RANDOM VECTORS IN THE ISOTROPIC POSITION

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Abstract. Let y be a random vector in $\mathbb{R}^n$, satisfying

$$\mathbb{E}\, y \otimes y = id.$$

Let M be a natural number and let $y_1, \dots, y_M$ be independent copies of y. We prove that for some absolute constant C

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \le C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left(\mathbb{E}\, \|y\|^{\log M} \right)^{1/\log M},$$

provided that the last expression is smaller than 1.

We apply this estimate to obtain a new proof of a result of Bourgain concerning the number of random points needed to bring a convex body into a nearly isotropic position.

1. Introduction

The problem we consider has arisen from a question in Computer Science. R. Kannan, L. Lovász and M. Simonovits [1] studied the problem of constructing a fast algorithm for calculating the volume of a convex body, To make this algorithm work they needed to bring the body into a certain "symmetric" position. More precisely, let K be a convex body in $\mathbb{R}^n$. We shall say that it is in the isotropic position if for any $x \in \mathbb{R}^n$

$$\frac{1}{\mathrm{vol}\,(K)} \int_K \langle x, y \rangle^2 \, dy = \|x\|^2$$

By $\|\cdot\|$ we denote the standard Euclidean norm.

The notion of isotropic position was extensively studied by V. Milman and A. Pajor [2]. Note that our definition is consistent with [1]. The normalization in [2] is slightly different.

If the information about the body K is uncomplete it is impossible to bring it exactly to the isotropic position. So, the definition of the isotropic position has to

This research was started when the author had a post doctoral position at MSRI. Research at MSRI is supported in part by NSF grant DMS-9022140. Research was also supported in part by NSF grant DMS-9706835.

be modified to allow a small error. We shall say that the body K is in ε-isotropic position if for any $x \in \mathbb{R}^n$

$$(1-\varepsilon) \cdot \|x\|^2 \leq \frac{1}{\text{vol }(K)} \int_K \langle x, y \rangle^2 \, dy \leq (1+\varepsilon) \cdot \|x\|^2 .$$

Let $\varepsilon > 0$ be given. Consider M random points $y_1, \ldots, y_M$ independently uniformly distributed in K and put

$$T = \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i .$$

If M is sufficiently large, than with high probability

$$\left\| T - \frac{1}{\text{vol}(K)} \int_K y \otimes y \right\|$$

will be small, so the body $T^{-1/2}K$ will be in ε-isotropic position. R. Kannan, L. Lovász and M. Simonovits ([1]) proved that it is enough to take

$$M = c\frac{n^2}{\varepsilon}$$

for some absolute constant c. This estimate was significantly improved by J. Bourgain [3]. Using rather delicate geometric considerations he has shown that one can take

$$M = C(\varepsilon) n \log^3 n.$$

Since the situation is invariant under a linear transformation, we may assume that the body K is in the isotropic position. Then the result of Bourgain may be reformulated as follows:

Theorem 0. [3] *Let K be a convex body in $\mathbb{R}^n$ in the isotropic position. Fix $\varepsilon > 0$ and choose independently M random points $x_1, \ldots, x_M \in K$,*

$$M \geq C(\varepsilon) n \log^3 n.$$

Then with probability at least $1 - \varepsilon$ for any $x \in \mathbb{R}^n$ one has

$$(1-\varepsilon) \|x\|^2 \leq \frac{1}{M} \sum_{i=1}^{M} \langle x, y \rangle^2 \leq (1+\varepsilon) \|x\|^2 .$$

We shall show that this theorem follows from a general result about random vectors in $\mathbb{R}^n$. Let y be a random vector. Denote by $\mathbb{E}X$ the expectation of a random variable X. We say that y is in the isotropic position if

$$\mathbb{E} y \otimes y = id. \tag{1.1}$$

If y is uniformly distributed in a convex body K, then this is equivalent to the fact that K is in the isotropic position.

We prove the following

Theorem 1. *Let $y \in \mathbb{R}^n$ be a random vector in the isotropic position. Let M be a natural number and let $y_1, \dots, y_M$ be independent copies of y. Then*

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq C \cdot \frac{\sqrt{\log n}}{\sqrt{M}} \cdot \left(\mathbb{E} \left\| y \right\|^{\log M} \right)^{1/\log M}, \tag{1.2}$$

provided that the last expression is smaller than 1.

Here and later C, c, etc. denote absolute constants whose values may vary from line to line.

Remark. Taking the trace of (1.1) we obtain that $\mathbb{E} \left\| y \right\|^2 = n$, so to make the right hand side of (1.2) smaller than 1, we have to assume that $M \geq cn \log n$.

The proof of Theorem 1 is based upon the estimate of a certain vector valued Rademacher series (the Lemma below). The author's proof of this estimate used the construction of a majorizing measure for a subgaussian process. After the first variant of this paper [4] was written, G. Pisier [5] found an alternative simpler proof based on the non–commutative Khinchine inequalities due to F. Lust-Piquard and himself [6]. We present this proof below. During the preparation of this paper the author was informed by G. Pisier that he included the proof of the Lemma in the upcoming book [7]. The original probabilistic proof can be found in [4].

Using Theorem 1 we prove a better estimate of the length of approximate John's decompositions [8] and thus improve the results about approximating a convex body by another one having a small number of contact points, obtained in [9]. Estimating the moment of the norm of random vector in a convex body, we obtain a different proof of Theorem 0 which gives also a better estimate.

Acknowledgment. I would like to thank Joram Lindenstrauss for helpful discussions and Gilles Pisier for the permission to present his proof of the Lemma.

2. Proof of the Theorem.

The proof of Theorem 1 consists of two steps. First we introduce a Rademacher series that majorizes the expectation of the norm in (1.2). Then we use the the Khinchine inequality in the Banach space C_p to obtain a bound for it.

The first step is relatively standard. Let $\varepsilon_1, \dots, \varepsilon_M$ be independent Bernoulli variables taking values $1, -1$ with probability $1/2$ and let $y_1, \dots, y_M, \quad \bar{y}_1, \dots, \bar{y}_M$ be independent copies of y. Denote $\mathbb{E}_y$, $\mathbb{E}_\varepsilon$ the expectation according to y and ε respectively. Since $y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i$ is a symmetric random variable, we have

$$\mathbb{E}_y \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \mathbb{E}_y \, \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \frac{1}{M} \sum_{i=1}^{M} \bar{y}_i \otimes \bar{y}_i \right\| =$$

$$\mathbb{E}_\varepsilon \, \mathbb{E}_y \, \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i (y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i) \right\| \leq 2 \, \mathbb{E}_y \, \mathbb{E}_\varepsilon \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\|.$$

To estimate the last expectation, we need the following Lemma, which generalizes Lemma 1 [10].

Lemma. *Let $y_1, \dots, y_M$ be vectors in $\mathbb{R}^n$ and let $\varepsilon_1, \dots, \varepsilon_M$ be independent Bernoulli variables taking values $1, -1$ with probability $1/2$. Then*

$$\mathbb{E} \left\| \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\| \le C \sqrt{\log n} \cdot \max_{i=1,\dots,M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\|^{1/2} .$$

We postpone the proof of the Lemma to the next section.

Applying the Lemma, we get

$$\begin{aligned} &\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \le \\ &C \cdot \frac{\sqrt{\log n}}{M} \cdot \left(\mathbb{E} \max_{i=1,\dots,M} \|y_i\|^2 \right)^{1/2} \cdot \left(\mathbb{E} \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\| \right)^{1/2} . \end{aligned} \tag{2.1}$$

We have

$$\begin{aligned} \left(\mathbb{E} \max_{i=1,\dots,M} \|y_i\|^2 \right)^{1/2} &\le \left(\mathbb{E} \left(\sum_{i=1}^{M} \|y_i\|^{\log M} \right)^{2/\log M} \right)^{1/2} \le \\ &M^{1/\log M} \cdot \left(\mathbb{E} \, \|y\|^{\log M} \right)^{1/\log M} . \end{aligned}$$

Thus, denoting

$$D = \mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| ,$$

we obtain by (2.1)

$$D \le C \cdot \frac{\sqrt{\log n}}{\sqrt{M}} \cdot \left(\mathbb{E} \, \|y\|^{\log M} \right)^{1/\log M} \cdot (D+1)^{1/2}.$$

If

$$C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left(\mathbb{E} \, \|y\|^{\log M} \right)^{1/\log M} \le 1,$$

we get

$$D \le 2C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left(\mathbb{E} \, \|y\|^{\log M} \right)^{1/\log M} ,$$

which completes the proof of Theorem 1.

3. Proof of the Lemma: non–commutative Khinchine inequality.

We present here a short proof of the Lemma found by G. Pisier.

Let $1 \le p \le \infty$. Denote by C_p^n the p-th Schatten class – the Banach space of the operators in $\mathbb{R}^n$, equipped with the norm

$$\|u\|_{C_p^n} = \left(\sum_{j=1}^{n} s_j^p(u) \right)^{1/p} ,$$

where $s_j(u)$ are the singular numbers of the operator u. Also, let $Q = \{-1; 1\}^{\mathbb{N}}$ and let μ be the Haar measure on Q. Let $\varepsilon_1, \dots, \varepsilon_M : Q \to \{-1; 1\}$ be the Rademacher functions: for $x \in Q$, $\varepsilon_j(x)$ is the j-th coordinate of x.

The proof is based on the following

Theorem. *(i) Assume $2 \le p < \infty$. Then there is a constant B_p such that for any finite sequence $\{X_j\}$ in C_p^n, one has*

$$\max\left\{ \left\| \left(\sum X_j^* X_j\right)^{1/2} \right\|_{C_p^n}, \left\| \left(\sum X_j X_j^*\right)^{1/2} \right\|_{C_p^n} \right\}$$
$$\le \left\| \sum \varepsilon_j x_j \right\|_{L_p(Q,\mu,C_p^n)}$$
$$\le B_p \max\left\{ \left\| \left(\sum X_j^* X_j\right)^{1/2} \right\|_{C_p^n}, \left\| \left(\sum X_j X_j^*\right)^{1/2} \right\|_{C_p^n} \right\}.$$

(ii) Assume $1 \le p \le 2$. Then there is a constant A_p such that for any finite sequence $\{X_j\}$ in C_p^n, one has

$$A_p |||\{X_j\}|||_p \le \left\| \sum \varepsilon_j X_j \right\|_{L_p(Q,\mu,C_p^n)} \le |||\{X_j\}|||_p,$$

where

$$|||\{x_j\}|||_p = \inf\left\{ \left\| \left(\sum Y_j^* Y_j\right)^{1/2} \right\|_{C_p^n} + \left\| \left(\sum Z_j Z_j^*\right)^{1/2} \right\|_{C_p^n} \;\middle|\; X_j = Y_j + Z_j \right\}.$$

The inequality (i) was obtained by Lust-Piquard [11]; the inequality (ii) was obtained later by Lust-Piquard and Pisier [6].

For $p = \log n$ we have

$$\|X\|_{C_p^n} \le \|X\| \le e \cdot \|X\|_{C_p^n}.$$

So, applying (i) for $X_j = y_j \otimes y_j$, we get

$$\mathbb{E} \left\| \sum_{i=1}^M \varepsilon_j y_j \otimes y_j \right\| \le e \cdot \left(\mathbb{E} \left\| \sum_{i=1}^M \varepsilon_j y_j \otimes y_j \right\|_{C_p^n}^p \right)^{1/p}$$
$$\le e \cdot B_p \left\| \left(\sum_{i=1}^M \|y_j\|^2 y_j \otimes y_j \right)^{1/2} \right\|_{C_p^n} \le e \cdot B_p \left\| \left(\sum_{i=1}^M \|y_j\|^2 y_j \otimes y_j \right)^{1/2} \right\|$$
$$\le e \cdot B_p \max_{j=1,\dots,M} \|y_j\| \cdot \left\| \sum_{i=1}^M y_j \otimes y_j \right\|^{1/2}.$$

To complete the proof we have to show that

$$B_p \le C \cdot \sqrt{p}. \tag{3.1}$$

The proof of (i) in [L-P] does not provide this estimate, so we have to dualize (ii) in order to get (3.1).

Let (Q', μ') be a copy of (Q, μ) and let $\varepsilon'_1, \dots, \varepsilon'_M$ be Rademacher functions on (Q', μ'). First, notice that for any finite sequence $\{X_j\} \subset C_p^n$

$$\left\| \sum \varepsilon_j X_j \right\|_{L_p(Q,\mu,C_p^n)} = \left\| \sum \varepsilon'_j \varepsilon_j X_j \right\|_{L_p(Q',\mu',L_p(Q,\mu,C_p^n))}.$$

So, by duality we have

$$B_p \le A_{p'} \cdot K(L_p(Q, \mu, C_p^n)),$$

where $K(E)$ is the K-convexity constant of a Banach space E [12], and p' is the conjugate of p. Since $L_p(Q, \mu, C_p^n)$ embeds isometrically into C_p^N for $N = n \cdot 2^n$, we have

$$K(L_p(Q, \mu, C_p^n)) \le K(C_p^N).$$

By [13] the K-convexity constant of a Banach space can be estimated by the type 2 constant, so

$$B_p \le A_{p'} \cdot C \cdot T_2(C_p^N) \tag{3.2}$$

and by [14]

$$T_2(C_p^N) \le C\sqrt{p}. \tag{3.3}$$

The estimate (3.1) follows now from the combination of (3.2), (3.3) and the uniformal boudedness of $A_{p'}$ for $1 \le p' \le 2$ [6, Cor. III.4].

Remark. Notice that actually we have proved the inequality

$$\left(\mathbb{E} \left\| \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\|^p \right)^{1/p} \le C \max\{\sqrt{\log n}, \sqrt{p}\} \cdot \max_{i=1,\dots,M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\|^{1/2} \tag{3.4}$$

which is formally stronger than the Lemma. However, it is easy to show that (3.4) is equivalent to the Lemma. Indeed, by a theorem of Talagrand [15], for any Banach space X and any vectors $u_1, \dots, u_M \in X$ one has

$$\left\| \sum \varepsilon_j u_j \right\|_{L_p(Q,\mu,X)} \le C \cdot \left(\left\| \sum \varepsilon_j u_j \right\|_{L_1(Q,\mu,X)} + \sqrt{p} \cdot \ell_2^w(\{u_j\}) \right),$$

where

$$\ell_2^w(\{u_j\}) = \sup_{\|u^*\|_{X^*} \le 1} \left(\sum \langle u^*, u \rangle^2 \right)^{1/2}.$$

For $u_j = y_j \otimes y_j$ we have

$$\ell_2^w(\{y_j \otimes y_j\}) = \sup_{\|u^*\|_{C_1^n} \le 1} \left(\sum_{i=1}^{M} \langle u^*, y_j \otimes y_j \rangle^2 \right)^{1/2}$$

$$= \sup_{\|x\| \le 1} \left(\sum_{i=1}^{M} \langle x \otimes x, y_j \otimes y_j \rangle^2 \right)^{1/2} \le \max_{i=1,\dots,M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\|^{1/2}.$$

So, the estimate for L_p-norm of $\sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i$ follows from that for L_1-norm.

4. Applications.

We turn now to the applications of Theorem 1. Applying Theorem 1 to the question of Kannan, Lovász and Simonovits, we obtain the following Corollary, which improves Theorem 0.

Corollary 4.1. *Let $\varepsilon > 0$ and let K be an n-dimensional convex body in the isotropic position. Let*

$$M \geq C \cdot \frac{n}{\varepsilon^2} \cdot \log^2 \frac{n}{\varepsilon^2}$$

and let $y_1, \dots, y_M$ be independent random vectors uniformly distributed in K. Then

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \varepsilon.$$

Proof. It follows from a result of S. Alesker [16], that

$$\mathbb{E} \exp \left(\frac{\|y\|^2}{c \cdot n} \right) \leq 2$$

for some absolute constant c. Then

$$\mathbb{E} \, \|y\|^{\log M} \leq \left(\mathbb{E} e^{\frac{\|y\|^2}{c \cdot n}} \right)^{1/2} \cdot \left(\mathbb{E} \left(\|y\|^{2 \log M} \cdot e^{-\frac{\|y\|^2}{c \cdot n}} \right) \right)^{1/2} \leq$$

$$\sqrt{2} \cdot \left(\max_{t \geq 0} t^{\log M} \cdot e^{-\frac{t}{c \cdot n}} \right)^{1/2} \leq (C \cdot n \cdot \log M)^{\frac{\log M}{2}}.$$

Corollary 4.1 follows from this estimate and Theorem 1. □

By a Lemma of Borell [17, Appendix III], most of the volume of a convex body in the isotropic position is concentrated within the Euclidean ball of radius $c\sqrt{n}$. So, it might be of interest to consider a random vector uniformly distributed in the intersection of a convex body and such a ball. In this case the previous estimate may be improved as follows.

Corollary 4.2. *Let $\varepsilon, R > 0$ and let K be an n-dimensional convex body in the isotropic position. Suppose that $R \geq c\sqrt{\log 1/\varepsilon}$ and let*

$$M \geq C_0 \cdot \frac{R^2 \cdot n}{\varepsilon^2} \cdot \log n. \tag{4.2}$$

Let $y_1, \dots, y_M$ be independent random vectors uniformly distributed in $(K \cap R\sqrt{n}) \cdot B_2^n$. Then

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \varepsilon.$$

Proof. Denote $a = R \cdot \sqrt{n}$ and let z be a random vector uniformly distributed in $K \cap aB_2^n$. Then for $x \in B_2^n$

$$\mathbb{E}\langle z, x\rangle^2 = \frac{\text{vol }(K)}{\text{vol }(K \cap aB_2^n)} \cdot \Big(\frac{1}{\text{vol }(K)} \int_K \langle y, x\rangle^2 \, dy - \frac{1}{\text{vol }(K)} \int_K \langle y, x\rangle^2 \cdot \mathbf{1}_{\{u \big| \|u\| \geq a\}}(y) \, dy \Big).$$

Since $R \geq c\sqrt{\log 1/\varepsilon}$, it follows from a result of S. Alesker [16] that

$$\frac{\text{vol }(K)}{\text{vol }(K \cap aB_2^n)} \leq 1 + e^{-ca^2/n} \leq 1 + \frac{\varepsilon}{4}.$$

The Khinchine type inequality for convex bodies [2, Sect. 1.4] implies

$$\frac{1}{\text{vol }(K)} \int_K \langle y, x\rangle^2 \cdot \mathbf{1}_{\{u \big| \|u\| \geq a\}}(y) \, dy \leq$$
$$\Big(\frac{1}{\text{vol }(K)} \int_K \langle y, x\rangle^4 \, dy \Big)^{1/2} \cdot \Big(\frac{1}{\text{vol }(K)} \int_K \mathbf{1}_{\{u \big| \|u\| \geq a\}}(y) \, dy \Big)^{1/2} \leq$$
$$Ce^{-ca^2/2n} \leq \frac{\varepsilon}{4}.$$

Thus for any $x \in B_2^n$

$$|\mathbb{E}\langle z, x\rangle^2 - 1| \leq \frac{\varepsilon}{2}.$$

Define a random vector

$$y = (\mathbb{E}\, z \otimes z)^{-1/2} z.$$

Then y is in the isotropic position and

$$\Big(\mathbb{E}\, \|y\|^{\log M} \Big)^{1/\log M} \leq \Big\| (\mathbb{E}\, z \otimes z)^{-1/2} \Big\| \cdot \Big(\mathbb{E}\, \|z\|^{\log M} \Big)^{1/\log M} \leq 2a,$$

so

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot 2a \leq \frac{\varepsilon}{2}$$

provided the constant C_0 in (4.2) is large enough. Thus,

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} z_i \otimes z_i - id \right\| \leq \mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \cdot \|\mathbb{E}\, z \otimes z\| + \|\mathbb{E}\, z \otimes z - id\| \leq \varepsilon.$$

□

The next application is connected to the approximation of a convex body by another one having a small number of contact points [9]. Let K be a convex body in $\mathbb{R}^n$ such that the ellipsoid of minimal volume containing it is the standard Euclidean ball B_2^n. Then by the theorem of John, there exist $N \leq (n+3)n/2$

points $z_1, \dots, z_N \in K, \quad \|x_i\| = 1$ and N positive numbers $c_1, \dots, c_N$ satisfying the following system of equations

$$id = \sum_{i=1}^{N} c_i \, z_i \otimes z_i \tag{4.3}$$

$$0 = \sum_{i=1}^{N} c_i \, z_i. \tag{4.4}$$

It was shown in [8] for convex symmetric bodies and in [9] in the general case, that the identity operator can be approximated by a sum of a smaller number of terms $x_i \otimes x_i$. We derive from Theorem 1 the following corollary, which improves Lemma 3.1 [9].

Corollary 4.3. *Let $\varepsilon > 0$ and let K be a convex body in $\mathbb{R}^n$, so that the ellipsoid of minimal volume containing it is B_2^n. Then there exist*

$$M \leq \frac{C}{\varepsilon^2} \cdot n \cdot \log n \tag{4.5}$$

contact points $x_1, \dots, x_M$ and a vector u, $\|u\| \leq \frac{C}{\sqrt{M}}$, so that the identity operator in $\mathbb{R}^n$ has the following representation

$$id = \frac{n}{M} \sum_{i=1}^{M} (x_i + u) \otimes (x_i + u) + S,$$

where

$$\sum_{i=1}^{M} (x_i + u) = 0 \tag{4.6}$$

and

$$\|S : \ell_2^n \to \ell_2^n\| < \varepsilon.$$

Proof. Let (4.3) be a decomposition of the identity operator. Let y be a random vector in $\mathbb{R}^n$, taking values $\sqrt{n} z_i$ with probability $c_i/\sqrt{n}$. Then, by (4.3), y is in the isotropic position. Obviously, for all $1 \leq p < \infty$

$$\left(\mathbb{E}\, \|y\|^p\right)^{1/p} = \sqrt{n}.$$

So, taking M as in (4.5), we obtain that for sufficiently large C

$$\left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \frac{\varepsilon}{2} \tag{4.7}$$

with probability greater than 3/4. Since by (4.4), $\mathbb{E} y = 0$ and $\|y\| = \sqrt{n}$, we have

$$\left\| \sum_{i=1}^{M} y_i \right\| \leq 2\sqrt{M} \tag{4.8}$$

with probability greater than 3/4. Take $y_1, \ldots, y_M$ for which (4.7) and (4.8) hold and put

$$x_i = \frac{1}{\sqrt{n}} \cdot y_i, \qquad u = -\frac{1}{M} \sum_{i=1}^{M} x_i.$$

Then (4.6) is satisfied and

$$\left\| \frac{n}{M} \sum_{i=1}^{M} (x_i + u) \otimes (x_i + u) - id \right\| \leq$$

$$\left\| \frac{n}{M} \sum_{i=1}^{M} x_i \otimes x_i - id \right\| + n \cdot \|u \otimes u\| \leq \frac{\varepsilon}{2} + \frac{4\, n}{M} \leq \varepsilon. \quad \square$$

Notice that in the previous construction of an approximate John's decomposition [9] the number of terms M was a random variable and the distribution of contact points $x_1, \ldots, x_M$ was completely untraceable. Unlike that situation, for the decomposition constructed above M is a constant and $x_1, \ldots, x_M$ are independent identically distributed random vectors, whose distribution is defined by the original John decomposition.

Substituting Lemma 3.1 [9] by Corollary 2.3 in the proof of Theorem 1.1 [9] we obtain the following

Corollary 4.3. *Let B be a convex body in $\mathbb{R}^n$ and let $\varepsilon > 0$. There exists a convex body $K \subset \mathbb{R}^n$, so that $d(K, B) \leq 1 + \varepsilon$ and the number of contact points of K with the ellipsoid of minimal volume containing it is less than*

$$M(n, \varepsilon) = \frac{C}{\varepsilon^2} \cdot n \cdot \log n.$$

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