8.1

Algorithm for QP thro' LCP

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Linear Complementarity Problem (LCP)

Input Data: $M_{n\times n}, q \in R^n$

Desired output: A $\quad w \in R^n, z \in R^n$ satisfying

$$\begin{aligned} w - Mz &= q \\ w, z &\geq 0 \\ w_j z_j &= 0, \quad \forall j \quad \text{(Complementarity)} \end{aligned}$$

$2n$ nonnegative variables forming n **Complementary pairs**. (w_j, z_j) is the jth pair. The complementarity cond. requires at least one variable in each pair be 0. No obj. func. to optimize!

Example: $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$

Other Concepts: Complementary pairs of column vectors; Complementary vector of variables; Complementary set of column vectors; Complementary Matrices; Complementary cones;

Complementary basic vectors; Complementary bases; degenerate, nondegenerate Complementary Cones; Complement of a variable; Complement of a Column Vector; Complementary Feasible Basic Vector.

Total Enumeration Method for LCP: $O(2^n)$.

Geometric Method for LCP When $n = 2$

Complementary Cones are the orthants when $M = I$.

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

The LCP Corresponding to QP

Consider min $cx + \frac{1}{2}x^T Dx$ s. to $Ax = b, x \geq 0$ where $A_{m\times n}$ & D is symmetric WLOG.

The KKT conds. for this QP form the LCP.

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C^T \\ -b \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \geq 0, \begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \qquad \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Convex QP corresponds to an LCP associated with a PSD matrix. Conversely an LCP with a PSD matrix is a convex QP.

Example: Minimum Distance Problem: Find nearest point to $P_0 = (-2, -1)^T$ in $< (1, 3)^T, (5, 4)^T, (5, 2)^T, (4, 0)^T >$.

The Complementary Pivot Algorithm

If $q \geq 0$, $(w = q, z = 0)$ is a complementary solution to LCP. So, assume $q \not\geq 0$.

Introduce artificial variable z_0 with column $-e = (-1, \ldots, -1)^T$.

w	z	z_0	q
I	$-M$	$-e$	q

$$w, z, z_0 \geq 0$$

$(w = q + \lambda e, z = 0, z_0 = \lambda)$ is feasible when λ is large, and forms an unbounded edge of the set of feasible solutions. Find smallest λ, λ_0 say, for which this sol. is feasible.

$$\lambda_0 = |\min\{q_i : i \in \{1, \ldots, n\}\}|$$

$(w = q + \lambda_0 e, z = 0, z_0 = \lambda_0)$ is an extreme point sol. associated with the basic vector $B_0 = (w_1, \ldots, w_{t-1}, z_0, w_{t+1}, \ldots, w_n)$ where t is s. th. $q_t = \min\{q_i : i = 1 \text{ to } n\}$.

The basic vector B_0 is called an **Almost Complementary Feasible Basic Vector (ACFBV)**. An ACFBV is feasible to above system and satisfies:

1. It has one basic variable from all but one complemaentary pair. That c. pair is called the **missing c. pair** in this ACFBV.

2. z_0 is a basic variable in it.

If w_t from missing pair is entered into B_0, it can be verified that we get the extreme half-line $\{(w = q + \lambda e, z = 0, z_0 = \lambda) : \lambda \geq \lambda_0\}$, called the **Initial AC Ray**.

The Algorithm: Bring z_t into B_0. Determine dropping basic variable by usual (lexico) min ratio test to maintain feasibility. Continue according to following rule.

C. P. Rule If at some stage z_0 becomes dropping variable, we are left with a CFBV, the corresponding BFS is a CFS of original LCP, terminate.

If z_0 does not become the dropping variable in the current ACFBV, bring the complement of the dropping variable in previous step.

If updated col. of entering variable has no positive entry, we get another extreme half-line called **the secondary AC ray**. In this case the algorithm has failed to solve this LCP, terminate.

If updated col. of entering variable has positive entries, determine dropping variable by (lexico) min ratio test and continue.

Two ways to terminate.

1. With a secondary AC ray. In this case, the algo. has failed to solve the problem.
2. With a complementary BFS.

Examples:

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} 3 \\ 5 \\ -9 \\ -5 \end{pmatrix}$$

$$M = \begin{pmatrix} -1 & 0 & -3 \\ 1 & -2 & -5 \\ -2 & -1 & -2 \end{pmatrix}, \quad q = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$$

Classes of Matrices in Study of LCP

Matrix $M_{n\times n}$ is said to be a:

Copositive matrix if $y^T My \geq 0$, $\forall y \geq 0$

Strict Copositive matrix if $y^T My > 0 \; \forall 0 \neq y \geq 0$

Copositive Plus matrix if copositive, and $y \geq 0, y^T My = 0 \Rightarrow y^T(M + M^T) = 0$.

P-Matrix if all principal subdeterminants are > 0.

Q- Matrix if LCP (q, M) has a solution for all q.

There are well over 50 other classes of matrices defined in the study of LCP.

Theorem: If M is a copositive plus matrix and the CP Algo. terminates in a secondary ray, then $w - Mz = q, (w, z) \geq 0$ is itself infeasible.

Corollary: C P Algo. processes all convex QP.

Theorem: CP Algo. solves LCP associated with a $P-$matrix.

Theorem: LCP (q, M) has a unique sol. $\forall q \in R^n$ iff M is a $P-$matrix.