A Problem in Enumerating Extreme Points, and an efficient Algorithm

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Abstract

We consider the problem of developing an efficient algorithm for enumerating the extreme points of a convex polytope specified by linear constraints. Murty and Chung [8] introduced the concept of a *segment* of a polytope, and used it to develop some steps for carrying out the enumeration efficiently until the convex hull of the set of known extreme points becomes a segment. That effort stops with a segment, other steps outlined in [8] for carrying out the enumeration after reaching a segment, or for checking whether the segment is equal to the original polytope, do not constitute an efficient algorithm.

Here we describe the central problem in carrying out the enumeration efficiently after reaching a segment. We then discuss two procedures for enumerating extreme points, the mukkadvayam checking procedure, and the nearest point procedure. We divide polytopes into two classes: Class 1 polytopes have at least one extreme point satisfying the property that there is a hyperplane H through that extreme point such that every facet of the polytope incident at that extreme point has relative interior point intersections with both sides of H; Class 2 polytopes have the property that every hyperplane through any any extreme point has at least one facet incident at that extreme point completely contained on one of its sides. We then prove that the procedures developed solve the problem efficiently when the polytope belongs to Class 2. The algorithm may also work when the polytope belongs to Class 1, but at the moment we do not have a proof that all its extreme points will be enumerated by the algorithm.

Key words Convex polytopes and their dual polytopes, FCFs(facetal constraint functions), enumeration of extreme points, adjacency, segments, mukkas, mukkadvayams, central problem and strategy for its solution, nearest points, linear and convex quadratic programs.

1 Introduction

We consider the problem of enumerating all the extreme points (also called vertices) of a convex polytope specified by a system of linear constraints.

For any matrix H, we denote by $H_{i.}, H_{.j}$ its ith row vector, jth column vector respectively. We will use the abbreviation "LP" for "linear program". Superscript T indicates transposition, i.e., for $y \in R^n$, y^T is its transpose. For any two sets C, D; we will denote the set of all elements in C which are not in D by $C \backslash D$.

If $\{a^1, \ldots, a^t\}$ is a set of points in R^n, we denote its convex hull by $< a^1, \ldots, a^t >$ or $< \{a^1, \ldots, a^t\} >$. The affine rank of $\{a^1, \ldots, a^t\}$ is defined to be the rank of $\{a^2 - a^1, \ldots, a^t - a^1\}$; it is the dimension of the affine space of $\{a^1, \ldots, a^t\}$.

A general system of linear constraints may consist of linear equations, inequalities and/or bounds on variables; when its set of feasible solutions is bounded (which is the case that we deal with in this paper), such a general system can be transformed into a system of the form (1) given below by simple transformations that preserve one-to-one correspondence between faces of the original and transformed systems. So, we consider the convex polytope K which is the set of feasible solutions of

$$Ax = b, \qquad x \geq 0 \tag{1}$$

where A is a matrix of order $m \times (n+m)$ and rank m. We assume that A, b are integer and that K is nonempty, bounded, and of dimension $n + m - m = n$. System (1) is said to be in **standard form**.

How to identify and eliminate redundant constraints in (1): For each $j = 1$ to $n + m$, solve the LP:

$$\begin{aligned} \text{Minimize} \quad & x_j \\ \text{subject to} \quad Ax &= b \\ x_i \geq 0 \qquad & \text{for all } i \in \{1, ..., j-1, j+1, ..., n+m\} \end{aligned}$$

If the minimum objective value in this LP is ≥ 0, then the inequality constraint $x_j \geq 0$ is a redundant inequality constraint in (1), eliminate it from (1) and make x_j an unrestricted variable. Now you can use one of the equality constraints in $Ax = b$ to eliminate the variable x_j from (1). This reduces both the number of variables and the number of equality constraints in (1) by 1, and results in another system in the same form as (1).

After this work is completed for all j, we will continue to denote the resulting system by (1), and its set of feasible solutions by K. Every constraint in the system is now nonredundant.

How to find the dimension of K and make sure that every inequality constraint in the system defines a facet of K: Solve $2(n+m)$ LPs to find $\alpha_j = \min\{x_j : \text{subject to } (1)\}$, $\beta_j = \max\{x_j : \text{subject to } (1)\}$; for all $j = 1$ to $n+m$. If $\alpha_j = \beta_j$ for any j, the variable x_j has this constant value over K, fix it at that value and eliminate it from (1), reducing the number of variables in (1) by 1. If $\alpha_j < \beta_j$ and $\alpha_j \neq 0$ for any j, take the transformation $x_j = \alpha_j +$ (new x_j) for each such j. The transformed system is again in the same form (1). K is bounded iff β_j is finite for all j, which we assume.

After this work is completed, we will continue to denote the system by (1), and its set of feasible solutions by K. The symbols $x, A, b, n, n+m$ will now refer to those of the final transformed system. We will now have $\alpha_j = 0 < \beta_j$ for all j, and the dimension of K is $n + m - m = n$. We will assume that these transformations have already been carried out. Now we will also have the property that the set of feasible solutions of

$$
\begin{aligned}
Ax &= b \\
x_j &= 0 \\
x_i &\geq 0, \quad i = 1, \ldots, j-1, j+1, \ldots, n+m.
\end{aligned}
$$

is a facet, denoted by F_j of K. Since the equation "$x_j = 0$" defines the facetal hyperplane containing the facet F_j, we will define "x_j" to be the **facetal constraint function (FCF)** corresponding to F_j.

Let $x_B = (x_{n+1}, \ldots, x_{n+m})$ be an arbitrarily selected feasible basic vector for (1), which will be fixed for the rest of the paper. Let $x_N = (x_1, \ldots, x_n)$. Then (x_B, x_N) is a basic, nonbasic partition of the variables in the vector x. The important property that holds in this (basic, nonbasic) partition is that B = corresponding basis = the $m \times m$ matrix consisting of column vectors of basic variables x_B in (1), is nonsingular and hence invertible. Let N be the $m \times n$ matrix consisting of column vectors of nonbasic variables in (1). Then, after rearranging variables, $Ax = b$ becomes $Bx_B + Nx_N = b$. So, (1) is equivalent to $x_B + B^{-1}Nx_N = B^{-1}b, \quad x_B, x_N \geq 0$. Eliminating the basic variables, K can be represented by the system of inequalities

$$
\begin{aligned}
B^{-1}Nx_N &\leq B^{-1}b \\
x_N &\geq 0
\end{aligned} \tag{2}
$$

in the space of the independent variables x_N in which it is of full dimension.

We will use this selected basic, nonbasic partition (x_B, x_N) of the variables in x in the following sections.

2 Historical Note

Extreme points of convex polyhedra were brought to prominence by George Dantzig when he introduced the simplex method for solving linear programs in 1947 [3]. This work of his is perhaps the most fundamental contribution in the annals of the study of convex polyhedra. Until then work on them remained the domain of abstract thinkers and their imagination. His work brought a new computational dimension to the study of convex polyhedra [9, 4].

Until Dantzig's work, visualization was only possible for 2-dimensional polyhedra. When you draw the picture of a 2-dimensional polyhedron, you can see (i.e., visualize) the adjacent extreme points of each extreme point, and thus get a "visual"-feel of the neighborhood of each of its extreme points. In higher dimensional polyhedra, we can do the same through computation using the primal simplex pivot steps. Thus, the techniques that originated in Dantzig's simplex method (in particular the wonderful computational tool of the primal simplex pivot step; and Phase I of the primal simplex method, the extension of the classical Gauss-Jordan method developed by the Chinese and Indians many centuries ago for solving linear equations into a method for solving linear inequalities) made it possible to compute any portion of a convex polyhedron that we want to see and visualize. One of the most important outcomes of Dantzig's work for mathematics is to bring about the possibility of clear visualization to convex polyhedra of dimension ≥ 3 through computation [9].

I will mention three example problems. On 2-dimensional polyhedra, answers to each of these three example problems can be found quickly and visually. In higher dimensions the same thing can be done through computation by solving a small number of linear and convex quadratic programming problems, illustrating this "visualization through computation": (1): Given a polytope Q specified through a system of linear inequalities in n variables, suppose we want to see whether Q is nonempty, and if it is nonempty see what its dimension is. All this can be accomplished through computation by solving a small number of linear programs. (2): Given a polytope P of full dimension in R^n specified through a system of linear inequality constraints, and a hyperplane H intersecting P specified by a linear equation, suppose we want to "see" how far H can move either left or right while still keeping a nonempty intersection with P. In each direction, this can be seen through computation by solving a single linear program. (3): Suppose we are given a subset S of extreme points of P. Suppose we want to "see" whether S is the set of all extreme points of P, and if not obtain an extreme point of P not in S. The algorithm discussed in this paper allows us to do that through computation, by solving a small number of linear and convex quadratic programs when K belongs to Class 2 defined in Section 8.

Ever since his work in 1947, the problem of enumerating all the extreme points of a convex polytope specified by linear constraints has been recognized as an important one.

There are several algorithms for this problem already, most of them are based on enumerating the feasible bases for (1), or faces of K of all dimensions. Public software programs implementing these algorithms are also available. But in all these algorithms, after computing r extreme points, the effort needed to compute the next one grows exponentially with r in the worst case.

An algorithm for this problem is said to be an efficient algorithm, or polynomial time algorithm, if it satisfies the following properties:

1. It should obtain the extreme points of K sequentially one after the other in a list.

2. When the list of known extreme points of K, $\{d^1, \ldots, d^r\}$, contains r extreme points, the effort needed in the algorithm to check whether it contains all the extreme points of K, and if it does not, to generate a new extreme point, should be bounded above by a polynomial in r and the size of system (1).

The goal of developing an efficient algorithm for this problem is more of a mathematical challenge, than a practical one. We will address this mathematical challenge.

The algorithm that we develop is based on solving a series of linear programs and convex quadratic programs , and checking the dimension of the optimal faces of some of the linear programs solved, which in turn requires solving more linear programs as discussed above. Proof of the polynomial time property of the algorithm is based on the well known fact that LPs and convex QPs can be solved by polynomial time algorithms. The reason why the proposed algorithm may not be practical while maintaining its polynomial time property, is its reliance on polynomial time algorithms for LP, convex QP which have not proven to be practical.

For the special case when (1) is the system of constraints in a pure network flow problem, Proven [10] has developed an efficient algorithm for this problem using the special properties of node-arc incidence matrices. We consider the case of general linear constraints.

In [5], the authors proved that the problem of enumerating all the extreme points of an unbounded convex polyhedron specified by linear constraints is hard, but the problem remains open if the polyhedron is bounded, i.e., if it is a polytope. We will consider this case, i.e., assume that K is a polytope in the sequel. Also, for the sake of simplicity, we will assume that exact arithmetic is used.

3 Relationship Between the Problems of Enumerating Extreme Points, and Facets

Consider a convex polytope P of dimension n in R^n. There are two different ways of representing it algebraically. These are:

1. Convex hull representation: Let $\{x^1, \ldots, x^\ell\}$ be the set of all extreme points of P. Then $P = < x^1, \ldots, x^\ell >$, the convex hull of its set of extreme points.

2. Halfspace intersection representation: P can be represented as the intersection of a finite number of halfspaces of R^n (i.e., as the set of feasible solutions of a system of linear inequalities).

These two different representations lead to two different enumeration problems associated with P. They are:

Extreme point enumeration problem: When P is given in halfspace intersection representation (i.e., as the set of feasible solutions of a system of linear inequalities), enumerate all the extreme points of P.

Facet enumeration problem: When P is given as the convex hull of its extreme points, enumerate all the facetal inequalities to represent it as the set of feasible solutions of a system of linear inequalities.

In the theory of convex polytopes, these two problems are known as the **duals** of each other. An algorithm for one of these problems can be used to solve the other problem directly. To show this, we denote by:

Procedure A: Any algorithm for enumerating the extreme points of P when it is specified as the set of feasible solutions of a nonredundant system of linear inequalities of full dimension.

Procedure B: Any algorithm for enumerating all the facetal inequalities to represent P when it is given as the convex hull of its extreme points.

How to enumerate all the facetal inequalities to represent P when the set of its extreme points $\{x^1, \ldots, x^\ell\}$ is given, using Procedure A

We assume that each x^t is an extreme point of $P = < x^1, \ldots, x^\ell >$ of full dimension n. Let $\bar{x} = (x^1 + \ldots + x^\ell)/\ell$, an interior point of P.

Take the transformation $x = y + \bar{x}$, let $y^t = x^t - \bar{x}$ for $t = 1$ to ℓ. Then $P = < x^1, \ldots, x^\ell > = < y^1, \ldots, y^\ell > + \bar{x}$. Also 0 is an interior point of $< y^1, \ldots, y^\ell >$.

Let $q = (q_1, \ldots, q_n)$ be a row vector of variables in R^n. Use Procedure A to generate all the extreme points of the polytope in q-space defined by the inequalities $qy^t \leq 1$, $t = 1$ to ℓ. Let $\{q^1, \ldots, q^M\}$ be the set of extreme points of this polytope. Then

$$< y^1, \ldots, y^\ell > \;=\; \{y : \quad q^s y \leq 1, \quad s = 1, ..., M\}$$

So, in terms of the space of variables $x \in R^n$ in which all the extreme points of the original P are specified,

$$P = \; < x^1, \ldots, x^\ell > \;=\; \{x : \quad q^s x \leq 1 + q^s \bar{x}, \quad s = 1, ..., M\}$$

with each inequality being a facetal inequality for P.

How to enumerate all the extreme points of $P = \{x : Qx \leq h\}$ using Procedure B

Let Q be of order $m \times n$. Assume that P is of full dimension and that each constraint in its representation given is nonredundant, i.e., that each inequality in this representation of P corresponds to a facet of P, and that the dimension of P is n. Find an interior point $\bar{x}$ of P by using the Low dimension step in Murty & Chung [8] (also discussed in the next section) to generate extreme points of P until the convex hull of generated extreme points is of full dimension. Take the transformation $x = y + \bar{x}$. Then $P = \{y : \quad Qy \leq h - Q\bar{x}\} + \bar{x}$. Also, since $\bar{x}$ is an interior point of P, $h - Q\bar{x} > 0$. For each $i = 1$ to m, define $a^i = Q_{i.}(1/(h_i - Q_{i.}\bar{x}))$ where $Q_{i.}$ is the i-th row vector of Q. Therefore $P = \{y : \quad a^i y \leq 1, \quad i = 1, ..., m\} + \bar{x}$.

Use Procedure B to generate a linear inequality representation of $< a^1, \ldots, a^m >$. Suppose it is $\{a : \quad ad^r \leq 1, \quad r = 1, ..., L\}$.

Then $\{d^1, \ldots, d^L\}$ is the set of extreme points of $\{y : \quad a^i y \leq 1, \quad i = 1, ..., m\}$.

So, the set of extreme points of P is $\{d^1 + \bar{x}, \ldots, d^L + \bar{x}\}$.

Here we will study the extreme point enumeration problem.

4 Key Facts Relating a Convex Polytope and Its Dual Polytope

The basic references on the properties of the dual polytope of a convex polytope, and on the relationships between the two of them are Grunbaum [4] in its latest edition, and Ziegler [12]. But here we will summarize the main facts that we will use in this paper.

The standard term **dual polytope** is called **polar polytope** in [12] and some other books; and they call the duality relationship between a polytope and its dual polytope as **polarity**. We will use the terms, dual polytope and duality, in this paper.

We consider a general polytope P of dimension n with 0 in its interior, represented by the system of inequalities

$$A_{t.}x \leq 1 \qquad t = 1 \text{ to } M$$

in $x \in R^n$, in which each inequality is nonredundant. Let $\{x^1, ..., x^N\}$ be the set of all its extreme points. Then its dual polytope Q in the space of variables $a \in R^n$ (here a is a row vector) is the set of feasible solutions of

$$ax^t \leq 1 \qquad t = 1, ..., N$$

and $\{A_{1.}, ..., A_{M.}\}$ is the set of all extreme points of Q. So, among the polytopes P, Q; extreme points of one of them are in one-to-one correspondence with facets (i.e., coefficient vectors of FCFs) of the other.

Also, let F be any nonempty proper face of P. Suppose $\{x_F^1, ..., x_F^s\}$ is the set of all extreme points of F. Let H be the face of Q defined by the system

$$\begin{aligned} ax_F^i &= 1 \qquad i = 1, ..., s \\ ax^t &\leq 1 \quad \text{for all } x^t \in \{x^1, ..., x^N\} \backslash \{x_F^1, ..., x_F^s\} \end{aligned}$$

Then H is said to be the face of Q corresponding to the face F of P. It can be verified that this correspondence between faces of P, Q is one-to-one. So, among P, Q every nonempty proper face of one of them corresponds to a unique nonempty proper face of the other under this correspondence. Also, the following relationship can be verified to hold:

(the dimension of a nonempty proper face of P) + (the dimension of the cooresponding face of Q) = $n - 1$.

For any face F of P let $\psi(F)$ denote the face of Q corresponding to F. Here is another important property.

Inclusion-reversing property: If F, G are two faces of P satisfying $F \subset G$, then $\psi(F) \supset \psi(G)$.

We will now derive a theorem that we will use in the proofs in Section 8.

Theorem 4.1: *Consider the pair of extreme points x^t, x^s of P; and the facets $\psi(x^t), \psi(x^s)$ of Q corresponding to them. Let $\Delta(x^t, x^s)$ denote the smallest dimensional face of P containing both x^t, x^s. Then $\psi(\Delta(x^t, x^s)) = \psi(x^t) \cap \psi(x^s)$.*

Proof: It is possible that there is no facet of P containing both x^t, x^s. In this case $\Delta(x^t, x^s) = P$ itself. In this case the face of Q corresponding to $\Delta(x^t, x^s) = P$

is the empty face $\emptyset$. Also, verify that $\psi(x^t) \cap \psi(x^s) = \emptyset$, so the result in the theorem holds for this case.

Now consider the case in which there is at least one facet of P containing both x^t, x^s. Let $F_1, ..., F_r$ be all the facets of P that contain both the extreme points x^t and x^s. Then $\Delta(x^t, x^s) = \cap_{i=1}^r F_i$. By the properties mentioned above, we see that $\psi(F_1), ..., \psi(F_r)$ are all the extreme points of Q contained in both its facets $\psi(x^t)$ and $\psi(x^s)$, so $\psi(x^t) \cap \psi(x^s)$ is the convex hull $< \{\psi(F_1), ..., \psi(F_r)\} >$. Again, by the above properties we see that the face of P corresponding to the face $\psi(x^t) \cap \psi(x^s)$ of Q is $\cap_{i=1}^r F_i = \Delta(x^t, x^s)$. □

5 Segments and Their Properties

Murty and Chung [8] introduced the concept of a *segment* of a polytope and used it to enumerate its extreme points and faces. They defined segments of various orders ranging from 1 to $(-2 +$ the dimension of the polytope$)$. Here we will only use segments of order 1, and hence we will refer to segments of order 1 as segments.

Definition 5.1: (Segment): A *segment* of a convex polytope K of dimension n is the convex hull, E, of a subset of its extreme points satisfying: **(i)** its dimension is the same as that of K, n, **(ii)** adjacency of extreme points on E coincides with that on K (i.e., every edge of E is also an edge of K).

In [8] the following two steps, LD and NS-1, have been developed for our problem (these names for the steps are new, these names have not been used in [8]).

LD (Low Dimension Step): Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K. If dimension$(\Gamma) <$ dimension(K), find the equation for a hyperplane containing Γ, in the space of the variables x_N in which K has full dimension. To do this, for $k = 1$ to r, let (d^k_B, d^k_N) be the partition of the vector d^k according to (x_B, x_N) partition selected in Section 1 for the variables x.

The dimension of $\{d^1, \ldots, d^r\}$ is the rank of the set of vectors $\{d^k_N - d^1_N : k = 2, \ldots, r\}$.

If this dimension is $< n$, the following homogeneous system of linear equations in variables f_N (written as a row vector)

$$f_N(d^k_N - d^1_N) = 0, \quad k = 2, \ldots, r$$

has a nonzero solution $f_N \neq 0$, find it. Let $\beta = f_N d^1_N$. Then all the extreme points $d^1, \ldots, d^r$ lie on the hyperplane represented by the equation $f_N x_N = \beta$ in the space of the independent variables x_N.

Now solve the two LPs, minimize $f_N x_N$, and maximize $f_N x_N$ subject to (1). One or both of these LPs will have as an optimum extreme point a point not in the current list. Call it d^{r+1}, add it to the list.

The application of the LD step can be repeated until the dimension of $\Gamma = < d^1, \ldots, d^r >$, the convex hull of the present list of known extreme points of K, becomes $= n =$ dimension of K. Once the dimension of Γ becomes $= n$, some new steps are needed to generate additional extreme points of K. One of these is the NS-1 step described next.

NS-1 (Non-Segment-1 Step): Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K satisfying dimension$(\Gamma) =$ dimension(K).

Let $p = (p_1, \ldots, p_{n+m})^T, q = (q_1, \ldots, q_{n+m})^T$ be a pair of extreme points of $\mathbf{K}$. They are adjacent on K iff rank of $\{A_{.j} : j$ such that either p_j or q_j or both are $> 0\}$ is $-1 +$ (its cardinality), nonadjacent on K iff the rank of this set is $< -1 +$ (its cardinality).

p, q are adjacent on $\Gamma = < d^1, \ldots, d^r >$ iff the following system in variables $c = (c_1, \ldots, c_{n+m})$ has a feasible solution, which can be checked by solving an LP.

$$\begin{aligned} c(p-q) &= 0 \\ c(p-d^k) &> 0, \quad \text{for all } k \text{ such that } d^k \neq p \text{ or } q. \end{aligned} \tag{3}$$

If a pair of extreme points of Γ; p, q say; which are adjacent on Γ, are not adjacent on K, then from the vector c satisfying (3); we get the supporting hyperplane $cx = cp$ of Γ that contains only extreme points p, q but no other extreme point of Γ, which is not a supporting hyperplane for K.

Now solve the LP: maximize cx over K to get an extreme point optimum solution. That extreme point may be a new extreme point of K not in the present list, but it could also be p or q itself. If it is a new extreme point of K not in the present list, add it to the list, and apply this NS-1 step again. If it is p or q, let $cp = \gamma$, now p, q are the only extreme points of K known at present on the face S of K determined by

$$\begin{aligned} Ax &= b \\ cx &= \gamma \\ x &\geq 0 \end{aligned}$$

Therefore by applying the LD step on S a new extreme point of K on S not in the present list can be determined in this case, add it to the list.

The application of NS-1 can be continued until at some stage the convex hull of the present list becomes a segment of K. To continue the enumeration then, some new steps are needed. One of those is LDF-1 given below.

LDF-1 (Low Dimension Facet Intersection Step): Let $\Gamma = \; < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K which is a segment of K. If there is a facet F of K satisfying dimension$(F \cap \Gamma) <$ dimension(F), apply the LD step on F and obtain a new extreme point of K on F and add it to the list.

The application of Steps NS-1, LDF-1 can again be continued until at some stage the conditions in the following definition hold for the extreme points in the present list (*mukka* is taken from a Telugu word meaning "a piece of an object which displays all the important properties of the orginal").

Definition 5.2 (Mukka): A *mukka* of a convex polytope K is the convex hull of a subset of extreme points of K which is a segment of K that has a full dimensional intersection with every facet of K.

When the convex hull of the present list of extreme points of K becomes a mukka of K, to continue the enumeration efficiently, some new steps are needed. These are discussed in the next section.

6 The Dual Polytope Step, to Continue the Enumeration

Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K which is a mukka of K. Here we will discuss how to use the dual polytope of Γ to continue the enumeration of extreme points of K at this stage. For details on dual polytopes of convex polytopes, see Section 4 and [4, 12]. I believe this is the first algorithmic use of the dual polytope of a convex polytope.

Cautionary note: The *duality* that we discuss in this paper refers is the duality between a convex polytope and its dual polytope mentioned in Section 4; not the "duality between linear programs" discussed in LP books.

The original polytope K in the space of nonbasic variables $x_N = (x_1, \ldots, x_n)^T$ selected in Section 1, has the representation

$$\begin{aligned} -Ix_N &\leq 0 \\ B^{-1}Nx_N &\leq B^{-1}b \end{aligned} \tag{4}$$

where I is the unit matrix of order n.

Let $\bar{d} = (d^1 + \ldots + d^r)/r$. The points $d^1, \ldots, d^r, \bar{d}$ correspond to $d^1_N, \ldots, d^r_N, \bar{d}_N$ in the space of nonbasic variables x_N. Translating the origin to $\bar{d}$, i.e., taking the transformation $x = y + \bar{d}$, in the space of independent variables $y_N = (y_1, \ldots, y_n)^T$, Γ becomes $\bar{d}_N + < y^1_N, \ldots, y^r_N >$; where $y^t = d^t - \bar{d} = (y^t_1, \ldots, y^t_{n+m})$, $y^t_N = (y^t_1, \ldots, y^t_n)^T$ for $t = 1$ to r, and $\bar{d}_N = (\bar{d}_1, \ldots, \bar{d}_n)^T$.

Carrying out the same transformation $x = y + \bar{d}$ on (4), we see that the representation of $\{y = x - \bar{d} : x \in K\}$ in the space of the y_N is

$$\begin{aligned} -Iy_N &\leq \bar{d}_N \\ B^{-1}Ny_N &\leq B^{-1}b - B^{-1}N\bar{d}_N \end{aligned} \tag{5}$$

Also, since 0 is an interior point of $< y^1, \ldots, y^r >$, we know that $\beta = (\beta_1, \ldots, \beta_m)^T = B^{-1}b - B^{-1}N\bar{d}_N > 0$, and $\bar{d}_N = (\bar{d}_1, \ldots, \bar{d}_n)^T > 0$. So, by dividing each constraint in (5) by its RHS constant and denoting the row vectors $q^i = (-I_{i.})/\bar{d}_i$ for $i = 1$ to n, and $q^{n+i} = (B^{-1}N)_{i.}/\beta_i$ for $i = 1$ to m, we can express (5) in the form (6) given below:

$$q^i y_N \leq 1 \quad \text{for } i = 1 \text{ to } n + m \tag{6}$$

In this representation of K in the space of the variables y_N, $q^i y_N$ is the FCF for the facet F_i corresponding to the i-th constraint in (6), and the vector q^i is its coefficient vector, it is known as the **FCF coefficient vector** corresponding to F_i in this representation. So, $\{y_N^1, \ldots, y_N^r\}$ is the present set of known extreme point solutions of (6), and $< y_N^1, \ldots, y_N^r >$ is a mukka for the set of feasible solutions of (6).

From Sections 3, 4 we know that $a_1 y_1 + \ldots + a_n y_n \leq 1$ represents a facet for $< y_N^1, \ldots, y_N^r >$ iff $a = (a_1, \ldots, a_n)$ is an extreme point solution of the system in variables a

$$a y_N^t \leq 1 \quad \text{for } t = 1 \text{ to } r \tag{7}$$

The set of feasible solutions of (7), Ω, is the **dual polytope** of $\Gamma = < y_N^1, ..., y_n^r >$, see Section 4 and [4, 12]. Every extreme point of Ω is the FCF coefficient vector in an inequality constraint with RHS constant 1 in a constraint representation of Γ.

When Γ is a mukka of K; the FCF coefficient vectors for K in its representation (6) (the points $q^1, \ldots, q^{n+m}$) are extreme point solutions of (7). Extreme points of (7) outside of the set $\{q^1, \ldots, q^{n+m}\}$ correspond to facets of Γ whose facetal hyperplanes are not facetal hyperplanes of K.

If $< q^1, \ldots, q^{n+m} >$ is not a mukka for the set of feasible solutions of (7) in the space of variables a, then the LD, NS-1, LDF-1 steps discussed in Section 5 can be applied to generate a new extreme point solution p for (7) which is not in the set $\{q^1, ..., q^{n+m}\}$. Then $p y_N \leq 1$ represents a new facet for $< y_N^1, \ldots, y_N^r >$ that does not correspond to a facet of the original polytope K. In this case an extreme point solution for the problem of maximizing $p x_N$ subject to (2), or (1) would lead to a new extreme point d^{r+1} for the original polytope K; add it to the list and then continue the enumeration of extreme points of K as in Section 5 again.

7 Mukkadvayams for K

The procedures discussed in Sections 5, 6, can help us to enumerate the extreme points of K until we get $\Gamma = < d^1, \ldots, d^r > =$ the convex hull of the present list of known extreme points of K satisfying the following properties 1 and 2.

1. Γ is a mukka of K.

2. Let $\bar{d} = (d^1 + \ldots + d^r)/r$; $y^t = d^t - \bar{d}$ for $t = 1$ to r; $\beta = (\beta_1, \ldots, \beta_m)^T = B^{-1}b - B^{-1}N\bar{d}_N$; $q^i = (-I_{i.})/\bar{d}_i$ for $i = 1$ to n, and $q^{n+i} = (B^{-1}N)_{i.}/\beta_i$ for $i = 1$ to m. Then $\{q^1, \ldots, q^{n+m}\}$, the set of FCF coefficient vectors of K in its representation (6) in the space of variables y_N, is a mukka for the set of feasible solutions Ω of the following system in variables $a = (a_1, \ldots, a_n)$

$$ay_N^t \leq 1 \quad \text{for } t = 1 \text{ to } r \tag{8}$$

Definition (Mukkadvayam) 7.1: At this stage when Γ satisfies both Properties 1 and 2 stated above, we will call it a *mukkadvayam* of K (*dvayam* is another Telugu word meaning *pair*, this name refers to the fact that not only is Γ a mukka of K at this stage, but $< q^1, \ldots, q^{n+m} >$, the convex hull of the FCF coefficient vectors in (6) defining K, is a mukka of the dual polytope Ω of Γ). Likewise, at this stage, $< q^1, \ldots, q^{n+m} >$ is a mukkadvayam of Ω.

It can be verified that until this stage, the enumeration is efficent. To continue the enumeration beyond this stage efficiently, some new steps are needed. At this stage, a face G of K is said to be

- a **complete face** of K for mukka Γ if $G \cap \Gamma = G$

- an **incomplete face** of K for mukka Γ if $G \cap \Gamma \neq G$

- a **star face** of K for mukka Γ if dimension$(G \cap \Gamma) <$ dimension(G).

Also, at this stage a row vector $p = (p_1, \ldots, p_n)$ which is an extreme point of Ω not contained in $\{q^1, \ldots, q^{n+m}\}$ is said to be

- a **star DPEP (dual polytope extreme point) vector** for Γ at this stage.

When $\Gamma \neq K$, there exist faces of K which are star faces for Γ (in fact every edge of K incident at an extreme point of K which is not in Γ, is a star face for Γ); and also star DPEP vectors for Γ (in fact if no star DPEP vector for Γ exists, then $q^1, \ldots, q^{n+m}$ are the only extreme point solutions for (8), which implies that Γ corresponds to the set of feasible solutions of (6), i.e., that $\Gamma = K$).

If a star face E of K for Γ can be found, then by applying the LD step on E for the known set $\{d^1, \ldots, d^r\} \cap E$ of extreme points on it, a new extreme point of E not in Γ can be found efficiently. Likewise if a star DPEP vector p for Γ can be found, then by maximizing px_N over K, a new extreme point of K not in Γ can be found. Hence efficient enumeration of extreme points becomes possible, if an efficient procedure can be developed to find either a star face of K with respect to the set of known extreme points of K, or a star DPEP vector for it. In fact in the process of generating a mukkadvayam for K, Steps NS-1, LDF-1 of Section 5 find and use star faces of K for the current set of extreme points Γ; and the dual polytope step of Section 6 finds and uses star DPEP vectors for Γ, to find a new extreme point of K not in Γ.

The central problem in enumerating extreme points efficiently: Given Γ, a mukkadvayam of K, develop a procedure for identifying a star face of K for Γ, or a star DPEP vector for Γ efficiently, or show that neither of them exist.

8 Mukkadvayam Checking Step

Note On this Section

In this section we will discuss the next step. We keep on applying this step as long as it is generating new extreme points of K not in the list, and stop using it only when a full application of it did not yield a new extreme point of K to add to the list. We also include in this section all the results and their proofs to show that at the end of this step the $\mathcal{F}_i = F_i \cap \Gamma$ for $i = 1$ to $n+m$ satisfy the property that the dimensions of $\mathcal{F}_i \cap \mathcal{F}_j$ and $F_i \cap F_j$ will be the same for all i, j; and a corresponding property holds for the dual polytope of Γ.

However, Hans Raj Tiwary [11] constructed a 5-dimensional counterexample to show that with all the steps developed so far including this mukkadvayam checking step, the procedure cannot guarantee that $\Gamma = K$ at termination.

We will classify polytopes of dimension n in R^n into two classes:

Class 1: The polytope K belongs to this class if it has at least one extreme point $\hat{y}_N$, and a hyperplane H through $\hat{y}_N$, such that every facet of K incident at $\hat{y}_N$ has relative interior point intersection with both sides of H.

Class 2: The polytope K belongs to this class if every hyperplane through any extreme point of K contains at least one facet of K incident at that extreme completely on one of its sides.

In Section 9 following this section we discuss the final step, and show in Section 10 that if the polytope K belongs to Class 2, then even without applying the mukkadvayam checking step of this section, the earlier steps together with the final step guarantee that all extreme points of K will be enumerated. In spite of this, the reason for including this mekkadvayam checking step and all the results on what can be guaranteed with it; is that I believe that when the mukkadvayam checking step is also included, all the extreme points of K can be guaranteed to be enumerated even when K belongs to Class 1. I do not have a proof of this yet.

Γ, the convex hull of the present list of extreme points of K, is a mukkadvayam when we come to this step. The work described in this section is based on a single procedure. In this step this procedure is applied on the dual polytope Ω of Γ, and on the original polytope K. We describe the application of this procedure on Ω, and on K separately below for clarity. If there are extreme points of K which are not in Γ, then these procedures may output a new extreme point of K not in Γ, we then add it to the list of extreme points generated, and go back to Section 5 to convert the augmented Γ into a mukkadvayam again.

Procedure on Ω

Let $\Gamma \;=\; < y_N^1, \ldots, y_N^r > =$ the convex hull of the present list of known extreme points of K in its representation (6) in the space of variables y_N, be a mukkadvayam of K. This is one step for checking whether $\Gamma = K$, if not this step may produce a new extreme point of K not in Γ. If a new extreme point is generated, we go back to Section 5 again to convert the new set of extreme points into a mukkadvayam again.

For a pair t, s in $\{1, ..., r\}$, define the following (in the LPs given, the variables are $a = (a_1, \ldots, a_n)$, the variables in (7) defining the dual polytope Ω of Γ).

$$\begin{aligned} \mu_{ts}(\Omega) = \quad & \text{Minimum value of } \; ay_N^s \\ & \text{subject to } \; ay_N^t \;=\; 1 \\ & \qquad ay_N^w \;\leq\; 1 \;\text{ for }\; w \in \{1, \ldots, t-1, t+1, \ldots, r\} \end{aligned}$$

$U_{ts}(\Omega) =$ optimum face of the above LP defining $\mu_{ts}(\Omega)$

$$\begin{aligned} \nu_{ts}(\Omega) = \quad \text{Maximum value of } & ay_N^s \\ \text{subject to } & ay_N^t \;=\; 1 \\ & ay_N^w \;\leq\; 1 \text{ for } w \in \{1, \ldots, t-1, t+1, \ldots, r\} \end{aligned} \tag{9}$$

$V_{ts}(\Omega)$ = optimum face of the above LP defining $\nu_{ts}(\Omega)$

$\mu_{ts}(\{q^1, \ldots, q^{n+m}\}) = \min\{ay_N^s : a \in \{q^i : i = 1, \ldots, n+m\} \text{ s. th. } q^i y_N^t = 1\}$

$U_{ts}(\{q^1, \ldots, q^{n+m}\})$ = convex hull of all the a that tie for the minimum defining $\mu_{ts}(\{q^1, \ldots, q^{n+m}\})$

$$\nu_{ts}(\{q^1, \ldots, q^{n+m}\}) = \max\{ay_N^s : a \in \{q^i : i = 1, \ldots, n+m \text{ s. th. } q^i y_N^t = 1\}\} \tag{10}$$

$V_{ts}(\{q^1, \ldots, q^{n+m}\})$ = convex hull of all the a that tie for the maximum defining $\nu_{ts}(\{q^1, \ldots, q^{n+m}\})$

If $\mu_{ts}(\Omega) < \mu_{ts}(\{q^1, \ldots, q^{n+m}\})$ [or $\nu_{ts}(\Omega) > \nu_{ts}(\{q^1, \ldots, q^{n+m}\})$], then an extreme point optimum solution for the LP defining $\mu_{ts}(\Omega)$ [$\nu_{ts}(\Omega)$] will lead to a star DPEP vector for Γ.

A note on checking the dimension of $V_{ts}(\Omega)$: The optimum face of a linear program is always a face of its set of feasible solutions. For an LP like (9), the system of constraints characterizing its optimum face is obtained from the system of constraints in (9) by changing some of the inequalities in it into equations. When (9) is solved, the complementary slackness optimality conditions in LP theory clearly identify which inequality constraints in (9) must be changed into equality constraints to get its optimum face. Then the dimension of this optimum face can be checked using the procedure discussed at the beginning of this paper.

If $\mu_{ts}(\Omega) = \mu_{ts}(\{q^1, \ldots, q^{n+m}\})$, but dimension$(U_{ts}(\Omega)) >$ dimension$(U_{ts}(\{q^1, \ldots, q^{n+m}\}))$, then $U_{ts}(\Omega)$ is a star face of Ω for its mukka $\{q^1, \ldots, q^{n+m}\}$. If $\nu_{ts}(\Omega) = \nu_{ts}(\{q^1, \ldots, q^{n+m}\})$, but dimension$(V_{ts}(\Omega)) >$ dimension$(V_{ts}(\{q^1, \ldots, q^{n+m}\}))$, then $V_{ts}(\Omega)$ is a star face of Ω for its mukka $\{q^1, \ldots, q^{n+m}\}$. In either of these cases, by applying the LD step on this star face for Ω we can find an extreme point of Ω not in $\{q^1, \ldots, q^{n+m}\}$, which leads to a star DPEP vector for Γ.

If a star DPEP vector p has been found, then find an an extreme point solution for the problem of maximizing px_N subject to (2), or (1); this will lead to a new extreme point d^{r+1} of the original polytope K not in the current list; add it to the list and then continue the enumeration of extreme points of K as in Section 5 again.

If $\mu_{ts}(\Omega) = \mu_{ts}(\{q^1, \ldots, q^{n+m}\})$, $\nu_{ts}(\Omega) = \nu_{ts}(\{q^1, \ldots, q^{n+m}\})$, dimension$(U_{ts}(\Omega))$ = dimension$(U_{ts}(\{q^1, \ldots, q^{n+m}\}))$, and dimension$(V_{ts}(\Omega))$ = dimension$(V_{ts}(\{q^1, \ldots, q^{n+m}\}))$ for all t, s; then this procedure has failed to find a star DPEP vector for Γ at this stage. In this case continue.

Procedure on K

Consider a facet F_i of K defined by the FCF $q^i y_N$ in its representation (6) in the space of variables y^N. For $i \in \{1, \ldots, n+m\}$, the FCF of the facet F_i of K is $q^i y_N$. $\Gamma, F_i \cap \Gamma$ are available to us as convex hulls of their extreme points. So, in the following, to optimize a linear function on Γ or $F_i \cap \Gamma$, we evaluate the linear function at each extreme point of the set, and select the best. Define:

$$
\begin{aligned}
\delta_{ij}(K) = \quad & \min\{q^j y_N : y_N \in F_i\} \\
\tau_{ij}(K) = \quad & \max\{q^j y_N : y_N \in F_i\} \qquad (11) \\
D_{ij}(K), T_{ij}(K) = \quad & \text{Optimum faces for the LPs defining } \delta_{ij}(K), \tau_{ij}(K) \text{ respectively} \\
\delta_{ij}(\Gamma) = \quad & \min\{q^j y_N : y^N \in F_i \cap \Gamma\} \\
\tau_{ij}(\Gamma) = \quad & \max\{q^j y_N : y^N \in F_i \cap \Gamma\} \qquad (12) \\
D_{ij}(\Gamma), T_{ij}(\Gamma) = \quad & \text{Optimum faces for the LPs defining } \delta_{ij}(\Gamma), \tau_{ij}(\Gamma) \text{ respectively, these sets are determined in terms of their extreme points.}
\end{aligned}
$$

Compute the $(n+m)\times(n+m)$ matrices $\delta(K) = (\delta_{ij}(K)), \tau(K) = (\tau_{ij}(K))$, $\delta(\Gamma) = (\delta_{ij}(\Gamma)), \tau(\Gamma) = (\tau_{ij}(\Gamma))$. For some i, j if either $\delta_{ij}(K) < \delta_{ij}(\Gamma)$ or dimension($D_{ij}(K)$) $>$ dimension($D_{ij}(\Gamma)$); then $D_{ij}(K)$ is a star face of K for Γ. Similarly if either $\tau_{ij}(K) > \tau_{ij}(\Gamma)$ or dimension($T_{ij}(K)$) $>$ dimension($T_{ij}(\Gamma)$); then $T_{ij}(K)$ is a star face of K for Γ at this stage. By applying the LD step on this star face using the known set of extreme points on it, a new extreme point of K not in the current list can be found; add it to the list and go back to Step 5 again to convert it into a mukkadvayam, and continue.

If $\delta(K) = \delta(\Gamma)$, $\tau(K) = \tau(\Gamma)$ and dimension($D_{ij}(K)$) = dimension($D_{ij}(\Gamma)$); and dimension($T_{ij}(K)$) = dimension($T_{ij}(\Gamma)$) then this Procedure has been unable to find a star face of K for Γ. In this case continue.

If both these procedures have failed to produce a new extreme point of K not in the present list, go to the final step described in the next section.

A Numerical Example

The two procedures in the Mukkadvayam checking step are actually the same principle applied either on the dual polytope Ω (dual polytope of Γ) and its mukka D; or on K and its mukka Γ. We will now provide a numerical example of the Procedure on K to illustrate this principle.

Example: Here we illustrate the application of the above Procedure on a polytope $K \subset R^4$ defined by the constraints

$$a_i(x) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 \leq 1, \quad i = 1, \ldots, 15$$

where $\{(a_{i1}, a_{i2}, a_{i3}, a_{i4}) : i = 1 \text{ to } 15\} = \{(1, 1, 1, -1),\ (1, 1, -1, 1),\ (1, 1, -1, -1),$ $(1, , -1, 1, 1), (1, -1, 1, -1), (1, -1, -1, 1), (1, -1, -1, -1), (-1, 1, 1, 1), (-1, 1, 1, -1),$ $(-1, 1, -1, 1), (-1, 1, -1, -1), (-1, -1, 1, 1), (-1, -1, 1, -1), (-1, -1, -1, 1), (-1, -1,$ $-1, -1)\}$. Here K is defined by a general system of linear constraints, which is not in standard form.

Let I denote the unit matrix of order 4. It can be verified that $\Gamma =$ convex hull of $\{I_{.j}, -I_{.j} : j = 1 \text{ to } 4\}$ is a mukka of K.

Let F_t be the facet of K on the facetal hyperplane defined by $a_t(x) = 1$, for $t =$ 1 to 15. $a_t(x)$ is the FCF corresponding to F_t for $t = 1$ to 15.

Consider the problem of maximizing $a_2(x)$, the FCF of F_2, on F_1 and on $F_1 \cap \Gamma =$ $< I_{.1}, I_{.2}, I_{.3}, -I_{.4} >$. The optimum objective value is 1 in both problems, but the dimension of the optimum face on F_1 is 2; while on $F_1 \cap \Gamma$, it is only 1. This shows that F_1 is an incomplete facet of K for Γ. A similar argument shows that the facets F_4 and F_8 are also incomplete facets of K for Γ.

Results on the Output Obtained By the Procedure Using Steps Discussed So Far

Let $\Gamma = < y_N^1, \ldots, y_N^r > =$ the convex hull of the present list of known extreme points of K in its representation (6) in the space of variables y_N, be a mukkadvayam of K, and suppose Γ is not $\neq K$. Then every facetal hyperplane of K is also a facetal hyperplane of Γ, but Γ must have some additional facetal hyperplanes which are not facetal hyperplanes of K; these additional facetal hyperplanes of Γ are called its **non-K-facetal hyperplanes**, and facetal hyperplanes of K are called the **K-facetal hyperplanes** of Γ. Also, the facets of Γ on these respective facetal hyperplanes are called its **non-K- facets, K-facets** respectively. Remember in this case both Γ and $K \backslash \Gamma$ have dimension n.

In the same way any face of Γ which is also a face of K is called a **K-face** of Γ, faces of Γ which are not faces of K are called **non-K-faces** of Γ. Let G be any non-K-facet of Γ. The following results can easily be verified at this stage.

1. There is an n-dimensional portion of $K \backslash \Gamma$ on the side of the facetal hyperplane corresponding to G not containing Γ, and the n-dimensional Γ on the other side.

2. Every non-K-facetal hyperplane of Γ has the property that all of Γ lies on one side of it, and an n-dimensional portion of $K \backslash \Gamma$ lies on the other side. This implies that a non-K-facet of Γ can never be a subset of a facet of K.

3. Since Γ is a mukka of K, all its vertices and edges, are vertices and edges of K. So the set of edges in any non-K-facet of Γ is a union of the sets of edges in several faces of K.

4. Since Γ is a mukka of K, the introduction of the inequality corresponding to G in the system of inequalities representing Γ, does not create any new extreme points, edges which are not extreme points, edges of K.

Let T be a face of K of dimension d, such that $T \cap G \neq \emptyset$, and $T \not\subset G$. Since Γ is a segment of K, when $d = 1$ or 2, $T \cap G$ must be a lower dimensional boundary face of T. When $d \geq 3$, there are two possibilities: (a) either $T \cap G$ is a lower dimensional boundary face of T, or (b) T contains d-dimensional portions on both sides of the facetal hyperplane corresponding to G.

Also, if T is a face of K of dimension 1 , and $T \cap \Gamma$ consists of more than a single extreme point of T; or if T is of dimension 2 and $T \cap \Gamma$ is more than a single extreme point or single edge of T; then T must be a subset of Γ, i.e., it must be a face of Γ.

5. Suppose a facet F_p of K has an intersection with G, which is an $(n-2)-$dimensional face of K. Then $F_p \cap G$ must be a facet of F_p, so, F_p is on one side of G. But since Γ is a mukka, this implies that F_p must be completely in Γ. But since $K \backslash \Gamma$ is n-dimensional, we again have a contradiction. This implies that any K-face of G must have dimension $\leq n-3$.

6. Suppose there is a d-dimensional face T of K which has d-dimensional portions on both sides of G with the property that $T \cap G$ has a $(d-2)$-dimensional face W of T. From properties 4 and 5, we know that $3 \leq d \leq (n-1)$; and since T has d-dimensional portions on both sides of G, W must be a common boundary facet of two facets of T, one lying on the Γ-side of G, and the other lying on the other side.

The following definitions and properties are used in deriving the results. For all $i = 1$ to $n+m$, $\mathcal{F}_i$ denotes the facet of Γ that is a subset of the facet F_i of K.

Definition 8.1: For any pair of extreme points y_N^s, y_N^t of Γ, define $\Delta(y_N^s, y_N^t)$ to be the smallest dimensional face of K that contains both y_N^s and y_N^t.

Definition 8.2: For any pair of extreme points q^i, q^j of $D = < q^1, ..., q^{n+m} >$ define $\Lambda(q^i, q^j)$ to be the smallest dimensional face of Ω that contains both of them.

Property 8.1: For every pair of extreme points y_N^s, y_N^t of Γ, $\Delta(y_N^s, y_N^t) \cap \Gamma$ is the smallest dimensional face of Γ containing both y_N^s, y_N^t and it has the same dimension as $\Delta(y_N^s, y_N^t)$. Also, if $\Delta(y_N^s, y_N^t)$ has dimension 1 or 2, then $\Delta(y_N^s, y_N^t) \cap \Gamma = \Delta(y_N^s, y_N^t)$, i.e., $\Delta(y_N^s, y_N^t)$ is a K-face of Γ.

Property 8.2: For every pair of extreme points q^i, q^j of D, $\Lambda(q^i, q^j) \cap D$ is the smallest dimensional face of D containing both q^i, q^j, and it has the same dimension as $\Lambda(q^i, q^j)$. Also, if $\Lambda(q^i, q^j)$ has dimension 1 or 2, then $\Lambda(q^i, q^j) \cap D = \Lambda(q^i, q^j)$, i.e., $\Lambda(q^i, q^j)$ is a face of D.

Theorem 8.1: *Let x, y be any two extreme points of a polytope P. Then the smallest dimension face of P containing both x, y is unique.*

Proof: Let the dimension of a smallest dimension face of P containing both x, y be d. Suppose P_1, P_2 are two faces of dimension d both of which contain both x and y. Then x, y are both contained in the face $P_1 \cap P_2$ of P which is a face of dimension $\leq d-1$, contradiction. So, the face of P Containing both x and y is unique. □

Theorem 8.2: *Let $\Gamma = \quad < y_N^1, ..., y_N^r >$ be a mukkadvayam of K. For some $t, s \in \{1, ..., r\}$, if the maximum objective value in (9), $\nu_{ts}(\Omega)$ is < 1, then the extreme point y_N^s of Γ is not contained on any of the facets of Γ incident at y_N^t, and conversely.*

If $\nu_{ts}(\Omega) = 1$, then y_N^s is contained on a facet of Γ incident at d^t, and conversely.

Proof: Extreme points of Ω defined by (8) are facetal vectors of Γ and vice versa. From this we see that extreme points of the set of feasible solutions of (9) are the facetal vectors of facets of Γ incident at y_N^t. So, $\nu_{ts}(\Omega) = 1$ iff there exists a facetal vector (a row vector), θ, of a facet of Γ containing y_N^t, satisfying $\theta y_N^s = 1$, but this shows that y_N^s satisfies the equation for that facet; hence $\nu_{ts}(\Omega) = 1$ iff there is a facet of Γ containing both y_N^t and y_N^s.

The same argument shows that $\nu_{ts}(\Omega) < 1$ iff y_N^s is not contained on any of the facets of Γ incident at y_N^t. □

Theorem 8.3: *For any $t, s \in \{1, ..., r\}$, $\nu_{ts}(\{q^1, \ldots, q^{n+m}\})$, the maximum objective value in (10), is = 1 iff y_N^s is contained on a K-facet of Γ incident at y_N^t. $\nu_{ts}(\{q^1, \ldots, q^{n+m}\}) < 1$ iff y_N^s is not contained on a K-facet of Γ incident at y_N^t.*

Proof: Follows from the arguments in the proof of Theorem 8.2, and the fact that $\{q^1, \ldots, q^{n+m}\}$ is the set of all FCF coefficient vectors corresponding to K-facets of Γ in the representation of K in (6) □

Theorem 8.4: *Suppose there exists vertices y_N^t, y_N^s of Γ such that y_N^s is not contained on any K-facet of Γ incident at y_N^t, but contained in a non-K-facet of Γ incident at y_N^t. Then $\Delta(y_N^s, y_N^t) = K$ is not a subset of Γ; and in this case $\nu_{ts}(\Omega) > \nu_{ts}(\{q^1, \dots, q^{n+m}\})$, so the Procedure on Ω in the Mukkadvayam checking step outputs a star DPEP vector for Γ.*

Proof: Here y_N^s is an extreme point of Γ incident to a non-K-facet of Γ incident at y_N^t, but not on any of the K-facets of Γ incident at y_N^t. By Theorems 8.2, 8.3 we then have $\nu_{ts}(\Omega) = 1$, and $\nu_{ts}(\{q^1, \dots, q^{n+m}\}) < 1$. The result in the theorem follows from this. □

Theorem 8.5: *Suppose there exists a pair of vertices y_n^t, y_N^s of Γ such that:* **1.** *y_N^s is contained on some K-facets of Γ incident at y_N^t, and also at least one non-K-facet G say, of Γ incident at y_N^t;* **2.** *$\Delta(y_N^t, y_N^s) \cap \Gamma$ is not the smallest dimensional face of Γ containing both y_N^t and y_N^s.*

Then with these t, s, both $\nu_{ts}(\Omega), \nu_{ts}(\{q^1, \dots, q^{n+m}\})$ are equal to 1. But the dimensions of the optimal faces in (9) and (10) are different. So the Procedure on Ω in the Mukkadvayam checking step outputs a star DPEP vector in this case.

Proof: Let the dimension of $\Delta(y_N^t, y_N^s)$ be h. Remember that $\Delta(y_N^t, y_N^s)$ is the smallest dimension face of K containing both y_N^t and y_N^s, so no facet of $\Delta(y_N^t, y_N^s)$ contains both y_N^t and y_N^s. Also since $\Delta(y_N^t, y_N^s)$ is not a subset of Γ, it must have h-dimensional portions on both sides of G. From condition **1** stated in the theorem, and Theorem 8.2 we see that that both $\nu_{ts}(\Omega), \nu_{ts}(\{q^1, \dots, q^{n+m}\})$ are equal to 1.

Suppose $F_{i_1}, ..., F_{i_v}$ are all the facets of K that contain both y_N^t and y_N^s. Then $\cap_{p=1}^{v} F_{i_p}$ is the smallest dimensional face of K that contains both y_N^t, y_N^s, i.e., it is $=$ $\Delta(y_N^t, y_N^s)$.

But since $\Delta(y_N^t, y_N^s) \cap \Gamma$ is not the smallest dimensional face of Γ containing both y_N^t, y_N^s; that smallest dimensional face of Γ containing both y_N^t, y_N^s must be a lower dimensional face of $\Delta(y^t_N, y^s_N) \cap \Gamma$; i.e., it is a face of $\Delta(y^t_N, y^s_N) \cap G$.

Since the dimensions of $\cap_{p=1}^{v} F_{i_p}$ and $G \cap (\cap_{p=1}^{v} F_{i_p})$ are different, from the correspondence between faces of Γ and its dual polytope Ω, we see that the dimensions of the optimal faces in (9) and (10) are different. So the Procedure on Ω in the Mukkadvayam checking step outputs a star DPEP in this case, and hence a new extreme point of K not in Γ. □

Theorem 8.6: *When Γ is a mukkadvayam of K, and the Procedure on Ω in the Mukkadvayam checking step does not produce a new extreme point of K not in Γ, then Property 8.1 stated above must hold for Γ.*

Proof: This follows from Theorems 8.4, 8.5; and the results listed under item 4 above. □

Here Γ is a mukka of K; and D, Ω (dual polytopes of K and Γ) are such that D is a mukka of Ω. Also, notice that in the Mukkadvayam checking step

the Procedure on Ω applies on Ω and its mukka D

the Procedure on K applies on K and its mukka Γ.

It can be verified that the two procedures are the duals of each other. Theorems 8.2 to 8.6 relate to the performance of the Procedure on Ω. Now we will discuss corresponding results regarding the performance of the Procedure on K.

Theorem 8.7: *For $i, j \in \{1, ..., n+m\}$, $\tau_{ij}(K)$ defined in (11) is 1 iff $F_i \cap F_j \neq \emptyset$; and $\tau_{ij}(K) < 1$ if $F_i \cap F_j = \emptyset$. Likewise $\tau_{ij}(\Gamma)$ is 1 iff $\mathcal{F}_i \cap \mathcal{F}_j \neq \emptyset$, $\quad < 1$ otherwise. So, the Procedure on K in the Mukkadvayam checking step outputs a star face of K for Γ if there exists a pair $i, j \in \{1, ..., n+m\}$ such that $F_i \cap F_j \neq \emptyset$, but $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.*

Proof: This theorem follows because $q^j y_N = 1$ iff $y_N \in K$ is on the facet F_j, it is < 1 otherwise. So, if there exists $i, j \in \{1, ..., n+m\}$ such that $F_i \cap F_j \neq \emptyset$, but

$\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$, then for that i, j we have $\tau_{ij}(K) = 1$, but $\tau_{ij}(\Gamma) < 1$, so, $T_{ij}(K)$ is a star face of K for Γ. □

Theorem 8.8: *If there exists $i, j \in \{1, ..., n+m\}$ such that $\mathcal{F}_i \cap \mathcal{F}_j \neq \emptyset$ and $\Lambda(q^i, q^j) \cap D$ is not the smallest dimensional face of D containing both q^i, q^j, then with these i, j both $\tau_{ij}(K)$ and $\tau_{ij}(\Gamma)$ are $= 1$, but the dimensions of the optimal faces of (11), (12) are different, and the Procedure on K in the Mukkadvayam checking step will output a star face of K For Γ.*

Proof: The proof of this theorem is similar to the proof of Theorem 8.5. It applies the same arguments as there, but on Ω and its mukkadvayam D instead of K and its mukkadvayam Γ. □

Theorem 8.9: *If the procedure on K in the mukkadvayam checking step does not output a star face of K For Γ, then Property 8.2 stated above must hold for the mukka D of Ω.*

Proof: This follows from Theorems 8.7 and 8.8, and the results corresponding to those in item 4 above to the mukka D of Ω. □

Theorem 8.10: *When Γ is a mukkadvayam of K, and the Mukkadvayam checking step does not produce a new extreme point of K not in Γ, then $\mathcal{F}_i \cap \mathcal{F}_j$ and $F_i \cap F_j$ have the same dimension for all i, j. Likewise $\mathcal{H}_t \cap \mathcal{H}_s$ and $H_t \cap H_s$ have the same dimension for all t, s.*

Proof: For any $i, j \in \{1, ..., n+m\}$, if $F_i \cap F_j \neq \emptyset$, then $T_{ij}(K) = F_i \cap F_j$, and $\Lambda(q^i, q^j)$ is the face of Ω corresponding to the face $\mathcal{F}_i \cap \mathcal{F}_j$ of Γ.

By Theorem 8.9, if the Procedure on K in the mukkadvayam checking step does not help in outputing a new extreme point of K not in Γ, then Property 8.2 must hold, i.e., $\Lambda(q^i, q^j) \cap D$ is the smallest dimesional face of D containing both q^i, q^j. Hence in this case $\Lambda(q^i, q^j) \cap D$ corresponds to the face $F_i \cap F_j$ of K. By Property 8.2, the dimensions of $\Lambda(q^i, q^j) \cap D$ and $\Lambda(q^i, q^j)$ are the same for all i, j. Therefore, $F_i \cap F_j$ and $\mathcal{F}_i \cap \mathcal{F}_j$ have the same dimension for all i, j.

Also at this stage $D = \ < q^1, \ldots, q^{n+m} >$ (the convex hull of the set of FCF coefficient vectors of K in its representation (6)) is a mukkadvayam of Ω. For $j = 1$ to r, let $H_j = \{a : ay_N^j = 1\}$ be the facet of Ω corresponding to the extreme point y_N^j of Γ. Let $\mathcal{H}_j$ be the facet of D that is a subset of H_j.

Now cosider a pair s, t in $\{1, ..., r\}$. y_N^s, y_N^t correspond to the facets H_s, H_t of Ω; and the facets $\mathcal{H}_s, \mathcal{H}_t$ of D.

By Theorem 4.1, the face $\Delta(y_N^s, y_N^t)$ of K corresponds to the face $\mathcal{H}_s \cap \mathcal{H}_t$ of D.

$\Delta(y_N^s, y_N^t) \cap \Gamma$ being the smallest dimensional face of Γ that contains both y_N^s, y_N^t corresponds to the face $H_s \cap H_t$ of Ω.

So, under Property 8.1, face $\mathcal{H}_s \cap \mathcal{H}_t$ of D, and $H_s \cap H_t$ of Ω, both have the same dimension for all t, s. $\square$

9 The Final Step

Here is the notation for the present entities that we are dealing with: K is the original polytope defined by (6) in the space of variables $y_N \in R^n$; F_i, $i = 1$ to $n + m$ its facets; $\{y_N^1, ..., y_N^r\}$, the present list of extreme points of K generated so far; $\Gamma = \ < y_N^1, ..., y_N^r >$, a mukka of K; $\mathcal{F}_i = F_i \cap \Gamma$ for $i = 1$ to $n + m$ are facets of Γ; Ω, the dual polytope of Γ defined by (7) in the space of variables $a = (a_1, ..., a_n)$; $D = \ < q^1, ..., q^{n+m} >$, convex hull of FCF coefficient vectors of K , a mukka of Ω; $H_1, ..., H_r$ the facets of Ω; and $\mathcal{H}_t = D \cap H_t$ the facet of D corresponding H_t of Ω for $t = 1$ to r.

From the mukkadvayam property we know that all $\mathcal{F}_i, \mathcal{H}_j$ are of dimension $n - 1$.

The work described in this section is based on a single procedure that is applied on K, and then on Ω. It consists of solving a series of convex quadratic programs. Here $||x - y||$ denotes the Euclidean distance between points x, y in R^n. Even though applying this procedure on K is sufficient; we discuss the application on both K and Ω for the sake of completeness.

Procedure on K

Do the following for each $i = 1$ to $n + m$.

For each extreme point y_N^t of Γ such that $y_N^t \notin F_i$ do the following: First find the optimum solutions of the following two convex quadratic programs:

$$\begin{aligned} \text{Minimize} \quad & ||y_N^t - y_N||^2 \\ \text{subject to} \quad & y_N \in F_i \end{aligned} \qquad (13)$$

and for simplicity let us denote the optimum solution of this problem by the symbol x_i^{*t}.

$$\begin{aligned} \text{Minimize} \quad & ||y_N^t - y_N||^2 \\ \text{subject to} \quad & y_N \in \mathcal{F}_i \end{aligned} \qquad (14)$$

and let us denote the optimum solution of this problem by the symbol x_i^{+t}. To formulate (14) as a regular quadratic program we can expresss each $y_N \in \mathcal{F}_i$ as $y_N = \sum(\alpha_s y_N^s :$ over $s \in \{1, ..., r\}$ such that $y_N^s \in \mathcal{F}_i)$, where the α_s are variables subject to the constraints $\sum(\alpha_s : s$ such that $y_N^s \in \mathcal{F}_i) = 1$, and all these α_s are ≥ 0; and solve the above quadratic program in terms of these α_s as the variables.

Now if $x_i^{*t} \neq x_i^{+t}$, this is an indication that $\mathcal{F}_i \neq F_i$.

Let Q be the smallest dimensional face of K containing x_i^{*t} (i.e., Q is the intersection of all the facets of K containing x_i^{*t}). In the affine space of Q find a feasible solution c, β to

$$\begin{aligned} cx_i^{*t} &= \beta \\ cy_N^t &\leq \beta \quad \text{for all } t = 1 \text{ to } r \text{ such that } y_N^t \in Q \end{aligned} \tag{15}$$

Such a c, β exist because $x_i^{*t} \notin \mathcal{F}_i$. So, if you maximize cy_N over $y_N \in Q$ for an extreme point optimum, you will get a new extreme point of F_i not in Γ, add it to the present list and return to Section 5 with the augmented list.

Even if $x_i^{*t} = x_i^{+t}$, check if (15) has a solution. If it does, for any solution c obtained for it, maximizing cy_N over $y_N \in Q$ to any extreme point optimum, may yield a new extreme point of Q not in Γ to add to the list and return to Section 5. If either (15) has no solution, or if it does and an extreme point optimum maximizing that cy_N over $y_N \in Q$ is already in the list, continue.

At the end of the procedure, if no new extreme point of K is produced by this procedure on K, go to the Procedure on Ω described next.

Procedure on Ω

This is the same procedure applied on Ω instead of K. For this procedure Ω, D, H_t, q^i, $\mathcal{H}_t$ replace K, Γ, F_i, y_N^t, $\mathcal{F}_i$ respectively in the above procedure. In this procedure, for each $t = 1, ..., r$; and each $i \in \{1, ..., n+m\}$ such that $q^i \notin H_t$ two quadratic forms of the form (13), (14) (to minimize $||q^i - q||^2$ over $q \in H_t$, and $q \in \mathcal{H}_t$ respectively) are solved. If the optimum solutions of the two quadratic programs are not the same, it is an indication that $\mathcal{H}_t \neq H_t$, in this case by solving another two LPs similar to those described above we can get a new star DPEP vector for Γ will be obtained. Even when the optimum solutions of the two quadratic programs are the same, by solving another two LPs similar to those described above we may get a new star DPEP vector for Γ. If a star DPEP vector for Γ is obtained, using it a new extreme point of K not in Γ can be obtained as described in Section 6, add it to the present list of extreme points of K, and return to Section 5 with the augmented list. Otherwise continue.

At the end of both the procedures, if no new extreme points of K are produced; conclude that the present list of extreme points of K includes all the extreme points of K and terminate.

10 Results

Let $\Gamma = < y_N^1, \ldots, y_N^r > =$ the convex hull of the present list of known extreme points of K in its representation (6) in the space of variables y_N, be a mukkadvayam of K, and suppose we are at the stage where the final step has been applied and it has terminated without producing any new extreme points of K not in the present list. As before, for $i = 1$ to $n + m$, F_i is the i-th facet of K, and $\mathcal{F}_i = F_i \cap \Gamma$.

Theorem 10.1: *If the final step does not output a new extreme point of K not in Γ, then $\Gamma = K$ if the polytope belongs to Class 2 discussed at the beginning of Section 8.*

Proof: We assume that K belongs in Class 2. So, if H is a hyperplane through an extreme point of K, then there is at least one facet of K incident at that extreme point completely contained on one side of H.

Suppose at some stage $\Gamma \neq K$. Let y_N^{r+1} be an extreme point of K which is not in Γ at that stage.

The main results on which the proof of this theorem will be based are the following elementary ones from 2-dimensional geometry which can easily be verified: Consider a triangle T with vertices V_1, V_2, V_3 drawn with the edge V_2V_3 horizontally and the point

V_1 underneath it. T is part of a cone C with vertex V_1 obtained by extending the lines joining V_1 to V_2, V_3 indefinitely. Let C_1, C_2 be the edges of C containing the points V_2, V_3 respectively. If the angle $V_3V_2V_1$ at V_2 in the triangle T is acute, then the nearest point (by Euclidean distance) to V_3 in C_1 lies on the corresponding edge V_1V_2 of T. Also, since at least one of the two angles $V_2V_3V_1$, $V_3V_2V_1$ at V_3, V_2 respectively of T must be acute; at least one of the vertices V_2, V_3 of T must satisfy the property that the nearest point to it on the edge of C not containing it lies in the corresponding edge of T. And finally, if the orthogonal projection of V_1 on the line joining V_2 and V_3 is in the edge V_2V_3 of T, then both vertices V_2, V_3 of T satisfy the above property.

Let $F_1, F_2, ..., F_k$ be all the facets of K incident at y_N^{r+1}; $H_1, ..., H_k$ the corresponding facetal hyperplanes; and $\mathcal{F}_1, ..., \mathcal{F}_k$ the corresponding facets of Γ. Let C_K denote the cone with vertex y_N^{r+1} and with $H_1, ..., H_k$ as its facetal hyperplanes. Let C_Γ denote the cone with vertex y_N^{r+1} generated by all the half-lines joining y_N^{r+1} to each extreme point of Γ and continuing in that direction. Clearly $C_\Gamma \subset C_K$, and C_K is a pointed cone.

Let P denote the nearest point (by Euclidean distance) to y_N^{r+1} in Γ. Let $\bar{\Gamma}$ denote the convex hull of $\Gamma \cup \{y_N^{r+1}\}$. For each $i \in \{1, ..., k\}$, $< (\mathcal{F}_i \cup \{y_N^{r+1}\}) >$ is a facet of $\bar{\Gamma}$ incident at y_N^{r+1} and these are all the facets of $\bar{\Gamma}$ incident at y_N^{r+1}.

We consider several cases depending on the location of P in Γ.

Case 1: P is not contained on any of the K-facets of Γ, but is a relative interior point of one of the non-K- facets of Γ separating Γ from y_N^{r+1}.

Let G be the non-K-facetal hyperplane of Γ containing P, and $\mathcal{B}_0$ the ball with y_N^{r+1} as center and P as a boundary point. Then G is the tangent plane to $\mathcal{B}_0$ at P; and G separates $\mathcal{B}_0$ from Γ. Then all the extreme points of Γ on G are the adjacent extreme points of y_N^{r+1} in $\bar{\Gamma}$.

Let G^+ be the half-space of G not containing y_N^{r+1}, $\Gamma \subset G^+$. The open half-space of G containing y_N^{r+1}, $G^- = R^n \backslash G^+$ satisfies $\bar{\Gamma} \backslash \Gamma \subset G^-$, and also $\bar{\Gamma} \backslash \Gamma \subset K \backslash \Gamma$.

Let y_N^1 be an extreme point of Γ on G that is the nearest to y_N^{r+1} among all the extreme points of G. Let H be the hyperplane through P orthogonal to the line segment $y_N^1 P$ and G; so H contains y_N^{r+1}. Let H^+, H^- be the half-spaces defined by H, containing y_N^1, not containing y_N^1 respectively.

Let H_1 be a facetal hyperplane of K incident at y_N^{r+1} that contains y_N^1. Let $\mathcal{B}_1$ be the ball with y_N^{r+1} as center and y_N^1 as a boundary point. $\mathcal{B}_1$, $\mathcal{B}_0$ are concentric spheres with $\mathcal{B}_0$ contained inside $\mathcal{B}_1$. The half-space H_1^+ of H_1 containing K contains the half-balls $\mathcal{B}_0^+$, $\mathcal{B}_1^+$ of $\mathcal{B}_0$, $\mathcal{B}_1$ respectively on the side of H_1 containing K. Let $\mathcal{B}_1^{++} = \mathcal{B}_1^+ \cap H^+$, $\mathcal{B}_1^{+-} = \mathcal{B}_1^+ \cap H^-$.

For any point $y_N \in \Gamma \cap G \cap H^-$, it can be verified that the hyperplane T through y_N orthogonal to the line segment $y_N^1 y_N$ has both the points y_N^1, y_N^{r+1} on the same side; which implies that the angle $y_N^1 y^N y_N^{r+1}$ at y_N in the triangle $y_N^1 y^N y_N^{r+1}$ is strictly acute.

This in turn implies, by the 2-dimensional results mentioned at the beginning of this proof, that the nearest point to y_N^1 on the half-line joining y_N^{r+1} to y_N and continuing in that direction, is on the line segment joining y_N^{r+1} and y_N but not including the point y_N, so it is in $\bar{\Gamma} \backslash \Gamma \subset K \backslash \Gamma$.

We will now consider two subcases.

Case 1.1: Suppose for some $i \in \{1, ..., k\}$, $\mathcal{F}_i$ not containing y_N^1 satisfies $\mathcal{F}_i \cap G \subset H^-$.

Let x_i^{+1} denote the nearest point in $\mathcal{F}_i$ to y_N^1, and let U denote the half-line joining y_N^{r+1} to x_i^{+1} and continuing in that direction. By the definition of x_i^{+1}, it must be the nearest point to y_N^1 on $U \cap \Gamma$.

Since $x_i^{+1} \in \mathcal{F}_i$, and $\mathcal{F}_i \subset G^+$, we know that $x_i^{+1} \in G^+$. So, U must intersect G.

Suppose U intersects G at x_i^{-1}. Clearly, $x_i^{-1} \in \Gamma \cap G \cap H^-$, and the half-open line segment U^- joining y_N^{r+1} to x_i^{-1} but not including x_i^{-1} is a subset of $K \backslash \Gamma$.

Then from the above facts we know that nearest point by Euclidean distance to y_N^1 on U is in U^-, which implies that the nearest point to y_N^1 in F_i is in $F_i \backslash \mathcal{F}_i$.

This shows that in this subcase, the final step carried out with the known extreme point y_N^1 and facet F_i of K will show that at this stage $\mathcal{F}_i \neq F_i$ and lead to a new extreme point in $F_i \backslash \mathcal{F}_i$ to add to the present list of extreme points.

Case 1.2: For all $i \in \{1, ..., k\}$ such that $\mathcal{F}_i$ does not not contain y_N^1 satisfies $\mathcal{F}_i \cap G$ is not a subset of H^-.

So in this case, for all $i \in \{1, ..., k\}$ such that $y_N^1 \notin \mathcal{F}_i$, H^+ contains a relative interior point of $G \cap \mathcal{F}_i$, which in turn implies that at least one extreme point of $G \cap \mathcal{F}_i$ is in the interior of H^+.

Now extend the line joining y_N^1 to P until it intersects the boundary of $G \cap \Gamma$ again; suppose it intersects it at a point $\bar{y}_N$. Since the line segment joining y_N^1 and $\bar{y}_N$ contains a relative interior point P of G, facets of Γ containing $\bar{y}_N$ do not contain y_N^1 and vice versa. We will now consider two subcases.

Case 1.2.1: $\bar{y}_N$ is itself an extreme point of Γ.

In this case the hyperplane through P orthogonal to the line segment $\bar{y}_N P$ is the hyperplane H itself discussed at the beginning of Case 1, but in this case what we called H^+ in Case 1 is the half-space of H not containing $\bar{y}_N$.

In this case all the facets $\mathcal{F}_i$ for $i \in \{1, ..., k\}$, G of Γ incident at $\bar{y}_N$ have a relative interior point intersection with H^+, i.e., at least one extreme point on it is in the interior of H^+. Since K belongs to Class 2, this implies that there must be at least one $j \in \{1, ..., k\}$ such that $\mathcal{F}_j \cap G \subset G \cap H^+$. In fact some $\mathcal{F}_j$ for $j \in \{1, ..., k\}$ incident at y_N^1 satisfies this property also. So, from the hypothesis of this case we conclude that this $\mathcal{F}_j$ does not contain $\bar{y}_N$.

Using this fact and arguments similar to those in Case 1 with $\bar{y}_N$ replacing y_N^1, we conclude that for all points $y_N \in \Gamma \cap G \cap H^+$, the hyperplane T through y_N orthogonal to the line segment $\bar{y}_N y_N$ has both the points $\bar{y}_N, y_N^{r+1}$ on the same side, which implies that the angle $\bar{y}_N y_N y_N^{r+1}$ at y_N in the triangle $\bar{y}_N y_N y_N^{r+1}$ is strictly acute, and consequently the nearest point to $\bar{y}_N$ in the half line joining y_N^{r+1} to y_N and cotinuing in that direction, is in the intersection of this half-line with $\bar{\Gamma} \backslash \Gamma \subset K \backslash \Gamma$. Continuing as in Case 1.1, we conclude that the nearest point to $\bar{y}_N$ in F_j is different from that in $\mathcal{F}_j$, and therefore that the final step carried out with the extreme point $\bar{y}_N$ and the facet F_j of K will generate a new extreme point of K not in Γ.

Case 1.2.2: $\bar{y}_N$ is not an extreme point of Γ.

Let S denote the smallest dimensional face of $\Gamma \cap G$ containing $\bar{y}_N$. Then $\bar{y}_N$ is a relative interior point of S, and it can be expressed as a convex combination of extreme points of S.

S is the intersection of G and all the $\mathcal{F}_i$ for $i \in \{1, ..., k\}$ that contain it as a subset. Let $S_I = \{i : i \in \{1, ..., k\}$ and S is a subset of $\mathcal{F}_i\}$. Since $y_N^1, \bar{y}_N$ are not both contained together on any $\mathcal{F}_i$ for $i \in \{1, ..., k\}$, we know that $y_N^1 \notin \mathcal{F}_i$ for all $i \in S_I$. Also, from the hypothesis we know that for all $i \in S_I$, $\mathcal{F}_i$ has a relative interior point intersection with the half-space H^+.

Let y_N^2 be an extreme point of S. Let H_2 be a facetal hyperplane of K incident at y_N^{r+1} containing y_N^2. As $y_N^2 \in \mathcal{B}_1^+ \cap H^-$, H_2 passes through B_1^{+-} defined earlier. Let H_2^+ be the half-space of H_2 containing K. Then $K \subset H_2^+ \cap H_1^+$.

Let $\bar{H}_2$ be the hyperplane through P orthogonal to the line segment $y_N^2 P$, and $\bar{H}_2$ also contains the line segment $P y_N^{r+1}$. So $\bar{H}_2$ is obtained by tilting H discussed earlier on the line segment $P y_N^{r+1}$ while keeping its orthogonality with G.

Let $\bar{H}_2^+, \bar{H}_2^-$ denote the half-spaces of $\bar{H}_2$ containing y_N^2, not containing it, respectively.

From the arguments under Case 1.2.1, we already know that in this case there exists a facet $\mathcal{F}_j$ of Γ incident at y_N^{r+1} and containing the point y_N^1, and satisfying

$\mathcal{F}_j \cap G \subset G \cap H^+$. So, we can select an extreme point y_N^2 on S such that $\bar{H}_2$ obtained by tilting H on the line Py_N^{r+1} satisfies the property that $\bar{H}_2^-$ contains this $\mathcal{F}_j \cap G$.

Repeating the arguments made under Case 1 with respect to y_N^1, but now with the point y_N^2, we conclude that for all $y_N \in \mathcal{F}_j \cap G$, the angle $y_N^2 y_N y_N^{r+1}$ at y_N in the triangle $y_N^2 y_N y_N^{r+1}$, is strictly acute.

So, continuing with arguments similar to those made earlier, we conclude that the final step carried out with this extreme point y_N^2 and the facet F_j of K will find a new extreme point on F_j to add to Γ.

Case 2: P is contained on a K-facet $\mathcal{F}_v$ say, of Γ.

Since $\Gamma \subset C_\Gamma$ discussed earlier, clearly $v \in \{1, ..., k\}$, and so $P \in \mathcal{F}_v$ and the half-open line segment $[y_N^{r+1}P)$ joining y_N^{r+1} to P but not containing P is a subset of $F_v \backslash \mathcal{F}_v$.

Again let $\mathcal{B}_0$ be the ball with y_N^{r+1} as center and P as a boundary point. Let L be the tangent plane to $\mathcal{B}_0$ at P in this case, so L is orthogonal to H_v; and denote the half-space of H_v containing K by H_v^+. Let $\bar{L}^+$ denote the open half-space of L on the side of Γ; and L^- the other closed half-space of L that contains y_N^{r+1}. Also $\bar{L}^+ \cup L = L^+$, the closure of $\bar{L^+}$.

If all the extreme points of $\mathcal{F}_v$ are in the open half-space $\bar{L}^+$, then $\mathcal{F}_v$, their convex hull will itself be contained in this open half-space, contradicting that $P \in \mathcal{F}_v$. So, $L \cap \mathcal{F}_v$ must have at least one extreme point of $\mathcal{F}_v$.

We will now consider several subcases.

Case 2.1: P contained in the relative interior of F_v, in the relative interior of an $(n-2)$-dimensional face of $\mathcal{F}_v$.

P must be contained in the $(n-2)$-dimensional intersection of $\mathcal{F}_v$ with some non-K-facet of Γ, let that non-K-facet of Γ be denoted by G. These facts imply that $G \cap \mathcal{F}_v = L \cap \mathcal{F}_v = S$ say. All the extreme points of this $(n-2)$-dimensional face S of

Γ are in L and in G.

Let y_N^1 be an extreme point of Γ in $L \cap \mathcal{F}_v$ and H be the hyperplane through P orthogonal to the line segment $y_N^1 P$, so H is orthogonal to L and H_v, and contains the line segment $P y_N^{r+1}$.

Let H^+ the half-space of H containing y_N^1; and H^- the other half-space of H. It can be verified that for any point $y_N \in L \cap H^-$, the angle $y_N^1 y_N y_N^{r+1}$ at y_N in the triangle $y_N^1 y_N y_N^{r+1}$ is acute. So, for some $i \in \{1, ..., k\}$ such that $y_N^1 \notin F_i$, if $F_i \cap L \subset H^-$, then this fact together with all the arguments used earlier leads to the conclusion that the nearest point to y_N^1 on the line joining y_N^{r+1} to y_N and continuing in that direction, for any $y_N \in F_i \cap L$ is on the side of L containing y_N^{r+1}.

Also, in this case for any $i \in \{1, ..., k\}$ such that $\mathcal{F}_i$ does not contain y_N^1, and any point $y'_N \in \mathcal{F}_i \cap G$, the line segment $y_N^{r+1} y'_N$ must intersect L at some point $y_N \in L \cap F_i$.

Using all these arguments together we conclude that the nearest point to y_N^1 in F_i for that i, is different from the nearest point to it in $\mathcal{F}_i$; again showing that in this case the final step will generate a new extreme point of K not in Γ.

If all the F_i not containing y_N^1 for $i \in \{1, ..., k\}$ have a relative interior point intersection with H^+, then extend the line joining y_N^1 to P until it intersects the relative boundary of $\mathcal{F}_v$ again. Suppose this point of intersection is $\bar{y}_N$.

If $\bar{y}_N$ is an extreme point of $\mathcal{F}_v$, using arguments similar to those in Case 1.2.1, we can again show that the final step carried out with $\bar{y}_N$ and a facet F_i containing y_N^1 for some $i \in \{1, ..., k\}$ will generate a new extreme point of K not in Γ.

If $\bar{y}_N$ is not an extreme point of $\mathcal{F}_v$, we can use arguments similar to those in Case 1.2.2, and again conclude that the final step will generate a new extreme point of K not in Γ.

Case 2.2: P is itself an extreme point of $\mathcal{F}_v$

Here let H be the hyperplane through P containing the line segment $P y_N^{r+1}$, and orthogonal to H_v, and L.

In this case, there must be at least one facet F_i for $i \in \{1, ..., k\}$ not containing P such that $\mathcal{F}_i$ is on one side of H. Since the line Py_N^{r+1} is orthogonal to L, it can be verified that the angle $Py_N y_N^{r+1}$ at y_N in the triangle $Py_N y_N^{r+1}$ is strictly acute for any $y_N \in L \cap \mathcal{F}_i$ for such an i. Using this and arguments similar to those in Case 2.1, we again conclude that the final step applied with P and the facet F_i for that i will generate a new extreme point of K not in Γ in this case also.

Case 2.3: P is in the relative interior of a face S of $\mathcal{F}_v$ of dimension between 2 and $n - 3$.

So in this case $S \subset L \cap \mathcal{F}_v$.

S must be the intersection of some non-K-facet of Γ, let G be one of those.

Let y_N^1 be an extreme point of S. Here we argue similar to that in Case 2.1 with S taking the role of $\mathcal{F}_v$. H, H^+, H^- are defined as in Case 2.1. If all the F_i for $i \in \{1, ..., k\}$ not containing y_N^1 have a relative interior point intersection with H^+, then we extend the line joining y_N^1 to P until it intersects the relative boundary of S again, call this point of intersection as $\bar{y}_N$; then continue the arguments similar to those in Case 2.1 and 1.2.2. Again, we can conclude that the final step will generate a new extreme point of K not in Γ in this case also.

Hence if the final step does not output a new extreme point of K not in Γ, then $\Gamma = K$, when K belongs to Class 2 defined in Section 8. $\square$

If the mukkadvayam $\Gamma \neq K$, these results show that the final step of Section 9 will generate a new extreme point of K to add to Γ, and continue the enumeration if K belongs to Class 2 defined in Section 8. Clearly the number of linear programs and convex quadratic programs to be solved in all these steps is bounded above by $O(r^3)$ and $O(r(n + m))$ respectively. So the procedure consisting of all the steps discussed above provides a polynomial time algorithm for extreme point enumeration when K

belongs to Class 2.

Even when K belongs to Class 1 defined in Section 8, I believe that the whole procedure including the mekkadvayam checking step of Section 8 will enumerate all the extreme points of K, but do not have a proof of it yet.

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