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Designing the Dynamics of Globally Coupled Oscillators

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A method for designing cluster states with prescribed stability is presented for coupled phase oscillator systems with all-to-all coupling. We determine criteria for the coupling function that ensure the existence and stability of a large variety of clustered configurations. We show that such criteria can be satisfied by choosing Fourier coefficients of the coupling function. We demonstrate that using simple trigonometric and localized coupling functions one can realize arbitrary patterns of stable clusters and that the designed systems are capable of performing finite state computation. The design principles may be relevant when engineering complex dynamical behavior of coupled systems, e.g. the emergent dynamics of artificial neural networks, coupled chemical oscillators and robotic swarms.

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§1. Introduction

Systems of coupled nonlinear oscillators can exhibit complex behavior that cannot be deduced by analyzing the dynamics of their elements but emerges from the interactions between the oscillators. Many different spatial, temporal, and spatiotemporal rhythmic patterns can arise including synchronization,[37] stable clustering[6] and sequential switching between cluster states.[21] Such dynamics appear in many different biological and physical systems including neural systems,[33),34] coupled chemical oscillators,[18] lasers[12] and Josephson junctions.[1] In many cases the corresponding biological or physical models are difficult to analyze due to the high number of variables and parameters. An effective way to describe these complex rhythms is by reducing the models to phase models[7] when the couplings between the oscillators are sufficiently weak. In these models the state of an oscillator is described by a scalar phase variable. When the oscillators are uncoupled their phases advance at constant 'velocities' (the oscillators' natural frequency) while different patterns can arise due to the coupling. In particular, a very rich variety of partially synchronized cluster states can be obtained by tuning the coupling functions.

Previous studies have mostly concentrated on understanding which cluster states (or what kind of sequences of cluster states) arise for a particular coupling function.[4)–6),37] This approach is particularly relevant when studying the behavior of neural ensembles where the coupling is prescribed by nature and the huge variety of emerging rhythmic patterns can be used to represent large amounts of information in a variety of ways.[33] In engineering problems one may instead want to design the dynamics such that particular cluster states emerge. This requires the investi-

gation of a converse mathematical problem: *Given a certain cluster configuration, what constraints have to be imposed on the coupling function to ensure the existence and stability of this cluster state?* The methodology presented here can be applicable for designing the emergent dynamics of neural networks[14),15),25),26)] and chemical oscillator systems[18),19),22)] as well as the collective motion of autonomous vehicles.[17),32),36)]

We consider N identical phase oscillators that are coupled identically to each other

$$\dot{\theta}_i = \omega + \frac{1}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j)\,, \tag{1·1}$$

where the dot denotes differentiation with respect to time t and $\theta_i \in \mathbb{T} = [0, 2\pi)$, $i = 1, \ldots, N$, i.e. the state space is the N-torus $\mathbb{T}^N$. We will use the vector notation $\theta = \mathrm{col}[\,\theta_1, \ldots, \theta_N\,]$. The coupling function $g\colon [0, 2\pi) \to \mathbb{R}$ is a 2π-periodic function having smoothness of at least C^2. For simplicity we absorb any 'coupling strength' parameter into g. (We remark that the form (1·1) arises from the theory of weakly connected networks[13)] and that many authors use $h(\varphi) = g(-\varphi)$ as coupling function.)

Observe that (1·1) has a continuous rotational S^1 symmetry, that is, the dynamics are invariant under the transformation

$$\theta \mapsto \theta + \varphi\,\mathrm{col}[\,1, \ldots, 1\,] \tag{1·2}$$

for any $\varphi \in \mathbb{T}$. The model also has a full permutation S_N symmetry, i.e. solutions are preserved under the transformation

$$\mathrm{col}[\,\theta_1, \ldots, \theta_N\,] \mapsto \mathrm{col}[\,\theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)}\,] \tag{1·3}$$

for any $\sigma \in S_N$; see Ref. 5).

We show that, by suitable choice of g, any arbitrarily complex clustering pattern can appear as an attractor of the system (1·1). More precisely, in §2 we give conditions on g and its derivatives for a cluster state to be present with given configuration and given stability. We use these results to design coupling functions that lead to stable clustering with prescribed phase relationships between clusters by representing g as a Fourier series[9),29)] and properly choosing the coefficients. Section 3 concentrates on simple harmonic and localized coupling functions that lead to clusters separated equally in phase. In this case, exploiting the coexistence of a large number of stable cluster states allows finite state computations to be performed.[3),14),15)] In particular, one may steer the system between different cluster configurations by applying sufficiently large perturbations to individual oscillators. Finally, in §4 we conclude and discuss some open problems.

We emphasize that the clustering behavior discussed in this paper is purely a dynamical phenomenon, driven by interactions between the oscillators through the coupling function g, and that there is no external inhomogeneity forced on the system. Such behavior cannot be obtained by using simple sinusoidal coupling (as in the Kuramoto model[23)]). One may impose additional inhomogeneity to the system

by replacing ω with ω_i in (1·1) but as long as the detuning is weak enough, analogous clustering behavior will be observed.

§2. Design of cluster states

Consider a cluster state with M clusters where $1 \leq M \leq N$. We define a corresponding M-cluster partition $\mathcal{A} = \{A_1, \ldots, A_M\}$ of $\{1, \ldots, N\}$ such that

$$\{1, \ldots, N\} = \bigcup_{p=1}^{M} A_p \,, \tag{2·1}$$

where the A_p are pairwise disjoint sets ($A_p \cap A_q = \emptyset$ if $p \neq q$). Note that summing up the cluster sizes $a_p = |A_p|$ gives

$$\sum_{p=1}^{M} a_p = N \,. \tag{2·2}$$

For a partition $\mathcal{A}$ we associate a subspace in the state space $\mathbb{T}^N$ that is defined by

$$\mathbb{T}^N_{\mathcal{A}} = \{\theta \in \mathbb{T}^N : \theta_i = \theta_j \;\Leftrightarrow\; \text{there is a } p \text{ such that } i, j \subset A_p\} \,, \tag{2·3}$$

and we say a given $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$. It is a simple consequence of S_N symmetry (1·3) that all such subspaces are invariant for the dynamics of (1·1), so the corresponding partitions may be called invariant partitions. We remark that invariant partitions play an important role in the operator theoretical approach of dynamical systems.[27)]

In order to analyze the dynamics in the subspace $\mathbb{T}^N_{\mathcal{A}}$ we restrict the model (1·1) onto that subspace. Denoting the phase of the p-th cluster by ψ_p, i.e. defining $\psi_p := \theta_i = \theta_j = \theta_k = \ldots$ such that $\{i, j, k, \ldots\} \subset A_p$ we obtain

$$\dot{\psi}_p = \omega + \frac{1}{N} \sum_{q=1}^{M} a_q \, g(\psi_p - \psi_q) \tag{2·4}$$

for $p = 1, \ldots, M$. The dynamics in such a subspace can be very complex, but initially we investigate simple clustering behavior

$$\psi_p = \Omega \, t + \phi_p \tag{2·5}$$

for $p = 1, \ldots, M$, where $\Omega \in \mathbb{R}^+$ and $\phi_p \in \mathbb{T}$. Since θ describes the phases of oscillators, (2·5) describes a periodic orbit in the state space of those oscillators. On this periodic orbit the oscillators are frequency-synchronized with frequency Ω and the clustering is due to the phases ϕ_p. We say that $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$ *as a periodic orbit* if (2·5) holds and all $\phi_p \pmod{2\pi}$ are different.

Substituting (2·5) into (2·4) gives

$$\Omega = \omega + \frac{1}{N} \sum_{q=1}^{M} a_q \, g(\phi_p - \phi_q) \tag{2·6}$$

for $p = 1, \ldots, M$. By subtracting the first equation ($p = 1$) from each of the following equations ($p = 2, \ldots, M$) we obtain

$$0 = \sum_{q=1}^{M} a_q \left(g(\phi_p - \phi_q) - g(\phi_1 - \phi_q) \right) \tag{2·7}$$

for $p = 2, \ldots, M$. For a given function g one may determine $M - 1$ phases out of ϕ_p, $p = 1, \ldots, M$ while one phase can be chosen arbitrarily. Then formula (2·6) determines the frequency Ω.

On the other hand, assuming that cluster phases are given

$$0 \leq \phi_1 < \phi_2 < \ldots < \phi_M < 2\pi \tag{2·8}$$

formula (2·7) provides us with $M - 1$ conditions for the function g. (Without loss of generality one may choose $\phi_1 = 0$.) We introduce the notation

$$g_{p,q} := g(\phi_p - \phi_q) \tag{2·9}$$

to give $M(M-1)$ different subscripts for $p, q = 1, \ldots, M$, $p \neq q$ and note that

$$g_{p,p} = g(0) =: g_0 \tag{2·10}$$

for any $p = 1, \ldots, M$.

We can state that $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$ *as a periodic orbit* if

$$0 = \sum_{q=1}^{M} a_q \left(g_{p,q} - g_{1,q} \right), \quad p = 2, \ldots, M\,, \tag{2·11}$$

as long as the corresponding frequency is non-zero, that is,

$$\Omega = \omega + \frac{1}{N} \sum_{q=1}^{M} a_q \, g_{p,q} \neq 0\,. \tag{2·12}$$

For typical ω note that $\Omega \neq 0$. Formula (2·11) is a system of $M - 1$ linear equations with $M(M-1)+1$ unknowns, namely the values of $g_{p,q}$ for $p, q = 1, \ldots, M$, $p \neq q$ and $g_{p,p} = g_0$ for $p = 1, \ldots, M$.

We say that $[\,\phi_1, \ldots, \phi_M\,]$ *has independent differences* if $p \neq q$ and $r \neq s$ imply $\phi_p - \phi_q \neq \mathrm{mod}(\phi_r - \phi_s, 2\pi)$ unless $p = r$ and $q = s$. In this case all $g_{p,q}$, $p \neq q$ can be considered as independent variables. Not having independent differences clearly decreases the number of independent variables $g_{p,q}$. Note that generic choices of $[\,\phi_1, \ldots, \phi_M\,]$ have independent differences, i.e. there are $M(M-1)+1$ independent variables.

The linear stability of cluster states was characterized in Refs. 7) and 29) for the special case of equally sized clusters that are equally separated in phase (called rotating blocks). Here we generalize these calculations for arbitrary partitions. Suppose that θ realizes a partition $\mathcal{A}$ as a periodic orbit, i.e. (2·11) is satisfied. In order to determine the stability of this periodic orbit we need to study:

- *Tangential stability*; stability to changes in the phases that *preserve* the cluster structure. This is given by the linearization within the cluster subspace $\mathbb{T}^N_{\mathcal{A}}$.
- *Transverse stability*; stability to changes in the phases that *split* at least one of the clusters into smaller groups. This is given by the linearization transverse to the cluster subspace.

To determine tangential stability we define $\chi_p = \psi_p - (\Omega\, t + \phi_p)$ and linearize (2·4) about the periodic orbit (2·5). Thus, we obtain the variational equation

$$\dot{\chi}_p = \frac{1}{N}\sum_{q=1}^{M} a_q\, g'(\phi_p - \phi_q)(\chi_p - \chi_q) = \sum_{q=1}^{M} T_{p,q}\,\chi_q\,, \tag{2·13}$$

with the matrix

$$T_{p,q} = \frac{1}{N}\left(\delta_{p,q}\Big(\sum_{r=1,r\neq p}^{M} a_r\, g'_{p,r}\Big) - (1-\delta_{p,q})\, a_q\, g'_{p,q}\right). \tag{2·14}$$

Here $\delta_{p,q}$ denotes the Kronecker delta and, as before, we introduce

$$g'_{p,q} := g'(\phi_p - \phi_q)\,. \tag{2·15}$$

In the case of independent differences between the ϕ_p's, this gives $M(M-1)$ different variables for $p, q = 1, \ldots, M$, $p \neq q$, while

$$g'_{p,p} = g'(0) =: g'_0 \tag{2·16}$$

for any $p = 1, \ldots, M$. The matrix T has a trivial zero eigenvalue since the sum of the elements in each row is zero, that is,

$$\sum_{q=1}^{M} T_{p,q} = 0 \quad \Rightarrow \quad \lambda_1^{\text{triv}} = 0\,. \tag{2·17}$$

This eigenvalue corresponds to the continuous rotational symmetry (1·2). The other $M-1$ non-trivial eigenvalues

$$\lambda_2^{\text{tang}}, \ldots, \lambda_M^{\text{tang}} \tag{2·18}$$

of T are complicated nonlinear functions of $g'_{p,q}$ where $p, q = 1, \ldots, M$, $p \neq q$, and these determine the tangential stability. One may show that (2·17) and (2·18) are eigenvalues for the linearization of (1·1) and calculate the corresponding eigenvectors whose form is given in (A·2) in Appendix A.

To determine transverse stability of the p-th cluster when $a_p > 1$ we define $\eta_i = \theta_i - (\Omega\, t + \phi_p)$ if $i \in A_p$ and $\eta_i = 0$ otherwise, for $i = 1, \ldots N$. Such perturbations do not break up the other clusters of the invariant partition $\mathcal{A}$ in (2·1). Linearizing (1·1) and relabelling the subscripts such that η_k, $k = 1, \ldots, a_p$ are the non-zero coordinates one obtains

$$\dot{\eta}_k = \sum_{\ell=1}^{a_p} S^{(p)}_{k,\ell}\,\eta_\ell\,, \tag{2·19}$$

with the matrix

$$S^{(p)}_{k,\ell} = \frac{1}{N}\left(\delta_{k,\ell}\left(\sum_{r=1}^{M} a_r\, g'_{p,r} - g'_0\right) - (1-\delta_{k,\ell})\, g'_0\right). \tag{2·20}$$

The real eigenvalue

$$\lambda^{\mathrm{tran}}_{M+p} = \frac{1}{N}\sum_{r=1}^{M} a_r\, g'_{p,r} \tag{2·21}$$

of $S^{(p)}$ has multiplicity $a_p - 1$ and it corresponds to splitting the p-th cluster. The corresponding (a_p-1)-dimensional eigenspace is spanned by the eigenvectors shown in (A·3) in Appendix A. (We remark that $S^{(p)}$ has an extra eigenvalue of multiplicity 1 but that is not an eigenvalue of the linearization of (1·1) and hence can be ignored). Only clusters containing more than one oscillator give transverse eigenvalues and we denote the number of these clusters by W where $0 \le W \le M$. Indeed, the total number of transverse eigenvalues is $\sum_{p=1}^{M}(a_p-1) = N - M$. We emphasize that it is essential to analytically decompose the state space into tangential and transverse subspaces to handle eigenvalues with multiplicity. In particular, many numerical software packages fail when analyzing systems with symmetry for this reason.

For stable cluster states both the tangential eigenvalues and the transverse eigenvalues have to be on the left-half complex plane, that is

$$\mathrm{Re}\,\lambda^{\mathrm{tang}}_p < 0\,, \quad p = 2,\ldots,M\,, \tag{2·22}$$

and

$$\lambda^{\mathrm{tran}}_{M+p} < 0\,, \quad p = 1,\ldots,W\,. \tag{2·23}$$

For example, choosing

$$\begin{aligned} &g'_{p,q} < 0\,, \quad p,q = 1,\ldots,M\,, \quad p \neq q\,,\\ &g'_0 < N \min_p\left\{\frac{R_p}{a_p}\right\}, \end{aligned} \tag{2·24}$$

with

$$R_p = \frac{1}{N}\sum_{r=1, r\neq p}^{M} (-a_r\, g'_{p,r}) > 0 \tag{2·25}$$

satisfies (2·22) and (2·23). This can be seen as follows. According to the Geršgorin Circle Theorem[38] the eigenvalues of T (2·14) are contained by the union of M discs each centered at $-R_p < 0$ and with radius R_p for $p = 1,\ldots,M$. This means that the disc

$$\left|z + \max_p\{R_p\}\right| \le \max_p\{R_p\}\,, \quad z \in \mathbb{C}\,, \tag{2·26}$$

(centered at $-\max_p\{R_p\} < 0$ with radius $\max_p\{R_p\}$) contains all eigenvalues of T. This disc is located on the left-half complex plane and is tangent to the imaginary axis at the origin. Since for any general choice of $g'_{p,q}$ we have $\mathrm{rank}(T) = M-1$, the only zero eigenvalue is the trivial one $\lambda^{\mathrm{triv}}_1 = 0$. Consequently, all nontrivial eigenvalues

of T have negative real part, implying tangential stability (2·22). Furthermore, using (2·24) in (2·21) gives transverse stability (2·23). Note that similar arguments may be used when considering networks with nonidentical couplings[11] and different topologies.[30]

In general, if there are independent differences between the ϕ_p's, then there are $M(M-1)+1$ independent variables that can be varied to satisfy the above stability conditions, namely the values of $g'_{p,q}$ for $p,q = 1,\dots,M$, $p \neq q$ and $g'_{p,p} = g'_0$ for $p = 1,\dots,M$. We note that the condition (2·23) is linear in $g'_{p,q}$ (see (2·21)), while the condition (2·22) is nonlinear in $g'_{p,q}$ (for $M \geq 3$).

We say that $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$ *as a stable periodic orbit* if the conditions (2·11), (2·22), and (2·23) are satisfied. Note that the equality (2·11) contains only the variables $g_{p,q}$ while the inequalities (2·22) and (2·23) contain only the variables $g'_{p,q}$, that is, the conditions for existence and conditions for stability are independent. Also notice that the total number of constraints given by (2·11), (2·22), and (2·23) is $2M-2+W$ where $0 \leq W \leq M$ and, if $[\,\phi_1,\dots,\phi_M\,]$ has independent differences, one may vary the $2M(M-1)+2$ variables $g_{p,q}, g'_{p,q}$, $p \neq q$ and g_0, g'_0 to satisfy these constraints.

2.1. *Example: designing stable 3-cluster states*

To illustrate the calculation above we consider a partition with three clusters. We present the following calculation for a general 3-cluster partition of N oscillators. Assume that $a_p > 1$, $p = 1,2,3$ and recall that $a_1 + a_2 + a_3 = N$; see (2·2). When θ realizes a 3-cluster partition as a periodic orbit, formula (2·11) gives

$$\begin{aligned} a_1(g_{2,1} - g_0) + a_2(g_0 - g_{1,2}) + a_3(g_{2,3} - g_{1,3}) &= 0\,, \\ a_1(g_{3,1} - g_0) + a_2(g_{3,2} - g_{1,2}) + a_3(g_0 - g_{1,3}) &= 0\,. \end{aligned} \tag{2·27}$$

The 7 variables $g_0, g_{1,2}, g_{2,1}, g_{2,3}, g_{3,2}, g_{1,3}, g_{3,1}$ can be varied to satisfy these 2 equations.

The matrix T of tangential stability (2·14) becomes

$$T = \frac{1}{N}\begin{bmatrix} a_2\,g'_{1,2} + a_3\,g'_{1,3} & -a_2\,g'_{1,2} & -a_3\,g'_{1,3} \\ -a_1\,g'_{2,1} & a_1\,g'_{2,1} + a_3\,g'_{2,3} & -a_3\,g'_{2,3} \\ -a_1\,g'_{3,1} & -a_2\,g'_{3,2} & a_1\,g'_{3,1} + a_2\,g'_{3,2} \end{bmatrix}, \tag{2·28}$$

which possesses the trivial eigenvalue $\lambda_1^{\mathrm{triv}} = 0$ and two tangential eigenvalues

$$\begin{aligned} \lambda_2^{\mathrm{tang}} &= \tfrac{1}{2}\big(\mu + \mathrm{i}\sqrt{\nu - \mu^2}\big)\,, \\ \lambda_3^{\mathrm{tang}} &= \tfrac{1}{2}\big(\mu - \mathrm{i}\sqrt{\nu - \mu^2}\big)\,, \end{aligned} \tag{2·29}$$

where

$$\begin{aligned} \mu &= \frac{1}{N}\big(a_2\,g'_{1,2} + a_3\,g'_{1,3} + a_1\,g'_{2,1} + a_3\,g'_{2,3} + a_1\,g'_{3,1} + a_2\,g'_{3,2}\big) = \mathrm{trace}(T)\,, \\ \nu &= \frac{4}{N^2}\big((a_1\,g'_{2,1} + a_3\,g'_{2,3})(a_1\,g'_{3,1} + a_2\,g'_{3,2}) - a_2\,g'_{3,2}a_3\,g'_{2,3} \\ &\qquad + (a_2\,g'_{1,2} + a_3\,g'_{1,3})(a_1\,g'_{3,1} + a_2\,g'_{3,2}) - a_1\,g'_{3,1}a_3\,g'_{1,3} \end{aligned}$$

$$+(a_2\,g'_{1,2}+a_3\,g'_{1,3})(a_1\,g'_{2,1}+a_3\,g'_{2,3})-a_1\,g'_{2,1}a_2\,g'_{1,2}\big)=4(\mathcal{T}_{11}+\mathcal{T}_{22}+\mathcal{T}_{33})\,. \tag{2·30}$$

Here $\mathcal{T}_{pp}$ is the determinant of the 2×2 matrix that is obtained by deleting the p-th row and p-th column of T. Note that $\nu-\mu^2$ is not necessarily positive, meaning that λ_2 and λ_3 can be real. One may show that

$$\mathrm{Re}\,\lambda_2^{\mathrm{tang}}<0,\quad \mathrm{Re}\,\lambda_3^{\mathrm{tang}}<0\quad\Leftrightarrow\quad \mu<0,\quad \nu>0\,. \tag{2·31}$$

The eigenvalues for transverse stability are given by (2·21):

$$\begin{aligned}\lambda_4^{\mathrm{tran}}&=\frac{1}{N}\big(a_1\,g'_0+a_2\,g'_{1,2}+a_3\,g'_{1,3}\big)\,,\\ \lambda_5^{\mathrm{tran}}&=\frac{1}{N}\big(a_1\,g'_{2,1}+a_2\,g'_0+a_3\,g'_{2,3}\big)\,,\\ \lambda_6^{\mathrm{tran}}&=\frac{1}{N}\big(a_1\,g'_{3,1}+a_2\,g'_{3,2}+a_3\,g'_0\big)\,.\end{aligned} \tag{2·32}$$

The 7 variables $g'_0, g'_{1,2}, g'_{2,1}, g'_{2,3}, g'_{3,2}, g'_{1,3}, g'_{3,1}$ can be varied to satisfy the 5 inequalities $\mu<0$, $\nu>0$, $\lambda_4^{\mathrm{tran}}<0$, $\lambda_5^{\mathrm{tran}}<0$, $\lambda_6^{\mathrm{tran}}<0$ obtained from (2·22), (2·23), and (2·31). For example, setting

$$\begin{aligned}&g'_{1,2},\ g'_{2,1},\ g'_{2,3},\ g'_{3,2},\ g'_{1,3},\ g'_{3,1}<0\,,\\ &g'_0<\min\left\{-\frac{a_2\,g'_{1,2}+a_3\,g'_{1,3}}{a_1},\,-\frac{a_1\,g'_{2,1}+a_3\,g'_{2,3}}{a_2},\,-\frac{a_1\,g'_{3,1}+a_2\,g'_{3,2}}{a_3}\right\},\end{aligned} \tag{2·33}$$

according to (2·24), ensures stability for 3-cluster states.

In this example the number of variables (14) was much larger than the number of constraints (7). One may impose further restrictions on g and still be able to satisfy these constraints. Appendix B shows how to design 3-cluster states when g is restricted to the class of odd functions.

2.2. *Realizing stable clustering by choosing harmonics*

The design principles presented above support a wide range of coupling functions since the values and derivatives of $g(\varphi)$ are only fixed at some special points. Since g is 2π-periodic one may use truncated Fourier series to approximate g. This approach has proved to be useful when analyzing the dynamics of globally coupled phase oscillator systems.[9),29)] Here we choose the coefficients to satisfy (2·11), (2·22), and (2·23). Recall that the total number of constraints is $2M-2+W$ where $0\le W\le M$ (such that the $M-1$ conditions (2·11) are equalities while the $M-1+W$ conditions (2·22) and (2·23) are inequalities).

The truncated Fourier expansion of g up to the L-th harmonic is given by

$$g(\varphi)=\sum_{\ell=1}^{L}\big(u_\ell\cos(\ell\varphi)+v_\ell\sin(\ell\varphi)\big)\,, \tag{2·34}$$

when g has zero mean. This contains $2L$ constants, namely u_ℓ, v_ℓ, $\ell=1,\ldots,L$. Thus, to satisfy the $2M-2+W$ conditions one needs to choose

$$L\ge M-1+W/2\,. \tag{2·35}$$

For example, one needs at least 2 harmonics to be able to design nontrivial 2-cluster partitions. More precisely, for $M = 2$, $a_1 > 1$, $a_2 > 1$ we have $L \geq 2$ and formulae (2·11), (2·22), and (2·23) become

$$\begin{aligned} &a_1(g_{2,1} - g_0) + a_2(g_0 - g_{1,2}) = 0\,, \\ &\lambda_2^{\text{tang}} = \frac{1}{N}\big(a_1\, g'_{2,1} + a_2\, g'_{1,2}\big) < 0\,, \\ &\lambda_3^{\text{tran}} = \frac{1}{N}\big(a_1\, g'_0 + a_2\, g'_{1,2}\big) < 0\,, \\ &\lambda_4^{\text{tran}} = \frac{1}{N}\big(a_1\, g'_{2,1} + a_2\, g'_0\big) < 0\,, \end{aligned} \tag{2·36}$$

which can be satisfied by varying the four coefficients u_1, v_1, u_2, v_2 in (2·34). To demonstrate this we fix the cluster phases $\phi_1 = 0$, $\phi_2 = \pi/2$ and the cluster sizes $a_1 = 6$, $a_2 = 9$ for $N = a_1 + a_2 = 15$ oscillators. Furthermore, we fix the eigenvalues $\lambda_2^{\text{tang}} = -1$, $\lambda_3^{\text{tran}} = -2$, $\lambda_4^{\text{tran}} = -3$ and by solving (2·36) we obtain $u_1 = 1/36$, $v_1 = -61/180$, $u_2 = 5/6$, $v_2 = 13/360$. The results are shown in Fig. 1. The designed coupling function g is shown in panel (a), the eigenvalues are displayed in panel (b) and the clustering is demonstrated by numerical simulation in panel (c) for randomly chosen initial conditions. (Due to periodicity the horizontal lines at 0 and 2π are identical and this will be the case for all simulation results shown below.) Observe that the system approaches the 2-cluster partition

$$\{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12, 13, 14, 15\}\}\,, \tag{2·37}$$

that is, 6 oscillators approach one cluster and 9 oscillators approach the other cluster according to our design.

Note that using harmonic functions to design stable M-cluster partitions may become complicated for $M \geq 3$ since the tangential eigenvalues (2·18) are nonlinear functions of the derivatives $g'_{p,q}$, $p, q = 1, \ldots, M$, $p \neq q$ (see, e.g. (2·29) and (2·30) for $M = 3$), and consequently they are nonlinear in u_ℓ, v_ℓ, $\ell = 1, \ldots L$. Furthermore, our design only ensures the existence and linear stability of the chosen cluster state. When using higher harmonics one may stabilize many other different cluster states (e.g. full synchrony) or more complicated attractors (e.g. heteroclinic cycles[5),20)]) which can have much larger basins of attractions than the chosen state.

Formula (2·35) gives a condition on the number of harmonics that allows one to design stable M-cluster states with any partition $\mathcal{A}$ of $\{1, \ldots, N\}$ with prescribed cluster phases (2·8). It would be interesting to determine a lower bound on the number of harmonics needed to realize a given partition $\mathcal{A}$ as a stable periodic orbit having no restrictions on the cluster phases. For example, in the case of only one harmonic, the only partitions that can be realized as hyperbolic periodic orbits are the trivial ones $\{1, 2, \ldots, N\}$ and $\{\{1\}, \{2\}, \ldots, \{N\}\}$; see Ref. 23). On the other hand, for two harmonics one may find additional 2-cluster and 3-cluster partitions such as $\{\{1, 2, \ldots, k\}, \{k+1, \ldots, N\}\}$ and $\{\{1, 2, \ldots, k\}, \{k+1\}, \{k+2, \ldots, N\}\}$; see Refs. 2), 4), and 20).

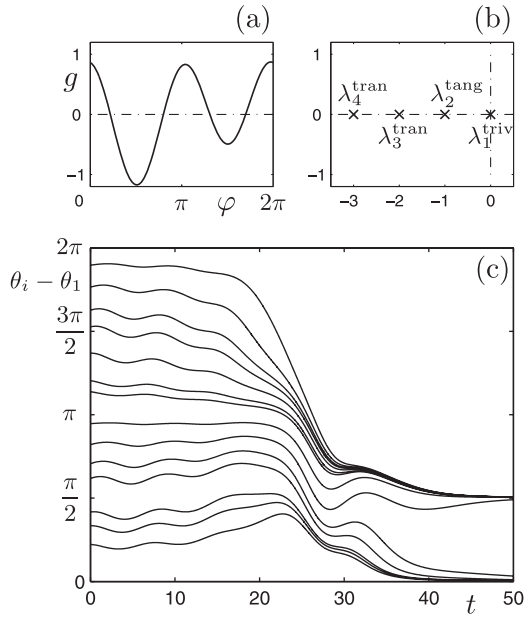

Fig. 1. Designing coupling function for stable 2-cluster partitions with cluster sizes $a_1 = 6$ and $a_2 = 9$ for $N = 15$ oscillators. The phase difference between the clusters is set to $\pi/2$. The designed coupling function (2·34) with $L = 2$ and Fourier coefficients $u_1 = 1/36$, $v_1 = -61/180$, $u_2 = 5/6$, $v_2 = 13/360$ is shown in panel (a) and the corresponding eigenvalues are displayed in panel (b). The numerical simulation results are shown in panel (c) where each curve corresponds to a particular phase difference $\theta_i - \theta_1$ for $i = 1, \ldots, 15$. Observe that the system converges to the designed 2-cluster partition (2·37).

§3. Equidistant clusters

As mentioned above, it may be complicated to determine the Fourier coefficients when the number of clusters M is large. However, if one only wants to design stable M-cluster partitions for specific cluster phases, much simpler coupling functions can be constructed. In this section we consider the case when the clusters are equally separated in phase, i.e.

$$\phi_p = (p-1)\, 2\pi/M \tag{3·1}$$

for $p = 1, \ldots, M$. We show that one can choose simple harmonic and localized coupling functions that result in stable M-cluster partitions with these cluster phases. Furthermore, by tuning the localized coupling functions one can obtain a stable chain of equidistant clusters with phases

$$\phi_p = (p-1)\, \xi\,, \qquad 0 < \xi \leq 2\pi/(M+1) \tag{3·2}$$

for $p = 1, \ldots, M$. We remark that the results shown below are a generalization of the example presented in Ref. 6).

3.1. *Equidistant clusters by harmonic coupling function*

For any M-cluster partition there is a stable periodic orbit realizing that partition for the coupling function

$$g(\varphi) = -\sin(M\varphi)\,, \tag{3·3}$$

shown in Fig. 2(a) for $M = 4$. For this function

$$g_{p,q} = 0\,, \quad g'_{p,q} = -M \tag{3·4}$$

for all $p, q = 1, \ldots, M$. Hence (2·11) is trivially satisfied and (2·12) gives $\Omega = \omega$. For tangential stability the matrix (2·14) simplifies to

$$T_{p,q} = -\frac{M}{N}\left(\delta_{p,q}\Big(\sum_{r=1,r\neq p}^{M} a_r\Big) - (1-\delta_{p,q})a_q\right). \tag{3·5}$$

One may show that (besides the trivial eigenvalue $\lambda_1^{\mathrm{triv}} = 0$) T possesses the tangential eigenvalues

$$\lambda_p^{\mathrm{tang}} = -\frac{M}{N}\sum_{r=1}^{M} a_r = -M \tag{3·6}$$

for $p = 2, \ldots, M$. That is, $-M$ is an eigenvalue of T with multiplicity $M-1$, implying tangential stability (2·22). The transverse eigenvalues (2·21) also evaluate to

$$\lambda_{M+p}^{\mathrm{tran}} = -\frac{M}{N}\sum_{q=1}^{M} a_q = -M\,, \tag{3·7}$$

for $p = 1, \ldots, W$, implying transverse stability (2·23). We note that the resulting periodic orbits are robust, that is, qualitatively the same behavior occurs for all perturbations on g that are sufficiently small in the C^2 norm.

Considering the coupling function (3·3) for $M = 4$, system (1·1) converges to a 4-cluster state. Figure 2(c) shows an example for $N = 15$ oscillators and randomly chosen initial condition that converges to the partition

$$\{\{1,2,3,4\},\{5,6,7\},\{8,9,10,11\},\{12,13,14,15\}\}\,. \tag{3·8}$$

The clusters are located at $\phi_p = (p-1)\,\pi/2$, $p = 1, 2, 3, 4$ according to our design (3·1). Note that different initial conditions may lead to 4-cluster states with different partitions (but with the same ϕ_p's). For certain initial conditions, some clusters may even be *empty* resulting in 3-, 2-, or 1-cluster states. In general, we can say that system (1·1) with coupling function (3·3) has many stable cluster states comprising all partitions of $\{1, \ldots, N\}$ into at most M-clusters located at (3·1).

3.2. *Equidistant clusters by localized coupling function*

Here we consider a localized coupling function that gives clustering into arbitrary partitions. For any M-cluster partition let us pick a smooth function $g(\varphi)$ and

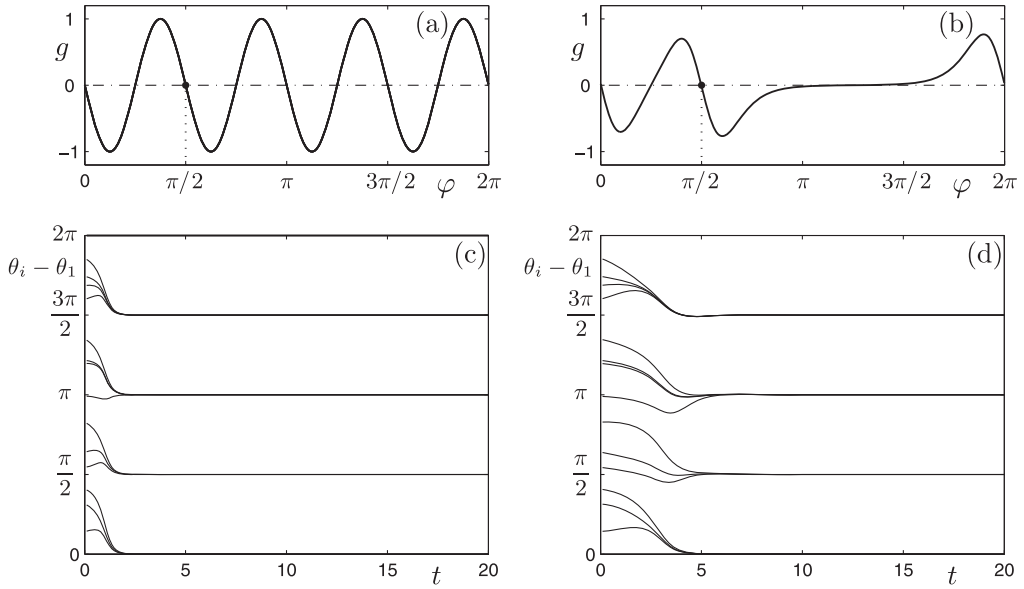

Fig. 2. The coupling functions (3·3) and (3·14, 3·15) are displayed in panels (a) and (b), respectively, for $M = 4$ and $\xi = 2\pi/M$. They both support stable 4-cluster states. Panels (c) and (d) show the corresponding time series of phases for a system of $N = 15$ coupled oscillators. Each curve corresponds to a particular phase difference $\theta_i - \theta_1$ for $i = 1, \ldots, 15$. Observe that the convergence is slower in case (d) suggesting that the tangential and transverse eigenvalues are closer to the imaginary axis.

$0 < \xi \leq 2\pi/M$ such that the following holds:

$$\begin{aligned} &g(0) = g(\xi) = 0\,, \\ &g'(0) = g'(\xi) = -M\,, \\ &g(\varphi) \equiv g'(\varphi) \equiv 0 \quad \text{for} \quad \frac{3\xi}{2} \leq \varphi < 2\pi - \frac{\xi}{2}\,. \end{aligned} \tag{3·9}$$

An example of a function approximately satisfying (3·9) is shown in Fig. 2(b) for $\xi = 2\pi/M$ and $M = 4$. Considering $\xi = 2\pi/M$ the p-th cluster only influenced by the $(p-1)$-st cluster for $p = 2, \ldots, M$ and the first cluster is only influenced by the M-th cluster. Introducing the notation $a_0 = a_M$, $g_{p,0} = g_{p,M}$, $g'_{p,0} = g'_{p,M}$, we can write

$$\begin{aligned} &g_{p,q} = 0\,, \\ &g'_{p,p} = g'_{p,p-1} = -M \quad \text{for} \quad p = 1, \ldots, M\,, \\ &g'_{p,q} = 0 \qquad\qquad\quad \text{for any other} \quad p, q\,. \end{aligned} \tag{3·10}$$

Hence again (2·11) is trivially satisfied and (2·12) gives $\Omega = \omega$. For tangential stability the matrix (2·14) becomes

$$T_{p,q} = -\frac{M}{N}\Big(\delta_{p,q}a_{p-1} - (1-\delta_{p,q})\delta_{q,p-1}a_{p-1}\Big)\,. \tag{3·11}$$

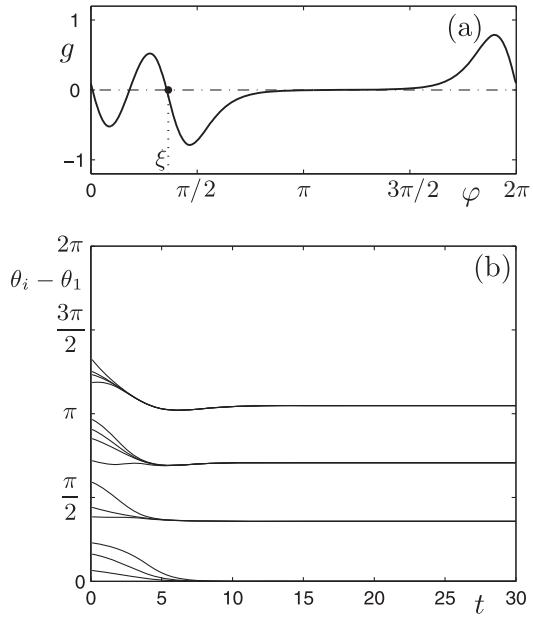

Fig. 3. Panel (a) shows the coupling function (3·14, 3·15) for $\xi = 2\pi/(M + 1.5)$ and $M = 4$. This function also supports stable 4-cluster states. The corresponding time series are shown in panel (b) for $N = 15$ coupled oscillators; each curve corresponds to a particular $\theta_i - \theta_1$ for $i = 1, \ldots, 15$.

Here the tangential eigenvalues are not easy to calculate. However, the Geršgorin Circle Theorem[38] still ensures tangential stability. Here $R_p = \frac{M}{N} a_{p-1}$ in (2·25), i.e. the eigenvalues of T are contained by the union of M discs, each centered at $-\frac{M}{N} a_p < 0$ and with radius $\frac{M}{N} a_p$ for $p = 1, \ldots, M$. That is, the disc

$$\left| z + \frac{M}{N} \max_p \{a_p\} \right| \leq \frac{M}{N} \max_p \{a_p\}\,, \quad z \in \mathbb{C}\,, \tag{3·12}$$

(centered at $-\frac{M}{N} \max_p\{a_p\} < 0$ with radius $\frac{M}{N} \max_p\{a_p\}$) contains all eigenvalues of T, implying tangential stability (2·22). The transverse eigenvalues (2·21) become

$$\lambda_{M+p}^{\text{tran}} = -\frac{M}{N}(a_p + a_{p-1}) < 0 \tag{3·13}$$

for $p = 1, \ldots, M$ implying transverse stability (2·23).

As an example of a coupling function approximately satisfying (3·9), we consider the function

$$g(\varphi) = f_M(\varphi) + f_M(\varphi - \xi)\,, \tag{3·14}$$

where f_M is a 2π-periodic 'wiggle function' with a negative slope at $\varphi = 0$, that is otherwise close to zero:

$$f_M(\varphi) = -2 \tanh\left(M \sin(\tfrac{\varphi}{2})\right) \operatorname{sech}^2\left(M \sin(\tfrac{\varphi}{2})\right) \cos(\tfrac{\varphi}{2})\,. \tag{3·15}$$

Function (3·14, 3·15) is shown in Fig. 2(b) on choosing $\xi = 2\pi/M$ and $M = 4$. Figure 2(d) shows the time evolution of phases for $N = 15$ oscillators with this coupling

for randomly chosen initial conditions that converges to a stable 4-cluster state with partition (3·8). The clusters are located at $\phi_p = (p-1)\,\pi/2$, $p = 1, 2, 3, 4$ according to our design (3·1). As for (3·3), there also are many simultaneously-stable cluster states with up to four clusters.

One may check that the localized function (3·9) may result in a stable chain of M equidistant clusters (3·2) when taking $0 < \xi \leq 2\pi/(M+1)$. Now the p-th cluster is only influenced by the $(p-1)$-st cluster for $p = 2, \ldots, M$ but the first cluster is not influenced by the M-th cluster. The existence and stability of M-cluster states in this context are discussed in Appendix C. Figure 3(a) shows the localized coupling function (3·14, 3·15) for $\xi = 2\pi/(M+1.5)$ and $M = 4$, while Fig. 3(b) displays the corresponding time evolution of phases approaching $\phi_p = (p-1)\,\xi$, $p = 1, 2, 3, 4$ according to our design (3·2). Note that considering (3·14, 3·15) with $\xi = 2\pi/(M+k)$, $k \in \mathbb{N}^+$, the $(M+k)$-cluster partitions are stabilized. In this case an emerging M-cluster partition is in fact a special case of an $(M+k)$-cluster partition with k empty clusters.

3.3. *Finite state computation — 'abacus dynamics'*

Here we show that exploiting the dynamics above one can perform finite state computation by using multiple valued logic. We present the dynamical principles that allow the implementation of such logic. Our approach differs from the methodology used in Refs. 14) and 15) where complicated time dependent signals were applied on a system of nonidentical oscillators to obtain clustering.

Notice that when stabilizing equidistant clusters by using coupling functions (3·3) or (3·9) the number of distinct stable M-cluster partitions is M^N (each of the N oscillators may be placed into any of the M clusters). By applying large enough perturbations to individual oscillators one may 'transfer' them from one cluster to another and so steer the system between different M-cluster partitions. (Indeed, small perturbations decay since the cluster states are stable.) Here we demonstrate this concept by using the harmonic function (3·3) but one may generalize it for localized functions (3·9).

First, we notice that together with the stable M-cluster partitions with equidistant clusters at $\phi_p = (p-1)\,2\pi/M$, $p = 1, \ldots, M$, unstable M-cluster partitions are created with clusters located halfway between the stable clusters at $\tilde{\phi}_p = (p-0.5)\,2\pi/M$, $p = 1, \ldots, M$; see the dashed horizontal lines in Fig. 4. A large enough perturbation to an oscillator can drive it 'to the other side' of an unstable cluster, i.e. into another M-cluster state. More precisely, if one resets the phase of an oscillator by an amount that is larger than π/M (the distance between a stable and the neighboring unstable cluster in phase) the oscillator joins the neighboring stable cluster. In this way one may steer the system between different M-cluster partitions similarly as one sets an abacus into different states by sliding the beads.

The process is demonstrated in Fig. 4 for $M = 4$ clusters and $N = 15$ oscillators where the same initial conditions are used as in Fig. 2(c). We simply reset the phases of chosen oscillators by 1 or -1 (that is larger in magnitude than $\pi/4$) and so transfer

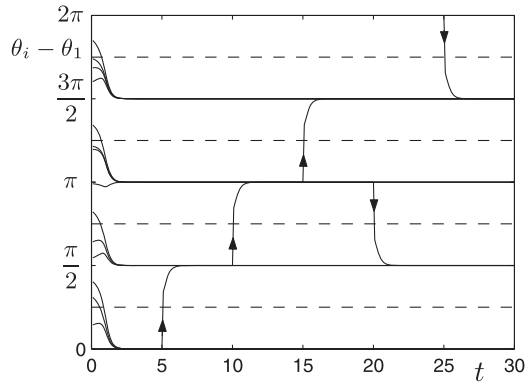

Fig. 4. Realizing the 'abacus dynamics' by transferring oscillators between $M = 4$ clusters for $N = 15$ oscillators. The same coupling function and initial conditions are used as in Fig. 2(c). The discrete path (3·16) is realized by resetting the phases of individual oscillators by ± 1 at $t = 5, 10, 15, 20, 25$. The horizontal dashed lines show the locations of unstable clusters: the applied resets must exceed the distance $\pi/4$ between stable and unstable clusters to lead to transitions.

them into other clusters. In particular, the discrete path

$$\begin{aligned}
&\{\{1, \underline{2}_{\rightarrow}, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10, 11\}, \{12, 13, 14, 15\}\}, \\
&\qquad\qquad \downarrow \\
&\{\{1, 3, 4\}, \{2, \underline{5}_{\rightarrow}, 6, 7\}, \{8, 9, 10, 11\}, \{12, 13, 14, 15\}\}, \\
&\qquad\qquad \downarrow \\
&\{\{1, 3, 4\}, \{2, 6, 7\}, \{5, 8, \underline{9}_{\rightarrow}, 10, 11\}, \{12, 13, 14, 15\}\}, \\
&\qquad\qquad \downarrow \\
&\{\{1, 3, 4\}, \{2, 6, 7\}, \{\underline{5}_{\leftarrow}, 8, 10, 11\}, \{9, 12, 13, 14, 15\}\}, \\
&\qquad\qquad \downarrow \\
&\{\{1, 3, \underline{4}_{\leftarrow}\}, \{2, 5, 6, 7\}, \{8, 10, 11\}, \{9, 12, 13, 14, 15\}\}, \\
&\qquad\qquad \downarrow \\
&\{\{1, 3\}, \{2, 5, 6, 7\}, \{8, 10, 11\}, \{4, 9, 12, 13, 14, 15\}\},
\end{aligned} \tag{3·16}$$

is realized between 4-cluster partitions where $\rightarrow$ and $\leftarrow$ denote phase resets by 1 and -1, respectively. Note that one may transfer oscillators between non-neighboring stable clusters by using larger resets. Since the cluster states are stable the above described computational scenario is robust against sufficiently small noise. For large numbers of oscillators the huge number of available partitions can facilitate large computations.

§4. Discussion

Our results demonstrate that the coupling function holds a lot of information about the possible dynamics of globally coupled oscillators: it is flexible enough to permit all possible clusterings to appear as attracting periodic orbits. Moreover, it

can be designed to ensure the existence and stability of specific cluster states and the designed systems may be capable of performing finite state computation.

We remark that the coupling function may also permit complex heteroclinic networks between cluster states of saddle type[2),4)] when the unstable manifold of a cluster state is contained in the stable manifolds of other cluster states. These state space structures may also be used for finite state computation and information encoding.[3),4),31)] However, designing the coupling function such that a particular heteroclinic network is realized is a difficult task: not only the existence and eigenvalue structure of specific saddle cluster states have to be ensured but also the existence of heteroclinic connections between them. We note that by tuning the coupling functions that permit heteroclinic networks one may find chaotic attractors.[4)] This shows that we have only scratched the surface of the dynamical complexity that may exist in globally coupled systems of identical oscillators.

Stabilizing multiple cluster states results in a robust system with enormous computational capabilities. The underlying network structure is however quite different from the VLSI (very-large-scale integration) design[24)] that our current day computers rely on, and also differs from CNN (cellular neural network) design.[35)] In VLSI and CNN systems large numbers of elements are connected to each other such that each element is linked to a few others. In contrast, high degree of connectivity is required to realize the abacus-like computation proposed in this paper. It shall be an interesting future research direction to invent computational algorithms for machines with this new architecture and also build them in the laboratory.

An interesting question arises when one wishes to build such systems: how sensitive they are with respect to time delays in the couplings (that are unavoidable in practice)? It is known that time delays introduce phase shifts to the coupling function of the phase models.[8),10),16)] Initial investigations show that as the delays are increased the designed cluster states may lose their stability so that the system would approach undesired cluster configurations. However, one may find windows of larger delays where the designed states are stable and so the system performs as it does for zero delay. We remark that to achieve certain emergent patterns in coupled oscillator systems, one may also design the coupling delays,[18),22)] the network topology,[25),26)] or/and the differences between individual oscillators.[14),15)] Exploiting such design principles may lead to the manifestation of brain-like neural computers of the future.[28)]

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Appendix A
Tangential and Transverse Eigenvectors

For the M-cluster partition

$$\mathcal{A} = \{\{1, \ldots, a_1\}, \{a_1 + 1, \ldots, a_1 + a_2\}, \ldots, \{N - a_M + 1, \ldots, N\}\} \tag{A·1}$$

one may show that the eigenvectors corresponding to the trivial eigenvalue (2·17) and the tangential eigenvalues (2·18) are given by

$$v_1^{\text{triv}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad v_p^{\text{tang}} = \begin{bmatrix} \alpha_{p,1} \\ \vdots \\ \alpha_{p,1} \\ \vdots \\ \alpha_{p,M} \\ \vdots \\ \alpha_{p,M} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} a_1 \\ \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} a_M \end{matrix} \tag{A·2}$$

for $p = 2, \ldots M$. The quantities $\alpha_{p,1}, \ldots, \alpha_{p,M}$ are complicated nonlinear functions of $g'_{p,q}$ where $p, q = 1, \ldots, M$, $p \neq q$. The structure of the tangential eigenvectors v_p^{tang} show that perturbations in these directions preserve the cluster structure.

Similarly for the p-th cluster with $a_p > 1$, the $(a_p - 1)$-dimensional eigenspace corresponding to the transverse eigenvalue (2·21) is spanned by

$$v_{M+p}^{\text{tran}} = c_{M+p,1} \begin{bmatrix} \vdots \\ 0 \\ 1 \\ \chi \\ \vdots \\ \chi \\ 0 \\ \vdots \end{bmatrix} + c_{M+p,2} \begin{bmatrix} \vdots \\ 0 \\ \chi \\ 1 \\ \vdots \\ \chi \\ 0 \\ \vdots \end{bmatrix} + \ldots + c_{M+p,a_p-1} \begin{bmatrix} \vdots \\ 0 \\ \chi \\ \vdots \\ 1 \\ \chi \\ 0 \\ \vdots \end{bmatrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} a_p \,, \tag{A·3}$$

where $\chi = -1/(a_p - 1)$ and the $a_p - 1$ scalars $c_{M+p,1}, \ldots, c_{M+p,a_p-1}$ are arbitrary. The structure of the vectors in v_{M+p}^{tran} corresponds to perturbations that split the p-th cluster by pulling one oscillator to the opposite directions than the others. For other M-cluster partitions with the same cluster sizes a_p, $p = 1, \ldots, M$ the tangential and transverse eigenvectors can be determined by permuting components of (A·2) and (A·3) appropriately.

Appendix B
Designing Stable 3-Cluster States for Odd g

Following the calculation shown in §2.1 we now consider the special case of odd coupling function $g(\varphi) = -g(-\varphi)$ that yields

$$g_0 = 0\,, \quad g_{1,2} = -g_{2,1}\,, \quad g_{2,3} = -g_{3,2}\,, \quad g_{1,3} = -g_{3,1}\,, \tag{B·1}$$

so Eq. (2·27) simplifies to

$$\begin{aligned} &(a_1 + a_2)\, g_{1,2} - a_3\, g_{2,3} + a_3\, g_{1,3} = 0\,, \\ &a_2\, g_{1,2} + a_2\, g_{2,3} + (a_1 + a_3)\, g_{1,3} = 0\,, \end{aligned} \tag{B·2}$$

with solution

$$\begin{bmatrix} g_{1,2} \\ g_{2,3} \\ g_{1,3} \end{bmatrix} = \text{const} \begin{bmatrix} a_3 \\ a_1 \\ -a_2 \end{bmatrix}. \tag{B·3}$$

Since the coupling function is odd, its the derivative is even $g'(\varphi) = g'(-\varphi)$ which results in

$$g'_{1,2} = g'_{2,1}\,, \quad g'_{2,3} = g'_{3,2}\,, \quad g'_{1,3} = g'_{3,1}\,. \tag{B·4}$$

Thus, (2·30) and (2·32) simplify to

$$\begin{aligned} \mu &= \frac{1}{N}\big((a_1 + a_2)g'_{1,2} + (a_2 + a_3)g'_{2,3} + (a_1 + a_3)g'_{1,3}\big)\,, \\ \nu &= \frac{4}{N}\big(a_1\, g'_{1,2}g'_{1,3} + a_2\, g'_{1,2}g'_{2,3} + a_3\, g'_{2,3}g'_{1,3}\big)\,, \\ \lambda_4^{\text{tran}} &= \frac{1}{N}\big(a_1\, g'_0 + a_2\, g'_{1,2} + a_3\, g'_{1,3}\big)\,, \\ \lambda_5^{\text{tran}} &= \frac{1}{N}\big(a_1\, g'_{1,2} + a_2\, g'_0 + a_3\, g'_{2,3}\big)\,, \\ \lambda_6^{\text{tran}} &= \frac{1}{N}\big(a_1\, g'_{1,3} + a_2\, g'_{2,3} + a_3\, g'_0\big)\,, \end{aligned} \tag{B·5}$$

and the conditions $\mu < 0$, $\nu > 0$, $\lambda_4^{\text{tran}} < 0$, $\lambda_5^{\text{tran}} < 0$, $\lambda_6^{\text{tran}} < 0$ can be satisfied by choosing, for instance,

$$\begin{aligned} &g'_{1,2} < 0\,, \quad g'_{2,3} < 0\,, \quad g'_{1,3} < 0\,, \\ &g'_0 < \min\left\{ -\frac{a_2\, g'_{1,2} + a_3\, g'_{1,3}}{a_1}, -\frac{a_1\, g'_{1,2} + a_3\, g'_{2,3}}{a_2}, -\frac{a_1\, g'_{1,3} + a_2\, g'_{2,3}}{a_3} \right\}. \end{aligned} \tag{B·6}$$

Appendix C
Chain of Equidistant Clusters by Localized Coupling Function

For a function satisfying (3·9) with $0 < \xi \le 2\pi/(M+1)$ we have

$$\begin{aligned} &g_{p,q} = 0\,, \\ &g'_{p,p} = -M \qquad \text{for} \quad p = 1, \ldots, M\,, \\ &g'_{p,p-1} = -M \quad \text{for} \quad p = 2, \ldots, M\,, \\ &g'_{p,q} = 0 \qquad \text{for any other} \quad p, q\,. \end{aligned} \tag{C·1}$$

Hence again (2·11) is trivially satisfied while (2·12) gives $\Omega = \omega$. For tangential stability the matrix (2·14) simplifies to

$$\begin{aligned} T_{1,q} &= 0\,, \\ T_{p,q} &= -\frac{M}{N}\Big(\delta_{p,q}a_{p-1} - (1-\delta_{p,q})\delta_{q,p-1}a_{p-1}\Big) \quad \text{for} \quad p = 2,\ldots,M\,. \end{aligned} \tag{C·2}$$

Since T is a lower triangular matrix, it has the eigenvalues

$$\begin{aligned} \lambda_1^{\text{triv}} &= 0\,, \\ \lambda_p^{\text{tang}} &= -\frac{M}{N}\,a_{p-1} < 0 \quad \text{for} \quad p = 2,\ldots,M\,, \end{aligned} \tag{C·3}$$

implying tangential stability (2·22). The transverse eigenvalues (2·21) evaluate to

$$\begin{aligned} \lambda_{M+1}^{\text{tran}} &= -\frac{M}{N}\,a_1 < 0\,, \\ \lambda_{M+p}^{\text{tran}} &= -\frac{M}{N}\,(a_p + a_{p-1}) < 0 \quad \text{for} \quad p = 2,\ldots,M\,, \end{aligned} \tag{C·4}$$

implying transverse stability (2·23).

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