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Can a finite number of discrete delays approximate stochastic delay?

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ABSTRACT

In stability analysis and control design for a system with stochastic delay, it is a question whether one can approximate the stochastic system, for instance in the sense of average, with a deterministic system that has a finite number of discrete delay terms with the same delays that appear in the stochastic system and the weight coefficients of these delayed terms are taken from the probability distribution function of the stochastic delay. In this note, we consider a linear system with stochastic delay and discuss conditions under which this approximation is valid and conditions where it is not. In particular, we assume that the delay has piece-wise constant realizations with constant dwelling time at each value and show that the above mentioned approximation loses its grounds when the delay dwelling time gets larger than the minimum delay in the system.

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1. Introduction

Consider the linear system

$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau(t)), \tag{1}$$

where the delay $\tau(t)$ stochastically changes in a finite set $\Omega = \{\tau_1, \tau_2, \ldots, \tau_K\}$ while it dwells at each value a fixed amount of time $t_{\rm d}$. In particular, the changes in the delay occur at times $nt_{\rm d}$, $n = 0, 1, 2, \ldots$, while new delay values are chosen according to the probability distribution $\mathbb{P}\big(\tau(t) = \tau_k\big) = w_k$, $k = 1, \ldots, K$.

Consider also the deterministic system

$$\dot{x}(t) = a\,x(t) + b\sum_{k=1}^{K} w_k\,x(t-\tau_k) \tag{2}$$

which contains, in the right hand side, a finite number of discrete (point) delay terms where the delays are the same as in the set Ω and the weight coefficients of the discrete delay terms are the same as the probability distribution w_k, $k = 1, \ldots, K$.

In analyzing the stability of the stochastic system (1) in engineering applications, it is a question whether one could consider the deterministic system (2) as an approximation for the mean of the stochastic system (1). For instance in connected vehicle systems, stochastic delays arise due to the random packet loss in wireless communication between vehicles [1]. Similarly in networked control systems, the communication delays in wireless communication channels may change stochastically in time [2,3]. In this paper, we show that the approximation of the mean of the stochastic system (1) by the deterministic system (2) can be completely misleading.

In particular, we consider the linear system (1) where $a, b \in \mathbb{R}$ and assume that the delay $\tau(t)$ can only take two delay values τ_1 and τ_2 where $0 < \tau_1 < \tau_2$; a sample realization of the delay is shown in Fig. 1a. The delay dwells in one value for a duration of $t_{\rm d}$ (dwelling time) and then switches to a new value based on the probability distribution $\mathbb{P}(\tau = \tau_1) = w_1$, $\mathbb{P}(\tau = \tau_2) = w_2$, where $w_1 + w_2 = 1$. Using this simplistic behavior for the delay, we aim to show that the dwelling time $t_{\rm d}$ can have a substantial effect on the stability of the mean of the stochastic system (1) that cannot be captured by the corresponding deterministic system (2). The use of scalar versions of systems (1) and (2) and the assumption that the delay can assume only two values are for the sake of the brevity of the notation and clarity in conveying the message of the paper. The results of the paper hold for the general vector case (*i.e.* $x \in \mathbb{R}^q$ and $a, b \in \mathbb{R}^{q\times q}$ where q is the dimension of vector x) with arbitrary $K \in \mathbb{N}$ delays in the set Ω. Our approach is to use a suitable time-discretization of the two systems (1) and (2) and compare the stability of these systems through comparing the spectra of the matrices emerging from the time-discretization of the two systems.

2. Discretization and approximation of spectra

In this section, we obtain a time discretization of both systems (1) and (2) which we will later use to compare the stability of

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these systems. To this aim, we first present the method of the discretization of a deterministic system with a single fixed delay using the semi-discretization technique developed in [4]. The semi-discretization technique is a well-known time discretization technique for delay differential equations and widely-used in engineering applications [4–6].

2.1. Discretization of a system with a single fixed delay

Consider the following deterministic system with a single fixed delay

$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau). \tag{3}$$

By substituting the trial solution $x(t) = \kappa \mathrm{e}^{\lambda t}$ in Eq. (3), we may obtain the corresponding characteristic equation

$$\lambda - a - b\,\mathrm{e}^{-\lambda\tau} = 0 \tag{4}$$

where λ is a characteristic root. System (3) is stable if and only if all characteristic roots are located in the left-half complex plane.

Now consider the mesh $t_i = i\Delta t$, $i = 0, 1, 2, \ldots$. Let $m = \lfloor \tau/\Delta t \rfloor$. Now in the time interval $[i\Delta t, (i+1)\Delta t]$, we use the approximation $x(t-\tau) \approx x(i\Delta t - m\Delta t)$ in (3) and solve the resulting ordinary differential equation $\dot{x}(t) = a\,x(t) + b\,x\big((i-m)\Delta t\big)$ to obtain

$$x\big((i+1)\Delta t\big) = \alpha\, x(i\Delta t) + \beta\, x\big((i-m)\Delta t\big), \tag{5}$$

where $\alpha = \mathrm{e}^{a\Delta t}$ and $\beta = \frac{b}{a}(\mathrm{e}^{a\Delta t} - 1)$, $\big(\beta = b\Delta t$ if $a = 0\big)$. Now forming an augmented state vector $X(i) = \big[x(i\Delta t),\ x\big((i-1)\Delta t\big),\ \ldots,\ x\big((i-m)\Delta t\big)\big]^{\mathrm{T}}$, (T denotes the transpose) that contains a history of the state values in the last m time steps, we arrive at

$$X(i+1) = T(\Delta t)X(i), \tag{6}$$

where

$$T(\Delta t) = \begin{bmatrix} \alpha & & & \beta \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(m+1)\times(m+1)}. \tag{7}$$

Note that all sub-diagonal elements are 1 and the 0 elements are not shown. The matrix $T(\Delta t)$ can be called the evolution matrix of system (3) since it is a finite-dimensional approximation of the infinite-dimensional solution operator of the linear system (3) [4].

Assume μ is an eigenvalue of $T(\Delta t)$. Then, as $\Delta t \to 0$, $\frac{1}{\Delta t}\ln\mu$ approaches a characteristic root given by (4); *i.e.* $\frac{1}{\Delta t}\ln\mu \to \lambda$. Therefore, one can obtain stable and unstable regions of system (3) in the parameter space by investigating the leading eigenvalues (largest eigenvalues in magnitude) of $T(\Delta t)$ that are calculated for a sufficiently small Δt value. Note that the larger the magnitude of μ the bigger the real part of $\frac{1}{\Delta t}\ln\mu$. See [4] for more details about the convergence properties of the semi-discretization method and [7] for more information on the discretization of the delay differential equations and approximating their spectra using other numerical techniques. In the next section, we investigate the difference in stability properties of systems (1) and (2) exploiting the semi-discretization of the two systems.

2.2. Stability of systems (1) and (2) using their finite-dimensional approximations

We first apply the discretization method described in the previous section to the stochastic system (1). We choose Δt such that

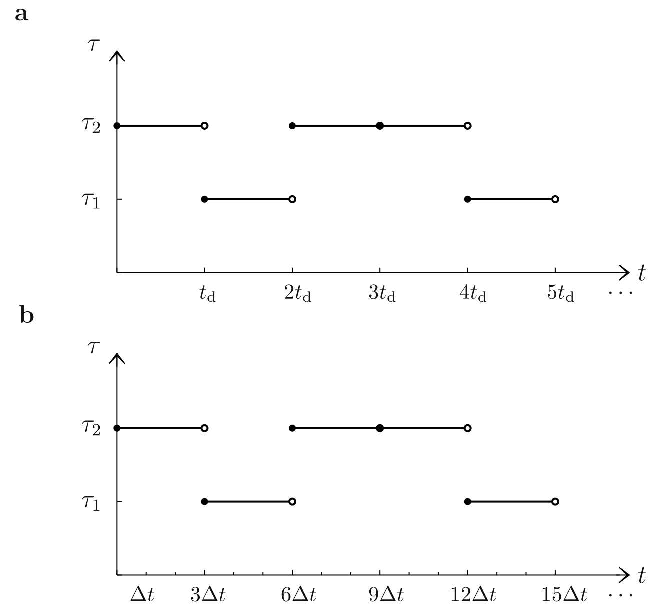

Fig. 1. (a) A sample path of the delay with two values τ_1 and τ_2 and dwelling time t_{d}. (b) The dwelling time t_{d} is discretized to ℓ time steps such that $t_{\mathrm{d}} = \ell\Delta t$ ($\ell = 3$ in this case).

$t_{\mathrm{d}} = \ell\Delta t$ where ℓ is an integer. We also assume $m_1 = \lfloor \tau_1/\Delta t \rfloor$ and $m_2 = \lfloor \tau_2/\Delta t \rfloor$ where $m_1 < m_2$. Using the augmented state vector $X(i) = \big[x(i\Delta t),\ x\big((i-1)\Delta t\big), \ldots, x\big((i-m_2)\Delta t\big)\big]^{\mathrm{T}}$, that contains a history of the state values in the last m_2 time steps (corresponding to the maximum delay), and similar to (6) and (7), the evolution matrix of the system $\dot{x}(t) = a\,x(t) + b\,x(t-\tau_1)$ is obtained as

$$T_1(\Delta t) = \begin{bmatrix} \alpha & & \overset{\text{column } m_1+1}{\overset{\downarrow}{\beta}} & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}_{(m_2+1)\times(m_2+1)}, \tag{8}$$

and the evolution matrix of the system $\dot{x}(t) = a\,x(t) + b\,x(t-\tau_2)$ is obtained as

$$T_2(\Delta t) = \begin{bmatrix} \alpha & & & \overset{\text{column } m_2+1}{\overset{\downarrow}{\beta}} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(m_2+1)\times(m_2+1)}, \tag{9}$$

where α and β are the same as in (5). Now recall that the delay changes every ℓ time steps; see Fig. 1b. Therefore, letting $\tilde{X}(n) = X(n\,\ell)$, the discretization of the stochastic system (1) is given by the stochastic map

$$\tilde{X}(n+1) = A(n)\tilde{X}(n), \tag{10}$$

$n = 0, 1, 2, \ldots$, where $A(n) = \big(T_k(\Delta t)\big)^{\ell}$ if $\tau(t) = \tau_k$ in the time interval $[n\ell\Delta t, (n+1)\ell\Delta t)$, and therefore $\mathbb{P}\Big(A(n) = \big(T_k(\Delta t)\big)^{\ell}\Big) = w_k$, $k = 1, 2$.

Now we take the expectation of both sides of (10). Since the probability distribution of the delay is fixed and is independent of the state, the matrix $A(n)$ is independent of $\tilde{X}(n)$. To see this, note that the matrix $A(n)$ only depends on the delay value in the time interval $[nt_{\rm d}, (n+1)t_{\rm d})$, *i.e.* if $\tau(t) = \tau_k$ in this time interval, then $A(n) = \big(T_k(\Delta t)\big)^{\ell}$. On the other hand, $\tilde{X}(n)$ depends on the delay values in the time intervals $[n't_{\rm d}, (n'+1)t_{\rm d})$, $n' = 0, 1, 2, \ldots, n-1$. Since the delay value in the time interval $[nt_{\rm d}, (n+1)t_{\rm d})$ is independent of the delay values in the previous time intervals, the matrix $A(n)$ is independent of $\tilde{X}(n)$. Thus, from (10),

$$\begin{aligned}\mathbb{E}\big[\tilde{X}(n+1)\big] &= \mathbb{E}\big[A(n)\tilde{X}(n)\big]\\ &= \mathbb{E}\big[A(n)\big]\mathbb{E}\big[\tilde{X}(n)\big].\end{aligned} \tag{11}$$

Let $\bar{\tilde{X}} = \mathbb{E}\big[\tilde{X}(n)\big]$ and $M_{\rm sd} = \mathbb{E}\big[A(n)\big]$. Then, (11) can be written as

$$\bar{\tilde{X}}(n+1) = M_{\rm sd}\bar{\tilde{X}}(n), \tag{12}$$

where

$$M_{\rm sd} = w_1\big(T_1(\Delta t)\big)^{\ell} + w_2\big(T_2(\Delta t)\big)^{\ell}. \tag{13}$$

System (12) describes the mean dynamics of system (10) that is a discretization of system (1). Therefore, one can analyze the stability of the mean of the stochastic system (1) by checking whether all the eigenvalues of matrix $M_{\rm sd}$ fall inside the unit circle (stable) or not (unstable). In practice, this is done by making Δt small enough to achieve convergence up to a desired accuracy.

Note that if $\ell > m_2$, *i.e.* if the dwelling time $t_{\rm d}$ is larger than the maximum delay in the system, we define the augmented state vector as $X(i) = \big[x(i\Delta t),\ x\big((i-1)\Delta t\big),\ \ldots,\ x\big((i-\ell)\Delta t\big)\big]^{\rm T}$, that contains the history of the state values in the last ℓ time steps. In this case, the evolution matrices $T_1(\Delta t)$ in (8) and $T_2(\Delta t)$ in (9) will have the same structure with the same places for elements α and β except that the sub-diagonal of 1's will extend further such that the size of the matrices becomes $(\ell+1)\times(\ell+1)$.

Now consider system (2) with two delays τ_1 and τ_2. Applying the discretization method described in Section 2.1 to system (2), and in the same fashion used to obtain matrices in (8) and (9), we obtain the evolution matrix for system (2) as

$$T_{\rm dd}(\Delta t) = \begin{array}{c} \text{column}: m_1+1 \quad m_2+1 \\ \downarrow \qquad\quad \downarrow \\ \begin{bmatrix} \alpha & & & w_1\beta & & w_2\beta \\ 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & & 1 & \end{bmatrix}_{(m_2+1)\times(m_2+1)} \end{array}. \tag{14}$$

To compare the stability of system (2) with that of the mean of system (1), we consider the ℓ-step evolution matrix $M_{\rm dd} = \big(T_{\rm dd}(\Delta t)\big)^{\ell}$ of system (2). Observe that $T_{\rm dd}(\Delta t) = w_1T_1(\Delta t) + w_2T_2(\Delta t)$. Therefore,

$$M_{\rm dd} = \big(w_1T_1(\Delta t) + w_2T_2(\Delta t)\big)^{\ell}. \tag{15}$$

Our goal is to demonstrate the effect of the delay dwelling time $t_{\rm d}$ on the spectra, and thus on the stability, of the mean of the stochastic system (1) and the corresponding deterministic, discrete-delay system (2). To this end, we compare the spectra of matrices $M_{\rm sd}$ defined in (13) (associated with the mean of system (1)) and $M_{\rm dd}$ defined in (15) (associated with system (2)) for a fixed time step Δt while changing ℓ (recall that $t_{\rm d} = \ell\Delta t$).

For $\ell = 1$ it is easy to check that $M_{\rm sd} = M_{\rm dd}$. In fact, by calculating $M_{\rm sd}$ and $M_{\rm dd}$ for different ℓ values, we observe that

$$M_{\rm sd} = M_{\rm dd} \quad \text{for} \quad \ell = 1, 2, \ldots, m_1+1. \tag{16}$$

For instance, for $\ell = 2$

$$M_{\rm sd} = M_{\rm dd} = \begin{array}{c} \text{column}: m_1+1 \qquad\qquad m_2+1 \\ \downarrow \qquad\qquad\qquad \downarrow \\ \begin{bmatrix} \alpha^2 & & & w_1\beta & \alpha w_1\beta & & w_2\beta & \alpha w_2\beta \\ \alpha & & & & w_1\beta & & & w_2\beta \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & & & 1 & & \end{bmatrix} \end{array}, \tag{17}$$

and for $\ell = 3$

$$M_{\rm sd} = M_{\rm dd} = \begin{bmatrix} \alpha^3 & & w_1\beta & \alpha w_1\beta & \alpha^2 w_1\beta & & w_2\beta & \alpha w_2\beta & \alpha^2 w_2\beta \\ \alpha^2 & & & w_1\beta & \alpha w_1\beta & & & w_2\beta & \alpha w_2\beta \\ \alpha & & & & w_1\beta & & & & w_2\beta \\ 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & & & 1 & & & \end{bmatrix}, \tag{18}$$

and so on. However, once ℓ gets larger than m_1+1 the equality (16) does not hold anymore, *i.e.* for $\ell > m_1+1$, $M_{\rm sd} \neq M_{\rm dd}$. Note that the validity of the equality (16) is independent of the Δt value. Moreover, observe that $\ell \le m_1+1 \Rightarrow \ell\Delta t \le (m_1+1)\Delta t \Rightarrow t_{\rm d} \le \tau_1 + \Delta t$. Since Δt can be made arbitrarily small, the condition for $M_{\rm sd} = M_{\rm dd}$ is given by

$$t_{\rm d} \le \tau_1. \tag{19}$$

Therefore if the delay dwelling time is less or equal than the smallest delay value, the spectra of (the mean of) the stochastic system (1) and the deterministic system (2) are equal and so are their stability properties. Consequently, the deterministic system (2) truly describes the mean of the stochastic system (1), in this case.

However, if $t_{\rm d} > \tau_1$, $M_{\rm sd} \neq M_{\rm dd}$ and therefore the spectrum of the mean of system (1) is different than that of system (2). Therefore, system (2) no longer describes the mean of system (1). As a matter of fact, in the case $t_{\rm d} > \tau_1$, as will be demonstrated through an example in Section 3, system (2) cannot even be considered a good approximation for the mean of system (1) as $t_{\rm d}$ increases. Note that this observation holds for any Δt values. Furthermore, the difference between matrices $M_{\rm sd}$ and $M_{\rm dd}$, and thus their spectra, does not vanish in the limit of $\Delta t \to 0$, as will be illustrated by an example in Section 3, cf. Fig. 3. This means that (the mean of) the stochastic system (1) and system (2), in the case $t_{\rm d} > \tau_1$, are truly different and the difference between them is not a result of time-discretization.

Before proceeding to the next section, we shall make some remarks.

Remark 1. We showed that the spectra of (the mean of) the stochastic-delay system (1) and the deterministic, discrete-delay system (2) are equal when $t_{\rm d} \le \tau_1$, using evolution matrices of the corresponding systems. This equality no longer holds, when $t_{\rm d} > \tau_1$. This observation can also be verified by taking the expectation of both sides of (1) that yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbb{E}[x(t)]\big) = a\,\mathbb{E}[x(t)] + b\,\mathbb{E}[x(t-\tau(t))] = a\,\mathbb{E}[x(t)] + b\sum_{k=1}^{K}\mathbb{P}(\tau(t)=\tau_k)\mathbb{E}\big[x(t-\tau_k)|\tau(t)=\tau_k\big]. \quad (20)$$

Now assume that $n\,t_\mathrm{d} \le t < (n+1)\,t_\mathrm{d}$ for some n. If the dwelling time is less than the minimum delay, *i.e.* $t_\mathrm{d} \le \tau_1$, then $\tilde{n}_k t_\mathrm{d} \le t-\tau_k < (\tilde{n}_k+1)t_\mathrm{d}$ for some $\tilde{n}_k < n$, $k = 1, 2$. In other words, $t-\tau_k$ will fall in one of the dwelling intervals before the current interval n; cf. Fig. 1a. Note that $x(t-\tau_k)$ depends only on the delay values up to the dwelling interval $\tilde{n}_k$. Hence, since the delays at different dwelling intervals are independent, $x(t-\tau_k)$ is independent of $\tau(t)$ and $\mathbb{E}\big[x(t-\tau_k)|\tau(t)=\tau_k\big] = \mathbb{E}\big[x(t-\tau_k)\big]$. Therefore, (20) reduces to

$$\dot{\bar{x}}(t) = a\,\bar{x}(t) + b\sum_{k=1}^{K} w_k\,\bar{x}(t-\tau_k), \quad (21)$$

where $\bar{x} = \mathbb{E}[x(t)]$ and $w_k = \mathbb{P}(\tau(t)=\tau_k)$. System (21) is indeed the same as system (2). Note that the reduction from (20) to (21) is not possible if $t_\mathrm{d} > \tau_1$, since we cannot get rid of the conditional expectation in (20).

Remark 2. The results of this paper are true if there are more than two delays in the set Ω, *i.e.* $K > 2$. In this case, we will have

$$M_\mathrm{sd} = \sum_{k=1}^{K} w_k\big(T_k(\Delta t)\big)^{\ell}, \quad (22)$$

and

$$M_\mathrm{dd} = \Big(\sum_{k=1}^{K} w_k T_k(\Delta t)\Big)^{\ell}. \quad (23)$$

Additionally, as was pointed out in the Introduction, the results of the paper hold for the vector case too, *i.e.* if $x\in\mathbb{R}^q$ and $a, b\in\mathbb{R}^{q\times q}$. In this case, $\alpha, \beta \in \mathbb{R}^{q\times q}$ in (5) and the sub-diagonal of 1's in matrices (7), (8), (9), (14) will be replaced by a block sub-diagonal of I_q's where I_q is the q-dimensional identity matrix.

Remark 3. We showed that the deterministic system (2) describes the mean of the stochastic system (1) when the delay dwelling time is less than the smallest delay in the system. As a result, if one is interested in the second moment stability of system (1) as true stability criteria, the stable region obtained from analyzing system (2) may be considered an estimate of the true stability region of the stochastic system (1), in the case when $t_\mathrm{d} < \tau_1$. This is owing to the fact that the stability of the mean of a stochastic system is a necessary condition for the stability of the second moment. On the other hand, one should note that even under the assumption of the delay dwelling time being less than the minimum delay, the true stable regions of system (1) can be quite smaller than the estimate ones obtained by analyzing system (2). See [8,9] for the second moment stability of system (1), and [10] for other stability criteria for similar stochastic systems.

3. An illustrative example

In this section, we consider the linear system (1) with two delays $\tau_1 = 0.4$ and $\tau_2 = 0.8$ with probability distribution function $w_1 = w_2 = 1/2$. Then, for different dwelling times $t_\mathrm{d} = 0.3$, $t_\mathrm{d} = 0.6$, $t_\mathrm{d} = 1$, and $t_\mathrm{d} = 2$, we construct stability charts in the (a, b) parameter space using the spectra of matrices M_sd (associated with the mean of the stochastic-delay system (1)) and M_dd (associated with the deterministic, discrete-delay system (2)). The goal is to demonstrate the effect of the parameter t_d on the fidelity of the approximation of system (1) by system (2).

Fig. 2(a–d) show the spectra of matrices M_dd (indicated by ○) and M_sd (indicated by ×) for $(a, b) = (-1.2, -5.5)$ (marked by P in the bottom panels). Note that only the first 10 leading eigenvalues are shown. The value $\Delta t = 0.005$ is used for all panels as it was found that this Δt value was small enough for the leading eigenvalues of matrices M_sd and M_dd to converge with a satisfying accuracy; cf. Fig. 3. When the dwelling time $t_\mathrm{d} = 0.3$ is less than the smallest delay $\tau_1 = 0.4$, the matrices M_dd and M_sd are equal and their spectra are the same, as shown in Fig. 2(a). In this case, the approximation of system (1) by system (2) is meaningful, since system (2) is indeed the average of the stochastic system (1). As Fig. 2(e) shows, the stable area of the mean of system (1), enclosed by the solid blue boundary, is the same as the stable area of system (2), enclosed by the dashed green boundary. The stable region is shaded light gray. Note that the crossing of the boundary from stable to unstable region is equivalent to the crossing of the unit circle by the leading eigenvalue from inside to outside.

When the dwelling time is $t_\mathrm{d} = 0.6$, we have $\tau_1 < t_\mathrm{d} < \tau_2$. In this case, the matrices M_dd and M_sd are different and their spectra, as shown in Fig. 2(b), are slightly different, since for $t_\mathrm{d} > \tau_1$, system (2) is no longer the average of system (1). Therefore, the stable area obtained through investigating the eigenvalues of M_dd (bounded by the dashed green curves) is different than the stable area obtained using matrix M_sd (bounded by the solid blue curves). The region shaded as dark gray is the parameter domain where the deterministic, discrete-delay system is unstable but the mean of the stochastic-delay system is stable.

As the dwelling time t_d gets larger relative to the delay values, the approximation of system (1) by system (2) gets worse. Fig. 2(c–d) show the cases $t_\mathrm{d} = 1$ and $t_\mathrm{d} = 2$ for which the dwelling time is larger than the maximum delay, $t_\mathrm{d} > \tau_2$. In these cases, the spectra of matrices M_sd and M_dd are very different. Correspondingly, the stable areas of system (2) and the mean of system (1), shown by Fig. 2(g–h), are significantly different. The region shaded by dark gray is again a parameter domain where the discrete-delay system (2) is unstable but the mean of the stochastic-delay system (1) is stable. On the other hand, the region shaded by red is where the discrete-delay system is stable but the mean of the stochastic-delay system is not.

The example given in this section is simple in the sense that the system is scalar and there are only two delay values. Despite this simplicity, we observed a big difference between stability properties of systems (1) and (2). One may expect that for vector-valued systems and a larger number of delay values, one might find bigger differences between the mean of system (1) and system (2). This observation may therefore be viewed as a warning if one wants to use a deterministic system with a finite number of discrete delay terms as an approximation for a stochastic-delay system.

We also emphasize that the difference observed between the mean of the stochastic system (1) and the deterministic system (2), in the $t_\mathrm{d} > \tau_1$ case, is not a result of the time-discretization of these systems. In other words, in the limit of $\Delta t \to 0$, this difference still exists. To illustrate this, in Fig. 3, we show the spectral radii of the two matrices M_sd and M_dd as functions of Δt for the parameters associated with point P in Fig. 2(h), *i.e.* $(a, b) = (-1.2, -5.5)$ and $t_\mathrm{d} = 2$. The spectral radii of M_sd and M_dd, shown by blue and green curves, respectively, are plotted versus $1/\Delta t$ where a logarithmic scale is used in the horizontal axis. One can see that the spectral radii converge to their limits as Δt decreases.

4. Conclusion

We showed that approximating (in an average sense) a system with stochastic delay using a deterministic system with a finite number of discrete delays is not always warranted. We considered

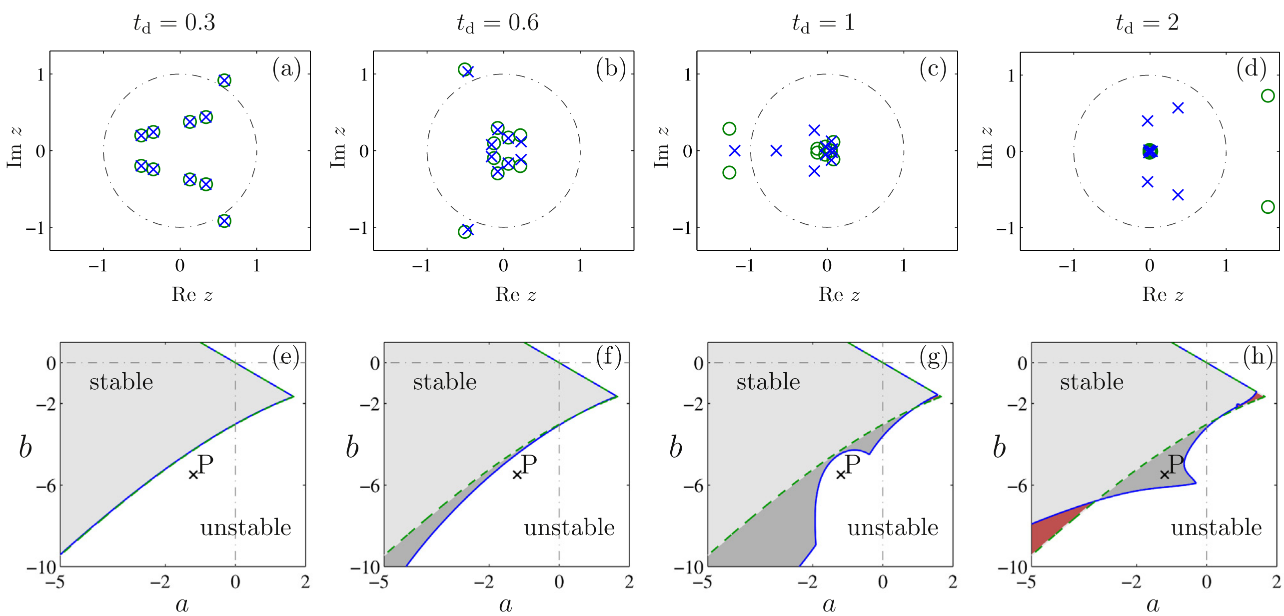

Fig. 2. (a–d) Comparison of the spectra of the matrix M_{dd} in (15) (indicated by ○), associated with the deterministic, discrete-delay system (2), and the spectra of the matrix M_{sd} in (13) (indicated by ×) associated with the mean of the stochastic-delay system (1), for different dwelling times t_d as indicated. The spectra are obtained for point P shown in the bottom panels and only the 10 largest eigenvalues in magnitude are shown in the complex plane. (e–f) Stable regions of the deterministic, discrete-delay system (2) obtained using the eigenvalues of the matrix M_{dd} in (15) (bounded by dashed green curves) and the mean of the stochastic-delay system (1) obtained using the eigenvalues of the matrix M_{sd} in (13) (bounded by solid blue curves), for different dwelling times. The light gray area is a parameter domain where both systems are stable. The dark gray area is where the mean of system (1) is stable but system (2) is not. The red area is where the mean of (1) is not stable but (2) is stable.

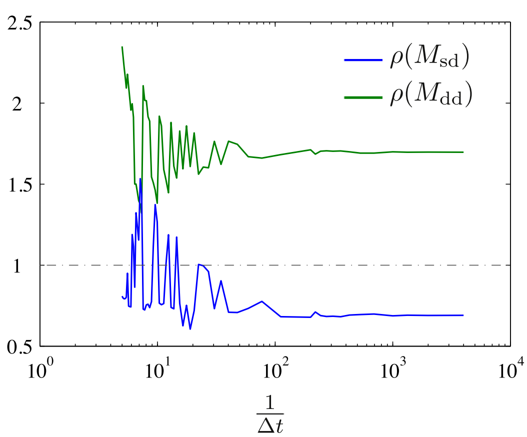

Fig. 3. The spectral radii of matrices M_{sd} and M_{dd}, denoted by $\rho(M_{sd})$ and $\rho(M_{dd})$, respectively, for different values of the time step Δt. Parameters correspond to point P in Fig. 2(h).

in particular a class of linear systems with stochastic delay where the delay trajectories were piece-wise constant functions of time and the delay dwelt at each value for a constant time. We showed that the aforementioned approximation is not valid if the dwelling time is larger than the minimum delay in the system. Furthermore, as the dwelling time increased, the approximation got worse. This shows that even a simplistic stochastic behavior of the delay can bring about non-intuitive consequences. This observation finds importance in stability analysis and control design in applications such as connected vehicle systems [1], wireless communication systems [2,3], and biological circuits [11] where stochastic delays may arise in the dynamics of the system.

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