4.1

# Theorems of Alternatives for Systems of Linear Constraints

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System of constraints is

| | |
|---|---|
| **Feasible** | if it has a feasible solution, i.e., one satisfying all constraints in it. |
| **Infeasible** | if it has no feasible solution. |

History: Beginning J. Farkas's famous paper (1901). These fundamental for deriving necessary optimality conditions for LP and NLP. Farkas was Prof. of theoretical physics at U. of Kolozsvar, Hungary, motivated by nonlinear min problems subject to inequality constraints in study of mechanical equilibrium first posed by J. Fourier in 1798.

General form: Every theorem of alternative (T. of A.) deals with a system of linear constraints, $(I)$ say. It constructs another system $(II)$ in different variables, but sharing the data with $(I)$. Statement usually says: "*Either* $(I)$ *has a feasible solution, or* $(II)$ *has a feasible solution, but not both*".

The FIE and the FII:

| | |
|---|---|
| The Fundamental Inconsistent Equation (FIE) | $0 = 1$ |
| The Fundamental Inconsistent Inequality (FII) | $0 \geqq 1$ |

Broadly, T. of A. for systems of equations says that an infeasible system of equations is equivalent to the FIE.

And a T. of A. for systems involving inequalities states that if the system is infeasible, it is equivalent to the FII.

T. of A. for systems of linear eqs.

Theorem: A system of linear eqs., $Ax = b$ is infeasible iff the FIE can be obtained as a linear combination of eqs. in it.

Example:

$$\begin{array}{rrrcr} x_1 & +x_2 & +x_3 & = & 2 \\ -x_1 & -x_2 & -x_3 & = & -1 \end{array}$$

Example:

$$A = \begin{pmatrix} 1 & -2 & 2 & -1 & 1 \\ -1 & 0 & 4 & -7 & 7 \\ 0 & -2 & 6 & -8 & 8 \end{pmatrix}, b = \begin{pmatrix} -8 \\ 16 \\ 6 \end{pmatrix}$$

T. of A. for linear eqs.: Given $A_{m\times n}, b_{m\times 1}$, either $(I)$ has a solution $x$, or $(II)$ has a solution $\pi = (\pi_1, \ldots, \pi_m)$ but not both.

$$\begin{array}{c} (I) \\ \hline Ax = b \\ \\ \end{array} \qquad \begin{array}{c} (II) \\ \hline \pi A = 0 \\ \pi b = 1 \end{array}$$

T. of A. for systems including linear inequalities:

Example:

$$\begin{array}{rrrcr} x_1 & +x_2 & +x_3 & \geqq & 2 \\ -x_1 & -x_2 & -x_3 & \geqq & -1 \end{array}$$

Valid Linear Combination of Linear Inequalities: Can multiply inequalities only by nonnegative scalars. Can add inequalities only if they are all in the same direction (i.e., either all are $\geqq$, or all are $\leqq$).

T. of A. for systems involving inequalities can be proved using **Tucker's Lemma**

Tucker's Lemma: $A_{m \times n}$ given. Consider following homogeneous systems.

$$Ax \geqq 0 \tag{1}$$

$$\pi A = 0 \quad , \quad \pi \geqq 0 \tag{2}$$

where $\pi = (\pi_1, \ldots, \pi_m)$, $x = (x_1, \ldots, x_n)^T$. There exist solutions $\bar{x}$ for (1); and $\bar{\pi}$ for (2) satisfying

$$A\bar{x} \quad + \quad (\bar{\pi})^T \quad > 0$$

Tucker Diagram:

$$\begin{array}{r|ccc|l}
 & x_1 & \ldots & x_n & \\
\hline
0 \leqq \pi_1 & a_{11} & \ldots & a_{1n} & \geqq 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 \leqq \pi_m & a_{m1} & \ldots & a_{mn} & \geqq 0 \\
\hline
 & = 0 & \ldots & = 0 &
\end{array}$$

Lemma says that there exist solutions $\bar{x}, \bar{\pi}$ to the two systems, which satisfy: for each $i = 1$ to $m$, either $\pi_i > 0$, or $A_{i.}\bar{x} > 0$, or both.

Farkas Lemma: Given $A_{m \times n}, b_{m \times 1}$, either $(I)$ has a solution $x$, or $(II)$ has a solution $\pi = (\pi_1, \ldots, \pi_m)$ but not both.

$$\begin{array}{cc} (I) & (II) \\ \hline Ax = b & \pi A \leqq 0 \\ x \geqq 0 & \pi b > 0 \end{array}$$

Optimality Condotions for LP from Farkas Lemma:

Theorem: Consider general LP: min $cx$ s. to $Ax \geqq b$ where $A_{m\times n}$.

If $\bar{x}$ is a feasible solution, it is optimal iff there exists a $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_m)$ satisfying:

$$\text{Dual feasibility} \quad \begin{cases} \bar{\pi}A = c \\ \bar{\pi} \geqq 0 \end{cases}$$

$$\text{C. S. Conds.} \quad \bar{\pi}_i(A_{i.}\bar{x} - b_i) = 0 \text{ for } i = 1 \text{ to } m$$

Example: Consider feasible solution $\bar{x} = (6, 0, -1, 0, 2)^T$ to LP:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|
| 1 | 1 | −1 | 2 | −1 | $\geqq$ | 5 |
| −2 | | 2 | −1 | 3 | $\geqq$ | −8 |
| 1 | | | | | $\geqq$ | 6 |
| | −3 | | 3 | | $\geqq$ | −5 |
| | | 5 | −1 | 7 | $\geqq$ | 7 |
| −3 | 1 | 3 | | 5 | | Minimize |

Motzkin's T. of A.:

$$\begin{array}{cc}
(I) & (II) \\
\overline{Ax > 0} & \overline{\pi A + \mu B + \gamma C = 0} \\
Bx \geqq 0 & \pi \geq 0, \mu \geqq 0 \\
Cx = 0 &
\end{array}$$

Gordon's T. of A.:

$$\begin{array}{cc}
(I) & (II) \\
\overline{Ax > 0} & \overline{\pi A = 0} \\
 & \pi \geq 0
\end{array}$$

Gale's T. of A.:

$$\begin{array}{cc}
(I) & (II) \\
\overline{Ax \geqq b} & \overline{\pi A = 0} \\
 & \pi b = 1 \\
 & \pi \geqq 0
\end{array}$$

Tucker's T. of A.:

$$\begin{array}{cc} (I) & (II) \\ \hline Ax \geq 0 & \pi A + \mu B + \gamma C = 0 \\ Bx \geqq 0 & \pi > 0, \mu \geqq 0 \\ Cx = 0 & \end{array}$$