Solutions for the take-home exam

Problem 1. Let H be a hyperplane in $\mathbf{P}^n$ and $p \in \mathbf{P}^n \smallsetminus H$.

i) The *projection* from $\mathbf{P}^n$ to H with center p is the map $\pi\colon \mathbf{P}^n \smallsetminus \{p\} \to H$ such that $\pi(q)$ is the intersection of the line spanned by p and q with H. Show that this is a morphism.
ii) Show that if X is a closed subvariety of $\mathbf{P}^n$ such that $p \not\in X$, then π restricts to a finite morphism $X \to H$ (this follows easily from the result stated in class saying that a proper morphism with finite fibers is finite; however, since we did not prove that result, you can't use it).

Solution. After a linear change of coordinates, we may assume that $H = (x_n = 0)$ and $p = [0, \ldots, 0, 1]$. We identify $\mathbf{P}^{n-1}$ with H via

$$[a_0, \ldots, a_{n-1}] \to [a_0, \ldots, a_{n-1}, 0].$$

If $q = [a_0, \ldots, a_n] \in \mathbf{P}^n \smallsetminus H$, then the line spanned by p and q is the set

$$\{[\lambda a_0, \ldots, \lambda a_{n-1}, b] \mid \lambda \in k^*, b \in k\}.$$

We thus see that the map $\pi\colon \mathbf{P}^n \smallsetminus \{p\} \to \mathbf{P}^{n-1}$ is given by

$$\pi\big([a_0, \ldots, a_n]\big) = [a_0, \ldots, a_{n-1}]$$

and it is now straightforward to check that π is a morphism. Indeed, $\mathbf{P}^{n-1}$ is covered by the affine open subsets $U_i = (x_i \neq 0) \simeq \mathbf{A}^{n-1}$, with $0 \leq i \leq n-1$, and $\pi^{-1}(U_i) = (x_i \neq 0) \simeq \mathbf{A}^n$. Via these isomorphisms, the induced map $\pi^{-1}(U_i) \to U_i$ gets identified with the map

$$\mathbf{A}^n \to \mathbf{A}^{n-1}, \quad (a_1, \ldots, a_n) \to (a_1, \ldots, a_{n-1}),$$

which is clearly a morphism. Since this holds for all i, we see that π is a morphism.

We now show that if X is a closed subvariety of $\mathbf{P}^n$ such that $p \not\in X$, then the induced morphism $\pi_X\colon X \to \mathbf{P}^{r-1}$ is finite. It is enough to show that if $U_i = (x_i \neq 0) \subseteq \mathbf{P}^{n-1}$, then for each i, with $0 \leq i \leq n-1$, the inverse image $\pi_X^{-1}(U_i)$ is affine and the induced homomorphism

$$\mathcal{O}(U_i) \to \mathcal{O}\big(\pi_X^{-1}(U_i)\big) \tag{1}$$

is a finite homomorphism. The fact that $\pi_X^{-1}(U_i)$ is affine is clear, since this is equal to $D_X^+(x_i)$, hence it is affine by a result proved in class. Moreover, we can identify the homomorphism (1) with

$$k[x_0, \ldots, x_{n-1}]_{(x_i)} = k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right] \to (S_X)_{(x_i)}, \tag{2}$$

where S_X is the homogeneous coordinate ring of X. Since $(S_X)_{(x_i)}$ is generated by $\frac{x_j}{x_i}$, with $0 \leq j \leq n$, in order to show that (2) is a finite homomorphism, it is enough to show that each $\frac{x_j}{x_i} \in (S_X)_{(x_i)}$ is integral over $k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right]$. This is clear if $j \leq n-1$, hence we only need to consider $\frac{x_n}{x_i}$. By hypothesis, we have $[0, \ldots, 0, 1] \not\in X$. Therefore there is a homogeneous polynomial f, say of degree d, in the ideal I_X corresponding to X such that x_n^d appears in f with nonzero coefficient. If $d = 0$, then X is empty, in which case the assertion to prove is trivial. If $d > 0$, we may assume that $f = x_n^d + \sum_{i=1}^d g_i(x_0, \ldots, x_{n-1})x_n^{d-i}$. Dividing by x_i^d, we thus conclude that

$$\left(\frac{x_n}{x_i}\right)^d + \sum_{i=1}^d g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right)\left(\frac{x_n}{x_i}\right)^{d-i} = 0 \quad \text{in} \quad (S_X)_{(x_i)},$$

hence $\frac{x_n}{x_i}$ is integral over $k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right]$. This gives our assertion.

Problem 2. Let $f\colon X \to Y$ be a dominant morphism of irreducible varieties. Show that if $\dim(X) = \dim(Y)$, then there is a non-empty open subset V of Y such that the induced morphism $f^{-1}(V) \to V$ is finite (in this case, one says that f is *generically finite*).

Solution. We may clearly replace Y by an affine open subset and X by the inverse image of this subset, in order to assume that Y is an affine variety. In fact, we may assume that X is affine as well. Indeed, let us choose an affine open subset U of X and suppose that we know the assertion in the proposition for the induced morphism $U \to Y$. In other words, we know that there is a non-empty open subset V of Y, such that the induced morphism $g\colon U \cap f^{-1}(V) \to V$ is finite. Note that if $Z = \overline{f(X \smallsetminus U)}$, then

$$\dim(Z) \leq \dim(X \smallsetminus U) < \dim(X) = \dim(Y),$$

hence Z is a proper closed subset of Y. If we take $V' = V \smallsetminus Z$, then V' is non-empty and the induced morphism $g^{-1}(V') = U \cap f^{-1}(V') \to V'$ is finite. However, it follows from the definition of X' that $f^{-1}(V') \subseteq U$, which implies that V' satisfies the requirement in the proposition.

Suppose now that both X and Y are affine varieties, and consider the homomorphism

$$f^{\#}\colon A = \mathcal{O}(Y) \to \mathcal{O}(X) = B$$

corresponding to f. Note that this is injective since f is dominant. Let $k(Y) = \mathrm{Frac}(A)$ be the field of rational functions of Y. The assumption that $\dim(X) = \dim(Y)$ implies that $\mathrm{Frac}(B)$ is algebraic, hence finite, over $\mathrm{Frac}(A)$ by the result proved in class, describing the dimension of an irreducible variety as the transcendence degree of its function field. Noether's Normalization lemma thus implies that $B \otimes_A k(Y)$ is a finite $k(Y)$-algebra. Let $b_1, \ldots, b_r \in B$ be generators of B as a k-algebra. Since each b_i is algebraic over $k(Y)$, we see that there is $f_i \in A$ such that $\frac{b_i}{1}$ is integral over A_{f_i}. This implies that if $f = \prod_i f_i$, then each $\frac{b_i}{1}$ is integral over A_f, hence $A_f \to B_f$ is a finite homomorphism. Therefore $V = D_Y(f)$ satisfies the assertion in the proposition.

Problem 3.

i) Give an example of an irreducible hypersurface of degree 3 in $\mathbf{P}^4$ that contains a 2-dimensional linear subspace.
ii) Show that any hypersurface as in i) is singular.
iii) Show that there are smooth hypersurfaces of degree 3 in $\mathbf{P}^5$ that contain a 2-dimensional linear subspace.

Solution. For part i), let $f \in k[x_0, x_1, x_2$ be a general degree 3 polynomial, that defines a smooth hypersurface C in $\mathbf{P}^2$. Note that this is automatically irreducible: any two irreducible components would meet, and the resulting intersection would be contained in the singular locus. For an explicit example, when $\mathrm{char}(k) \neq 3$, one can take $f = x_0^3 + x_1^3 + x_2^3$. If H is the hypersurface defined by $f = 0$ in $\mathbf{P}^4$, then H is clearly irreducible and for every point $[a_0, a_1, a_2] \in C$, it contains the 2-dimensional linear subspace

$$\{[a_0, a_1, a_2, b, c] \in \mathbf{P}^4 \mid b, c \in k\}.$$

The assertions in ii) and iii) are special cases of the following more general assertions: consider hypersurfaces in $\mathbf{P}^n$ of degree $d \geq 2$. We will show that the following hold:

a) If X is a smooth such hypersurface containing a linear subspace $\Lambda \subseteq \mathbf{P}^n$ of dimension r, then $r \leq \frac{n-1}{2}$.
b) If $\Lambda \subseteq \mathbf{P}^n$ is a linear subspace of dimension $r \leq \frac{n-1}{2}$, then a general hypersurface containing Λ is smooth.

After a suitable choice of coordinates on $\mathbf{P}^n$, we may assume that Λ is the linear subspace defined by

$$x_{r+1} = \ldots = x_n = 0.$$

Suppose that X is the hypersurface defined by a homogeneous polynomial F, of degree d. If X contains Λ, then we can write

$$F = \sum_{i=1}^{n-r} x_{r+i} f_i, \tag{3}$$

for some $f_i \in k[x_0, \ldots, x_n]$, homogeneous of degree $d-1$. For every i, with $1 \leq i \leq n-r$, consider the homogeneous polynomials of degree $d-1$

$$g_i(x_0, \ldots, x_r) = f_i(x_0, \ldots, x_r, 0, \ldots, 0).$$

If $n - r \leq r$, then a repeated application of a result proved in class[1] implies that there is a point $[u_0, \ldots, u_r] \in \mathbf{P}^r$ such that

$$g_i(u_0, \ldots, u_r) = 0 \quad \text{for} \quad 1 \leq i \leq n-r.$$

In other words, there is a point $p \in \Lambda$ such that $f_i(p) = 0$ for all $1 \leq i \leq n-r$. In this case, it follows from (3) that $F(p) = 0$ and $\frac{\partial F}{\partial x_j}(p) = 0$ for $0 \leq j \leq n$, hence p is a singular point of X. We thus deduce that if X is smooth, then $n - r \geq r + 1$, giving a).

[1] A special case of this result says that if X is a closed subset of $\mathbf{P}^n$, with $\dim(X) \geq 1$, and H is a hypersurface in $\mathbf{P}^n$, then $X \cap H$ is nonempty, of dimension equal to $\dim(X) - 1$.

Suppose now that $r \leq \frac{n-1}{2}$ and consider the subset W of $\mathbf{P}^{N_d}$ consisting of those $[F]$ such that Λ is contained in the zero-locus $(F=0)$ (recall that $\mathbf{P}^{N_d}$ is the projective space of lines in the vector space $k[x_0,\ldots,x_n]_d$). Therefore W consists of those $[F]$ such that $F \in (x_{r+1},\ldots,x_n)$, which is a linear subspace in $\mathbf{P}^{N_d}$, of codimension $\binom{r+d}{d}$. Let U be the subset of W consisting of those $[F]$ such that there is no $p \in \mathbf{P}^n$, with

$$F(p) = 0 = \frac{\partial F}{\partial x_i}(p) \quad \text{for} \quad 0 \leq i \leq n. \tag{4}$$

Every such F generates a radical ideal and the corresponding degree d hypersurface contains Λ and is smooth. We need to show that U is open and non-empty.

Let us consider the set $\mathcal{Y}_W$ of pairs $(p,[F]) \in \mathbf{P}^n \times W$ such that (4) holds. This is a closed subset of $\mathbf{P}^n \times W$, hence it is a projective variety. Let $\alpha\colon \mathcal{Y}_W \to \mathbf{P}^n$ and $\beta\colon \mathcal{Y}_W \to W$ be the morphisms induced by the two projections. Since $U = W \smallsetminus \beta(\mathcal{Y}_W)$, it follows that U is open in W, and it is enough to show that $\beta(\mathcal{Y}_W) \neq W$.

We now describe the fiber $\alpha^{-1}(p)$ for $p \in \mathbf{P}^n$. Suppose first that $p \in \Lambda$. We may choose coordinates such that $p = [1,0,\ldots,0]$. The conditions in (4) are equivalent with the fact that the coefficients of $x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$ in F are 0. Since $F \in (x_{r+1},\ldots,x_n)$, we see that $\alpha^{-1}(p) \hookrightarrow W$ is a linear subspace of codimension $n-r$. Suppose now that $p \not\in \Lambda$, in which case we may choose coordinates such that $p = [0,\ldots,0,1]$, in which case the conditions in (4) are equivalent with the fact that the coefficients of $x_n^d, x_n^{d-1}x_{n-1}, \ldots, x_n^{d-1}x_0$ are 0. We thus see that in this case $\alpha^{-1}(p) \hookrightarrow W$ is a linear subspace of codimension $n+1$. We deduce from the theorems on fiber dimensions that

$$\dim\left(\alpha^{-1}(\Lambda)\right) = \dim(\Lambda) + \dim(W) - (n-r) = \dim(W) + (2r-n)$$

and

$$\dim\left(\alpha^{-1}(\mathbf{P}^n \smallsetminus \Lambda)\right) = \dim(\mathbf{P}^n \smallsetminus \Lambda) + \dim(W) - (n+1) = \dim(W) - 1.$$

Since by assumption we have $2r-n \leq -1$, we deduce that $\dim(\mathcal{Y}_W) = \dim(W) - 1$, hence $\dim\left(\beta(\mathcal{Y}_W)\right) \leq \dim(\mathcal{Y}_W) < \dim(W)$. This completes the proof of b).

Problem 4. Fix a positive integer n and let

$$\mathcal{N}_n = \{A \in M_n(k) \mid A \,\text{is nilpotent}\}.$$

i) Show that $\mathcal{N}_n$ is a closed subset of $M_n(k)$. Show that, in fact, it is the zero-locus of n regular functions on $M_n(k)$.
ii) Show that $\mathcal{N}_n$ is irreducible and $\dim(\mathcal{N}_n) = n^2 - n$. Hint: use the description of nilpotent endomorphisms of k^n in terms of the existence of a suitable flag in k^n.

Solution. Recall that a matrix $A \in M_n(k)$ is nilpotent if and only if $A^n = 0$. Since the entries of A^n are homogeneous polynomials of degree n in the entries of A, it follows that $\mathcal{N}_n$ is a closed subset of $M_n(k)$, preserved by the standard k^*-action on $M_n(k)$ (in other words, $\mathcal{N}_n$ is the affine cone over a projective variety $\mathcal{N}_n^{\rm proj}$ in the projective space $\mathbf{P}$ of lines in $M_n(k)$, isomorphic to $\mathbf{P}^{n^2-1}$).

In fact, we can define $\mathcal{N}_n$ by only n equations. Indeed, a matrix A is nilpotent if and only if its characteristic polynomial $\det(A - \lambda I)$ is equal to $(-\lambda)^n$. If we write

$$\det(A - \lambda I) = \sum_{i=0}^{n} (-1)^i p_i(A) \lambda^i,$$

then $p_n(A) = 1$ and for each i, with $0 \le i \le n-1$, $p_i(A)$ is a homogeneous polynomial of degree $n - i$ in the entries of A. We thus see that $\mathcal{N}_n$ is the zero-locus of the ideal $(p_0, \ldots, p_{n-1})$. This proves i)

Our next goal is to show that $\mathcal{N}_n$ is irreducible and compute its dimension. In order to apply our irreducibility criterion, it is more convenient to work with the corresponding projective variety $\mathcal{N}_n^{\rm proj}$.

The key observation is the following: a matrix $A \in M_n(k)$ is nilpotent if and only if there is a complete flag of subspaces

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n = V,$$

with $\dim_k(V_i) = i$ and $A(V_i) \subseteq V_{i-1}$ for $1 \le i \le n$ (where we put $V_0 = 0$). Indeed, it is clear that if we have such a flag, then $A^n = 0$. Conversely, if $A^n = 0$, let $W_i = A^{n-i}(k^n)$. It follows from definition that

$$W_0 = 0 \subseteq W_1 \subseteq \ldots \subseteq W_n = k^n$$

and $A(W_i) \subseteq W_{i-1}$ for $1 \le i \le n$. If we refine this sequence of subspaces to a complete flag, this flag will satisfy the required conditions.

Motivated by this, we define

$$Z = \big\{ \big([A], (V_1, \ldots, V_n)\big) \in \mathbf{P} \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1} \,\text{for}\, 1 \le i \le n \big\}$$

(where in the above formula we make the convention that $V_0 = \{0\}$). It is not hard to check that Z is a closed subset of $\mathbf{P} \times \mathrm{Fl}(k^n)$. In particular, we see that Z is a projective variety. The projections of $\mathbf{P} \times \mathrm{Fl}(k^n)$ onto the two components induce proper morphisms

$$\pi_1 \colon Z \to \mathbf{P} \quad \text{and} \quad \pi_2 \colon Z \to \mathrm{Fl}(k^n).$$

Let us consider the fiber of π_2 over a flag $V_\bullet = (V_1, \ldots, V_n)$. If we choose a basis $e_1, \ldots, e_n$ such that each V_i is generated by $e_1, \ldots, e_i$, it follows that $\pi_2^{-1}(V_\bullet)$ is isomorphic to the the subvariety of $\mathbf{P}$ consisting of classes of nonzero strictly upper-triangular matrices, hence it is isomorphic to $\mathbf{P}^{\frac{n(n-1)}{2}-1}$. Since $\mathrm{Fl}(k^n)$ is irreducible, of dimension $\frac{n(n-1)}{2}$ (by a problem in HW10), it follows from our irreducibility criterion that Z is an irreducible variety, of dimension $n^2 - n - 1$ (by the second theorem on fiber dimensions).

Consider now the morphism $\pi_1 \colon Z \to \mathbf{P}$, whose image is $\mathcal{N}_n^{\mathrm{proj}}$. This implies that $\mathcal{N}_n^{\mathrm{proj}}$ is irreducible. We next show that over a non-empty open subset of $\mathcal{N}_n^{\mathrm{proj}}$, each fiber of π_1 consists of just one point. Note that if $A \in M_n(k)$ is a nilpotent matrix, then its rank is $\leq n-1$. Let $\mathcal{U}_n^{\mathrm{proj}}$ be the open subset of $\mathcal{N}_n^{\mathrm{proj}}$ consisting of matrices of rank $n-1$. Note that this is a non-empty subset: for example, the nilpotent matrix $(a_{i,j})$ with $a_{\ell,\ell-1} = 1$ for $2 \leq \ell \leq n$ and all other $a_{i,j}$ equal to 0 has rank $n-1$. We note that if $[A] \in \mathcal{U}_n^{\mathrm{proj}}$, then $\pi^{-1}\big([A]\big)$ has only one element: if $(V_1, \ldots, V_n)$ is a flag in k^n such that $A(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq n$, then $V_i = A^{n-i}(V)$ for all i. Indeed, the condition on the flag implies that $A^{n-i}(k^n) \subseteq V_i$ and the condition on the rank of A implies easily, by descending induction on i, that $\dim_k A^{n-i}(k^n) = i$. Therefore $A^{n-i}(k^n) = V_i$ for $1 \leq i \leq n$.

Since π_1 has finite fibers over $\mathcal{U}_n$, we deduce from the first theorem on fiber dimensions that

$$\dim(\mathcal{N}_n^{\mathrm{proj}}) = \dim(Z) = n^2 - n - 1.$$

We thus conclude that $\mathcal{N}_n$ is an irreducible variety of dimension $n^2 - n$.