Supplementary Materials of "Information Ratio Test for Model Misspecification on Parametric Structures in Stochastic Diffusion Models"

This document is prepared to provide the detailed and complete proof of the lemma in section 2.2. The proof of the lemma is sketched in the appendix due to the space limitations. The main theorem follows immediately from the lemma.

Proposition 1. *If conditions (a) and (b) in the appendix are satisfied, then*

$$\hat{S}(\hat{\theta}_n, \tilde{\gamma}) \xrightarrow{pr} S(\theta_0, \gamma_0), \;\; n \to \infty,$$
$$\hat{V}(\hat{\theta}_n, \tilde{\gamma}) \xrightarrow{pr} V(\theta_0, \gamma_0), \;\; n \to \infty.$$

Proof of Proposition 1. Rewrite $\hat{S}(\theta, \gamma)$ and $\hat{V}(\theta, \gamma)$ as follows:

$$\hat{S}(\theta, \gamma) \triangleq \frac{1}{n} \sum_{k=1}^{n} g_s(X_{(k-1)\Delta}, X_{k\Delta}; \theta, \gamma),$$
$$\hat{V}(\theta, \gamma) \triangleq \frac{1}{n} \sum_{k=1}^{n} g_v(X_{(k-1)\Delta}, X_{k\Delta}; \theta, \gamma).$$

By conditions (a) and (b), we have $\hat{\theta} \xrightarrow{pr} \theta_0$ and $\tilde{\gamma} \xrightarrow{pr} \gamma_0$. Applying the uniform law of large number (Theorem 4.1 in Wooldridge (1994)), the conclusions of proposition 1 are proved. ■

Furthermore, note that under the null hypothesis of correct model specification, $S(\theta_0, \gamma_0) = V(\theta_0, \gamma_0)$. Thus, by condition (d) and Slutsky's theorem, we have

$$tr\left(S^{-1}(\hat{\theta}, \tilde{\gamma}) V(\hat{\theta}, \tilde{\gamma})\right) \xrightarrow{pr} p, \;\; \text{as} \;\; n \to \infty.$$

Now, we provide a detailed and complete proof the lemma.

Proof of Lemma . In this proof, we apply the uniform law of large number theorem multiple times, and readers are refer to Theorem 4.1 of Wooldridge (1994) for the detail.

We consider the case that the null hypothesis H_0 is true. Given that $\hat{S}(\hat{\theta}, \tilde{\gamma})$ converges to the positive definite sensitivity matrix $S(\theta_0, \gamma_0)$ in probability, we can obtain the following exact expansion:

$$\begin{aligned}
\hat{S}^{-1}(\hat{\theta}, \tilde{\gamma}) &= S^{-1}(\theta_0, \gamma_0) + \hat{S}^{-1}(\hat{\theta}, \tilde{\gamma}) - S^{-1}(\theta_0, \gamma_0) \\
&= S^{-1}(\theta_0, \gamma_0) + S^{-1}(\theta_0, \gamma_0)\left(S(\theta_0, \gamma_0) - \hat{S}(\hat{\theta}, \tilde{\gamma})\right)\hat{S}^{-1}(\hat{\theta}, \tilde{\gamma}) \\
&= S^{-1}(\theta_0, \gamma_0) + S^{-1}(\theta_0, \gamma_0)\left(S(\theta_0, \gamma_0) - \hat{S}(\hat{\theta}, \tilde{\gamma})\right)S^{-1}(\theta_0, \gamma_0) \\
&\quad + \{S^{-1}(\theta_0, \gamma_0)(S(\theta_0, \gamma_0) - \hat{S}(\hat{\theta}, \tilde{\gamma}))\}^2 \hat{S}^{-1}(\hat{\theta}, \tilde{\gamma})
\end{aligned}$$

It follows that the information ratio statistic R_n can be represented as follows:

$$\begin{aligned}
\sqrt{n}\{R_n - p\} &= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat{\theta}, \tilde{\gamma})\hat{V}(\hat{\theta}, \tilde{\gamma}) - I_p\right\} \\
&= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat{\theta}, \tilde{\gamma})\hat{V}(\hat{\theta}, \tilde{\gamma}) - S^{-1}(\theta_0, \gamma_0)V(\theta_0, \gamma_0)\right\} \\
&= \mathrm{tr}\left\{S^{-1}(\theta_0, \gamma_0)\sqrt{n}\left(\hat{V}(\hat{\theta}, \tilde{\gamma}) - V(\theta_0, \gamma_0)\right)\right\} \\
&\quad + \mathrm{tr}\left\{S^{-1}(\theta_0, \gamma_0)\hat{V}(\hat{\theta}, \tilde{\gamma})S^{-1}(\theta_0, \gamma_0)\sqrt{n}\left(S(\theta_0, \gamma_0) - \hat{S}(\hat{\theta}, \tilde{\gamma})\right)\right\} \\
&\quad + \mathrm{tr}\left\{\hat{S}^{-1}(\hat{\theta}, \tilde{\gamma})\hat{V}(\hat{\theta}, \tilde{\gamma})S^{-2}(\theta_0, \gamma_0)\sqrt{n}\left(S(\theta_0, \gamma_0) - \hat{S}(\hat{\theta}, \tilde{\gamma})\right)^2\right\}.
\end{aligned}$$

For $1 \leq i, j \leq p$, expanding the (i, j)-the element of $\hat{S}(\hat{\theta}, \tilde{\gamma})$ around (θ_0, γ_0), we obtain

$$\begin{aligned}
&\sqrt{n}\left(\hat{S}(\hat{\theta}, \tilde{\gamma})_{ij} - S(\theta_0, \gamma_0)_{ij}\right) \\
= \ &\sqrt{n}(\hat{S}(\theta_0, \gamma_0)_{ij} - S(\theta_0, \gamma_0)_{ij}) + \frac{\partial}{\partial \theta^T}\hat{S}(\theta_0, \tilde{\gamma})_{ij}\sqrt{n}(\hat{\theta} - \theta_0) + \frac{\partial}{\partial \gamma^T}\hat{S}(\theta_0, \gamma_0)_{ij}\sqrt{n}(\tilde{\gamma} - \gamma_0) + o_p(1)
\end{aligned}$$

By conditions (a)-(c) and the uniform law of large number theorem (Theorem 4.1 in Wooldridge (1994)), there exist matrices $M_{s,\theta}^{ij}$ and $M_{s,\gamma}^{ij}$ such that

$$\begin{aligned}
&\frac{\partial}{\partial \theta^T}\hat{S}(\theta_0, \tilde{\gamma})_{ij} \xrightarrow{pr} M_{s,\theta}^{ij}, \\
&\frac{\partial}{\partial \theta^T}\hat{S}(\theta_0, \gamma_0)_{ij} \xrightarrow{pr} M_{s,\gamma}^{ij}.
\end{aligned}$$

In the meanwhile, expanding $G_n(\hat{\theta}, \tilde{\gamma})$ around (θ_0^T, γ_0^T), we have

$$G_n(\hat{\theta}, \tilde{\gamma}) = G_n(\theta_0, \gamma_0) + G_{n,\gamma}(\check{\theta}, \check{\gamma})(\tilde{\gamma} - \gamma_0) + G_{n,\theta}(\check{\theta}, \tilde{\gamma})(\hat{\theta} - \theta_0),$$

where $\check{\theta}$ lies between θ_0 and $\hat{\theta}$ and $\check{\gamma}$ lies between γ_0 and $\tilde{\gamma}$. Here $G_{n,\theta}(\cdot)$ and $G_{n,\gamma}(\cdot)$ are the first-order partial derivatives of $G_n(\cdot)$ with respect to θ and γ, respectively. By conditions (a) and (b), applying the uniform law of large number theorem, we have $G_{n,\theta}(\check{\theta},\tilde{\gamma})$ and $G_{n,\gamma}(\check{\theta},\check{\gamma})$ converge to some constant matrices, denoted by $K_{g,1}$ and $K_{g,2}$, respectively.

Now, denote martingale estimating equation (5) by $H_n(\gamma) = \frac{1}{n}\sum_{k=1}^{n} h(X_{(k-1)\Delta}, X_{k\Delta};\gamma)$. Because $\tilde{\gamma}$ is the root of equation (5), the mean-value theorem implies,

$$H_n(\tilde{\gamma}) = H_n(\gamma_0) + H_{n,\gamma}(\check{\gamma})(\tilde{\gamma}-\gamma_0),$$

where $\check{\gamma}$ lies between γ_0 and $\tilde{\gamma}$ and $H_{n,\gamma}(\cdot)$ is the first-order derivative of H_n w.r.t.γ. Under conditions (a) and (b), applying the uniform law of large number, there exist a matrix K_h, such that $H_{n,\gamma}(\check{\gamma}) \stackrel{pr}{\to} K_h$. Thus, we have

$$\begin{aligned}\sqrt{n}\,(\tilde{\gamma}-\gamma_0) &= -K_h^{-1}\frac{1}{\sqrt{n}}\sum_{k=1}^{n} h(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1) \\ &\triangleq \frac{1}{\sqrt{n}}\sum_{k=1}^{n} f_\gamma(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1).\end{aligned}$$

Using similar arguments to those given above, we obtain

$$\begin{aligned}\sqrt{n}(\hat{\theta}-\theta_0) &= -G_{n,\theta}^{-1}(\check{\theta},\tilde{\gamma})\sqrt{n}G_n(\theta_0,\gamma_0) + G_{n,\theta}^{-1}(\check{\theta},\tilde{\gamma})G_{n,\gamma}(\check{\theta},\check{\gamma})H_{n,\gamma}^{-1}(\check{\gamma})\sqrt{n}H_n(\gamma_0) \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\Big(-K_{g,1}^{-1}g(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + K_{g,1}^{-1}K_{g,2}K_h h(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0)\Big) + o_p(1) \\ &\triangleq \frac{1}{\sqrt{n}}\sum_{k=1}^{n} f_\theta(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sqrt{n}\left(\hat{S}(\hat{\theta},\tilde{\gamma})_{ij} - S(\theta_0,\gamma_0)_{ij}\right) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} g_s(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) - S(\theta_0,\gamma_0) \\ &\quad + \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_{s,\theta}^{ij} f_\theta(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) \\ &\quad + \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_{s,\gamma}^{ij} f_\gamma(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1) \\ &\triangleq \frac{1}{\sqrt{n}}\sum_{k=1}^{n} h_S^{ij}(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1),\end{aligned}$$

Again, using similar argument, we have

$$\begin{aligned}
& \sqrt{n}\left(\hat{V}(\hat{\theta},\tilde{\gamma})_{ij} - V(\theta_0,\gamma_0)_{ij}\right) \\
= & \sqrt{n}(\hat{V}(\theta_0,\gamma_0)_{ij} - V(\theta_0,\gamma_0)_{ij}) + \frac{\partial}{\partial\theta^T}\hat{V}(\theta_0,\tilde{\gamma})_{ij}\sqrt{n}(\hat{\theta}-\theta_0) + \frac{\partial}{\partial\gamma^T}\hat{V}(\theta_0,\gamma_0)_{ij}\sqrt{n}(\tilde{\gamma}-\gamma_0) + o_p(1) \\
= & \frac{1}{\sqrt{n}}\sum_{k=1}^{n} g_v(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) - V(\theta_0,\gamma_0) \\
& + \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_{v,\theta}^{ij} f_\theta(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) \\
& + \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_{v,\gamma}^{ij} f_\gamma(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1) \\
\triangleq & \frac{1}{\sqrt{n}}\sum_{k=1}^{n} h_V^{ij}(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + o_p(1),
\end{aligned}$$

where $M_{v,\theta}^{ij}$ and $M_{v,\gamma}^{ij}$ are some matrices such that

$$\begin{aligned}
& \frac{\partial}{\partial\theta^T}\hat{V}(\theta_0,\tilde{\gamma})_{ij} \xrightarrow{pr} M_{v,\theta}^{ij}, \\
& \frac{\partial}{\partial\theta^T}\hat{V}(\theta_0,\gamma_0)_{ij} \xrightarrow{pr} M_{v,\gamma}^{ij}.
\end{aligned}$$

By conditional (a)-(c), applying the uniform law of large number and the martingale central limit theorem in Billingsley (1961), we have $\sqrt{n}\left(\hat{S}(\hat{\theta},\tilde{\gamma}) - S(\theta_0,\gamma_0)\right)^2 = O_p(\frac{1}{\sqrt{n}})$. Therefore, we can reach the following expression:

$$\sqrt{n}\left\{R_n - p\right\} = \text{tr}\left\{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} h_{R_n}(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0)\right\} + o_p(1)$$

where $h_{R_n} = S^{-1}(\theta_0,\gamma_0)\left\{h_S(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0) + h_V(X_{(k-1)\Delta}, X_{k\Delta};\theta_0,\gamma_0)\right\}$, h_S and h_V are $p\times p$ matrices with element h_S^{ij} and h_V^{ij}, respectively, $i,j = 1,...,p$. Using the Billingsley (1961)'s martingale central limit theorem, we obtain

$$\sqrt{n}\left\{R_n - p\right\} \xrightarrow{L} N(0,\sigma_R^2).$$

where σ_R^2 is the asymptotic variance, which can be consistently estimated by

$$\hat{\sigma}_R^2 = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{p} h_{R_n}^i\right)^2$$

where $h_{R_n}^i$ is the i-th diagonal element of matrix h_{R_n}.

■

References

Billingsley, P. (1961). The Lindeberg-Lévy theorem for martingales. *Proc. Amer. Math. Soc*, 12:788–792.

Wooldridge, J. M. (1994). Estimation and inference for dependent processes. In Engle, R. and McFadden, D., editors, *Handbook of Econometrics (IV)*. Elsevier Science.