Chaotic Dynamics of Falling Disks

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(June 3, 1997)

We investigate experimentally the behavior of falling disks in a fluid and identify several dynamical regimes as a function of the moment of inertia and Reynolds number: steady falling, periodic oscillating, chaotic, and continuous tumbling. One-dimensional iteration maps of the disk angles at turning points were constructed to explore the evolution of the dynamics, as boundaries between these regimes are crossed. We obtain the first experimental evidence of a progression from fixed-point to chaotic motion via a type-V intermittency, characterized by a discontinuity in the iteration map.

PACS numbers: 47.52.+j, 05.45.+b

In a certain lucrative bar trick, a naive bystander is asked to try to drop a playing card into a hat at his feet. The idea is demonstrated to him by the crafty challenger, who holds the card vertically and drops it over the hat. Each time the bystander drops the card in this manner, it flutters wildly in some apparently random direction and completely misses the hat. The challenger bets that he can hit the hat with just one drop. Bets are placed. The challenger then steps up and drops the card with a *horizontal* initial orientation, and it falls obediently into the hat. This simple trick illustrates several fundamental aspects of the dynamics of a falling thin plate. First, as shown by the behavior of the card when dropped vertically, the motion of such an object can be very chaotic. Here, one of the key features of chaotic dynamics—a severe sensitivity to the precise initial conditions—is what makes the scheme unprofitable for the bystander. Second, the severity of this dependence can itself be a function of the value of the initial condition. Thus, a card dropped with an initially horizontal orientation remains nearly so; for vertical distances typical for this trick, initially different trajectories diverge only slightly, reaping the challenger his profit.

Understanding the motion of bluff objects falling in a viscous medium is of great importance, and has applications in many disciplines, including meteorology [1], sedimentology [2], aerospace engineering [3], and chemical engineering [4]. The study of this problem goes back to at least Newton, who observed complex motion of objects falling in both air and water [3]. The problem was also studied by Maxwell, who discussed the motion of a falling strip of paper in a beautiful 1854 article [5]. More recently, Aref and Jones [6] have studied numerical solutions of Kirchhoff's equations for an object moving (in the absence of gravity and viscosity) in an ideal fluid. Tanabe and Kaneko [7] have taken a different approach, using a simplified model of a falling one-dimensional (1D) piece of paper, including within this model effects of lift and viscosity, but neglecting the inertial effects of the fluid. Both [6] and [7] find that the motion of moving or falling bodies may be chaotic.

Given the difficulty of studying this problem theoretically, many workers have chosen to examine experimental systems [3,4,8,9]. Of particular note is the work of Refs. [8] and [9]. These groups studied the behavior of disks falling in various liquids, and found a wide variety of different types of motion: steady-falling, periodic-oscillating, continuous tumbling, and "glide-tumbling" behaviors. Perhaps most interesting is this last type of motion. Here, the disk would oscillate back and forth several times as it fell, increasing its amplitude each oscillation until it completely turned over. It would then tumble several times "in an apparently random manner," and when the disks finally hit the container bottom, "Their ultimate location ...could never be predicted [8]." These statements were made several years before the development of the ideas of deterministic chaos, and yet they may contain at least the seeds of these ideas.

No experiments on this systems appear to have been done since the advent of modern chaos theory. The theoretical study of Ref. [6] neglects effects of fluid viscosity and gravity, while that of Ref. [7] focuses on 1D objects and ignores the important effect of fluid inertia. This somewhat unclear state of affairs has motivated us to perform systematic experimental studies of the motion of disks falling in a liquid, applying modern tools of dynamical systems.

The experiments are in principle simple. Steel and lead disks with diameters d = 5.1–18.0 mm and thicknesses t = 0.076–1.63 mm were dropped in water and water/glycerol mixtures from heights ranging from $\approx$ 0.3–1 m; some paper disks were also dropped in air, as discussed later. This combination of disk sizes and densities, and fluid viscosities, allowed us to explore a very large region of parameter space. Several different types of motion were observed, as shown in Fig. 1. These sequences were obtained by imaging from the side using a video camera; the digitized images were measured, and computer-drawn images are shown. Figure 1(a) shows the steady-falling regime observed at low Reynolds' number R: A disk, dropped with *any* initial orientation quickly settles down to a steady fall with horizontal orientation. Figure 1(b) shows the periodic-oscillating motion observed at higher R and low dimensionless moment of inertia I^*. Such disks oscillate with a well-defined period, again settling into this pattern after initial transients have died down. Figure 1(c) shows the chaotic motion characteristic when both R and I^* are moderately large. Typically, a disk in this regime will begin to oscillate with larger and larger amplitude until its angle is so high that it actually flips. It then tumbles several times and then suddenly jumps back to oscillating behavior. The number of tumbles and time between tumbling behaviors appear to the eye to be random. The disk may also suddenly change its overall direction of motion. Finally, Fig. 1(d) shows the tumbling motion found at very large I^*. Here, the disk turns continuously end-over-end while drifting in one direction.

What system parameters determine the type of motion observed? There are five material parameters: the disk diameter d, thickness t, and density ρ, as well as the fluid density ρ_f and kinematic viscosity ν. The parameter space implied by these five parameters is of course quite large. Fortunately, it is possible to reduce this number to only three independent dimensionless ratios [8]. The geometry of the disks may be characterized by the dimensionless moment of inertia $I^* = I_{\text{disk}}/\rho_f d^5 = \pi \rho t / 64 \rho_f d$. I^* is thus the ratio of the moment of inertia of a disk to a quantity proportional to the moment of inertia of a sphere of liquid with the same diameter. That we take the ratio to a *sphere* of liquid emphasizes the importance of the momentum of the liquid in determining the motion of the disk [8]. For small I^*, we expect the effects of the fluid to be important; at large I^* the moment of inertia of the disk dominates. A second dimensionless quantity is the Reynolds number $R = Ud/\nu$. Here, U is the mean vertical disk velocity. The final independent dimensionless quantity is the thickness-to-diameter ratio t/d. However, for the disks we consider here, $t/d \ll 1$, and we expect therefore that this ratio plays no role in the disks' dynamics.

To investigate the dependence of the disks' behavior on these two dimensionless parameters I^* and R, we have dropped a large number of disks and characterized their motion as steady falling, periodic oscillating, chaotic or tumbling. It was generally obvious to the eye to which of these types of motion any particular drop corresponded. We may then plot a "phase" diagram, indicating a disk's behavior as a function of I^* and R (Fig. 2). Also plotted on this diagram is the data of Refs. [8] and [9]; their data and ours agree quite well. We note that there is a well-defined boundary between each regime. This implies that the two dimensionless quantities I^* and R do in fact characterize a disk/liquid combination; it further implies that ignoring the small but finite thickness of these disks is justified.

Mapping out a phase diagram by direct observation allows a compact overview of the disks' behavior. However, it does not give us any physical insight as to *why* the disks fall as they do. To explore this question more fully, we have made direct observations of the trajectories of disks as they fall. A video camera and frame grabber allowed us to obtain side-view images of falling disks at a frame rate of 60 Hz. From such images, it is possible to obtain the two Cartesian coordinates of a disk's center in the vertical plane perpendicular to the direction of observation, as well as the direction in space of the symmetry axis of the disk. The Cartesian coordinate parallel to the direction of observation, and the angle of rotation around the symmetry axis, could not be recorded. A simple analysis of this motion would seem difficult, then, since the disk has six degrees of freedom, of which we record four.

Interestingly, we have found it possible to describe this complex behavior by a simple one-dimensional map. The single variable required is the angle θ between the disk's normal and the vertical, so that a horizontal disk has $\theta = 0$. To understand this, imagine dropping a disk from rest in a fluid, starting at an angle θ_i. As seen in Fig. 1(a), the disk will glide downwards and to the side, level off, and then begin to tilt in the other direction. Just when it has reached the extremum of its motion [arrow in Fig. 1(a)] the disk is almost stationary in all variables except for its vertical coordinate. Thus there is a well-defined final angle θ_f for a disk dropped from rest with initial angle θ_i. Since the motion is deterministic, this final angle θ_f of the disk must be a function f of its initial conditions—the initial angle equaling θ_i and the other coordinates and velocities being zero. Thus it must be true that $\theta_f = f(\theta_i)$.

We may now ask the interesting question as to whether this final angle θ_f may serve as the *initial* angle θ_i for the next oscillation, with the *next* θ_f determined from the same mapping $f(\theta_i)$ as above. This will be possible to the extent to which this angle alone completely specifies the initial conditions, even for a disk which has fallen quite far. How might this be so? At an extremum, it appears (cf. Fig. 1) that the angular velocities are essentially zero. So, too, are the horizontal translational velocities. The vertical velocity, while not zero, very rapidly reaches a terminal value which is about the same for all subsequent extrema; one could argue that this velocity, in dimensionless form, is simply the Reynolds number and is hence a *parameter*, not a dynamical variable, of the motion. Certainly the disk's center-of-mass coordinates are irrelevant if we consider the fluid to be infinite. We also expect from symmetry considerations that effects of rotations around the disk's axis will be small. Thus, the only relevant *disk* coordinate is the angle θ between the disk's plane and the horizontal.

The effect of the infinite degrees of freedom of the *fluid* is, of course, much more difficult to consider. A detailed study of the fluid dynamics as the disk approaches a given θ_f would undoubtedly reveal vortex shedding, turbulence, and other complex motion. However, the question we are considering is only whether the disk's subsequent motion is determined by its initial angle. Without knowing the details of the motion of the fluid, we can imagine that for a repetitive motion like the periodic oscillation of a disk, the motion of the fluid in the vicinity of the disk also repeats itself. Thus we can consider the fluid motion to be *included* in our description of the state of the system, which we describe by the single variable θ_f. This state is still determined by the previous θ_i, even thought it is determined by extremely complicated dynamics. We can thus think of $f(\theta_i)$ as a single-valued function, and can map it out empirically.

This idea of including the fluid dynamics in some variable which ostensibly describes the *disk* is not a novel one. For the related case of a body moving in an *ideal* fluid, Kirchhoff showed (see, e.g., Ref. [6]) that the equations of motion for such a body can be written so that the infinite degrees of freedom of the fluid are eliminated. Rather than explicitly including the fluid in the equations of motion, one can in effect ignore the fluid but think of the object as having a different (tensor) mass from its physical (scalar) mass. Its mass is transformed by way of the *added mass tensor*, whose elements depend only on the shape of the body.

In Fig. 3, we show eight iteration maps of θ_f vs θ_i, for several disk/liquid combinations. These maps were constructed as follows. A disk was dropped from rest with some initial angle θ_0. From the resulting digitized motion, a sequence of angles at the extrema of motion was measured. Each angle serves as an initial angle θ_i for the next final angle θ_f. By starting out at several different values of θ_0, a rather wide range of angles could be mapped out. To avoid transients, we did not include the initial angle from which the disk was dropped at rest in constructing these maps, making small initial angles sometimes difficult to achieve—the angle after even the first iteration was often already large.

Maps shown in Fig. 3(a)–(d) correspond to those points in the phase diagram of Fig. 2 labeled *a*–*d*. We see that the later form a vertical slice in phase space which cuts from the periodic oscillation region of low I^* through the boundary with the chaotic region of high I^*. The progression of the maps indicates how this change in dynamical behavior proceeds. In Fig. 3(a), we see that, near an initial angle of about $\theta_0 \approx 60°$, the map cuts across the diagonal line $\theta_f = \theta_i$ with a slope less than one. θ_0 thus represents a *stable fixed point*, and the motion will always settle down to periodic oscillations with an extremum angle of about $60°$. As we increase I^*, we come to point *b* of Fig. 2, which is near the border between periodic and chaotic behavior. In the iteration map of Fig. 3(b), we see that there appears to be a stable fixed point, although it is very near to $90°$. Thus, at the end of each oscillation, the disk is nearly vertical, and is quite near to actually flipping over. Proceeding further in I^*, we come to point *c* of Fig. 2, which is now clearly in the chaotic region. The map of Fig. 3(c) shows how this comes about. The map no longer crosses the diagonal, so there are no fixed points. Instead, a disk dropped with any initial angle very rapidly increases its successive angles until $\theta \geq 90°$. We see from the map, however, that there is no final angle defined for initial angles greater than about $80°$; this is because for initial angles greater than this the disk will actually flip over. We find that it then tumbles several times; this number is unpredictable. Then the disk suddenly is *reinjected* into the map at some low angle θ, and the process begins again. This progression from oscillations near a fixed point to chaotic behavior has been termed intermittency [10]. Finally, for very large I^* [Fig. 3(d)] the map is very far from the diagonal line, and the motion is strongly chaotic. Oscillation angles rapidly build towards $90°$, and the disk alternates rapidly between tumbling and oscillating. We have schematically indicated this complicated tumbling behavior by the rapidly fluctuating curve for $\theta \gtrsim 67°$ in Fig. 3(d).

A similar route to chaos is observed when taking horizontal slices through the phase diagram, as in runs *e*–*h* in Fig. 2. At low values of R (point *e*), the behavior is steady falling (Fig. 1a). Figure 3(e) shows the corresponding map. The map appears to be heading towards the origin with slope less than one, i.e., there is a stable fixed point at zero angle. Thus the disk is driven towards a state where all angles are zero, which is the steady falling situation. Point *f* of Fig. 2 appears to be right on the boundary between steady falling and periodic motion. In the corresponding iteration map [Fig. 3(f)], it is indeed difficult to judge whether the map goes through the origin without crossing the diagonal (a stable fixed point at $0°$), or leaves the origin with a slope greater than one and recrosses the diagonal at some low angle θ_0 (a stable fixed point at θ_0). By point *g* of Fig. 2, however, we are in the periodic regime, and the iteration map clearly crosses the diagonal at $\theta_0 \approx 70°$. Finally, the phase-space slice cuts through the periodic-chaotic boundary, and point *h* is chaotic, as shown by the map of Fig. 3(h). Again, a disk at any initial angle quickly moves into the tumbling regime, only to be reinjected into the oscillating regime at some apparently random time.

Pomeau and Manneville classify three types of intermittency depending on the way fixed points become unstable, i.e., in the way the eigenvalues of the map cross the unit circle [10]. The great majority of experimental systems exhibiting intermittency appear to be of one of these three types. This classification, however, does not cover maps that are not differentiable around fixed points [11]. Here, a fixed point can lose its stability by colliding with a discontinuity in the map. When the critical control parameter ϵ vanishes, the fixed point is stable in one direction of the 1D phase space and unstable in the other.

After passing the point of discontinuity, the iteration is reinjected "stochastically" into the laminar region. We observe this "type V" intermittency in our experiments.

Examining the progression from periodic to chaotic behavior described by the maps of Fig. 3, we see that for periodic maps, as in Fig. 3(a), the disk will fall forever oscillating at a stable fixed point; for the borderline map of Fig. 3(b), the angle of oscillation is very nearly 90°. As we enter the chaotic regime, a discontinuity indeed develops in the map [Fig. 3(c),(d)] at that value of $\theta_i = \theta_c$ which yields $\theta_f = 90°$. For $\theta_i > \theta_c$, the disk will tumble several times before being reinjected at some angle less than θ_c. Since this tumbling is deterministic, we expect that the map still exists—but is extremely complicated—for angles $\theta_i > \theta_c$. This part of the map is shown schematically in Fig. 3(d) as a rapidly fluctuating line. In this sense the map is essentially discontinuous at θ_c.

In principle, the time the disk remains in the oscillating or "laminar" regime should diverge, possibly logarithmically [11], as the control parameter—say, I^*—is varied past some I^*_c which reduces θ_c below 90°. We have not been able to quantitatively study any such divergence, probably because of the difficulty in accurately tuning I^* to values near I^*_c. Imperfections in the disks could well mask the details of this transition.

Finally, we turn to the final regime, that of continuous tumbling, observed at very large values of I^*. This regime cannot be analyzed in the same way as the previous three, since the disk never comes to rest and hence no values of θ_i are defined. It is actually not clear if this tumbling behavior is periodic in nature. The analysis of Maxwell [12] indicates that this might be so; however, direct video observation on lead disks shows some departure from periodicity [Fig. 1(d)]. Also, Ref. [9] has shown that the velocity of a tumbling disk's center of mass can vary by large ($\approx 60\%$) amounts from the mean value, in an apparently unpredictable manner. These two observations suggest that the rotational period in the tumbling regime is not constant. It is thus possible that the tumbling regime is also characterized by chaotic dynamics, although we have no definitive information on this point. It is also possible to access some extremely high values of I^* using paper disks in air, with its very low value of ρ_f. It appears to the eye, at least, that the rotational period of these is quite constant. It is possible, then, that at very high values of I^* there is another *periodic* tumbling regime.

In conclusion, we have studied the motion of falling disks, constructing a phase diagram which shows sharply-defined regimes of steady-falling, periodic, chaotic, and tumbling behavior. The apparently complex behavior of the disks can surprisingly be reduced to a series of 1D maps. In particular, the maps develop a discontinuity as the border between periodic and chaotic behavior is crossed. This discontinuity leads to an unusual type-V intermittency transition between the two behaviors, which we observe here for the first time.

We thank Prof. D.T. Walker for providing the video equipment. This work was supported in part by NSF REU grant number PHY-9300567.

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FIG. 1. Trajectories of falling disks, as captured on video and rendered by computer. (a) Steady falling, (b) periodic oscillating, (c) chaotic, and (d) tumbling trajectories. The arrow in (a) points to an extremum or turning point of angular motion.

FIG. 2. Phase diagram showing the behavior of the disks as a function of the two dimensionless parameters I^* and R. Closed symbols represent disks with steady falling or chaotic behavior, and open symbols periodic or tumbling behavior. Gray symbols represent cases judged to lie on the border between two regimes. The lettered points, enlarged for clarity, correspond to iteration maps (a)–(h) of Fig. 3.

FIG. 3. Iteration maps of falling disks. Sequence (a)–(d) represents a roughly vertical slice through I^*-R space, and (e)–(h) a horizontal slice. The curves are guides to the eye. In (d), the map is extended *schematically* as a rapidly fluctuating curve for angles above $\theta_c \approx 67°$.