Contents

Chapter 6

Single Commodity Flows with Additional Linear Constraints

Here we consider the single commodity flow problem (5.3) in the directed connected pure network $\mathrm{G} = (\mathcal{N}, \mathcal{A}, \ell, k, c, V)$ with $|\mathcal{N}| = n, |\mathcal{A}| = m$, in which the flow vector is required to satisfy additional linear constraints. Select a node, say n, and fix the root node at it, and eliminate the flow conservation equation at it as the redundant constraint. After this our problem is of the form: find $f = (f_{ij} : (i, j) \in \mathcal{A})$ to

$$\begin{aligned}
\text{Minimize } \textstyle\sum(c_{ij} f_{ij} : \text{ over } (i,j) \quad &\in \quad \mathcal{A}) \\
\text{Subject to } -f(i,\mathcal{N}) + f(\mathcal{N}, i) \quad &= \quad -V_i, \text{ for each } i \neq n \quad (6.1) \\
\sum_{(i,j)} a_{ijr} f_{ij} = (\text{ or } \quad &\leqq \quad) \;\; b_r, \text{ for } r = 1 \text{ to } \rho \quad (6.2) \\
\ell_{ij} \leqq f_{ij} \leqq k_{ij} \quad &, \quad\; \text{for all } (i,j) \in \mathcal{A} \qquad (6.3)
\end{aligned}$$

Because of (6.2) this problem cannot be solved by network methods alone. To solve it by the bounded variable simplex method, we have to deal with bases of order $n - 1 + \rho$ which could be very large even if ρ is small. We discuss an efficient special implementation (belonging to the area of **structured LP**) of the bounded variable primal simplex method for this problem due to Chen and Saigal [1977] that exploits the special structure of this problem. It uses spanning trees in G to handle the flow conservation constraints (6.1), and maintains the inverse of a working basis of order ρ (as opposed to that of maintaining the inverse of bases of order $(n-1+\rho)$ in the usual simplex algorithm) to handle the additional linear constraints (6.2). It is very convenient to solve this problem as long as ρ is not very large. If ρ is large, it may be better to solve this problem directly by large scale implementations of interior point methods mentioned in Section 5.11.

We denote the arcs in $\mathcal{A}$ by $e_1, \ldots, e_m$; and by ℓ_t, k_t, c_t, f_t, the lower bound, capacity, unit cost coefficient, and flow amount, respectively, on e_t for $t = 1$ to m. ρ_0, ρ_1 are the numbers of equality, inequality constraints in (6.2). Introduce the nonnegative slack variables $f_{m+1}, \ldots, f_{m+\rho_1}$ and convert all the inequality constraints in (6.2) into equations. For $t = 1$ to ρ_1, let $A_{.t}$ (a column vector of the unit matrix of order ρ) denote the column vector of f_{m+t} in the system (6.2) after it is transformed into a system of equations. Let $H_{.t}$ be the column vector of f_t in (6.1) for $t = 1$ to m, and let $H_{.t} = 0 \in \mathbb{R}^{n-1}$ for $t = m+1$ to $m+\rho_1$. Define $\ell_t = 0, k_t = \infty, c_t = 0$ for $t = m+1$ to $m+\rho_1$. Let ℓ, k, c, f denote the vectors in $\mathbb{R}^{m+\rho_1}$ consisting of the associated quantities, $\gamma = (-V_1, \ldots, -V_{n-1})^T, b = (b_1, \ldots, b_\rho)^T$, $H = (H_{.1} \ldots H_{.m+\rho_1}), A = (A_{.1} \ldots A_{.m+\rho_1})$. Then the problem (6.1), to (6.3) becomes the following LP

$$\begin{aligned}
\text{Minimize} \quad & cf \\
\text{Subject to} \quad & Hf = \gamma && (6.4)\\
& Af = b && (6.5)\\
\ell \leqq\ & f \leqq k && (6.6)
\end{aligned}$$

By the results in Chapter 1 any basis for just the system of conservation equations (6. 4) corresponds to a spanning tree in G and vice versa. If $\{e_{p_1}, \ldots, e_{p_{n-1}}\}$ are the in-tree arcs in a spanning tree $\mathbb{T}$ of G arranged in this order, $H^1 = (H_{.p_1} \ldots H_{.p_{n-1}})$ is the basis for (6.4) associated with $\mathbb{T}$.

For $t = m+1$ to $m+\rho_1$, since $H_{.t} = 0, (H^1)^{-1} H_{.t} = 0$. For $1 \leqq t \leqq m$, let e_t be an out-of-tree arc wrt $\mathbb{T}$, and let $(\lambda_{1t}, \ldots, \lambda_{n-1,t})^T$

be the in-tree arc-fundamental cycle incidence vector associated with it. Then from Chapter 1 we know that $(\lambda_{1t}, \ldots, \lambda_{n-1,t})^T = (H^1)^{-1} H_{.t}$, i.e., $H_{.t} = \sum_{u=1}^{n-1} \lambda_{u,t} H_{.p_u}$. As an example, the in-tree arc-fundamental cycle incidence vector of the out-of-tree arc e_8 wrt the spanning tree $\mathbb{T}$ with in-tree arcs e_1, e_2, e_3, e_4 arranged in this order, in Figure 6.1 is $(1, -1, -1, 1)^T$. $H_{.t}$ is the column vector of f_t among the top 4 rows in Tableau 6.1. It can be verified that $H_{.8} = H_{.1} - H_{.2} - H_{.3} + H_{.4}$.

We will assume that the constraints in (6.4), (6.5) are linearly independent. Hence if B is a basis for (6.4), (6.5), it has a natural row partition as in (6.7). Since the submatrix consisting of the first $(n-1)$ rows of B is of full row rank, the nonzero columns in it (i.e., those associated with the flow variables, and not the slack variables discussed earlier) correspond to arcs in a connected subnetwork which contains a spanning tree, say $\mathbb{T}$, of G (in general, there may be several, choose one and call it $\mathbb{T}$). Once the choice of $\mathbb{T}$ is made, basic variables associated with arcs in $\mathbb{T}$ are called **key basic variables** and the corresponding columns in B are called **key basic columns**; the remaining ρ basic variables are called **nonkey basic variables** and their columns in B are called **nonkey basic columns**. The spanning tree $\mathbb{T}$ itself is called the **key tree** in this step. Notice that the choice of key basic variables in a basic vector for (6.4), (6.5) may not be unique, it depends on the key tree selected in the subnetwork corresponding to basic flow variables. All key basic variables are always flow variables, the slack variables in the additional linear constraints will either be nonkey basic variables or nonbasic variables in every step. With this, B has been partitioned as below.

$$
\begin{array}{cccc}
 & \begin{array}{c} n-1 \text{ key} \\ \text{basic cols.} \end{array} & \begin{array}{c} \rho \text{ nonkey} \\ \text{basic cols.} \end{array} & \\
B = & \left(\begin{array}{c} H^1 \\ \ldots \\ A^1 \end{array} \right. & \left. \begin{array}{c} \vdots \quad H^2 \\ \ldots \\ \vdots \quad A^2 \end{array} \right) & \begin{array}{l} n-1 \text{ rows from (6.4)} \\ \\ \rho \text{ rows from (6.5)} \end{array}
\end{array}
\qquad (6.7)
$$

Let the key and nonkey basic variables be $(f_{p_1}, \ldots, f_{p_{n-1}})$ and $(f_{p_n}, \ldots, f_{p_{n-1+\rho}})$, arranged in some order. When ordered this way,

f_{p_r} is known as the r**-*th* key basic variable**, for $r = 1$ to $n-1$; and $f_{p_{g+n-1}}$ is known as the g**-*th* nonkey basic variable**, for $g = 1$ to ρ. Let $\lambda_{.u} = (\lambda_{1,u}, \ldots, \lambda_{n-1,u})^T$ be 0 if $f_{p_{u+n-1}}$ is a slack variable, or the in-tree arc-fundamental cycle incidence vector of $e_{p_{u+n-1}}$ wrt $\mathbb{T}$ otherwise. Then $\lambda = (\lambda_{.1} \ldots \lambda_{.\rho}) = (H^1)^{-1}H^2$ is known as the $\lambda - matrix$ corresponding to the present basis B, and the key, nonkey choice. So, the rth row $\lambda_{r.}$ of λ is associated with the key basic arc e_{p_r}, and its gth column $\lambda_{.g}$ is associated with the nonkey basic variable $f_{p_{g+n-1}}$. As discussed above we have

$$H_{.p_{g+n-1}} = \lambda_{1,g} H_{.p_1} + \ldots + \lambda_{n-1,g} H_{.p_{n-1}} \tag{6.8}$$

All entries in the λ- matrix are 0, or ± 1, so it can be stored very compactly (it is only necessary to store the two sets of cells in which the entries are $+1$, -1 respectively).

For carrying out the computations in the simplex algorithm in the step in which B is the present basis, we will make use of the following square upper triangular matrix M of order $(n-1+\rho)$ known as the **transformation matrix**. Here, for any t, I_t is the unit matrix of order t. Also, verify that BM has the form given below.

$$M = \begin{pmatrix} I_{n-1} & \vdots & -\lambda \\ \cdots & & \cdots \\ 0 & \vdots & I_\rho \end{pmatrix} \tag{6.9}$$

$$BM = \begin{pmatrix} H^1 & \vdots & 0 \\ \cdots & & \cdots \\ A^1 & \vdots & A^2 - A^1\lambda \end{pmatrix} \tag{6.10}$$

The square matrix in the lower right corner of BM, $W = A^2 - A^1\lambda$ of order ρ is called the **working basis** in this step. It depends on the key, nonkey choice. W must be nonsingular, as otherwise the set of last ρ columns in (6.10) will be a linearly dependent set, a contradiction since B and M are both nonsingular. The special implementation of the bounded variable primal simplex method discussed in this chapter for solving (6.4) to (6.6), carries out all the computations in the step when B is the present basis, using the key tree, and the inverse of the working basis W (either the explicit inverse, or the inverse in some convenient product or factorization form). After each pivot step it updates the tree labels for the key tree, the λ-matrix, and the inverse of the working basis, very efficiently.

EXAMPLE 6.1

Tableau 6.1

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}		
0	0	0	−1	−1	0	0	−1	−1	0	=	$-V_1$
0	0	1	1	0	0	−1	0	0	0	=	$-V_2$
−1	0	−1	0	1	1	0	0	0	0	=	$-V_3$
0	−1	0	0	0	−1	1	1	0	0	=	$-V_4$
...	...	...	...	...	...	...	...	...	...		...
1	2	3	−1	1	−1	0	2	1	0	=	13
0	1	2	−2	2	0	1	1	2	0	=	7
1	−1	−1	3	−1	1	−1	2	2	1	=	35

$$B = \left(\begin{array}{cccc|ccc} & \textit{key} & & & & \textit{nonkey} & \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 2 & 3 & -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -2 & 2 & 1 & 0 \\ 1 & -1 & -1 & 3 & -1 & 2 & 1 \end{array}\right) = \left(\begin{array}{c|c} H^1 & H^2 \\ \hline A^1 & A^2 \end{array}\right)$$

We will illustrate the derivation of the working basis in this example. Only data relevant to this illustration are given in this example. Consider the network in Figure 6.1, with arcs numbered e_1 to e_9. f_r denotes the flow amount on arc e_r for $r = 1$ to 9. Node 5 is the root node. There are 3 additional linear constraints on the variables in this problem, two equations; and one inequality, the slack variable associated with which is called f_{10}. The conservation equation corresponding to the root node 5 is omitted as a redundant constraint, the remaining conservation equations, and the additional linear constraints are given in Tableau 6.1.

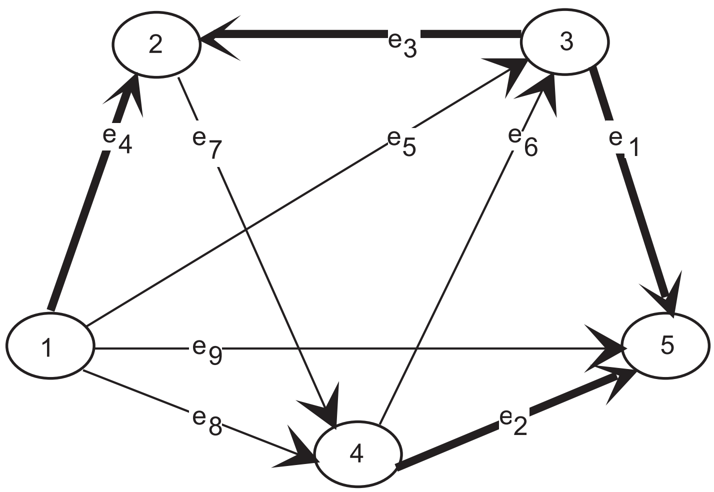

Figure 6.1:

Let $(f_1, f_2, f_3, f_4, f_5, f_8, f_{10})$ be the basic vector under consideration for this system. Choose (f_1, f_2, f_3, f_4) as the key basic variables, and (f_5, f_8, f_{10}) as the nonkey basic variables in that order. The associated basis B, partitioned as in (6.7) is given above.

The subnetwork corresponding to the basic flow variables in this basic vector consists of the arcs $e_1, e_2, e_3, e_4, e_5, e_8$. The key tree is marked with thick arcs in Figure 6.1. In this example we have

$$\lambda = (H^1)^{-1}H^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$W = A^2 - A^1\lambda = \begin{pmatrix} 5 & 7 & 0 \\ 6 & 6 & 0 \\ -5 & -4 & 1 \end{pmatrix}, \; W^{-1} = \begin{pmatrix} -1/2 & 7/12 & 0 \\ 1/2 & -5/12 & 0 \\ -1/2 & 5/4 & 1 \end{pmatrix}$$

W is the working basis in this example, and its inverse, W^{-1}, is given above.

To solve (6.4) to (6.6) the bounded variable primal simplex algorithm deals with partitions of the variables in this problem of the form (**B, L, U**) where **B** is the vector of basic variables associated with a basis B; say, **L** is the vector of all nonbasic variables whose values are fixed equal to their lower bound in (6.6) in the current basic solution, and **U** is the vector of all nonbasic variables for which the capacity is finite, and the values of these variables are fixed equal to their capacity. Given the partition (**B, L, U**), the primal basic solution associated with it is obtained by fixing the values of all the variables in **L, U** at the respective bounds as mentioned above, in (6.4), (6.5); and then solving the remaining system for the values of the basic variables in **B** (this will lead to a system of the form (6.12) given below). The partition is a **feasible partition** if the values of all the basic variables in **B** in the associated basic solution are within their bounds. The primal simplex algorithm for (6.4) to (6.6) is initiated with a feasible partition (**B, L, U**) which can be generated, if the problem is feasible, by applying the same algorithm on a Phase I problem set up as discussed later on. In the algorithm, after each pivot step, the new primal BFS is obtained by updating the one from the previous step. We will now discuss how all the computations in a step of this algorithm can be carried out efficiently using a key tree selected in the basic subnetwork and the associated working basis inverse.

How to Carry Out All the Computations in a Pivot Step Using the Key Tree and the Working Basis Inverse

Let **B, L, U** be the present feasible partition for the problem, where **B** is the vector of basic variables associated with the basis B for (6.4), (6.5). Let $\mathbb{T}$ be the key tree selected in the basic subnetwork as discussed above, resulting in the partition of the basis B as in (6.7), and the λ- matrix λ and working basis W. To carry out the computations in this step, we need to solve systems of equations of the following forms.

$$\pi B = c_B \tag{6.11}$$

$$By = d \tag{6.12}$$

c_B is the row vector of the original cost coefficients of the basic variables, and the solution π of (6.11) is the dual basic solution associated with the basis B. Once π is obtained, the relative cost coefficient of the nonbasic variable f_t wrt B is

$$\overline{c}_t = c_t - \pi \begin{pmatrix} H_{.t} \\ \cdots \\ A_{.t} \end{pmatrix}$$

The present feasible partition (**B, L, U**) and the associated BFS are optimal if $\overline{c}_t \geqq 0$ for all t such that f_t is in **L**, and $\overline{c}_t \leqq 0$ for all t such that f_t is in **U**. If this optimality criterion is violated, any nonbasic variable violating it is **eligible to be the entering variable** in this step, one of them is selected as the entering variable for a pivot step in the present partition. Then, we need to compute its updated column vector, which is the solution y of (6.12), where d is its original column vector. This updated column vector is the pivot column for the pivot step. Using the pivot column, and the values of the basic variables in the current BFS, the minimum ratio test of the bounded variable primal simplex algorithm is carried out to determine the dropping variable in this pivot step. If the dropping variable is the same as the entering variable, it moves from **L** or **U** where it is contained in the present partition, to the other set. In this case these nonbasic sets are revised appropriately, the primal BFS is updated, and the algorithm moves to the next step with the same basis. Since there is no change in **B** in this case, the key tree, working basis inverse, and the dual basic solution all remain unchanged. On the other hand, if the dropping variable is not the same as the entering variable, it will be a basic variable in **B**, which should be replaced by the entering variable in this pivot step. In this case we need to update the key tree, the λ-matrix, and the working basis inverse, to get the corresponding things for the next partition. We will now discuss efficient procedures for doing all this work.

To Solve (6.11)

Let $\pi^1 = (\pi_1, \ldots, \pi_{n-1}), \pi^2 = (\pi_n, \ldots, \pi_{n-1+\rho})$. $\pi = (\pi^1, \pi^2)$ is the vector of dual variables corresponding to (6.4), (6.5) in that order. Let M be the transformation matrix discussed in (6.9). Compute $c_B M = \nu = (\nu_1, \ldots, \nu_{n-1+\rho})$. Let $\nu^1 = (\nu_1, \ldots, \nu_{n-1}), \nu^2 = (\nu_n, \ldots, \nu_{n-1+\rho})$. Multiplying both sides of (6.11) by the nonsingular transformation matrix M leads to $\pi B M = c_B M = \nu$. Using (6.10) this reduces to

$$\pi^1 H^1 + \pi^2 A^1 = \nu^1 \quad (6.13)$$

$$\pi^2 W = \nu^2 \quad (6.14)$$

Hence $\pi^2 = \nu^2 W^{-1}$, it can be computed directly using ν^2 and the available W^{-1}. Let $h = \nu^1 - \pi^2 A^1$. Then from (6.13), $\pi^1 H^1 = h$. H^1 is the coefficient matrix of the flow variables on in-tree arcs in $\mathbb{T}$ in the flow conservation equations (6.4). So, π^1 is the vector of node prices of non-root nodes in $\mathbb{T}$, determined as in Section 5.4, with h as the vector of cost coefficients for in-tree arcs in $\mathbb{T}$, and 0 as the node price for the root node. So, π^1 can be determined by back substitution, beginning at the root node n, and going down in increasing order of level in the rooted tree $\mathbb{T}$ as described in Section 5.4.

As an example consider the basis B associated with the key basic vector (f_1, f_2, f_3, f_4) and the nonkey basic vector (f_5, f_8, f_{10}) discussed in Example 6.1. Suppose $c_B = (c_1, c_2, c_3, c_4, c_5, c_8, c_{10}) = (3, -4, 7, 5, 6, -2, 0)$. In this example, we have $\nu = c_B M = (3, -4, 7, 5, 4, 3, 0)$ using the λ derived in Example 6.1. So, $\nu^1 = (3, -4, 7, 5)$ and $\nu^2 = (4, 3, 0)$. Hence $\pi^2 = (\pi_5, \pi_6, \pi_7) = (4, 3, 0)W^{-1} = (-1/2, 13/12, 0)$. So, $h = \nu^1 - \pi^2 A^1 = (7/2, -49/12, 19/3, 20/3)$. We show the key tree with h as the arc cost coefficients in Figure 6.2.

Fixing the price of the root node 5 to 0, the equations to solve to get $\pi^1 = (\pi_1, \pi_2, \pi_3, \pi_4)$ are:

$$\begin{aligned} 0 - \pi_4 &= -49/12 \\ 0 - \pi_3 &= 7/2 \\ \pi_2 - \pi_3 &= 19/3 \end{aligned}$$

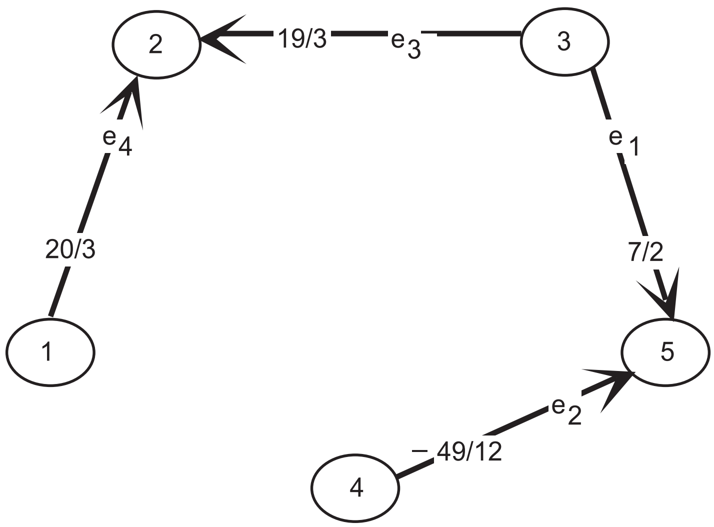

Figure 6.2:

$$\pi_2 - \pi_1 \quad = \quad 20/3$$

yielding the solution $\pi^1 = (-23/6, 17/6, -7/2, 49/12)$. So, the solution for (6.11) in this example is $\pi = (-23/6, 17/6, -7/2, 49/12, -1/2, 13/12, 0)$.

To Solve (6.12)

Define $\xi = (\xi_1, \dots, \xi_{n-1+\rho})^T$ as a column vector of new variables, with $\xi^1 = (\xi_1, \dots, \xi_{n-1})^T$, $\xi^2 = (\xi_n, \dots, \xi_{n-1+\rho})^T$. Define $d^1 = (d_1, \dots, d_{n-1})^T$, $d^2 = (d_n, \dots, d_{n-1+\rho})^T$; (d^1, d^2) is a partition of d corresponding to the partition (ξ^1, ξ^2) of ξ. We will first solve the system of equations $(BM)\xi = d$, i.e.,

$$H^1\xi^1 \quad = \quad d^1 \tag{6.15}$$

$$W\xi^2 \quad = \quad d^2 - A^1\xi^1 \tag{6.16}$$

H^1 is the coefficient matrix of the flow variables on the in-tree arcs in $\mathbb{T}$ in the flow conservation equations (6.4). Hence, the solution ξ^1

for (6.15) is the vector of flows on in-tree arcs in $\mathbb{T}$ to satisfy exogenous flow amounts of $p_i = d_i$ at non-root nodes $i = 1$ to $n-1$, and $-(d_1 + \ldots + d_{n-1})$ at the root node n. Since H^1 is a triangular matrix, the solution ξ^1 of (6.15) can be obtained by back substitution as discussed in Section 5.4, in $(n-1)$ steps. In each step we find an in-tree arc e incident to a non-root terminal node i of the remaining tree at this stage. Make the flow amount on e equal to $+p'_i$ or $-p'_i$ depending on whether i is the tail or head node of e, where p'_i is the present updated exogenous flow amount at node i. If j is the other node on e, update the exogenous flow amount of j by adding (subtracting) the flow amount on e to (from) it depending on whether e is directed into (or out of) j. Now delete e from the tree and go to the next step if any in-tree arcs remain.

Once the solution ξ^1 for (6.15) is obtained, we obtain ξ^2 from (6.16) to be $W^{-1}(d^2 - A^1\xi^1)$, which is computed using the available W^{-1}.

Once ξ^1, ξ^2 are both determined, we compute the solution y for (6.12) from $y = M\xi$.

As an example, consider the basis B associated with the key basic vector (f_1, f_2, f_3, f_4) and the nonkey basic vector (f_5, f_8, f_{10}) discussed in Example 6.1. Suppose we need to solve (6.12) with $d = (0, -1, 0, 1, 0, 1, -1)^T$. Here $d^1 = (0, -1, 0, 1)^T$ and $d^2 = (0, 1, -1)^T$. We show the key tree with exogenous flow amounts of d^1 at the non-root nodes, and exogenous flow amount of $-(d_1 + d_2 + d_3 + d_4) = 0$ at root node 5 in Figure 6.3.

Following the procedure discussed above, we first select the non-root terminal node 1 and make $\xi_4 = 0$, and delete e_4. Next, we select node 2, make $\xi_3 = 1$, delete e_3, and revise the exogenous flow amount at node 3 to -1. Next we select node 3 and make $\xi_1 = -1$, delete e_1, and revise the exogenous flow amount at the root node 5 to -1. Next we select node 4, and make $\xi_2 = 1$. So, in this example, the solution for (6.15) is $\xi^1 = (-1, 1, 1, 0)^T$

We have $d^2 - A^1\xi^1 = (4, 4, -4)^T$. So, $\xi^2 = W^{-1}(4, 4, -4)^T = (1/3, 1/3, -1)^T$.

Hence, the solution for (6.15), (6.16) in this example is $\xi = (1, -1, -1, 0, 1/3, 1/3, -1)^T$. Therefore, the solution for (6.12) in this example is $y = M\xi = (2/3, -2/3, -1/3, -2/3, 1/3, 1/3, -1)^T$.

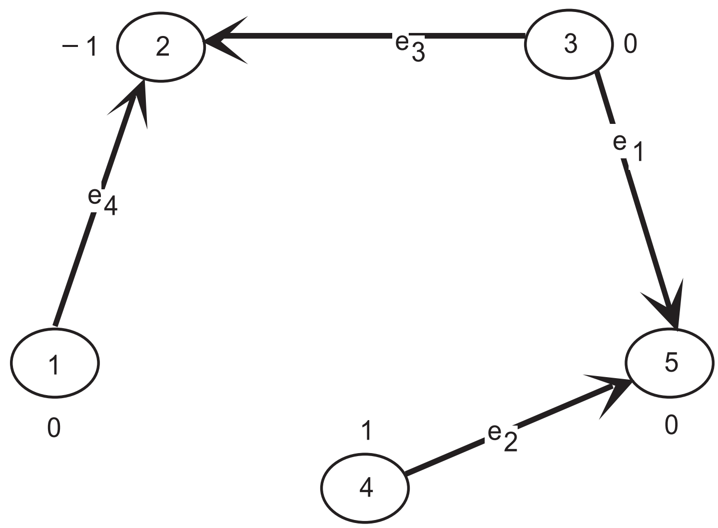

Figure 6.3:

Updating the Key Tree, λ-Matrix, and the Working Basis Inverse in a Pivot Step

Let B be the basis associated with the basic vector $(f_{p_1}, \ldots, f_{p_{n-1+\rho}})$ at the beginning of a pivot step, with $(f_{p_1}, \ldots, f_{p_{n-1}})$ as the key basic vector, the key tree $\mathbb{T}$, the λ-matrix λ, and the working basis W. Suppose the key, nonkey partition of B, is the one given in (6.7). Let f_s be the entering variable into this basic vector in this pivot step, with $(H_{.s}, A_{.s})$ as its original column vector in (6.4), (6.5). For updating the key tree, the λ-matrix, and the working basis inverse in this pivot step, we consider three cases depending on which present basic variable is the dropping variable.

Case 1 : The Dropping Variable Is a Nonkey Basic Variable

Let the dropping variable be the gth nonkey basic variable, $f_{p_{g+n-1}}$. The entering variable f_s replaces $f_{p_{g+n-1}}$ as the gth nonkey basic variable in this pivot step. By definition, the present $W_{.g} = A_{.p_{g+n-1}} - A^1\lambda_{.g}$. Compute $\delta = A_{.s} - A^1(H^1)^{-1}H_{.s}$. Let $\overline{\delta} = W^{-1}\delta$.

To update the λ-matrix in this case, replace $\lambda_{.g}$ by $(H^1)^{-1}H_{.s}$. To update the working basis, replace its gth column by δ. To update the working basis inverse, put the column $\overline{\delta}$ by the side of the present W^{-1}, and perform a pivot step with the gth row as the pivot row and $\overline{\delta}$ as the pivot column. This pivot step updates W^{-1} into the new working basis inverse. Because there is no change in the key basic variables, the key tree remains unchanged.

As an example, consider the basic vector $(f_1, f_2, f_3,\ f_4, f_5, f_8,\ f_{10})$ with $(f_1,\ f_2,\ f_3,\ f_4)$ as the key part, for the problem discussed in Example 6.1. Suppose f_7 is the entering variable into this basic vector, replacing the second nonkey basic variable, f_8, from it. The column vector δ defined above, is $A_{.7} - A^1(H^1)^{-1}H_{.7} = (4, 4, -4)^T$ in this example, and $\overline{\delta} = W^{-1}\delta = (1/3,\ 1/3,\ \text{-}\ 1)^T$. The new working basis is obtained by replacing $W_{.2}$ in the present W by δ. So, it is

$$\text{New working basis } = \begin{pmatrix} 5 & 4 & 0 \\ 6 & 4 & 0 \\ -5 & -4 & 1 \end{pmatrix}$$

For updating the working basis inverse, the pivot column $\overline{\delta}$ is entered on the right-hand side following the present working basis inverse in the tableau given below. The pivot row is row 2, and the pivot element is in a box. So, the new working basis inverse is the matrix on the left-hand side bottom of the tableau given below. To update λ, replace $\lambda_{.2}$ from the present λ by $(H^1)^{-1}h_{.7} = (1, -1, -1, 0)^T$.

Working basis inverse			Pivot col. $\overline{\delta}$
−1/2	7/12	0	1/3
1/2	−5/12	0	$\boxed{1/3}$
−1/2	5/4	1	−1
−1	1	0	0
3/2	−5/4	0	1
1	0	1	0

The new λ-matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Case 2 : The Dropping Variable Is an Essential Key Basic Variable

A key basic variable is said to be **essential** if none of the present nonkey basic variables can replace it as a key basic variable in the present basic vector, **inessential** otherwise.

Consider the uth key basic variable f_{p_u} associated with the in-tree arc e_{p_u} in the key tree $\mathbb{T}$. Let $[\mathbf{X}, \overline{\mathbf{X}}]$ be the fundamental cutset corresponding to e_{p_u} in $\mathbb{T}$. e_{p-u} can be replaced by any of the out-of-tree arcs in the cutset $[\mathbf{X}, \overline{\mathbf{X}}]$ to yield another spanning tree in G. So, e_{p_u} is an essential key basic arc (and f_{p_u} is an essential key basic variable) iff none of the present nonkey basic variables $f_{p_n}, \ldots, f_{p_{n-1+\rho}}$ is the flow variable on an arc in the cutset $[\mathbf{X}, \overline{\mathbf{X}}]$, which happens iff $\lambda_{u.} = 0$.

Here we consider the case where the dropping variable in this pivot step is an essential key basic variable, say f_{p_u}. So in the present λ-matrix, $\lambda_{u.} = 0$. The entering variable f_s must therefore be a flow variable corresponding to an arc in the fundamental cutset of e_{p_u}. Since $\lambda_{u.} = 0$, this change in the basic vector leaves the λ-matrix unchanged. The working basis remains unchanged and so does its inverse. So the only updating to be done in this case is to update the tree labels for replacing the in-tree arc e_{p_u} by e_s, which is carried out as described in Section 5.4.

As an example, consider basic vector $(f_1, f_2, f_3, f_4, f_5, f_9, f_10)$ for the problem given in Example 6.1, with (f_1, f_2, f_3, f_4) as the key basic variables, and (f_5, f_9, f_10) as the nonkey basic variables. The λ-matrix in this case is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

The key tree is marked with thick lines in Figure 6.1. Suppose f_8 is the entering variable into this basic vector, and the second key basic variable f_2 is the dropping variable. Since $\lambda_{2.} = 0$ in this case, f_2 is an essential key basic variable. f_8 replaces f_2 as the second key basic variable in this pivot step, but there is no change in λ or W.

Case 3 : The Dropping Variable Is an Inessential Key Basic Variable

Here we consider the case where the dropping variable is an inessential key basic variable f_{p_r}. So, $\lambda_{r.} \neq 0$. Let $[\mathbf{X}, \overline{\mathbf{X}}]$ be the fundamental cutset of the in-tree arc e_{p_r} in the key tree $\mathbb{T}$. In this case the updating will be done in two stages.

In Stage 1, a nonkey basic variable which is a flow variable for an arc in the cutset $[\mathbf{X}, \overline{\mathbf{X}}]$ is selected to replace f_{p_r} as the rth key basic variable. The gth nonkey basic variable $f_{p_{g+n-1}}$ is eligible to be selected for this if $\lambda_{rg} \neq 0$. Suppose the nonkey basic variable $f_{p_{u+n-1}}$ has been selected for this. This operation just rearranges the basic vector as $(f_{p_1}, \ldots, f_{p_{r-1}}, f_{p_{u+n-1}}, f_{p_{r+1}}, \ldots, f_{p_{u-2+n}}, f_{p_r}, f_{p_{u+n}}, \ldots, f_{p_{\rho+n-1}})$. In this order the vector corresponds to the basis $\hat{B}$ that is obtained from B by interchanging its rth and $(u+n-1)$th column vectors. Even though $\hat{B}$ is just the same as B except for the rearrangement of two of its columns, since the key basic variables are different, there will be a change in the key tree, the λ-matrix and the working basis.

In the original basis B, each nonkey column in H^2 can be expressed as a linear combination of the key columns in H^1, with the coefficients coming from the λ-matrix. This relationship is expressed in the first tableau given below. The left hand part of this tableau is $-\lambda^T$, followed on the right by the unit matrix of order ρ. To get the λ-matrix corresponding to the new key basic vector, we need to perform a pivot step in this tableau with the column vector under $H_{.P_r}$ as the pivot column, and the uth row as the pivot row. This leads to the second tableau at the bottom. The matrix under the columns headed with $H_{.p_1}, \ldots, H_{.p_{r-1}}, H_{.p_{u+n-1}}, H_{.p_{r+1}}, \ldots, H_{.p_{n-1}}$, in that order in this second tableau is $-(\hat{\lambda})^T$ where $\hat{\lambda}$ is the new λ-matrix. Since all the $\lambda_{ij}, \hat{\lambda}_{ij}$ are 0, ± 1, this pivot step can be carried out very efficiently.

Let $W_{.g}, \hat{W}_{.g}$ denote the gth column of the original and new working bases respectively, for $1 \leqq g \leqq \rho$. Using the formulas for $\hat{\lambda}$ in terms of λ, and the definition of the working bases, we get the formula for $\hat{W}_{.g}$ given below.

$H_{.p_1}$	...	$H_{.p_r}$	...	$H_{.p_{n-1}}$	$H_{.p_n}$	...	$H_{.p_{u+n-1}}$	...	$H_{.p_{\rho+n-1}}$	
$-\lambda_{11}$	...	$-\lambda_{r1}$	...	$-\lambda_{n-1,1}$	1	...	0	...	0	0
⋮		⋮		⋮	⋮		⋮		⋮	⋮
$-\lambda_{1u}$	...	$\boxed{-\lambda_{ru}}$	...	$-\lambda_{n-1,u}$	0	...	1	...	0	0
⋮		⋮		⋮	⋮		⋮		⋮	⋮
$-\lambda_{1\rho}$	...	$-\lambda_{r\rho}$	...	$-\lambda_{n-1,\rho}$	0	...	0	...	1	0
$-\hat{\lambda}_{11}$	...	0	...	$-\hat{\lambda}_{n-1,1}$	1	...	$-\hat{\lambda}_{r1}$	...	0	0
⋮		⋮		⋮	⋮		⋮		⋮	⋮
$-\hat{\lambda}_{1u}$	...	1	...	$-\hat{\lambda}_{n-1,u}$	0	...	$-\hat{\lambda}_{ru}$	...	0	0
⋮		⋮		⋮	⋮		⋮		⋮	⋮
$-\hat{\lambda}_{1\rho}$	...	0	...	$-\hat{\lambda}_{n-1,\rho}$	0	...	$-\hat{\lambda}_{r\rho}$	...	1	0

$$\hat{W}_{.g} = \begin{cases} W_{.g} - (\lambda_{rg}/\lambda_{ru})W_{.u}, & \text{for } g \neq u \\ (-1/\lambda_{ru})W_{.u}, & \text{for } g = u \end{cases}$$

From this it can be verified that $(\hat{W})^{-1} = Q^{-1}W^{-1}$, where Q^{-1} is the elementary matrix of order $\rho \times \rho$ that differs from the unit matrix of order ρ in just its uth row, given below. Notice that the λs in Q^{-1} are those from the λ-matrix corresponding to the original partition of the basis as in B. Hence, updating the inverse of the working basis in this Stage 1 consists of multiplying the present working basis inverse on the left by Q^{-1}.

$$Q^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ -\lambda_{r1} & -\lambda_{r2} & \dots & -\lambda_{r,u-1} & -\lambda_{ru} & -\lambda_{r,u+1} & \dots & -\lambda_{r\rho} \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ & & & & \text{col. } u & & & \end{pmatrix} \quad \text{row } u$$

As an example for this Stage 1, consider the basis for the problem in Example 6.1 with the key basic vector (f_1, f_2, f_3, f_4), and the nonkey basic vector (f_5, f_8, f_{10}). Suppose we have to replace the third key basic variable f_3 by a nonkey basic variable. In the above notation, $r = 3$ (since f_3 is the third key basic variable here), and we verify that the first two entries in $\lambda_{3.}$ are nonzero. Either the first or the second nonkey basic variable can replace f_3 as a key basic variable, suppose we select the first one, f_5. Thus, in the above notation $u = 1$ (since f_5 is the first nonkey basic variable presently). After f_5 replaces f_3 as the third key basic variable, the new key and nonkey basic vectors will be (f_1, f_2, f_5, f_4), (f_3, f_8, f_{10}), respectively. The λ-matrix changes to $\hat{\lambda}$ given below. And the matrix Q^{-1} for this operation is also given below. The working basis inverse changes to $(\hat{W})^{-1} = Q^{-1}W^{-1}$.

$$\hat{\lambda} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; (\hat{W})^{-1} = \begin{pmatrix} 0 & 1/6 & 0 \\ 1/2 & -5/12 & 0 \\ -1/2 & 5/4 & 1 \end{pmatrix}$$

Since there is a change in the key tree, the tree labels for storing it are updated too, as discussed in Section 5.4.

Once Stage 1 is completed, the dropping variable f_{p_r} is a nonkey basic variable in the new key, nonkey partition of the basis. Replacing f_{p_r} by the entering variable f_s is now accomplished as discussed in Case 1, during Stage 2. This completes the updating process in this Case 3.

Setting Up the Phase I Problem

First, select an initial partition $(\mathbb{T}_0, \mathbf{L}_0, \mathbf{U}_0)$ in G for the constraints (6.4), (6.6) as in Section 5.4, ignoring the additional linear constraints (6.5). Let f^0 be the basic flow vector in G associated with $(\mathbb{T}_0, \mathbf{L}_0, \mathbf{U}_0)$. If f^0 violates the bounds on some of the in-tree arcs in $\mathbb{T}_0$, define, as in Section 5.4, the type 1, 2 arcs $\mathbf{K}_1, \mathbf{K}_2$; modify the bounds on them so that f^0 satisfies the modified bounds, and define the Phase I cost coefficients c_t^* on arcs in $\mathcal{A}$. Check whether f^0 satisfies the additional linear constraints (6.5). To each constraint in (6.5) violated by f^0, add or subtract a nonnegative artificial variable as appropriate, so that the constraint becomes satisfied by giving a suitable nonnegative value to that artificial variable. This completes the construction of the Phase I problem. The objective function to be minimized in Phase I is $\sum_{t=1}^{m} c_t^* f_t$ + the sum of all the artificial variables introduced. We get an initial feasible partition for the Phase I problem from $(\mathbb{T}_0, \mathbf{L}_0, \mathbf{U}_0)$ by including all the artificial variables as basic variables together with the flow variables associated with in-tree arcs in $\mathbb{T}_0$. Hence, the Phase I problem is a problem of the same form as (6.4) to (6.6) with an initial feasible partition, so it can be solved by the primal simplex algorithm beginning with this initial partition, using the special implementation discussed here. During Phase I, after each pivot step we revise the data and the sets $\mathbf{K}_1, \mathbf{K}_2$ using the current primal solution, as in Section 5.4. If Phase I terminates with a feasible partition for the original problem, we then solve it beginning with this feasible partition, by the special implementation of the primal simplex algorithm again.

The special implementation of the primal simplex method for (6.4) to (6.6) discussed here can be used conveniently even if the network G is large, as long as the number of additional linear constraints, ρ is not very large. It has proven to be practically efficient for solving problems of this type.

6.1 Exercises

6.1 Find a minimum cost feasible flow vector in the network in Figure 6.4, satisfying the additional linear constraints

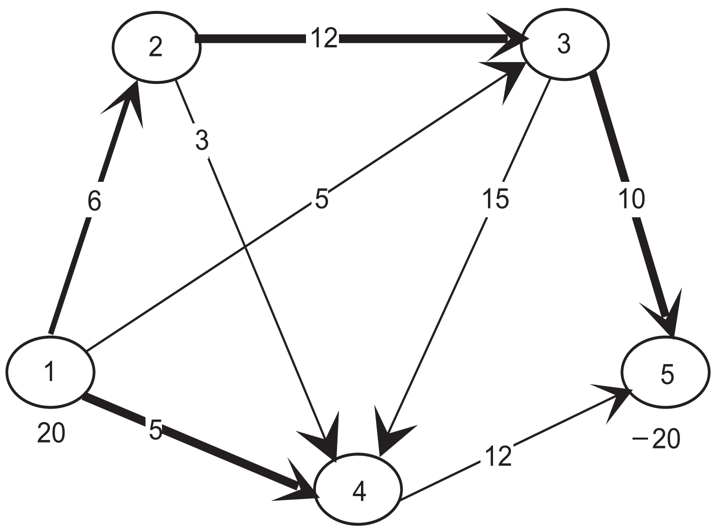

Figure 6.4: All lower bounds are 0, and capacities are 12. c_{ij} entered on arc (i, j). V_i is entered by the side of node i if it is nonzero.

$$\begin{aligned} f_{12} + f_{13} + f_{14} + f_{35} + 2f_{45} &\leqq 50 \\ 2f_{13} + f_{14} + f_{34} + 2f_{35} + f_{45} &\leqq 50 \end{aligned}$$

by the special implementation of the primal simplex method discussed in this chapter. Use the spanning tree marked with thick arcs in Figure 6.4 to initiate the algorithm.

6.2 Consider the special case of the problem (6.1) to (6.3), in which (6.2) contains only one constraint. Under appropriate rank assumptions, show that any basis for this problem corresponds to a quasitree as defined in Chapter 8. Develop an efficient special version of the algorithm discussed in this chapter to solve this problem. (Glover, Karney, Klingman, and Russell [1978])

Comment 6.1 Many real world network applications lead to minimum cost flow problems with additional linear constraints. The networks encountered in these applications are usually very large, thus making it computationally difficult to solve the whole problem as an LP directly by the simplex method. Here we discussed a variant of a structured linear programming technique of Chen and Saigal [1977] developing a special implementation of the revised simplex method that fully exploits the predominant network structure in this problem. This implementation leads to significant gains in computational efficiency and reduction in memory requirements. It is useful for handling problems in which the number of additional linear constraints is small (up to a few hundreds). It is based on the GUB techniques of Dantzig and Van Slyke [1967]. Other methods for handling additional linear constraints in network models are discussed by Barr, Farhangian, and Kennington [1986], and McBride [1985].

If the number of additional linear constraints is itself large, this implementation may not offer any particular advantage for solving the problem. In this case one may consider solving the overall problem as an LP using some of the recently developed interior point methods.

6.2 References

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