proof of Theorem B. We first observe that

$$ab \leq Cn \log n.$$

Indeed, let $e \in S^{n-1}$ be such that $\|e\|_K = b$ and let γ be a standard normal variable. Then

$$cb = E\|\gamma e\| \leq E\|g\|_K.$$

Similarly,

$$ca \leq E\|g\|_{K^\circ}.$$

Multiplying these inequalities, we obtain

$$ab \leq CE\|g\|_K E\|g\|_{K^\circ} \leq Cn \log n.$$

The last inequality follows from the fact that M is minimal for K, and Pisier's inequality [Pi].

Assume now that $b = 1$. Let $t = [\log n]$. For $s = 1, 2, \ldots, t$ put $k_s = [(1 - 1/s)n]$ and let $E_s = E_{k_s}$ be a subspace from our flag. Then by Theorem 3.1 we have $(1/a_s)|x| \leq \|x\|_K \leq |x|$ on E_s, where

$$a_s \leq r((n-k_s)/n) \leq \frac{c}{M} s^{1/2} \log\left(\frac{2s}{M}\right) \leq s \cdot \frac{c}{M} \log\left(\frac{2}{M}\right) =: s \cdot c(M).$$

Now, Lemma 3.2 shows that

$$\begin{aligned} \frac{|K|}{|D_n|} \leq \left(Ca\sqrt{t}\right)^{n-k_t} \frac{|K \cap E_t|}{|D_{k_t}|} &\leq (Cn \log^2 n)^{n/\log n} \frac{|K \cap E_t|}{|D_{k_t}|} \\ &\leq C^n \frac{|K \cap E_t|}{|D_{k_t}|} \end{aligned}$$

and

$$\begin{aligned}\frac{|K\cap E_{s+1}|}{|D_{k_{s+1}}|} &\le (Ca_s s)^{k_{s+1}-k_s}\,\frac{|K\cap E_s|}{|D_{k_s}|}\\ &\le \big(Cc(M)s^2\big)^{k_{s+1}-k_s}\,\frac{|K\cap E_s|}{|D_{k_s}|},\end{aligned}$$

for all $s=1,2,\dots,t$. Since $a_1\le c(M)$, we have $|K\cap E_1|/|D_{k_1}|\le c(M)^{k_1}$. Hence, multiplying the inequalities above, we get

$$\frac{|K|}{|D_n|}\le C^n(Cc(M))^{k_t-k_1}c(M)^{k_1}\prod_{s=2}^{t}s^{2(k_{s+1}-k_s)}.$$

By the definition of k_s,

$$\prod_{s=2}^{t}s^{2(k_{s+1}-k_s)}\le \exp\left(cn\cdot\sum_{s=2}^{t}\frac{\log s}{s^2}\right)\le e^{cn},$$

therefore

$$\left(\frac{|K|}{|(1/b)D_n|}\right)^{1/n}\le C_1c(M).$$

The left hand side inequality is an immediate consequence of Hölder's inequality:

$$\begin{aligned}\left(\frac{|K|}{|(1/b)D_n|}\right)^{1/n} &= b\left(\int_{S^{n-1}}\|x\|_K^{-n}\sigma(dx)\right)^{1/n}\\ &\ge b\left(\int_{S^{n-1}}\|x\|_K\right)^{-1}=\frac{b}{M}.\end{aligned}$$

□

4 Gauss-John Position

We prove Theorem C. Consider the following optimization problem:

$$F(T)=\int_{\mathbb{R}^n}\|T^{-1}x\|_K\,d\gamma(x)\to\min \tag{4.1}$$

under the constraint

$$H_x(T)=|Tx|^2-1\le 0\quad\text{for}\quad x\in K.$$

Assume that the body K is in the Gauss-John position, namely the minimum in (4.1) is attained for $T=I$. Let W be the set of the contact points of K: $W=\partial K\cap\partial S^{n-1}$. First we apply an argument of John [J] to show that we can consider only finitely many constraints. Since the paper [J] is not easily available, we shall sketch the argument. Let T be a self-adjoint operator and let $T_s = I + sT$. We shall prove that if

$$\frac{d}{ds} H_x(T_s)|_{s=0} < 0$$

for every $x \in W$, then

$$\frac{d}{ds} F(T_s)|_{s=0} \geq 0.$$

Indeed, assume that

$$a = \sup_{x \in W} \frac{d}{ds} H_x(T_s)|_{s=0} < 0.$$

Let W_ε be an ε-neighborhood of W: $W_\varepsilon = \{x \in K | \operatorname{dist}(x, W) < \varepsilon\}$. There exists an $\varepsilon > 0$ such that

$$\frac{d}{ds} H_x(T_s)|_{s=0} < \frac{a}{2}$$

for every $x \in W_\varepsilon$. So, there exists $s_0 > 0$ such that for any $0 < s < s_0$ and any $x \in W_\varepsilon$

$$H_x(T_s) < H_x(I) \leq 0.$$

On the other hand, if $x \in K \setminus W_\varepsilon$ then

$$H_x(T_s) \leq |H_x(T_s) - H_x(I)| + H_x(I).$$

Here,

$$|H_x(T_s) - H_x(I)| = \left| |T_s x|^2 - |x|^2 \right| \leq \|T\|(1 + \|T\|)s$$

and

$$\sup_{x \in K \setminus W_\varepsilon} H_x(I) = \sup_{x \in K \setminus W_\varepsilon} |x|^2 - 1 < 0$$

since $K \setminus W_\varepsilon$ is compact.

Thus for a sufficiently small s we have $H_x(T_s) < 0$ for all $x \in K$. So, since I is the solution of the minimization problem (4.1),

$$\frac{d}{ds} F(T_s)|_{s=0} \geq 0.$$

Since $\frac{d}{ds} H_x(T_s)|_{s=0} = \langle \nabla H_x(I), T \rangle$ and $\frac{d}{ds} F(T_s)|_{s=0} = \langle \nabla F(I), T \rangle$, this means that the vector $-\nabla F(I)$ cannot be separated from the set $\{\nabla H_x(I) | x \in W\}$ by a hyperplane. By Carathéodory's theorem, there exist $M \leq n(n+1)/2$ contact points $x_1 \dots x_M \in W$ and numbers $\lambda_1 \dots \lambda_M > 0$ such that

$$-\nabla F(I) = \sum_{i=1}^{M} \lambda_i \nabla H_{x_i}(I) = \sum_{i=1}^{M} \lambda_i x_i \otimes x_i. \tag{4.2}$$

Now we have to calculate $\nabla F(I)$.

We have

$$F(T) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|T^{-1}x\|_K e^{-|x|^2/2} dx$$
$$= (2\pi)^{-n/2} \det T \cdot \int_{\mathbb{R}^n} \|x\|_K e^{-|Tx|^2/2} dx,$$

so,

$$\nabla F(I) = \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} \|x\|_K e^{-|x|^2/2} dx \right) I$$
$$-(2\pi)^{-n/2} \int_{\mathbb{R}^n} \|x\|_K e^{-|x|^2/2} x \otimes x dx$$
$$= \int_{\mathbb{R}^n} (I - x \otimes x) \cdot \|x\|_K d\gamma(x).$$

Combining it with (4.2) we obtain

$$\int_{\mathbb{R}^n} (I - x \otimes x) \cdot \|x\|_K d\gamma(x) + \sum_{i=1}^{M} \lambda_i x_i \otimes x_i = 0.$$

Taking the trace, we get

$$\mathrm{Tr} \left(\int_{\mathbb{R}^n} (I - x \otimes x) \cdot \|x\|_K d\gamma(x) \right)$$
$$= n \int_{\mathbb{R}^n} \|x\|_K d\gamma(x) - \int_{\mathbb{R}^n} |x|^2 \cdot \|x\|_K d\gamma(x)$$
$$= n \int_0^\infty r^n e^{-n^2/2} dr \int_{S^{n-1}} \|\omega\|_K dm(\omega) - \int_0^\infty r^{n+2} e^{-n^2/2} dr \int_{S^{n-1}} \|\omega\|_K dm(\omega)$$
$$= - \int_{\mathbb{R}^n} \|x\|_K d\gamma(x).$$

Finally, putting $\lambda_i = c_i \int_{\mathbb{R}^n} \|x\|_K d\gamma(x)$, we obtain the decomposition

$$\int_{\mathbb{R}^n} (I - x \otimes x) \|x\|_K d\gamma(x) = \int_{\mathbb{R}^n} \|x\|_K d\gamma(x) \left(\sum_{i=1}^{M} c_i x_i \otimes x_i \right),$$

where $\sum_{i=1}^{M} c_i = 1$. This completes the proof of Theorem C. □

We proceed to compare D_n with the John ellipsoid in the Gauss-John position:

Proposition. *Let K be a symmetric convex body in $\mathbb{R}^n$ which is in the Gauss-John position. Then,*

(i) $(2/\pi)^{1/2} n^{-1} D_n \subset K \subset D_n$;

(ii) *It may happen that* $\frac{c\sqrt{\log n}}{n} D_n$ *is not contained in* K.

Proof. (i) Let T_0 be an operator which puts K into the maximal volume position. Then $\min F(T) \le F(T_0) \le n$. From the other side, if there exists $y \in S^{n-1}$ such that $\|y\|_{K^\circ} < (2/\pi)^{1/2} n^{-1}$ then

$$\int_{\mathbb{R}^n} \|x\|_K d\gamma(x) \ge \int_{\mathbb{R}^n} |\langle x, y/\|y\|_{K^\circ}\rangle| d\gamma(x) \ge (2/\pi)^{1/2} \cdot 1/\|y\|_{K^\circ} > n.$$

(ii) Let $K = B_1^{n-1} + [-e_n, e_n]$. Let T be a positive self-adjoint operator such that TK is in the Gauss-John position. We first prove that T is a diagonal operator. Let $G \subset O(n)$ be the group generated by the operators $U_i = I - 2e_i \otimes e_i,\ i = 1, \dots, n$ and let m be the uniform measure on G. Notice that $U_i K = K$ for every i. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_{TK} d\gamma(x) &= \int_G \int_{\mathbb{R}^n} \|Ux\|_{TU(K)} d\gamma(x) dm(U) \\ &\ge \int_{\mathbb{R}^n} \left\| \left(\int_G U^{-1} T^{-1} U \ dm(U) \right) x \right\| d\gamma(x).\end{aligned}$$

Put

$$W = \int_G U^{-1} T^{-1} U dm(U) = \mathrm{diag}(T^{-1}).$$

We claim that

$$W \ge \big(\mathrm{diag}(T)\big)^{-1}.$$

Indeed, since for any i

$$\langle e_i, \mathrm{diag}(T^{-1}) e_i \rangle = \langle e_i, (T^{-1}) e_i \rangle$$

and

$$\langle e_i, \big(\mathrm{diag}(T)\big)^{-1} e_i \rangle = \langle e_i, T e_i \rangle^{-1},$$

the claim follows from the fact that for any $\theta \in S^{n-1}$

$$\langle \theta, T^{-1} \theta \rangle \cdot \langle \theta, T\theta \rangle \ge 1.$$

Let $S = \mathrm{diag}(T)$. Since $W \ge S^{-1}$, we have

$$F(T) = \int_{\mathbb{R}^n} \|x\|_{TK} d\gamma(x) \ge \int_{\mathbb{R}^n} \|Wx\|_K d\gamma(x) \ge \int_{\mathbb{R}^n} \|S^{-1} x\|_K d\gamma(x) = F(S).$$

[Notice that since $TK \subset D_n$,

$$SK = \left(\int_G U^{-1} T U \ dm(U) \right) (K) \subset D_n,$$

so the restrictions of the optimization problem (4.1) are satisfied.]

Let now $G' \subset O(n)$ be the group generated by the operators $U_{ij} = I - e_i \otimes e_i - e_j \otimes e_j + e_i \otimes e_j + e_j \otimes e_i$ for $i, j = 1, \dots, n-1,\ i \neq j$. Arguing the same way we can show that there exist $a, b > 0$ such that $F(S) \geq F(T_0)$, where

$$T_0 = a \left(\sum_{i=1}^{n-1} e_i \otimes e_i \right) + b e_n \otimes e_n \ .$$

Since the vertices of $T_0 K$ are contact points,

$$a^2 + b^2 = 1.$$

We have

$$\|x\|_{T_0 K} = \max \left(a^{-1} \sum_{i=1}^{n-1} |x_i|, b^{-1} |x_n| \right).$$

Denote $\|x\|_1 = \sum_{i=1}^{n-1} |x_i|$ and let $t = t(x) = (b/a) \cdot \|x\|_1$. Then,

$$\begin{aligned}
\psi(b) &= \int_{\mathbb{R}^n} \|x\|_{T_0 K} d\gamma(x) \\
&= \int_{\mathbb{R}^{n-1}} \left(\frac{1}{\sqrt{2\pi}} \int_{-t}^{t} a^{-1} \|x\|_1 e^{-x_n^2/2} dx_n + \frac{2}{\sqrt{2\pi}} \int_t^\infty b^{-1} x_n e^{-x_n^2/2} dx_n \right) d\gamma(x) \\
&= \int_{\mathbb{R}^{n-1}} \left(\frac{2}{a} \|x\|_1 \Phi(t) + \frac{2}{\sqrt{2\pi}} b^{-1} e^{-t^2/2} \right) d\gamma(x),
\end{aligned}$$

where $\Phi(t) = (1/\sqrt{2\pi}) \int_0^t e^{-u^2/2} du$. We have to show that $b \leq \frac{c\sqrt{\log n}}{n}$. We may assume that $b \geq c/n$. Putting $a = (1-b^2)^{1/2}$ and differentiating, we get after some calculations

$$\frac{d}{db} \psi(b) = \int_{\mathbb{R}^{n-1}} \left(2a^{-3} b \|x\|_1 \Phi(t) - \frac{2}{\sqrt{2\pi}} b^{-2} e^{-t^2/2} \right) d\gamma(x) \ .$$

Since $b \geq c/n$ and $\|x\|_1 \geq Cn$ with probability at least $1/2$, we have $\Phi(t) > c$ with probability $1/2$, for some absolute constant $c > 0$. So,

$$\frac{d}{db} \psi(b) \geq \bar{c} - C b^{-2} \exp(-c n^2 b^2),$$

which is positive when $b \geq c\sqrt{\log n}/n$. □

Remark. The dual problem

$$f(T) = \int_{\mathbb{R}^n} \sup_{y \in TK} \langle x, y \rangle d\gamma(x) \to \max$$

under the constraint

$$h_x(T) = |Tx|^2 - 1 \leq 0 \quad \text{for} \quad x \in K$$

is very different. The examples suggest that the matrix T for which the maximum is attained may be singular.

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