EXTREMAL DISTANCES BETWEEN SECTIONS OF CONVEX BODIES

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Abstract. Let K, D be convex centrally symmetric bodies in $\mathbb{R}^n$. Let $k < n$ and let $d_k(K, D)$ be the smallest Banach–Mazur distance between k-dimensional sections of K and D. Define

$$\Delta(k, n) = \sup d_k(K, D),$$

where the supremum is taken over all $n-$dimensional convex symmetric bodies K, D. We prove that for any $k < n$

$$\Delta(k, n) \sim_{\log n} \begin{cases} \sqrt{k} & \text{if } k \le n^{2/3} \\ \frac{k^2}{n} & \text{if } k > n^{2/3}, \end{cases}$$

where $A \sim_{\log n} B$ means that $1/(C \log^a n) \cdot A \le B \le (C \log^a n) \cdot A$ for some absolute constants $C, a > 0$.

1. Introduction.

Let K and D be n-dimensional convex centrally symmetric bodies. Define the distance between K and D as follows

$$d(K, D) = \inf\{a \cdot b \mid \frac{1}{a} K \subset TD \subset bK\}.$$

Here the infimum is taken over all invertible linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$. This definition corresponds to the classical Banach-Mazur distance between Banach spaces. Estimating distances for different classes of convex bodies is an important and difficult problem. In many cases such estimates require elaborate geometric and probabilistic techniques (see the book [TJ] for further references).

A celebrated theorem of John [J] states that the distance between a convex symmetric body K and the Euclidean ball B_2^n is bounded by $\sqrt{n}$. This implies that the distance between two $n-$dimensional convex symmetric bodies does not exceed n. The problem whether this estimate can be improved remained open for a long time. It was finally solved by Gluskin [Gl1], who constructed random convex polytopes K

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and K' such that

$$d(K, K') \geq cn.$$

with probability close to 1. More precisely, let $N > n$ and let $g_1, \ldots, g_N$, $g'_1, \ldots, g'_N$ be independent standard Gaussian vectors in $\mathbb{R}^n$. Define random polytopes

$$K = K(\omega) = \text{abs.conv}(\sqrt{n}e_1, \ldots, \sqrt{n}e_n, g_1, \ldots, g_N),$$
$$K' = K(\omega') = \text{abs.conv}(\sqrt{n}e_1, \ldots, \sqrt{n}e_n, g'_1, \ldots, g'_N),$$

where the abs.conv(A) stands for the convex hull of A and $-A$. Then for N proportional to n there are absolute constants $c, c' > 0$ such that

$$\mathbb{P}\left(d(K(\omega), K(\omega')) \geq cn\right) \geq 1 - e^{-c'n}.$$

This construction, called below Gluskin's polytopes, became later a major source of counterexamples in local theory of Banach spaces (see extensive survey [M-TJ1]).

However, in spite of being very far from each other, Gluskin's polytopes possess sections of proportional dimension which are close. Indeed, the volume ratio theorem [Sz1], [Sz-TJ] implies that with probability close to 1 the body $K(\omega)$ has a section of dimension at least $n/2$ whose distance to the Euclidean ball is bounded.

This leads to the following general question: *How large can the distance between two sections of given convex bodies be?*

A problem of this type was recently considered by Mankiewicz and Tomczak-Jaegerman [M-TJ2]. They found a precise estimate of the distance between *random* k-dimensional sections of two convex symmetric bodies in terms of the average distance of a $k/2$-dimensional section of each body to a ball. Notice that while the distance between random sections is large, the minimal distance may be much smaller.

To make the formulation of the question above more concrete we introduce a new distance. Let $k < n$ and define the distance d_k between $n-$dimensional convex bodies K and D by

$$d_k(K, D) = \inf d(K \cap E, D \cap F),$$

where the infimum is taken over all subspaces $E, F \subset \mathbb{R}^n$ of dimension k. Finding a precise estimate of $d_k(K, D)$ for given bodies K and D seems to be a very hard problem. Indeed, even in the case when K is the n-dimensional cube the precise order of $\sup_D d_k(K, D)$ is still unknown [Sz4], [B-Sz], [Sz-T], [G]. We consider here a simpler problem of finding the "d_k-diameter" of the Minkowski compactum (also known as Banach–Mazur compactum). Namely, define

$$\Delta(k, n) = \sup d_k(K, D),$$

where the supremum is taken over all $n-$dimensional convex symmetric bodies K, D. We are interested in obtaining an estimate for $\Delta(k,n)$, which would be precise up to logarithmic terms. To simplify the notation, we shall write $A \preceq_{\log n} B$ if $A \le CB \cdot \log^{\alpha} n$ for some absolute constants C, α. If $A \preceq_{\log n} B$ and $B \preceq_{\log n} A$, we shall write $A \sim_{\log n} B$.

Since any k-dimensional section of an n-dimensional convex symmetric body is a convex symmetric body in $\mathbb{R}^k$, $\Delta(k,n) \le k$. Also from Dvoretzky's theorem it follows that

$$\Delta(c \log n, n) \le C.$$

Here and below C, C', c etc. denote absolute constants whose value can change from line to line. Also, all dimensions and indexes are integer numbers. If the dimension appearing in some formula is not integer, we assume that the integer part is taken.

Denote by B_p^n the unit ball of the space l_p^n:

$$B_p^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\Big|\; \sum_{j=1}^{n} |x_j|^p \le 1\}.$$

A lower estimate of $d_k(B_2^n, B_\infty^n)$ was independently obtained by Bourgain, Lindenstrauss and Milman [BLM], Gluskin [Gl3] and Carl and Pajor [CP]. They proved that for any $k < n$

$$d_k(B_2^n, B_\infty^n) \ge c\sqrt{\frac{k}{\log\left(1 + \frac{n}{k}\right)}}. \tag{1.1}$$

This result implies that for any $k < n$

$$\sqrt{k} \preceq_{\log n} \Delta(k,n) \le k.$$

It turns out that the upper estimate is exact up to logarithmic terms for k proportional to n, while the lower one is exact for small k. The main result of this paper is the following

Theorem 1.1. *For any $k < n$,*

$$\Delta(k,n) \sim_{\log n} \begin{cases} \sqrt{k} & \text{if } k \le n^{2/3} \\ \frac{k^2}{n} & \text{if } k > n^{2/3}. \end{cases}$$

A quantity similar to d_k was studied by Bourgain and Milman [BM]. Instead of the distances between sections, they considered the distance between a section and a projection. Namely they considered the distance

$$\widetilde{d}_k(K, D) = \inf d(K \cap E, P_F D),$$

where the infimum is taken over the subspaces $E, F \subset \mathbb{R}^n$, $\dim E = \dim F = k$. Here P_F denotes the orthogonal projection onto F. They showed that for any K, D

$$\widetilde{d}_{n/2}(K, D) \le C\sqrt{n} \cdot \log^2 n.$$

This estimate together with Theorem 1.1 shows that the distances d_k and $\widetilde{d}_k$ are essentially different.

We denote by K° the polar of a convex body $K \subset \mathbb{R}^n$:

$$K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \le 1 \quad \text{for all } y \in K\}.$$

Since the Banach–Mazur distance between two bodies is equal to the distance between their polars, we can write $\Delta(k, n)$ as

$$\Delta(k, n) = \sup_{K,D} \delta_k(K, D),$$

where $\delta_k(K, D)$ denotes the minimal distance between k-dimensional *projections* of K and D. Then the upper estimate of $\Delta(k, n)$ is contained in the following

Theorem 1.2. *Let $K, D \subset \mathbb{R}^n$ be convex symmetric bodies and let $k \le n$. There exist projections QK and $Q'D$ of K and D of dimension k such that*

$$d(QK, Q'D) \le C \cdot \max\left(\frac{k^2}{n},\ \sqrt{k}\log n\right).$$

We prove this theorem in three steps. In section 2 we show that for an n-dimensional body K there exists a linear operator Q of rank $m \ge cn/\log n$ such that $QK \subset B_2^n$ and QK contains an octahedron $r_1 B_1^m$ and a ball $r_2 B_2^n$ for some relatively large r_1, r_2 depending on K. To achieve this we project K on a subspace on which the John ellipsoid and the ℓ-ellipsoid are proportional. Then we apply Vershynin's theorem [V] to construct a farther projection of K which contains a copy of B_1^m inside.

In section 3 we find a k-dimensional coordinate projection of QK which is contained in the convex hull of the octahedron $\rho(K)B_1^k$ and the ball $\sqrt{k/n}B_2^k$ for some $\rho(K)$. The projection has to be coordinate to preserve the the octahedron inside QK. We use a random coordinate projection. This leads to estimating the supremum of a Rademacher random process, which is done by applying Talagrand's comparison theorem ([L-T], Theorem 4.12).

We conclude the proof of the Theorem 1.2 in Section 4. Given n-dimensional bodies K and D we consider the body QK constructed in section 2 and the body $Q'D$ constructed similarly. We show that either projections of QK and $Q'D$ onto random k-dimensional coordinate subspaces or projections onto random subspaces uniformly distributed over the Grassmanian satisfy Theorem 1.2.

Section 5 contains a technical result about Gluskin polytopes which is necessary to prove the lower estimate of Theorem 1.1. The estimate itself is established in section 6. We prove the following

Theorem 1.3. *For any* $k < n$

$$\Delta(k,n) \geq c \cdot \max\left(\frac{k^2}{n\log\log n}, \sqrt{\frac{k}{\log(1+n/k)}}\right).$$

Finally, Theorem 1.1 follows from Theorems 1.2 and 1.3.

We study also the question of existence of two convex symmetric bodies whose sections of all dimensions are far apart. More precisely, do there exist two bodies $K, D \subset \mathbb{R}^n$ such that for any $k < n$, $\delta_k(K,D) \sim_{\log n} \Delta(k,n)$? We show in Section 6 that while *two* such bodies do not exist, there exist *three* bodies $K, D_1, D_2 \subset \mathbb{R}^n$ for which

$$\max(\delta_k(K,D_1), \delta_k(K,D_2)) \sim_{\log n} \Delta(k,n)$$

for all k.

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2. Approximation from the inside.

We start with introducing some notation. Let K be an n-dimensional convex symmetric body. Denote $d_K = d(K, B_2^n)$. Let F be a finite dimensional Euclidean space and let g_F be the standard Gaussian vector in F. In case $F = \mathbb{R}^n$ we write g instead of g_F. For a convex symmetric body $K \subset F$ denote

$$\ell(K) = \mathbb{E}\,\|g_F\|_K = \mathbb{E}\sup_{x\in K^\circ}\langle g_F, x\rangle.$$

In the geometric language $\ell(K)/\sqrt{n}$ is approximately equal to the mean width of the body K°. We shall repeatedly use below the following well known observation. Let E be a linear subspace of F and let $K \subset F$ be a convex symmetric body. Then

$$\ell(K\cap E) \leq \ell(K) \quad \text{and } \ell(P_E K) \leq \ell(K). \tag{2.1}$$

Indeed, the standard Gaussian vector $g_F \in F$ can be decomposed as $g_F = g_E + g_{E^\perp}$, where g_E and $g_{E^\perp}$ are independent standard Gaussian vectors in the spaces E and $E^\perp$. Hence

$$\ell(K) = \mathbb{E}_{g_E}\mathbb{E}_{g_{E^\perp}} \|g_E + g_{E^\perp}\|_K \geq \mathbb{E}_{g_E} \left\|g_E + \mathbb{E}_{g_{E^\perp}} g_{E^\perp}\right\|_K = \mathbb{E}_{g_E} \|g_E\|_K .$$

The second inequality is a dual of the first one.

We need two fundamental results from the local theory of Banach spaces. The first one, known as the MM^*-estimate, is obtained by combining results of Pisier [P], Theorem 2.5 and Figiel and Tomczak-Jaegermann [F-TJ].

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a convex symmetric body. There exists an invertible self-adjoint linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\ell(TK) \leq C\sqrt{n} \cdot \log d_K, \quad \ell((TK)^\circ) \leq \sqrt{n}.$$

The second result, known as the low M^*-estimate, was proved by Milman ([M-S1], Theorem 4.8, see also [P-TJ]).

Theorem 2.2. *Let $L \subset \mathbb{R}^n$ be a convex symmetric body. There exists a subspace E of dimension $s \geq n/2$ such that*

$$L \cap E \subset C\frac{\ell(L^\circ)}{\sqrt{n}} \cdot B_2^s.$$

Combining the MM^* and the low-M^* estimates and using the inequality (2.1) we obtain

Corollary 2.3. *Let $K \subset \mathbb{R}^n$ be a convex symmetric body. There exists an operator $V : \mathbb{R}^n \to \mathbb{R}^n$ of rank $s \geq n/2$ such that*

$$\ell(VK) \leq \sqrt{n}, \quad \ell((VK)^\circ) \leq C\sqrt{n} \cdot \log d_K.$$

and $cB_2^s \subset VK$.

Proof. Let T be the linear operator from Theorem 2.1, applied to the body K° and let $L = TK^\circ$. Then $\ell(L^\circ) \leq \sqrt{n}$. Applying Theorem 2.2 to L, we find a subspace E such that

$$TK^\circ \cap E \subset C \cdot B_2^s.$$

Let P be the orthogonal projection onto E. Then taking polars of both parts of the previous inclusion we get

$$PT^{-1}K \supset C^{-1}B_2^s$$

Denote $V = PT^{-1}$. Then (2.1) implies

$$\ell(VK) \leq \ell(T^{-1}K) = \ell(L^\circ) \leq \sqrt{n}$$

and

$$\ell((VK)^\circ) = \ell(L \cap E) \leq \ell(L) \leq C\sqrt{n}\log d_K.$$

□

The main result of this section is the following

Theorem 2.4. *Let $K \subset \mathbb{R}^n$ be a convex symmetric body. There exists an*

$$l \geq \frac{n}{8}$$

and a linear operator $Q : \mathbb{R}^n \to \mathbb{R}^l$ such that

$$\operatorname{conv}\left(cB_1^l,\ \frac{c \cdot \ell((QK)^\circ)}{\sqrt{n} \log d_K} B_2^l \right) \subset QK \subset B_2^l \tag{2.2}$$

and

$$\ell(QK) \cdot \ell((QK)^\circ) \leq Cn \log d_K. \tag{2.3}$$

Proof. To construct a linear image of the body K which contains a ball and satisfies the estimate (2.3) we use Corollary 2.3. Then to embed an octahedron we shall use a theorem of Vershynin (Theorem 3.3 [V]). However, the last theorem requires a specific position of the body. Namely, the ellipsoid of minimal volume containing the body should be equal to the Euclidean unit ball. The properties of the minimal volume ellipsoid, called the John ellipsoid of a body, are discussed in the survey [G-M].

To apply both results simultaneously we have to combine two Euclidean structures: one, given by Corollary 2.3 and another, given by the John ellipsoid. More precisely, let V be the operator from Corollary 2.3 and let $\mathcal{E}$ be the ellipsoid of minimal volume containing VK. Define a positive self-adjoint operator $T : \mathbb{R}^s \to \mathbb{R}^s$ such that $TB_2^s = \mathcal{E}$. Let

$$K_1 = T^{-1}VK.$$

Then the ellipsoid of minimal volume containing K_1 is $T^{-1}\mathcal{E} = B_2^s$. We shall find a projection of the body K_1, which contains a large Euclidean ball and satisfies (2.3). We need the following well known lemma.

Lemma 2.5. *Let $T : \mathbb{R}^s \to \mathbb{R}^s$ be a self-adjoint linear operator. Then there exists $\lambda \in \mathbb{R}$ and a subspace $F \subset \mathbb{R}^s$ of dimension at least $s/2$ such that $T|_F = \lambda U$, where U is an isometry.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ be the eigenvalues of T and let $e_1, \ldots, e_s$ be corresponding eigenvectors. Assume first that s is even. Choose λ so that $\lambda_{s/2} \geq \lambda \geq \lambda_{s/2+1}$. Then for any $j \leq s/2$ the two-dimensional subspace span (e_j, e_{s-j}) contains a unit vector v_j such that $\|Tv_j\| = \lambda$. Notice that the vectors $v_1, \ldots, v_{s/2}$ are mutually orthogonal. The vectors $Tv_1, \ldots, Tv_{s/2}$ are mutually orthogonal as well. Set

$$F = \text{span } (v_1, \ldots, v_{s/2}).$$

Let $x = \sum_{j=1}^{s/2} a_j v_j$ be a vector in F. Then

$$\|x\| = \left(\sum_{j=1}^{s/2} a_j^2 \right)^{1/2}$$

and

$$\|Tx\| = \left(\sum_{j=1}^{s/2} a_j^2 \, \|Tv_j\|^2 \right)^{1/2} = \lambda \cdot \left(\sum_{j=1}^{s/2} a_j^2 \right)^{1/2}.$$

If s is odd, the proof can be modified in an obvious way. Set $\lambda = \lambda_{(s+1)/2}$. For $1 \le j < (s+1)/2$ choose v_j as above and put $v_{(s+1)/2} = e_{s+1}/2$. Then $F = \text{span}\,(v_1, \dots, v_{(s+1)/2})$ satisfies the conditions of the Lemma. □

We continue the proof of Theorem 2.4. Set $L = (VK)^\circ$. Let $E = TF$. Since U is an isometry, the Gaussian vector g_E is distributed like $\lambda^{-1} T g_F$. Hence, by (2.1)

$$\begin{aligned} \mathbb{E}\,\|g_E\|_{TL\cap E} &= \lambda^{-1}\mathbb{E}\,\|Tg_F\|_{T(L\cap F)} = \lambda^{-1}\mathbb{E}\,\|g_F\|_{L\cap F} \\ &\le \lambda^{-1}\mathbb{E}\,\|g\|_L \le \lambda^{-1} \cdot C\sqrt{n}\log d_K \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathbb{E}\sup_{x\in TL\cap E}\langle g_E, x\rangle &= \lambda^{-1}\mathbb{E}\sup_{x\in T(L\cap F)}\langle Tg_F, x\rangle = \lambda^{-1}\mathbb{E}\sup_{y\in L\cap F}\langle Tg_F, Ty\rangle \\ &= \lambda\mathbb{E}\sup_{y\in L\cap F}\langle g_F, y\rangle \le \lambda\mathbb{E}\sup_{y\in L}\langle g, y\rangle \le \lambda\cdot C\sqrt{n}. \end{aligned}$$

Thus,

$$\ell(TL\cap E)\cdot\ell((TL\cap E)^\circ) \le \ell(L)\cdot\ell(L^\circ) \le Cn\log d_K.$$

Also, since $L \subset CB_2^s$, (2.4) implies

$$TL\cap E \subset C\lambda B_2^s \cap E \subset \frac{C\sqrt{n}\log d_K}{\mathbb{E}\,\|g_E\|_{TL\cap E}} B_2^s \cap E. \tag{2.5}$$

Notice that $(TL\cap E)^\circ = P_E(T^{-1}L^\circ) = P_E(T^{-1}VK) = P_E(K_1)$, where P_E is the orthogonal projection on E. The previous inequality reads

$$\ell(P_E K_1)\cdot\ell((P_E K_1)^\circ) \le Cn\log d_K. \tag{2.6}$$

Dualizing the inclusion (2.5) we get

$$P_E K_1 \supset \frac{\mathbb{E}\,\|g_E\|_{TL\cap E}}{C\sqrt{n}\log d_K} B_2^s \cap E = \frac{\ell((P_E K_1)^\circ)}{C\sqrt{n}\log d_K} B_2^s \cap E. \tag{2.7}$$

To establish (2.2) we shall use the fact that the John ellipsoid of K_1 is B_2^s. Then by the classical John's theorem [J] there exist positive numbers $c_1, \dots, c_N$ and points $x_1, \dots, x_N \in S^{s-1} \cap \partial K_1$, called contact points, which form the following decomposition of the identity operator in $\mathbb{R}^s$:

$$id = \sum_{i=1}^{N} c_i x_i \otimes x_i.$$

We shall find a subspace F of E such that some of the projections of x_i to F form a system which is equivalent to an orthonormal basis.

Applying [V], Theorem 3.3 to the operator P_E, we find a set $\sigma \subset J$ of cardinality $l \geq \dim E/2 \geq s/4$ such that

$$\|P_E x_j\|_2 \geq C\sqrt{\frac{\dim E}{n}} = C' \quad \text{for all } j \in \sigma \tag{2.8}$$

and the system $(P_E x_j / \|P_E x_j\|_2)_{j\in\sigma}$ is C-equivalent to an orthonormal basis of ℓ_2^l i.e.

$$C^{-1} \|(a_j)_{j\in\sigma}\|_{B_2^\sigma} \leq \left\| \sum_{j\in\sigma} a_j \frac{P_E x_j}{\|P_E x_j\|} \right\|_2 \leq C \|(a_j)_{j\in\sigma}\|_{B_2^\sigma}$$

for all $(a_j)_{j\in\sigma}$.

Let $P' : \mathbb{R}^s \to \mathbb{R}^s$ be the orthogonal projection onto the space $H = \mathrm{span}(P_E x_j)_{j\in\sigma}$. Since $x_j \in K_1$,

$$\mathrm{abs.conv}(P' x_j)_{j\in\sigma} \subset P' K_1 \subset B_2^s \cap H.$$

Let $S : H \to \mathbb{R}^l$ be the operator defined by

$$S \frac{P' x_j}{\|P' x_j\|_2} = e_j \quad \text{for } j \in \sigma.$$

Since the system $(P_E x_j / \|P_E x_j\|_2)_{j\in\sigma}$ is C-equivalent to an orthonormal basis,

$$\left\| S : (H, \|\cdot\|_2) \to \ell_2^l \right\| \leq C \text{ and } \left\| S^{-1} : \ell_2^l \to (H, \|\cdot\|_2) \right\| \leq C.$$

We have

$$\|P' x_j\|_2 \cdot e_j = S^{-1} P' x_j \in S^{-1} P' K_1,$$

so (2.8) implies that

$$c B_1^l \subset S^{-1} P' K_1.$$

We have constructed a linear image of K_1, which is also a linear image of K, that contains an octahedron of the required size. It remains to check that it contains a Euclidean ball of the required radius and the inequality (2.3) holds for it.

Define the operator $Q : \mathbb{R}^n \to \mathbb{R}^l$ by $Q = 1/C \cdot S^{-1}P'T^{-1}V$. Since $K_1 = T^{-1}VK$, we have $QK = 1/C \cdot S^{-1}P'K_1$. Hence,

$$c'B_1^l \subset QK$$

and $K_1 \subset B_2^s$ implies

$$QK \subset 1/C \cdot S^{-1}P'B_2^s \subset B_2^l.$$

To estimate $\ell(QK)$ and $\ell((QK)^\circ)$ we use the following Lemma, which follows from Slepian's inequality [P], p. 69.

Lemma 2.6. *Let $K \subset \mathbb{R}^m$ be a convex symmetric body. Then for any linear operator $S : \mathbb{R}^m \to \mathbb{R}^m$*

$$\mathbb{E}\left\|Sg\right\|_K \le \|S : \mathbb{R}^m \to \mathbb{R}^m\| \cdot \mathbb{E}\left\|g\right\|_K .$$

Proof. For $z \in K^\circ$ define Gaussian random variables $X_z = \langle Sg, z\rangle$ and $Y_z = \|S\| \cdot \langle g, z\rangle$. Since $\|S^*\| = \|S\|$,

$$\begin{aligned}\mathbb{E}|X_z - X_{z'}|^2 &= \mathbb{E}\langle g, S^*(z-z')\rangle^2 = \left\|S^*(z-z')\right\|^2 \\ &\le \left\|S\right\|^2 \cdot \left\|z-z'\right\|^2 = \mathbb{E}|Y_z - Y_{z'}|^2.\end{aligned}$$

Now Slepian's inequality implies

$$\mathbb{E}\left\|Sg\right\|_K = \mathbb{E}\sup_{z\in K^\circ} X_z \le \mathbb{E}\sup_{z\in K^\circ} Y_z = \|S\| \cdot \mathbb{E}\left\|g\right\|_K .$$

□

If a linear operator $S : \mathbb{R}^m \to \mathbb{R}^m$ is invertible, then by Lemma 2.6

$$\ell(SK) = \mathbb{E}\left\|S^{-1}g\right\|_K \le \left\|S^{-1}\right\| \ell(K).$$

Since $P' = P'P_E$, we have by (2.1) and the previous inequality

$$\begin{aligned}\ell((QK)^\circ) &\le 1/C \cdot \ell(S(P'K_1)^\circ) \le 1/C \left\|S^{-1}\right\| \cdot \ell((P'P_EK_1)^\circ) \\ &\le \ell((P_EK_1)^\circ),\end{aligned}$$

and (2.2) follows from (2.7).

It remains to check (2.3). The inequality (2.6) implies

$$\begin{aligned}\ell(QK)\ell((QK)^\circ) &\le \|S\|\, \ell(P'P_EK_1) \cdot \left\|S^{-1}\right\| \ell((P'P_EK_1)^\circ) \\ &\le C^2\ell(P_EK_1) \cdot \ell((P_EK_1)^\circ) \le C'n\log d_K.\end{aligned}$$

□

3. Random coordinate projection.

Let $P : \mathbb{R}^l \to \mathbb{R}^l$ be an orthogonal projection onto a random m-dimensional subspace. It is well-known that such projection "shrinks" the distances by the "shrinking factor" $\sqrt{m/l}$. Namely, for any point $x \in \mathbb{R}^l$, $\|Px\| \le C\sqrt{m/l}\,\|x\|$ with probability close to 1. Combining this fact with measure concentration, one obtains the following Lemma, which is essentially contained in [M-S2].

Lemma 3.1. *Let $K \subset B_2^l$ be a convex symmetric body. Let $m < l$ and let $P : \mathbb{R}^l \to \mathbb{R}^l$ be a random projection of rank m. Then with probability close to 1 the following holds.*

(i): *If $\ell(K^\circ) > C\sqrt{m}$, then $d(PK, B_2^m) \le 2$.*
(ii): *If $\ell(K^\circ) \le C\sqrt{m}$, then $PK \subset c\sqrt{m/l} \cdot B_2^m$.*

If we apply a random projection to the body QK constructed in the previous section, the structure of ℓ_1-ball inscribed in QK will be destroyed. To preserve it we consider a different type of randomness, namely random *coordinate* projection.

Let us introduce some notation. Let $0 < \delta < 1$ and let $\delta_1 \dots \delta_n$ be i.i.d. Bernoulli random variables with expectation δ. Let $I = \{j \mid \delta_j = 1\}$ be a random subset of $\{1 \dots n\}$. In this model the cardinality of I is a random variable, which is highly concentrated about its mean δn.

Define a random coordinate projection $P_I : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_I(x_1 \dots x_n) = (\delta_1 x_1 \dots \delta_n x_n).$$

There is an essential difference between random projection considered in Lemma 3.1 and random coordinate projection. Indeed, Lemma 3.1 shows that the diameter of the body K decreases under random projection. However, if we consider $K = B_1^n$, then $P_I K = B_1^I = B_1^n \cap \mathbb{R}^I$ for any set I, so the diameter of a projection remains 1.

To modify Lemma 3.1 for random coordinate projections we have to introduce a new characteristic of a convex body, which plays the role of $\ell(K^\circ)$. Let $\varepsilon_1 \dots \varepsilon_n$ be i.i.d. Rademacher random variables and let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$. For a convex symmetric body $K \subset \mathbb{R}^n$ denote

$$\rho(K) = \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j e_j \right\|_K = \mathbb{E} \sup_{x \in K^\circ} \sum_{j=1}^n \varepsilon_j \cdot x_j.$$

It is easy to check (see [M-S1]) that $\rho(K^\circ) \le c\ell(K^\circ)$.

We prove that a random projection of K onto an m-dimensional coordinate subspace is contained in the convex hull of the octahedron $C\rho(K^\circ)B_1^m$ and the ball of radius $C\sqrt{m/n}$.

More precisely, we prove the following

Theorem 3.2. *Let $K \subset B_2^n$ be a closed set and let $0 < \delta < 1$. Let P_I be a random coordinate projection defined above. Then with probability at least* $9/10$

$$P_I K \subset c \cdot \left(\rho(K^\circ) B_1^I + \sqrt{\delta} B_2^I \right).$$

Remark 3.3. It can be shown that the probability above is exponentially close to 1. However since we do not use this fact here, we shall present the proof elsewhere.

Remark 3.4. The coefficients $\rho(K^\circ)$ and $\sqrt{\delta}$ are exact. Consider the case $\rho(K^\circ) \le \sqrt{\delta n}$ (the other case is less interesting, since $\rho(K^\circ) \ge \sqrt{\delta n}$ implies that $\sqrt{\delta} B_2^n \subset \rho(K^\circ) B_1^n$).

Let $M \in [1, \sqrt{\delta n}]$. Then there exists a convex symmetric body $K \subset B_2^n$ with $\rho(K^\circ) \le M$ having the following property. Let s be the integer part of δn. Assume that there exist $a < \sqrt{s}$, $b < 1$ and a coordinate projection P_I of rank s such that

$$P_I K \subset \text{conv}\,(a B_1^n \cup b B_2^n) \cap \mathbb{R}^I \tag{3.1}$$

Then $a \ge M - 1$ and $b \ge \sqrt{\delta}$.

Indeed, let m be the integer part of M^2. Denote

$$K_1 = \frac{1}{\sqrt{m}} \text{conv} \left\{ \sum_{j \in J} \varepsilon_j e_j \mid \text{card}\,(J) = m,\ \varepsilon_j \in \{+1, -1\} \right\},$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Put

$$K = \text{conv}\,(K_1, \pm \frac{1}{\sqrt{n}} \sum_{j=1}^n e_j).$$

Then $K \subset B_2^n$ and $\rho(K^\circ) \le \sqrt{m} + 1$. Let P_I be a coordinate projection of rank s. Since $M \le \sqrt{\delta n}$, $m \le s$. Thus

$$P_I K \supset \frac{1}{\sqrt{m}} \text{conv} \left\{ \sum_{j \in J} \varepsilon_j e_j \mid J \subset I,\ \text{card}\,(J) = m,\ \varepsilon_j \in \{+1, -1\} \right\}.$$

Since $b < 1$, (3.1) implies that $P_I K \subset a B_1^n \cap \mathbb{R}^I$, so $a \ge \sqrt{m}$.

Similarly,

$$u = \frac{1}{\sqrt{n}} \sum_{j \in I} e_j \in P_I K$$

and $\|u\|_1 = \sqrt{s}$. Since $a < \sqrt{s}$, (3.1) implies that $u \in b B_2^n \cap \mathbb{R}^I$. Thus, $b \ge \|u\| = \sqrt{s/n}$.

Proof of Theorem 3.2. The proof uses Gine–Zinn type argument based on the following Talagrand's comparison theorem for Rademacher processes (Theorem 4.12 [L-T]).

Theorem 3.5. *Let $\alpha_j : \mathbb{R}_+ \to \mathbb{R}_+,\ j = 1, \ldots, N$ be 1-Lipschitz functions such that $\alpha_j(0) = 0$. Let $\varepsilon_1 \ldots \varepsilon_N$ be Rademacher random variables. Then for any bounded set T in $\mathbb{R}^n$*

$$\mathbb{E} \sup_{t \in T} \sum_{j=1}^{N} \varepsilon_j \alpha_j(t_j) \le 2\mathbb{E} \sup_{t \in T} \sum_{j=1}^{N} \varepsilon_j t_j.$$

Denote

$$\nu = \frac{\delta}{\rho(K^\circ)}.$$

We shall decompose each $x \in K$ as $x = \varphi(x) + \psi(x)$ for some functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}^n$. The coordinates of x whose absolute value is less than ν will be included into $\varphi(x)$, while the coordinates, whose absolute value is greater than ν will be distributed between $\varphi(x)$ and $\psi(x)$. The concrete form of $\varphi(x)$ and $\psi(x)$ is chosen to have a good control of Lipschitz constants of functions $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ which appear in the proof below.

Define a function $\tau : \mathbb{R} \to [0, 1]$ by

$$\tau(t) = \begin{cases} 1, & |t| \le \nu \\ \sqrt{\frac{\nu}{|t|}}, & \text{otherwise.} \end{cases}$$

For $x = (x_1 \ldots x_n) \in \mathbb{R}^n$ denote

$$\varphi(x) = (\tau(x_1) \cdot x_1, \ldots, \tau(x_n) \cdot x_n)$$
$$\psi(x) = ((1 - \tau(x_1)) \cdot x_1, \ldots, (1 - \tau(x_n)) \cdot x_n).$$

Notice that if $x \in B_2^n$, then $\varphi(x) \in B_2^n$ and $\psi(x) \in B_2^n$ as well. Since $x = \varphi(x) + \psi(x)$, it is enough to prove that

$$P_I \varphi(K) \subset C\sqrt{\delta} B_2^n$$

and

$$P_I \psi(K) \subset C\rho(K^\circ) B_1^n$$

with high probability. Consider the set $\psi(K)$ first. We shall prove that

$$\mathbb{E} \sup_{x \in \psi(K)} \|P_I x\|_1 \le C\rho(K^\circ). \tag{3.2}$$

Then the probability estimate will follow from Chebychev's inequality.

We apply the standard symmetrization argument (see [L-T] for example). Let $\delta_1' \dots \delta_n'$ be independent copies of $\delta_1 \dots \delta_n$. We have

$$\mathbb{E} \sup_{x \in \psi(K)} \|P_I x\| = \mathbb{E} \sup_{x \in \psi(K)} \sum_{j=1}^n \delta_j |x_j|$$
$$\leq \delta \cdot \sup_{x \in \psi(K)} \sum_{j=1}^n |x_j| + \mathbb{E} \sup_{x \in \psi(K)} \sum_{j=1}^n (\delta_j - \delta_j') \cdot |x_j|$$

Let $x \in \psi(K)$ and let $y \in K$ be such that $\psi(y) = x$. Then $x \in B_2^n$, so

$$\sum_{j=1}^n |x_j| \leq \left(\sum_{j=1}^n x_j^2 \right)^{1/2} \cdot |\text{supp } x|^{1/2}$$
$$\leq (\text{card } \{j \mid |y_j| \geq \nu\})^{1/2} \leq \frac{1}{\nu} \|y\| \leq \frac{1}{\nu}. \tag{3.3}$$

Let $\varepsilon_1 \dots \varepsilon_n$ be Rademacher random variables independent of $\delta_1 \dots \delta_n$, $\delta_1' \dots \delta_n'$. Denote the expectation with respect to $\delta_1, \dots, \delta_n, \delta_1', \dots, \delta_n'$ by $\mathbb{E}_\delta$ and the expectation with respect to $\varepsilon_1, \dots, \varepsilon_n$ by $\mathbb{E}_\varepsilon$. Since the random variables $\delta_j - \delta_j'$ are symmetric,

$$\mathbb{E}_\delta \sup_{x \in \psi(K)} \sum_{j=1}^n (\delta_j - \delta_j') \cdot |x_j| = \mathbb{E}_\varepsilon \mathbb{E}_\delta \sup_{x \in \psi(K)} \sum_{j=1}^n \varepsilon_j (\delta_j - \delta_j') \cdot |x_j| . \tag{3.4}$$

Fix $\delta_1, \dots, \delta_n, \delta_1', \dots, \delta_n'$ and consider the expectation with respect to $\varepsilon_1, \dots, \varepsilon_n$. Let

$$J(\delta) = \{j \mid \delta_j - \delta_j' \neq 0\}.$$

Then (3.4) reads as

$$\mathbb{E}_\delta \mathbb{E}_\varepsilon \sup_{y \in K} \sum_{j=1}^n \varepsilon_j \cdot \alpha_j(y_j)$$

where $\alpha_j(t) = (1 - \tau(t)) \cdot |t|$ for $j \in J(\delta)$ and $\alpha_j(t) = 0$ otherwise. Notice that $\alpha_j(t) = 0$ when $|t| \leq \nu$. For any $j \in J(\delta)$ and any t such that $|t| > \nu$ we get

$$|\alpha_j'(t)| = \left| (1 - \tau(t)) \cdot \text{sign}(t) - \tau'(t) \cdot |t| \right| \leq 1 + \frac{1}{2} \sqrt{\nu} |t|^{-1/2} \leq \frac{3}{2} .$$

This means that the Lipshcitz constant of α_j does not exceed 3/2. Theorem 3.5 implies

$$\mathbb{E}_\varepsilon \sup_{y \in K} \sum_{j=1}^n \varepsilon_j \cdot \alpha_j(y_j) \leq 3 \mathbb{E}_\varepsilon \sup_{y \in K} \sum_{j=1}^n \varepsilon_j \cdot y_j = 3\rho(K^\circ),$$

so (3.4) is bounded by $3\rho(K^\circ)$ as well.

Combining this with (3.3) and using the definition of ν, we obtain

$$\mathbb{E}_\delta \sup_{x\in\psi(K)} \sum_{j=1}^n \delta_j |x_j| \le C \cdot \left(\frac{\delta}{\nu} + \rho(K^\circ)\right) \le C' \rho(K^\circ).$$

This completes the proof of (3.2).

Now let us consider the set $\varphi(K)$. As before, the bound for probability follows from the estimate of expectation. We shall show that

$$\mathbb{E} \sup_{x\in\varphi(K)} \|P_I x\|_2^2 \le C \cdot \delta.$$

Proceeding as above, we get

$$\mathbb{E}_\delta \sup_{x\in\varphi(K)} \sum_{j=1}^n \delta_j x_j^2 \le \delta \cdot \sup_{x\in\varphi(K)} \sum_{j=1}^n x_j^2 + \mathbb{E}_\delta \mathbb{E}_\varepsilon \sup_{x\in\varphi(K)} \sum_{j=1}^n \varepsilon_j (\delta_j - \delta_j') \cdot x_j^2$$

Notice that $K \subset B_2^n$ implies $\varphi(K) \subset B_2^n$, so the first term is bounded by δ.

The second term can be rewritten as

$$\nu \cdot \mathbb{E}_\delta \mathbb{E}_\varepsilon \sup_{y\in K} \sum_{j=1}^n \varepsilon_j \cdot \beta_j(y_j),$$

where $\beta_j(t) = 1/\nu \cdot (\tau(t)\cdot t)^2$ if $t \in J(\delta)$ and $\beta_j(t) = 0$ otherwise. Let $j \in J(\delta)$. For $|t| < \nu$ we have

$$|\beta_j'(t)| = |(t^2/\nu)'| \le 2,$$

while if $|t| > \nu$ then $\beta_j(t) = |t|$, so $|\beta_j'(t)| = 1$. So, the Lipschitz constant of β_j is 2 and $\beta_j(0) = 0$. Applying Theorem 3.5 we obtain

$$\begin{aligned}\nu \mathbb{E}_\varepsilon \sup_{y\in K} \sum_{j=1}^n \varepsilon_j \cdot \beta_j(y_j) &\le 2\nu \cdot \mathbb{E} \sup_{y\in K} \sum_{j=1}^n \varepsilon_j \cdot y_j \\ &= 2\nu \cdot \rho(K^\circ) = C\delta,\end{aligned}$$

which completes the proof of Theorem 3.2. □

4. Upper estimate.

Let $K, D \subset \mathbb{R}^n$ be convex symmetric bodies. We shall show that

$$\delta_k(K, D) \le C \max\left(\frac{k^2}{n},\ \sqrt{k}\log d_K,\ \sqrt{k}\log d_D\right).$$

Since the distance between an n-dimensional convex symmetric body and a ball does not exceed $\sqrt{n}$, this estimate implies Theorem 1.2. Notice that we can assume that $k < n/8$, since otherwise the estimate above is trivial. To prove this estimate we apply Theorem 2.4 to find projections K_1 of K and D_1 of D which contain an octahedron and a ball of certain sizes inside. The next step depends upon the values of $\ell(K_1^\circ)$ and $\ell(D_1^\circ)$. If $\ell(K_1^\circ)$ and $\ell(D_1^\circ)$ are small, we use Theorem 3.2 to find coordinate projections of K_1 and D_1 which are contained in a convex hull of an octahedron and a ball. Then the distance between these projections can be bounded in terms of the sizes of the inscribed and superscribed octahedra and balls. However, if the product of $\ell(K_1^\circ)$ and $\ell(D_1^\circ)$ is large, this method does not provide the necessary estimate for $\delta_k(K, D)$. In this case instead of a projection on a random coordinate subspace we consider a projection on a random subspace uniformly distributed over the Grassmanian.

Let $l = n/8$. By our assumption $k < l$. Let $K_1 = QK$, where $Q : R^n \to \mathbb{R}^l$ is the linear operator constructed in Theorem 2.4. Then

$$\operatorname{conv}\left(cB_1^l,\ \frac{c \cdot \ell(K_1^\circ)}{\sqrt{n}\log d_K} B_2^l\right) \subset K_1 \subset B_2^l$$

and

$$\ell(K_1) \cdot \ell((K_1)^\circ) \leq Cn \log d_K.$$

Recall that $\rho(K_1^\circ) \leq C\ell(K_1^\circ)$. Let $\delta = k/l$. Applying Theorem 3.2, we find a projection $K_2 = PK_1$ of dimension k such that

$$\operatorname{conv}\left(cB_1^k,\ a(K_1)B_2^k\right) \subset K_2 \subset C\left(\ell(K_1^\circ) \cdot B_1^k + \sqrt{\frac{k}{l}} \cdot B_2^k\right),$$

where

$$a(K_1) = \max\left(\frac{\ell(K_1^\circ)}{\sqrt{n}\log d_K},\ \frac{c}{\sqrt{k}}\right).$$

The estimate for the radius of the inscribed ball follows from an elementary observation

$$\frac{1}{\sqrt{k}} B_2^k \subset B_1^k.$$

Similarly, we can define the bodies D_1 and D_2 so that

$$\operatorname{conv}\left(cB_1^l,\ \frac{c \cdot \ell(D_1^\circ)}{\sqrt{n}\log d_D} B_2^l\right) \subset D_1 \subset B_2^l$$

and

$$\operatorname{conv}\left(cB_1^k,\ a(D_1)B_2^k\right) \subset D_2 \subset C\left(\ell(D_1^\circ) \cdot B_1^k + \sqrt{\frac{k}{l}} \cdot B_2^k\right),$$

where

$$a(D_1) = \max\left(\frac{\ell(D_1^\circ)}{\sqrt{n}\log d_D},\ \frac{c}{\sqrt{k}}\right).$$

To cover the body K_2 by a homothetic copy of the body D_2 it is enough to cover the octahedron $\ell(K_1^\circ) \cdot B_1^k$ by a copy of the octahedron contained in D_2 and the ball $\sqrt{k/l} \cdot B_2^k$ by a copy of a ball contained in D_2. This means that

$$K_2 \subset \max\left(\sqrt{\frac{k}{l}} a^{-1}(D_1),\ c \cdot \ell(K_1^\circ)\right) D_2. \tag{4.1}$$

Similarly,

$$D_2 \subset \max\left(\sqrt{\frac{k}{l}} a^{-1}(K_1),\ c \cdot \ell(D_1^\circ)\right) K_2. \tag{4.2}$$

Assume that $\ell(K_1^\circ) \le \ell(D_1^\circ)$.

We shall consider three cases.

Case 1. *Assume that* $\ell(D_1^\circ) \le k/\sqrt{l}$.

Since $a(K_1) \ge \frac{c}{\sqrt{k}}$, (4.2) implies that

$$D_2 \subset \max\left(\sqrt{\frac{k}{l}} \cdot \frac{\sqrt{k}}{c},\ c \cdot \ell(D_1^\circ)\right) K_2 = C\frac{k}{\sqrt{l}} K_2.$$

Similarly, (4.1) reads

$$K_2 \subset C\frac{k}{\sqrt{l}} D_2,$$

so

$$d(K_2, D_2) \le C\frac{k^2}{l} \le C'\frac{k^2}{n}.$$

Case 2. *Assume that* $k/\sqrt{l} < \ell(D_1^\circ)$ *and* $\ell(K_1^\circ) \cdot \ell(D_1^\circ) \le \sqrt{k} \log d_D$.

Since

$$a^{-1}(K_1) \le C\sqrt{k}, \quad \text{and } a^{-1}(D_1) \le C\frac{\sqrt{n}\log d_D}{\ell(D_1^\circ)},$$

the inclusions (4.1) and (4.2) become

$$K_2 \subset \max\left(C\sqrt{k} \cdot \frac{\log d_D}{\ell(D_1^\circ)},\ c \cdot \ell(K_1^\circ)\right) D_2$$

$$D_2 \subset \max\left(C\frac{k}{\sqrt{l}},\ c \cdot \ell(D_1^\circ)\right) K_2 = c \cdot \ell(D_1^\circ) \cdot K_2.$$

Thus we get that

$$d(K_2, D_2) \le \max\left(C\sqrt{k}\log d_D, C\ell(K_1^\circ) \cdot \ell(D_1^\circ)\right). \tag{4.3}$$

Since $\ell(K_1^\circ) \cdot \ell(D_1^\circ) \le \sqrt{k}\log d_D$, we are done.

Case 3. *Assume that*

$$\ell(K_1^\circ) \cdot \ell(D_1^\circ) \geq \sqrt{k} \log d_D. \tag{4.4}$$

Then instead of random coordinate projections we consider orthogonal projections onto random k-dimensional subspaces uniformly distributed over the Grassmannian. Let $D_3 = PD_1$ be a random projection of D_1. If $\ell(D_1^\circ) \geq C_0\sqrt{k}$, then by Lemma 3.1 (i)

$$d(D_3, B_2^k) \leq 2,$$

so any k-dimensional projection of K_1 satisfies

$$d(D_3, P'K_1) \leq 2\sqrt{k}.$$

Assume now that $\ell(D_1^\circ) \leq C_0\sqrt{k}$. Let $K_3 = P'K_1$ be a random projection of K_1 of rank k. Since $\ell(K_1^\circ) \leq \ell(D_1^\circ)$, Lemma 3.1 (ii) implies that

$$\frac{c\ell(K_1^\circ)}{\sqrt{n}\log d_K} B_2^k \subset K_3 \subset c\sqrt{\frac{k}{l}} B_2^k,$$

so

$$d(K_3, B_2^k) \leq c\frac{\sqrt{k}}{\sqrt{l}} \cdot \frac{\sqrt{n}\log d_K}{\ell(K_1^\circ)} \leq C\frac{\sqrt{k}\log d_K}{\ell(K_1^\circ)}.$$

The same argument shows that

$$d(D_3, B_2^k) \leq C\frac{\sqrt{k}\log d_D}{\ell(D_1^\circ)}$$

and thus

$$d(K_3, D_3) \leq C\frac{k \log d_K \cdot \log d_D}{\ell(K_1^\circ) \cdot \ell(D_1^\circ)}. \tag{4.5}$$

Now the inequality (4.4) implies

$$d(K_3, D_3) \leq C\sqrt{k} \log d_K,$$

so the proof of Theorem 1.2 is complete. □

Remark 4.1. Let X_D be the normed space whose unit ball is D. Denote by $\mathcal{K}(X_D)$ the K-convexity constant of the space X_D (see [P], Ch. 2). The proof above yields a stronger statement:

$$\delta_k(K, D) \leq C \max\left(\frac{k^2}{n},\ \sqrt{k}\cdot\mathcal{K}(X_K),\ \sqrt{k}\cdot\mathcal{K}(X_D)\right).$$

Remark 4.2. The proof shows that there is no pair of bodies such that

$$\delta_k(K, D) \sim_{\log n} \begin{cases} \sqrt{k} & \text{if } k \leq n^{2/3} \\ \frac{k^2}{n} & \text{if } k > n^{2/3}. \end{cases}$$

for all $k < n$. Indeed, assume that such bodies K and D exist and let K_1, D_1 be as in the proof of the Theorem. Then (4.5) shows that $\ell(K_1^\circ) \cdot \ell(D_1^\circ) \preceq_{\log n} \sqrt{k}$ for all $k \leq n^{2/3}$. For small k it implies

$$\ell(K_1^\circ) \cdot \ell(D_1^\circ) \preceq_{\log n} 1. \tag{4.6}$$

Notice that since $cB_1^l \subset K_1$, $\ell(K_1^\circ) \geq c\sqrt{\log n}$. Similarly, $\ell(D_1^\circ) \geq c\sqrt{\log n}$. Combining this with (4.6), we show that there exist absolute constants C, a such that

$$\ell(K_1^\circ) \leq C \log^a n \quad \text{and } \ell(D_1^\circ) \leq C \log^a n.$$

However this means that the conditions of Case 1 are satisfied for any $k \geq \sqrt{l} \cdot C \log^a n = C' \sqrt{n} \log^a n$, so

$$d(K_2, D_2) \leq C \frac{k^2}{n}.$$

If $k = n^\beta$ for $1/2 < \beta < 2/3$, this is significantly less than $\sqrt{k}$.

Remark 4.3. The statement of Theorem 1.2 holds for non-symmetric convex bodies as well if one assumes that $k < (1-\varepsilon)n$ for some $\varepsilon > 0$. Indeed, the only part of the proof that has to be modified is that of Theorem 2.4. The main difference between the symmetric and the non-symmetric settings is that in the later the MM^*-estimate is unknown. However, for any n-dimensional convex body one can find a projection of dimension at least $n/2$, whose MM^* is bounded by $C \log^2 n$ (see [R]). Assume that $k \leq n/16$. Then using Theorem 1 [R] instead of Theorem 2.1, one can complete the proof of Theorem 2.4 with minimal modifications. If $n/16 < k \leq (1-\varepsilon)n$ then Theorem 1 and Theorem 5 [R] imply that $\delta_k(K, D) \leq C(\varepsilon) n \log^\alpha n \sim_{\log n} k^2/n$.

5. Linear mappings of Gluskin polytopes.

To prove Theorem 1.3 we need some results about Gluskin polytopes. These polytopes, introduced in a seminal paper of Gluskin [Gl1], were later used to provide extremal examples to many problems in asymptotic geometric analysis [Gl2], [Sz2], [Sz4] etc. Basic properties of Gluskin polytopes as well as some useful techniques can be found in the extensive survey of Mankiewicz and Tomczak-Jaegermann [M-TJ1].

Recall some definitions. Let $N > n$ and let $g_1, \dots, g_N$ be independent standard Gaussian vectors in $\mathbb{R}^n$. The Gluskin polytope is a random polytope are defined as

$$K = K(\omega) = \text{abs.conv}(\sqrt{n}e_1, \dots, \sqrt{n}e_n, g_1, \dots, g_N).$$

It is convenient to include a copy of the standard basis in the definition of these polytopes, although for some constructions it is not necessary [M-TJ1].

There exists a constant C such that $\|g_j\|_2 \le C\sqrt{n}$ with probability at least $1 - e^{-cn}$. This means that the set

$$\Omega_0 = \{\omega \mid B_2^n \subset K(\omega) \subset C\sqrt{n} B_2^n\}.$$

satisfies $\mathbb{P}(\Omega_0) \ge 1 - Ne^{-cn}$.

Let us introduce some notation. For a linear operator $T : \mathbb{R}^n \to \mathbb{R}^k$ denote by $s_1(T) \ge \cdots \ge s_k(T)$ the singular numbers of T. Let $\mathcal{P}_k$ be the set of all orthogonal projections of rank k equipped with the metric $r(P, P') = \|P - P'\|$.

To prove Theorem 1.3 we have to show that an operator which is nicely invertible on a subspace of a large dimension does not map one Gluskin polytope into another with high probability. More precisely, we prove the following technical result.

Theorem 5.1. *Let $N \ge Cn \log n$ and let $c \log n \le k \le n$. For any $Q \in \mathcal{P}_k$ denote*

$$\mathcal{A}_Q = \{T : \mathbb{R}^n \to Q\mathbb{R}^n \mid s_{k/2}(T) \ge 1\}. \tag{5.1}$$

Let $D = K(\omega')$ for some $\omega' \in \Omega_0$. Then there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(\omega \in \Omega_0 \mid \exists Q \in \mathcal{P}_k \ \exists T \in \mathcal{A}_Q \ TK(\omega) \subset \frac{c_1 k}{\sqrt{n} \cdot \sqrt{\log(N/n)}} QD\right)$$
$$\le \exp(-c_2 N k).$$

Proof. The proof of the Theorem consists of three steps. Set

$$\lambda_0 = \frac{c_1 k}{\sqrt{n}\sqrt{\log N/n}}.$$

First we show that that for a *fixed* $Q \in \mathcal{P}_k$ and a *fixed* $T \in \mathcal{A}_Q$ the probability that $TK(\omega) \subset \lambda_0 QD$ is exponentially small. Then we construct a t-net $\mathcal{M}$ in $\mathcal{P}_k$ of small cardinality. For each $Q \in \mathcal{M}$ we build a τ-net $\mathcal{N}_Q$ in some subset of the set $\mathcal{A}_Q$, whose cardinality is small as well. From the probabilistic estimate of Step 1 it will follow that the probability that there exists a $Q \in \mathcal{M}$ and an operator $T \in \mathcal{N}_Q$ such that $TK(\omega) \subset \lambda_0 QD$ remains exponentially small. Finally, we use approximation to show that if for some $Q \in \mathcal{P}_k$ and some $T \in \mathcal{A}_Q$ $TK(\omega) \subset \lambda_0/4\, QD$, then there exist a projection $Q_0 \in \mathcal{M}$ and an operator $T \in \mathcal{N}_{Q_0}$ for which $T_0 K(\omega) \subset \lambda_0 Q_0 D$.

Step 1. *Individual estimate.*

Let Q be a fixed orthogonal projection of rank k and let $T \in \mathcal{A}_Q$ be a fixed linear operator. We have to prove that for some constant $c_1 > 0$ appearing in the definition of λ_0

$$\mathbb{P}\left(\omega \in \Omega_0 \mid TK(\omega) \subset \lambda_0 QD\right) \le e^{-Nk}.$$

We use the following Lemma, which is similar to Lemma 4 [M-TJ1].

Lemma 5.2. *Let $V \subset \mathbb{R}^k$ be a convex set and let $T : \mathbb{R}^n \to \mathbb{R}^k$ be an operator such that $s_{k/2}(T) \ge 1$. Then there exists a subspace F of dimension $k/2$ such that*

$$\mathbb{P}\left(\omega \mid TK(\omega) \subset \lambda V\right) \le \left((\sqrt{2\pi} \cdot \lambda)^{k/2} \cdot \mathrm{vol}(P_F V)\right)^N.$$

Here N is the number of vertices of $K(\omega)$.

Proof. We present the proof of the Lemma for the sake of completeness. Let

$$T = \sum_{l=1}^{k} s_l x_l \otimes y_l$$

be the polar decomposition of T. Here $s_1 \ge \ldots \ge s_k$ are singular numbers of T. Denote $F = \mathrm{span}(x_1, \ldots, x_{k/2})$ and $E = \mathrm{span}(y_1, \ldots, y_{k/2})$. Let g be the standard Gaussian vector in $\mathbb{R}^{k/2}$. Replacing the Gaussian density by its maximal value, one obtains

$$\mathbb{P}\left(g \in L\right) \le (2\pi)^{-k/4} \mathrm{vol}(L)$$

for any body $L \subset \mathbb{R}^{k/2}$ [M-TJ1]. Since $\|(T|_E)^{-1}\| \le 1$, we get

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega_0 \mid TK(\omega) \subset \lambda V\right) \le \mathbb{P}\left(\omega \in \Omega_0 \mid P_F TK(\omega) \subset \lambda P_F V\right) \\
&\le \prod_{j=1}^{N} \mathbb{P}\left(P_F T g_j(\omega) \in \lambda P_F V\right) = \prod_{j=1}^{N} \mathbb{P}\left(T P_E g_j(\omega) \in \lambda P_F V\right) \\
&= \prod_{j=1}^{N} \mathbb{P}\left(P_E g_j(\omega) \in \lambda (T|_E)^{-1} P_F V\right) \le \prod_{j=1}^{N} (2\pi)^{-k/4} \mathrm{vol}(\lambda (T|_E)^{-1} P_F V) \\
&\le (2\pi)^{-Nk/4} \lambda^{Nk/2} \left(\mathrm{vol}(P_F V)\right)^N.
\end{aligned}$$

□

We apply Lemma 5.2 to the set $V = QD$. Let F be the subspace from Lemma 5.2 and let P_F be the orthogonal projection onto F. To estimate the volume of $P_F QD$ we use the following Lemma, which was proved independently by Carl and Pajor [CP] and Gluskin [Gl3].

Lemma 5.3. *Let $x_1, \dots, x_M$ be vectors in $\mathbb{R}^m$ of length at most 1. Then*

$$\text{vol}(\text{abs.conv}(x_1, \dots, x_M)) \le \left(C \frac{\sqrt{\log(2 + M/m)}}{m} \right)^m .$$

Since $\omega' \in \Omega_0$, the set $P_F QD$ is the absolute convex hull of $N + n$ vectors in $\mathbb{R}^{k/2}$ whose norms do not exceed $C\sqrt{n}$. By Lemma 5.3

$$\text{vol}(P_F QD) \le \left(C \frac{\sqrt{n}\sqrt{\log(3 + N/n)}}{k/2} \right)^{k/2} .$$

Combining this with Lemma 5.2, we obtain

$$\mathbb{P}\left(\omega \in \Omega_0 \mid TK(\omega) \subset \lambda_0 QD \right) \le \left(C' \lambda_0 \frac{\sqrt{n}\sqrt{\log N/n}}{k/2} \right)^{Nk/2} .$$

From the definition of λ_0 it follows that the expression above is equal to $(C' \cdot c_1)^{Nk}$. Then with an appropriate choice of c_1 we get

$$\mathbb{P}\left(\omega \in \Omega_0 \mid TK(\omega) \subset \lambda_0 QD \right) \le e^{-Nk}.$$

Step 2. *Construction of ε-nets.*

We need two results on ε-nets. The first one was proved by Szarek [Sz3].

Lemma 5.4. *Let $0 < t < 1$. Then the set $\mathcal{P}_k$ admits a t-net $\mathcal{M}$ of cardinality at most*

$$|\mathcal{M}| \le \left(\frac{C}{t} \right)^{nk} .$$

The second lemma deals with the nets in the set of operators which map an octahedron into a given convex body. The proof of Step 1 uses only the fact that the Gaussian vectors $g_1, \dots, g_N$ belong to $K(\omega)$. In the rest of the proof we are going to use also that $\sqrt{n} e_j \in K(\omega)$ for $j = 1, \dots, n$, so $TK(\omega) \subset D$ implies $T(\sqrt{n} B_1^n) \subset D$.

Lemma 5.5. *Let $k \ge c \log n$. Let $Q \in \mathcal{P}_k$ and let $B \subset Q\mathbb{R}^n$ be a convex symmetric body. Let $\tau > 0$ and assume that $\tau\sqrt{n} Q B_2^n \subset B$. Define*

$$\mathcal{A}(B) = \{ T : \mathbb{R}^n \to Q\mathbb{R}^n \mid T(\sqrt{n} B_1^n) \subset B \}.$$

Then any subset $\mathcal{A}'$ of $\mathcal{A}(B)$ admits a τ-net $\mathcal{N}$ in the operator norm of cardinality at most

$$|\mathcal{N}| \le \left(\frac{C\sqrt{k}}{\tau\sqrt{n}} \right)^{nk} \cdot \big(\text{vol}(B) \big)^n .$$

Proof. The matrix of any operator $T : \mathbb{R}^n \to Q\mathbb{R}^n$ can be considered as a vector in $\mathbb{R}^{nk}$ equipped with the standard Euclidean structure. Let

$$\mathcal{E} = \{T : \mathbb{R}^n \to Q\mathbb{R}^n \mid TB_2^n \subset QB_2^n\} \subset \mathbb{R}^{nk}.$$

Since $\tau\sqrt{n}QB_2^n \subset B$, $\tau\mathcal{E} \subset \mathcal{A}(B)$. Now a standard volumetric estimate yields that any subset $\mathcal{A}'$ of $\mathcal{A}(B)$ admits a τ-net of cardinality

$$|\mathcal{N}| \le 3^{nk} \cdot \left(\frac{\operatorname{vol}(\mathcal{A}(B))}{\operatorname{vol}(\tau\mathcal{E})} \right). \tag{5.2}$$

Notice that

$$\mathcal{A}(B) = \{T : \mathbb{R}^n \to Q\mathbb{R}^n \mid Te_j \in \frac{1}{\sqrt{n}} B \text{ for } j = 1, \dots, n\},$$

so

$$\operatorname{vol}(\mathcal{A}(B)) = (\operatorname{vol}(B/\sqrt{n}))^n.$$

It remains to estimate the volume of $\mathcal{E}$. Let $G(\omega) : \mathbb{R}^n \to \mathbb{R}^k$ be the standard $n \times k$ Gaussian matrix. Recall that $k \ge c \log n$. Then for some constant c

$$\mathbb{P}\left\{\omega \mid \left\|G(\omega) : \ell_2^n \to \ell_2^k\right\| \le c\sqrt{k}\right\} \ge 1/2.$$

This follows from the fact that $\|g\|_{\ell_2^k} \le c\sqrt{k}$ for the standard gaussian vector $g \in \mathbb{R}^k$ with probability exponentially close to 1 (see [M-TJ1] for details).

Replacing the Gaussian density by its maximal value, we get

$$\operatorname{vol}(\mathcal{E}) \ge \left(\frac{2\pi}{c^2 k} \right)^{nk/2} \cdot \mathbb{P}\left\{\omega \mid G(\omega) \in c\sqrt{k}\mathcal{E}\right\} \ge \left(\frac{c'}{k} \right)^{nk/2}.$$

Substituting this estimate into (5.2) completes the proof of Lemma 5.5. □

Now we pass to the construction of nets. Let $t = \alpha/n$, where $\alpha < 1$ is a constant to be chosen later. By Lemma 5.4 we can construct a t-net $\mathcal{M} \subset \mathcal{P}_k$ of cardinality at most $(C/t)^{nk}$. For any $Q \in \mathcal{M}$ set $B = \lambda_0/4 \cdot QD$. As before, Lemma 5.3 yields

$$\operatorname{vol}(B) \le \left(\frac{\lambda_0}{4} \cdot C \frac{\sqrt{n}\sqrt{\log N/n}}{k} \right)^k.$$

Let $\tau = \beta\lambda_0/\sqrt{n}$, where the constant β will be chosen later. For $\beta < 1/4$ we have $\tau\sqrt{n}QB_2^n \subset B$. Let

$$\mathcal{A}'_Q = \mathcal{A}_Q \cap \mathcal{A}(B),$$

where $\mathcal{A}_Q$ is defined in (5.1). Then Lemma 5.5 implies that there exists a τ-net $\mathcal{N}_Q \subset \mathcal{A}'_Q$ whose cardinality does not exceed

$$\left(C \frac{\sqrt{n}\sqrt{\log N}}{\beta \sqrt{k}} \right)^{nk}.$$

From the Step 1 it follows that the set

$$\Omega' = \{\omega \in \Omega_0 \mid \exists Q \in \mathcal{M} \ \exists T \in \mathcal{N}_Q \ TK(\omega) \subset \lambda_0 QD\}$$

has the probability

$$\begin{aligned} \mathbb{P}(\Omega') &\le |\mathcal{M}| \cdot \sup_{Q \in \mathcal{M}} |\mathcal{N}_Q| \cdot \exp(-Nk) \\ &\le \left(\frac{Cn}{\alpha} \right)^{nk} \cdot \left(\frac{\sqrt{n}\sqrt{\log N}}{\beta \sqrt{k}} \right)^{nk} \cdot e^{-Nk} \\ &\le \left(\frac{Cn^{3/2}\sqrt{\log N}}{\alpha\beta} \right)^{nk} \cdot e^{-Nk}. \end{aligned}$$

Since $N \ge Cn \log n$, the last expression does not exceed $e^{-c'Nk}$.

Step 3. *Approximation.*

Choose now $\omega \in \Omega_0 \setminus \Omega'$. Assume that there exist $Q \in \mathcal{P}_k$ and $T \in \mathcal{A}_Q$ such that

$$TK(\omega) \subset \lambda_0/8 \cdot QD. \tag{5.3}$$

We shall show that in this case there exist $Q_0 \in \mathcal{M}$ and $T_0 \in \mathcal{N}_{Q_0}$ such that $T_0 K(\omega) \subset \lambda_0 \cdot Q_0 D$, which contradicts the choice of ω.

Notice that $B_2^n \subset \sqrt{n} B_1^n \subset K(\omega)$ and $D \subset C\sqrt{n} B_2^n$, so the assumption (5.3) implies

$$\|T\| \le C\sqrt{n}\lambda_0/8 \le C \frac{c_1 k}{8\sqrt{\log N/n}} \le C'n.$$

Let $Q_0 \in \mathcal{M}$ be such that

$$\|Q - Q_0\| \le \alpha/n.$$

Then using $B_2^n \subset D \subset C\sqrt{n} B_2^n$, we obtain that

$$\begin{aligned} Q_0 Q D &\subset Q_0 D + Q_0(Q - Q_0) D \subset Q_0 D + C\sqrt{n} \cdot Q_0 (Q - Q_0) B_2^n \\ &\subset Q_0 D + \frac{\alpha}{n} \cdot C\sqrt{n} \cdot Q_0 B_2^n \subset 2 Q_0 D. \end{aligned}$$

Hence by (5.3)

$$Q_0 T K(\omega) \subset \lambda_0/4 \cdot Q_0 D,$$

which means that $Q_0 T \in \mathcal{A}(B)$, where as in Step 2, $B = \lambda_0/4 \cdot Q_0 D$. Also,

$$s_{k/2}(Q_0 T) \ge s_{k/2}(T) - \|(id - Q_0) T\|$$

and

$$\begin{aligned}\|(id - Q_0)T\| &= \|(id - Q_0)QT\| \le \|(id - Q_0)Q\| \cdot \|T\| \\ &\le \|Q - Q_0\| \cdot C'n \le C'\alpha.\end{aligned}$$

Here we used the fact that since Q is an orthogonal projection,

$$\|(id - Q_0)Q\| = \|(Q - Q_0)Q\| \le \|Q - Q_0\|.$$

Choose $\alpha < 1/2C'$. Then

$$s_{k/2}(Q_0T) \ge 1 - C'\alpha \ge 1/2,$$

hence $2Q_0T \in \mathcal{A}'_{Q_0}$. Recall that $\mathcal{N}_{Q_0}$ is a $\tau = \beta\lambda_0/\sqrt{n}$-net for $\mathcal{A}'_{Q_0}$, so we can choose $T_0 \in \mathcal{N}_{Q_0}$ for which

$$\|2Q_0T - T_0\| \le \beta\lambda_0/\sqrt{n}.$$

Since $K \subset C\sqrt{n}B_2^n$ and $B_2^n \subset D$, we have

$$\begin{aligned}T_0K &\subset 2Q_0TK + (2Q_0T - T_0)K \subset \lambda_0/2 \cdot Q_0D + C\sqrt{n}(2Q_0T - T_0)Q_0B_2^n \\ &\subset \lambda_0/2 \cdot Q_0D + \beta C \cdot \lambda_0 Q_0 B_2^n \subset \lambda_0 Q_0 D,\end{aligned}$$

if β is chosen so that $\beta C < 1/2$. This means that $\omega \in \Omega'$.

We proved that if $\omega \in \Omega_0 \setminus \Omega'$ then (5.3) cannot hold for any $Q \in \mathcal{P}_k$ and $T \in \mathcal{A}_Q$. The proof of Theorem 5.1 is complete. □

6. Lower estimate.

In this section we present the proof of Theorem 1.3. Actually, we shall prove a stronger result. Remark 4.2 shows that a pair of bodies K and D such that $\delta_k(K, D) \succ_{\log n} \Delta(k, n)$ for all $k < n$ does not exist. However, we show below that there exist *three* bodies K, D_1, D_2 such that for any $k < n$, $\max(\delta_k(K, D_1), \delta_k(K, D_2)) \succ_{\log n} \Delta(k, n)$. More precisely, we prove the following result, which implies Theorem 1.3.

Theorem 6.1. *There exist n-dimensional convex symmetric bodies K, D such that for any $k < n$*

(i):

$$\delta_k(K, D) \ge \frac{ck^2}{n \log\log n};$$

(ii):

$$\delta_k(K, B_2^n) \ge c'\sqrt{\frac{k}{\log(1 + n/k)}}.$$

Proof. Notice that in the estimate (i) it is enough to consider $k \geq \sqrt{n}$. We shall show that some Gluskin polytopes K, D satisfy (i) and (ii).

Choose $N = C'n \log n$, where $C' > C$ from Theorem 5.1. Let $\sqrt{n} \leq k \leq n$. Denote

$$\Omega'_k = \left(\omega \in \Omega_0 \mid \exists Q \in \mathcal{P}_k \ \exists T \in \mathcal{A}_Q \ TK(\omega) \subset \frac{c_1 k}{\sqrt{n} \cdot \sqrt{\log \log n}} QD \right).$$

Then by Theorem 5.1

$$\mathbb{P} \left(\bigcup_{k=\sqrt{n}}^{n} \Omega'_k \right) \leq \sum_{k=\sqrt{n}}^{n} \exp(-c_2 \cdot C'n \log n \cdot k) \leq 2 \exp(-c' n^{3/2} \log n).$$

Choose $\omega, \omega' \in \Omega_0 \setminus \cup_{k=\sqrt{n}}^{n} \Omega'_k$ and set $K = K(\omega), D = K(\omega')$.

Fix a $k \leq n$ and let $P, Q \in \mathcal{P}_k$. Denote $E = P\mathbb{R}^n$, $F = Q\mathbb{R}^n$ and let $T : E \to F$ be an invertible linear operator. We have to estimate $\|T : PK \to QD\| \cdot \|T^{-1} : QD \to PK\|$ from below. After an appropriate renorming we may assume that $s_{k/2}(T) \geq 1$ and $s_{k/2}(T^{-1}) \geq 1$. Then $s_{k/2}(TP) \geq 1$ and since $\omega \notin \Omega'_k$,

$$\|T : PK \to QD\| \geq \frac{c_1 k}{\sqrt{n} \cdot \sqrt{\log \log n}}.$$

Similarly, $s_{k/2}(T^{-1}Q) \geq 1$, so

$$\left\|T^{-1} : QD \to PK\right\| \geq \frac{c_1 k}{\sqrt{n} \cdot \sqrt{\log \log n}}.$$

Combining these two estimates we obtain (i).

Now assume that $k < n^{2/3}$. Notice that K is the image of B_1^n under a certain linear mapping $V : \mathbb{R}^{N+n} \to \mathbb{R}^n$. Thus the estimate (1.1) implies that

$$\begin{aligned} \delta_k(K, B_2^n) &= d_k(K^\circ, B_2^n) \geq d_k(B_\infty^{N+n}, B_2^{N+n}) \\ &\geq c \sqrt{\frac{k}{\log\left(1 + \frac{N+n}{k}\right)}} \geq c' \sqrt{\frac{k}{\log n/k}}. \end{aligned}$$

□

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