SOME NP-COMPLETE PROBLEMS IN LINEAR PROGRAMMING

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Degeneracy checking in linear programming is NP-complete. So is the problem of checking whether there exists a basic feasible solution with a specified objective value.

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1. Degeneracy testing

Consider the general linear program in standard form, with integer data

$$\begin{aligned} \text{minimize} \quad & Z(x) = cx, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0 \end{aligned} \tag{1}$$

where A is a matrix of order $m \times n$ and rank m. This problem is said to be *degenerate*, if there exists a basis B for (1) satisfying the property that at least one component in the vector $B^{-1}b$ is zero. See [1–10,13]. Degeneracy in linear programming was studied extensively, because of the problem of cycling that it can introduce in the simplex algorithm, thereby preventing the simplex algorithm from terminating in a finite number of steps unless special measures are taken to resolve degeneracy [1–10,13]. If (1) is degenerate, the point b must be in a subspace of $\mathbf{R}^m$ spanned by some subset of $(m-1)$ column vectors of A. Therefore, when A, b, c are allowed to be real or rational, in a statistical sense, (1) will be nondegenerate almost always. Also, even if (1) is degenerate, when b is modified to $b(\varepsilon) = b + (\varepsilon, \varepsilon^2, \ldots, \varepsilon^m)^{\mathrm{T}}$, there exists an $\varepsilon_1 > 0$ such that whenever $0 < \varepsilon < \varepsilon_1$, the modified problem is nondegenerate. Thus, a minor perturbation will make (1) nondegenerate, and methods for resolving degeneracy in the simplex and other pivotal algorithms have been developed based on such perturbations [2,3,7,10,13]. In spite of all these statistical arguments, it has been observed that most linear programming models constructed in practical applications tend to be degenerate [3].

Here we study the computational complexity of checking whether a given instance of (1) is degenerate. We will first discuss a combinatorial optimization problem. Let $a = \{a_1, \ldots, a_m\}$, $b = \{b_1, \ldots, b_n\}$ be two given finite sets of positive integers. The term *equal partial sums* denotes the combinatorial optimization problem: given the sets a, b, find whether there exist subsets I, J satisfying $\emptyset \neq I \subset \{1, \ldots, m\}$, $\emptyset \neq J \subset \{1, \ldots, n\}$, such that $\Sigma_{i \in I} a_i = \Sigma_{j \in J} b_j$.

Lemma. *The problem equal partial sums is NP-complete.*

Proof. Clearly, the problem equal partial sums is in NP. Consider the subset sum problem: given positive integers $d_1, \ldots, d_p$; d_0, check whether there exists a subset $I \subset \{1, \ldots, p\}$ satisfying $\Sigma_{i \in I} d_i = d_0$. Here, is $\Sigma_{i=1}^{p} d_i = d_0$, $I = \{1, \ldots, p\}$ provides an answer to the subset sum problem in the affirmative, so without any loss of generality we can assume that $\Sigma_{i=1}^{p} d_i > d_0$. Let $\alpha = 1 + \Sigma_{i=1}^{p} d_i$. In this case, the subset sum problem is equivalent to the equal partial sums problem with $a = \{d_1, \ldots, d_p\}$, $b = \{d_0, \alpha\}$. Thus, the subset sum problem is a special case of the equal partial sums problem. Since the subset sum problem is NP-complete [11,12], these facts imply that so is the equal partial sums problem.

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Theorem. *Degeneracy testing is NP-complete.*

Proof. If a basis B for (1) exists exhibiting degeneracy, a nondeterministic algorithm can select this basis one column at a time in at most m steps. Thus, degeneracy testing is in NP.

Consider the special case of (1), known as the transportation problem, in which the constraints are of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1 \text{ to } m,$$
$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1 \text{ to } n,$$
$$x_{ij} \geq 0 \quad \text{for all } i, j \qquad (2)$$

where $a_1,\ldots,a_m$; $b_1,\ldots,b_n$ are given positive integers satisfying $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. It is known [8,9,13] that (2) is degenerate iff there exists proper subsets $\emptyset \neq I \subset \{1,\ldots,m\}$, $\emptyset \neq J \subset \{1,\ldots,n\}$ satisfying $\sum_{i\in I} a_i = \sum_{j\in J} b_j$. Thus checking whether (2) is degenerate is equivalent to the equal partial sums problem with $a = \{a_1,\ldots,a_m\}$, $b = \{b_1,\ldots,b_n\}$. By lemma, these facts clearly imply that degeneracy testing is NP-complete.

A *degenerate feasible basis* for (1) is a basis B for (1) satisfying $B^{-1}b \geq 0$, and at least one component of $B^{-1}b$ is zero. It is possible for (1) to be degenerate, and yet there may not exist a degenerate feasible basis for (1). For the special case of the transportation problem (2), it can be shown that a degenerate feasible basis exists iff the equal partial sums problem with $a = \{a_1,\ldots,a_m\}$, $b = \{b_1,\ldots,b_n\}$ has a solution. This leads to the following.

Corollary 1. *The problem of checking whether there exists a degenerate feasible basis for* (1), *is NP-complete.*

Corollary 2. *Degeneracy testing is NP-complete even for the special case of the transportation problem.*

2. Extreme point with a specified objective value

Given the LP (1) with integer data, and a rational number θ expressed as a ratio in smallest terms, this problem is to check whether there exists a basic feasible solution of (1) at which the objective function assumes the value of θ. Clearly this problem is in NP and it can be shown to be NP-complete by showing the problem of testing for a degenerate feasible basis in (1) to be a special case of it. We now show that extreme point with a specified objective value problem is NP-complete even for the special case of Assignment problem:

Consider the subset sum problem with data $d_1,\ldots,d_p$; d_0 discussed above. Let $C = (c_{ij})$ be a $2p \times 2p$ matrix:

$$C = \left(\begin{array}{c|c} d_1 d_1 \ldots d_1 & \\ d_2 d_2 \ldots d_2 & 0 \\ d_p d_p \ldots d_p & \\ \hline 0 & 0 \end{array}\right).$$

The last p columns of C are zero. The first p columns of C are all equal to $(d_1, d_2,\ldots, d_p, 0, 0,\ldots,0)^T \in \mathbb{R}^{2p}$. Clearly, the answer to the subset sum problem is in the affirmative iff there exists an assignment of order $2p$ for which the objective value, with C as the cost matrix is d_0. Since the assignments are the extreme points associated with the assignment problem, this shows that the subset sum problem is a special case of the extreme point with a specified objective value problem. So the extreme point with a specified objective value problem is NP-complete, even when restricted to the assignment problem.

3. Singular principal submatrix problem

Given a square, nonsingular, integer matrix, A, consider the problem of checking whether there exists a singular principal submatrix of A. This problem is clearly in NP. To show that it is NP-complete, consider again the subset sum problem with data $d_1, d_2,\ldots d_p$; d_0, as discussed before. If $d_0 \geq \sum_{i=1}^{p} d_i$, the problem becomes trivial. So, without loss of generality, let $d_0 < \sum_{i=1}^{p} d_i$.

Now let us define a square, non-singular matrix, A, as follows:

$$A = \begin{bmatrix} d_0 & d_1 & d_2 & \ldots & d_p \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Clearly, the answer to the subset sum problem is affirmative iff there exists a principal submatrix of A, which is singular. This leads to the following:

Corollary: *Given two square, nonsingular, integer matrices A, B of order n, let $A.j$, $B.j$ denote the jth column vectors of A, B resp. Then, checking whether there exists a set of columns $\{D.1, \ldots, D.n\}$, which is linearly dependent, with $D.j \in \{A.j, B.j\}$, is NP-complete.*

4. Bilinear problem

The problem, considered, is

$$\begin{aligned} \text{minimize} \quad & y^T Dz + p^T y + q^T z, \\ \text{subject to} \quad & By = d, \\ & Ez = \theta, \\ & y \geq 0,\ z \geq 0. \end{aligned} \tag{3}$$

We show below that degeneracy testing in (1) can be posed as a bilinear problem of the type (3).

Define 0–1 variables

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ in a solution for (1)}, \\ 0 & \text{if } x_j = 0. \end{cases}$$

Then the standard trick of transforming (1), using 0–1 variables y_j to count the number of positive variables, x_j, in a solution in (1) is well known. This leads to a system, sav (4) of linear constraints in x, y, including $0 \leq y_j \leq 1$ for all j. To make sure that all y_j are either 0 or 1, make the objective function equal to $\alpha \Sigma_{j=1}^{n} y_j(1 - y_j)$, where α is a suitably large positive penalty parameter.

Now, put two sets of system (4) together. In one call the variables as x', y. In the other, call them x'', z. Call this combined system as (5).

The question: Does (1) have a feasible solution with number of positive $x_j \leq m - 1$ is equivalent to the following:

Does (5) have a feasible solution in which all y_j, z_j are integer and $y = z$?

This problem is the same as that of minimizing

$$\sum_{j=1}^{n} y_j + \alpha \sum_{j=1}^{n} y_j(1 - y_j) + \alpha \sum_{j=1}^{n} z_j(1 - z_j) + \alpha \sum_{j=1}^{n} (y_j - z_j)^2 \tag{6}$$

subject to (5).

On simplification, (6) becomes

$$-2\alpha \sum_{j=1}^{n} y_j z_j + \alpha \left(\sum_{j=1}^{n} (y_j + z_j) \right) + \sum_{j=1}^{n} y_j. \tag{7}$$

Which is clearly a bilinear objective function. So, the bilinear problem (1) is NP-hard.

5. Some open problems

Here is a problem related to the Hirsch Conjecture whose status is unknown. Given a convex polyhedron specified by linear inequalities with integer data

$Ax \geq b$

and two extreme points x^1, x^2 on it and a positive integer α, the problem is to check whether there exists an edge path in the polyhedron between x^1, x^2, containing α or less edges. The computational complexity of this problem is still unknown.

A second problem of interest is the following: Suppose we are given an integer matrix A of order $m \times n$. The problem of finding a maximum cardinality linearly independent subset of column vectors of A can of course be solved efficiently, using pivot step in at most $O(n^3)$ time. The complementary problem of finding a minimum cardinality linearly dependent subset of column vectors of A seems to be hard in general. A specific problem of interest is, given that rank of A is m, checking whether there exists a subset of m columns of A, which is linearly dependent. The problem is simple when A is unimodular. But its computational complexity is not known in general.

References

[1] E.M.L. Beale, "Cycling in the dual simplex algorithm", *Naval Res. Logist. Quart.* **2**, 269–276 (1955).

[2] A. Charnes, "Optimality and degeneracy in linear programming", *Econometrica* **20** (2), 160–170 (1952).

[3] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ (1963).

[4] B.J. Gassner, "Cycling in the transportation problem", *Naval Res. Logist. Quart.* **11** (1), 43–58 (1964).

[5] A.J. Hoffman, "Cycling in the simplex algorithm", National Bureau of Standards Report 2974, December 1953.

[6] M.M. Kostreva, "Cycling in linear complementarity problems", *Math. Programming* **10** (1), 127–130 (1979).

[7] K.T. Marshall and J.W. Suurballe, "A note on cycling in the simplex method", *Naval Res. Logist. Quart.* **16** (1), 121–137 (1969).

[8] A. Orden, "The transshipment problem", *Management Sci.* **2** (3), 276–285 (1956).

[9] M. Simonnard, *Linear Programming*, translated from French by W.S. Jewell, Prentice-Hall, Englewood Cliffs, NJ (1966).

[10] P. Wolfe, "A technique for resolving degeneracy in linear programming", *J. SIAM* **11**, 205–211 (1963).

[11] M.R. Garey and D.S. Johnson, *Computers and Intractability, a Guide to the Theory of NP-Completeness*, Freeman, San Francisco (1979).

[12] R.M. Karp, "Reducibility among combinatorial problems", in: R.E. Miller and J.W. Thatcher, Eds., *Complexity of Computer Computations*, Plenum Press, New York (1972) 85–103.

[13] K.G. Murty, *Linear Programming*, Wiley, New York, to appear.