Lecture notes for Math 631 & 632: Introduction to algebraic geometry

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Contents

CHAPTER 1

Affine and quasi-affine varieties

The main goal in this chapter is to establish a correspondence between various geometric notions and algebraic ones. Some references for this chapter are [**Har77**, Chapter I], [**Mum88**, Chapter I], and [**Sha13**, Chapter I].

1.1. Algebraic subsets and ideals

Let k be a fixed algebraically closed field. We do not make any assumption on the characteristic. Important examples are $\mathbf{C}$, $\overline{\mathbf{Q}}$, and $\overline{\mathbf{F}_p}$, for a prime integer p.

For a positive integer n we denote by $\mathbf{A}^n$ the *n-dimensional affine space*. For now, this is just a set, namely k^n. We assume that n is fixed and denote the polynomial ring $k[x_1, \ldots, x_n]$ by R. Note that if $f \in R$ and $u = (u_1, \ldots, u_n)$, we may evaluate f at u to get $f(u) \in k$. This gives a surjective ring homomorphism

$$k[x_1, \ldots, x_n] \to k, \quad f \to f(u),$$

whose kernel is the (maximal) ideal $(x_1 - u_1, \ldots, x_n - u_n)$.

Our goal in this section is to establish a correspondence between certain subsets of $\mathbf{A}^n$ (those *defined by polynomial equations*) and ideals in R (more precisely, radical ideals). A large part of this correspondence is tautological. The non-trivial input will be provided by Hilbert's Nullstellensatz, which we will be prove in the next section.

Definition 1.1.1. Given a subset $S \subseteq R$, the *zero-locus of S* (also called *the subset of $\mathbf{A}^n$ defined by S*) is the set

$$V(S) := \{u \in \mathbf{A}^n \mid f(u) = 0 \text{ for all } f \in S\}.$$

An *algebraic subset* of $\mathbf{A}^n$ is a subset of the form $V(S)$ for some subset S of R.

Example 1.1.2. Any linear subspace of k^n is an algebraic subset; in fact, it can be written as $V(S)$, where S is a finite set of linear polynomials (that is, polynomials of the form $\sum_{i=1}^n a_i x_i$). More generally, any translation of a linear subspace (that is, an *affine subspace*) of k^n is an algebraic subset.

Example 1.1.3. A union of two lines in $\mathbf{A}^2$ is an algebraic subset (see Proposition 1.1.6). For example, the union of the two coordinate axes can be written as $V(x_1x_2)$.

Example 1.1.4. Another example of an algebraic subset of $\mathbf{A}^2$ is the hyperbola

$$\{u = (u_1, u_2) \in \mathbf{A}^2 \mid u_1u_2 = 1\}.$$

Remark 1.1.5. Recall that if S is a subset of R and I is the ideal of R generated by S, then we can write

$$I = \{g_1f_1 + \ldots + g_mf_m \mid m \geq 0, f_1, \ldots, f_m \in S, g_1, \ldots, g_m \in R\}.$$

It is then easy to see that $V(S) = V(I)$. In particular, every algebraic subset of $\mathbf{A}^n$ can be written as $V(I)$ for some ideal I in R.

We collect in the following proposition the basic properties of taking the zero locus.

PROPOSITION 1.1.6. *The following hold:*

1) $V(R) = \emptyset$*; in particular, the empty set is an algebraic subset.*
2) $V(0) = \mathbf{A}^n$*: in particular,* $\mathbf{A}^n$ *is an algebraic subset.*
3) *If* I *and* J *are ideals in* R *with* $I \subseteq J$*, then* $V(J) \subseteq V(I)$*.*
4) *If* $(I_\alpha)_\alpha$ *is a family of ideals in* R*, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right) = V\left(\sum_\alpha I_\alpha\right).$$

5) *If* I *and* J *are ideals in* R*, then*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

PROOF. The assertions in 1)–4) are trivial to check. Note also that the inclusions

$$V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(I \cdot J)$$

follow directly from 3). In order to show that $V(I \cdot J) \subseteq V(I) \cup V(J)$, we argue by contradiction: suppose that $u \in V(I \cdot J) \smallsetminus \big(V(I) \cup V(J)\big)$. We can thus find $f \in I$ such that $f(u) \neq 0$ and $g \in J$ such that $g(u) \neq 0$. In this case $fg \in I \cdot J$ and $(fg)(u) = f(u)g(u) \neq 0$, a contradiction with the fact that k is a domain. $\square$

An important consequence of the assertions in the above proposition is that the algebraic subsets of $\mathbf{A}^n$ form the closed subsets for a topology of $\mathbf{A}^n$. This is the *Zariski topology* on $\mathbf{A}^n$.

The Zariski topology provides a convenient framework for dealing with algebraic subsets of $\mathbf{A}^n$. However, we will see that it has a lot less subsets than one is used to from the case of the usual Euclidean space (over $\mathbf{R}$ or over $\mathbf{C}$).

We now define a map in the other direction, from subsets of $\mathbf{A}^n$ to ideals in R. Given a subset W of $\mathbf{A}^n$, we put

$$I(W) := \{f \in R \mid f(u) = 0 \text{ for all } u \in W\}.$$

It is straightforward to see that this is an ideal in R. In fact, it is a radical[1] ideal: indeed, since k is a reduced ring, if $f(u)^q = 0$ for some positive integer q, then $f(u) = 0$. We collect in the next proposition some easy properties of this definition.

PROPOSITION 1.1.7. *The following hold:*

1) $I(\emptyset) = R$.
2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$*, then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$*.*
3) *If* $W_1 \subseteq W_2$*, then* $I(W_2) \subseteq I(W_1)$*.*

PROOF. All assertions follow immediately from definition. $\square$

[1]An ideal I in a ring R is *radical* if whenever $f^q \in I$ for some $f \in R$ and some positive integer q, we have $f \in I$. A related concept is that of a *reduced* ring: this is a ring such that whenever $f^q = 0$ for some $f \in R$ and some positive integer q, we have $f = 0$. Note that an ideal I is radical if and only if R/I is a reduced ring.

We have thus set up two maps between subsets of $\mathbf{A}^n$ and ideals in R and we are interested in the two compositions. Understanding one of these compositions is tautological, as follows:

PROPOSITION 1.1.8. *For every subset Z of $\mathbf{A}^n$, the set $V\big(I(Z)\big)$ is equal to the closure $\overline{Z}$ of Z, with respect to the Zariski topology. In particular, if Z is an algebraic subset of $\mathbf{A}^n$, then $V\big(I(Z)\big) = Z$.*

PROOF. We clearly have
$$Z \subseteq V\big(I(Z)\big),$$
and since the right-hand side is closed by definition, we have
$$\overline{Z} \subseteq V\big(I(Z)\big).$$
In order to prove the reverse inclusion, recall that by definition of the closure of a subset, we have
$$\overline{Z} = \bigcap_W W,$$
where W runs over all algebraic subsets of $\mathbf{A}^n$ that contain Z. Every such W can be written as $W = V(J)$, for some ideal J in R. Note that we have $J \subseteq I(W)$, while the inclusion $Z \subseteq W$ gives $I(W) \subseteq I(Z)$. We thus have $J \subseteq I(Z)$, hence $V\big(I(Z)\big) \subseteq V(J) = W$. Since $V\big(I(Z)\big)$ is contained in every such W, we conclude that
$$V\big(I(Z)\big) \subseteq \overline{Z}.$$
□

The interesting statement here concerns the other composition. Recall that if J is an ideal in a ring R, then the set
$$\{f \in R \mid f^q \in J \text{ for some } q \geq 1\}$$
is a radical ideal; in fact, it is the smallest radical ideal containing J, denoted $\mathrm{rad}(J)$.

THEOREM 1.1.9 (Hilbert's Nullstellensatz). *For every ideal J in R, we have*
$$I\big(V(J)\big) = \mathrm{rad}(J).$$

The inclusion $J \subseteq I\big(V(J)\big)$ is trivial and since the right-hand side is a radical ideal, we obtain the inclusion
$$\mathrm{rad}(J) \subseteq I\big(V(J)\big).$$
This reverse inclusion is the subtle one and this is where we use the hypothesis that k is algebraically closed (note that this did not play any role so far). We will prove this in the next section, after some preparations. Assuming this, we obtain the following conclusion.

COROLLARY 1.1.10. *The two maps $I(-)$ and $V(-)$ between the algebraic subsets of $\mathbf{A}^n$ and the radical ideals in $k[x_1, \ldots, x_n]$ are inverse, order-reversing bijections.*

REMARK 1.1.11. It follows from Corollary 1.1.10 that via the above bijection, the minimal nonempty algebraic subsets correspond to the maximal ideals in R. It is clear that the minimal nonempty algebraic subsets are precisely the points in $\mathbf{A}^n$. On the other hand, given $a = (a_1, \ldots, a_n) \in \mathbf{A}^n$, the ideal $I(u)$ contains the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$, hence the two ideals are equal. We thus deduce that *every* maximal ideal in R is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. We will see in the next section that the general statement of Theorem 1.1.9 is proved by reduction to this special case.

EXERCISE 1.1.12. Show that the closed subsets of $\mathbf{A}^1$ are $\mathbf{A}^1$ and its finite subsets.

EXERCISE 1.1.13. Show that if W_1 and W_2 are algebraic subsets of $\mathbf{A}^n$, then

$$I(W_1 \cap W_2) = \mathrm{rad}\big(I(W_1) + I(W_2)\big).$$

EXERCISE 1.1.14. For m and $n \geq 1$, let us identify $\mathbf{A}^{mn}$ with the set of all matrices $B \in M_{m,n}(k)$. Show that the set

$$M^r_{m,n}(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \leq r\}$$

is a closed algebraic subset of $M_{m,n}(k)$.

EXERCISE 1.1.15. Show that the following subset of $\mathbf{A}^3$

$$W_1 = \{(t, t^2, t^3) \mid t \in k\}$$

is a closed algebraic subset, and describe $I(W_1)$. Can you do the same for

$$W_2 = \{(t^2, t^3, t^4) \mid t \in k\}?$$

How about

$$W_3 = \{(t^3, t^4, t^5) \mid t \in k\}?$$

EXERCISE 1.1.16. For an arbitrary commutative ring R, one can define the *maximal spectrum* $\mathrm{MaxSpec}(R)$ of R, as follows. As a set, this is the set of all maximal ideals in R. For every ideal J in R, we put

$$V(J) := \{\mathfrak{m} \in \mathrm{MaxSpec}(R) \mid J \subseteq \mathfrak{m}\}$$

and for every subset $S \subseteq \mathrm{MaxSpec}(R)$, we define

$$I(S) := \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

i) Show that $\mathrm{MaxSpec}(R)$ has a structure of topological space in which the closed subsets are the subsets of the form $V(I)$, for an ideal I in R.
ii) Show that for every subset S of $\mathrm{MaxSpec}(R)$, we have $V\big(I(S)\big) = \overline{S}$.
iii) Show that if R is an algebra of finite type over an algebraically closed field k, then for every ideal J in S, we have $I\big(V(J)\big) = \mathrm{rad}(J)$.
iv) Show that if $X \subseteq \mathbf{A}^n$ is a closed subset, then we have a homeomorphism $X \simeq \mathrm{MaxSpec}(R/J)$, where $R = k[x_1, \ldots, x_n]$ and $J = I(X)$.

1.2. Noether normalization and Hilbert's Nullstellensatz

The proof of Hilbert's Nullstellensatz is based on the following result, known as *Noether's normalization lemma*. As we will see, this has many other applications.

Before stating the result, we recall that a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module. It is straightforward to check that a composition of two finite homomorphisms is again finite. Moreover, if $A \to B$ is a finite homomorphism, then for every homomorphism $A \to C$, the induced homomorphism $C = A \otimes_A C \to B \otimes_A C$ is finite. For details about finite morphisms and the connection with integral morphisms, see Appendix A. One property that we will need is that if $A \hookrightarrow B$ is an injective finite homomorphism, with A and B domains, then A is a field if and only if B is a field (see Proposition A.2.1).

REMARK 1.2.1. If $A \hookrightarrow B$ is an injective, finite homomorphism between two domains, and $K = \mathrm{Frac}(A)$ and $L = \mathrm{Frac}(B)$, then the induced injective homomorphism $K \hookrightarrow L$ is finite. Indeed, by tensoring the inclusion $A \hookrightarrow B$ with K, we obtain a finite, injective homomorphism $K \hookrightarrow K \otimes_A B$ between domains. Note that $K \otimes_A B$ is a ring of fractions of B, hence the canonical homomorphism $K \otimes_A B \to L$ is injective. Since K is a field, it follows that $K \otimes_A B$ is a field, and thus $K \otimes_A B = L$. In particular, we see that $[L : K] < \infty$.

THEOREM 1.2.2. *Let k be a field and A a finitely generated k-algebra which is an integral domain, with fraction field K. If* $\mathrm{trdeg}(K/k) = n$*, then there is a k-subalgebra B of A, such that*

1) *B is isomorphic as a k-algebra to $k[x_1, \ldots, x_n]$, and*
2) *The inclusion $B \hookrightarrow A$ is finite.*

PROOF. We only give the proof when k is infinite. This will be enough for our purpose, since in all our applications the field k will always contain an algebraically closed (hence infinite) field. For a proof in the general case, see [**Mum88**].

The fact that k is infinite will be used via the following property: for every nonzero polynomial $f \in k[x_1, \ldots, x_r]$, there is $\lambda \in k^r$ such that $f(\lambda) \neq 0$. When $r = 1$, this follows from the fact that a nonzero polynomial in one variable has at most as many roots as its degree. The general case then follows by an easy induction on r.

Let $y_1, \ldots, y_m \in A$ be generators of A as a k-algebra. In particular, we have $K = k(y_1, \ldots, y_m)$, hence $m \geq n$. We will show, by induction on m, that we can find a change of variable of the form

$$y_i = \sum_{j=1}^{n} b_{i,j} z_j, \quad \text{for} \quad 1 \leq i \leq m, \quad \text{with} \quad \det(b_{i,j}) \neq 0,$$

(so that we have $A = k[z_1, \ldots, z_m]$) such that the inclusion $k[z_1, \ldots, z_n] \hookrightarrow A$ is finite. Note that this is enough: if $B = k[z_1, \ldots, z_n]$, then it follows from Remark 1.2.1 that the induced field extension $\mathrm{Frac}(B) \hookrightarrow K$ is finite. Therefore we have

$$n = \mathrm{trdeg}(K/k) = \mathrm{trdeg}\big(k(z_1, \ldots, z_n)/k\big),$$

hence $z_1, \ldots, z_n$ are algebraically independent.

If $m = n$, there is nothing to prove. Suppose now that $m > n$, hence $y_1, \ldots, y_m$ are algebraically dependent over k. Therefore there is a nonzero polynomial $f \in k[x_1, \ldots, x_m]$ such that $f(y_1, \ldots, y_m) = 0$. Suppose now that we write

$$y_i = \sum_{j=1}^{m} b_{i,j} z_j, \quad \text{with} \quad b_{i,j} \in k, \, \det(b_{i,j}) \neq 0.$$

Let $d = \deg(f)$ and let us write

$$f = f_d + f_{d-1} + \ldots + f_0, \quad \text{with} \quad \deg(f_i) = i \quad \text{or} \quad f_i = 0.$$

By assumption, we have $f_d \neq 0$. If we write

$$f = \sum_{\alpha \in \mathbf{Z}_{\geq 0}^m} c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

then we have

$$0 = f(y_1, \ldots, y_m) = \sum_{\alpha} c_\alpha (b_{1,1}z_1 + \ldots + b_{1,m}z_m)^{\alpha_1} \cdots (b_{m,1}z_1 + \ldots + b_{m,m}z_m)^{\alpha_m}$$

$$= f_d(b_{1,m}, \ldots, b_{m,m}) z_m^d + \text{lower degree terms in } z_m.$$

Since we assume that k is infinite, we may choose the $b_{i,j}$ such that

$$\det(b_{i,j}) \cdot f_d(b_{1,m}, \ldots, b_{m,m}) \neq 0.$$

In this case, we see that after this linear change of variable, the inclusion

$$k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite, since the right-hand side is generated as a module over the left-hand side by $1, y_m, \ldots, y_m^{d-1}$. Note that by Remark 1.2.1, the induced extension

$$k(y_1, \ldots, y_{m-1}) \hookrightarrow k(y_1, \ldots, y_m)$$

is finite, hence $\operatorname{trdeg}\big(k(y_1, \ldots, y_{m-1})/k\big) = n$. By induction, we can do a linear change of variable in $y_1, \ldots, y_{m-1}$, after which the inclusion

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}]$$

is finite, in which case the composition

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite. This completes the proof of the theorem. □

We will use Theorem 1.2.2 to prove Hilbert's Nullstellensatz in several steps.

Corollary 1.2.3. *If k is a field, A is a finitely generated k-algebra, and $K = A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in A, then K is a finite extension of k.*

Proof. Note that K is a field which is finitely generated as a k-algebra. It follows from the theorem that if $n = \operatorname{trdeg}(K/k)$, then there is a finite injective homomorphism

$$k[x_1, \ldots, x_n] \hookrightarrow K.$$

Since K is a field, it follows that $k[x_1, \ldots, x_n]$ is a field, hence $n = 0$. Therefore K/k is finite. □

Corollary 1.2.4. *(Hilbert's Nullstellensatz, weak version) If k is an algebraically closed field, then every maximal ideal $\mathfrak{m}$ in $R = k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$, for some $a_1, \ldots, a_n \in k$.*

Proof. It follows from Corollary 1.2.3 that if $K = R/\mathfrak{m}$, the field extension K/k is finite. Since k is algebraically closed, the canonical homomorphism $k \to K$ is an isomorphism. In particular, for every i there is $a_i \in R$ such that $x_i - a_i \in \mathfrak{m}$. Therefore we have $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$ and since both ideals are maximal, they must be equal. □

We can now prove Hilbert's Nullstellensatz, in its strong form.

Proof of Theorem 1.1.9. It follows from Corollary 1.2.4 that given any ideal $\mathfrak{a}$ of R, different from R, the zero-locus $V(\mathfrak{a})$ of $\mathfrak{a}$ is nonempty. Indeed, since $\mathfrak{a} \neq R$, there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. By Corollary 1.2.4, we have

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n) \quad \text{for some} \quad a_1, \ldots, a_n \in k.$$

In particular, we see that $a = (a_1, \ldots, a_n) \in V(\mathfrak{m}) \subseteq V(J)$. We will use this fact in the polynomial ring $R[y] = k[x_1, \ldots, x_n, y]$; this is *Rabinovich's trick*.

It is clear that for every ideal J in R we have the inclusion

$$\mathrm{rad}(J) \subseteq I\big(V(J)\big).$$

In order to prove the reverse inclusion, suppose that $f \in I\big(V(J)\big)$. Consider now the ideal $\mathfrak{a}$ in $R[y]$ generated by J and by $1 - fy$. If $\mathfrak{a} \neq R[y]$, we have seen that there is $(a_1, \ldots, a_n, b) \in V(\mathfrak{a})$. By definition of $\mathfrak{a}$, this means that $g(a_1, \ldots, a_n) = 0$ for all $g \in J$ (that is, $(a_1, \ldots, a_n) \in V(J)$) and $1 = f(a_1, \ldots, a_n)g(b)$. In particular, we have $f(a_1, \ldots, a_n) \neq 0$, contradicting the fact that $f \in I\big(V(J)\big)$.

We thus conclude that $\mathfrak{a} = R$. Therefore we can find $f_1, \ldots, f_r \in J$ and $g_1, \ldots, g_{r+1} \in R[y]$ such that

$$\sum_{i=1}^{r} f_i(x)g_i(x, y) + \big(1 - f(x)y\big)g_{r+1}(x, y) = 1. \tag{1.2.1}$$

We now consider the R-algebra homomorphism $R[y] \to R_f$ that maps y to $\frac{1}{f}$. The relation (1.2.1) gives

$$\sum_{i=1}^{r} f_i(x)g_i\big(x, 1/f(x)\big) = 1$$

and after clearing the denominators (recall that R is a domain), we see that there is a positive integer N such that $f^N \in (f_1, \ldots, f_r)$, hence $f \in \mathrm{rad}(J)$. This completes the proof of the theorem. $\square$

1.3. The topology on the affine space

In this section we begin making use of the fact that the ring $k[x_1, \ldots, x_n]$ is Noetherian. Recall that a (commutative) ring R is Noetherian if the following equivalent conditions hold:

i) Every ideal in R is finitely generated.
ii) There is no infinite strictly increasing sequence of ideals of R.
iii) Every nonempty family of ideals of R has a maximal element

For this and other basic facts about Noetherian rings and modules, see Appendix B. A basic result in commutative algebra is Hilbert's basis theorem: if R is a Noetherian ring, then $R[x]$ is Noetherian (see Theorem B.2.1). In particular, since a field k is trivially Noetherian, a recursive application of the theorem implies that every polynomial algebra $k[x_1, \ldots, x_n]$ is Noetherian.

As in the previous sections, we fix an algebraically closed field k and a positive integer n. The fact that the ring $R = k[x_1, \ldots, x_n]$ is Noetherian has two immediate consequences. First, since every ideal is finitely generated, it follows that for every algebraic subset $W \subseteq \mathbf{A}^n$, there are finitely many polynomials $f_1, \ldots, f_r$ such that $W = V(f_1, \ldots, f_r)$. Second, we see via the correspondence in Corollary 1.1.10 that there is no infinite strictly decreasing sequence of closed subsets in $\mathbf{A}^n$.

Definition 1.3.1. A topological space X is *Noetherian* if there is no infinite strictly decreasing sequence of closed subsets in X.

We have thus seen that with the Zariski topology $\mathbf{A}^n$ is a Noetherian topological space. This implies that every subspace of $\mathbf{A}^n$ is Noetherian, by the following

LEMMA 1.3.2. *If X is a Noetherian topological space and Y is a subspace of X, then Y is Noetherian.*

PROOF. If we have a infinite strictly decreasing sequence of closed subsets of Y

$$F_1 \supsetneq F_2 \supsetneq \ldots,$$

consider the corresponding sequence of closures in X:

$$\overline{F_1} \supseteq \overline{F_2} \supseteq \ldots.$$

Since F_i is closed in Y, we have $\overline{F_i} \cap Y = F_i$ for all i, which implies that $\overline{F_i} \neq \overline{F_{i+1}}$ for every i. This contradicts the fact that X is Noetherian. $\square$

REMARK 1.3.3. Note that every Noetherian topological space is quasi-compact: this follows from the fact that there is no infinite strictly increasing sequence of open subsets.

EXAMPLE 1.3.4. The real line $\mathbf{R}$, with the usual Euclidean topology, is not Noetherian.

We now introduce an important notion.

DEFINITION 1.3.5. A topological space X is *irreducible* if it is nonempty and whenever we write $X = X_1 \cup X_2$, with both X_1 and X_2 closed, we have $X_1 = X$ or $X_2 = X$. We say that X is *reducible* when it is not irreducible.

REMARK 1.3.6. By passing to complements, we see that a topological space is irreducible if and only if it is nonempty and for every two nonempty open subsets U and V, the intersection $U \cap V$ is nonempty (equivalently, every nonempty open subset of X is dense in X).

REMARKS 1.3.7.

1) If Y is a subset of X (with the subspace topology), the closed subsets of Y are those of the form $F \cap Y$, where F is a closed subset of X. It follows that Y is irreducible if and only if it is nonempty and whenever $Y \subseteq Y_1 \cup Y_2$, with Y_1 and Y_2 closed in X, we have $Y \subseteq Y_1$ or $Y \subseteq Y_2$.
2) If Y is an irreducible subset of X and if $Y \subseteq Y_1 \cup \ldots \cup Y_r$, with all Y_i closed in X, then there is i such that $Y \subseteq Y_i$. This follows easily by induction on r.
3) If Y and F are subsets of X, with F closed, then $Y \subseteq F$ if and only if $\overline{Y} \subseteq F$. It then follows from the description in 1) that Y is irreducible if and only if $\overline{Y}$ is irreducible.
4) If X is irreducible and U is a nonempty open subset of X, then it follows from Remark 1.3.6 that U is dense in X. Since X is irreducible, it follows from 3) that U is irreducible.

In the case of closed subsets of $\mathbf{A}^n$, the following proposition describes irreducibility in terms of the corresponding ideal.

PROPOSITION 1.3.8. *If $W \subseteq \mathbf{A}^n$ is a closed subset, then W is irreducible if and only if $I(W)$ is a prime ideal in R.*

PROOF. Note first that $W \neq \emptyset$ if and only if $I(W) \neq R$. Suppose first that W is irreducible and let $f, g \in R$ be such that $fg \in I(W)$. We can then write

$$W = \big(W \cap V(f)\big) \cup \big(W \cap V(g)\big)$$

Since both subsets on the right-hand side are closed and W is irreducible, it follows that we have either $W = W \cap V(f)$ (in which case $f \in I(W)$) or $W = W \cap V(g)$ (in which case $g \in I(W)$). Therefore $I(W)$ is a prime ideal.

Conversely, suppose that $I(W)$ is prime and we write $W = W_1 \cup W_2$, with W_1 and W_2 closed. Arguing by contradiction, suppose that $W \neq W_i$ for $i = 1, 2$, in which case $I(W) \subsetneq I(W_i)$, hence we can find $f_i \in I(W_i) \smallsetminus I(W)$. On the other hand, we have $f_1 f_2 \in I(W_1) \cap I(W_2) = I(W)$, contradicting the fact that $I(W)$ is prime. $\square$

Example 1.3.9. Since R is a domain, it follows from the proposition that $\mathbf{A}^n$ is irreducible.

Example 1.3.10. If $L \subseteq \mathbf{A}^n$ is a linear subspace, then L is irreducible. Indeed, after a linear change of variables, we have $R = k[y_1, \ldots, y_n]$ such that $I(L) = (y_1, \ldots, y_r)$ for some $r \geq 1$, and this is clearly a prime ideal in R.

Example 1.3.11. The union of two lines in $\mathbf{A}^2$ is a reducible closed subset.

Proposition 1.3.12. *Let X be a Noetherian topological space. Given a closed, nonempty subset Y, there are finitely many irreducible closed subsets $Y_1, \ldots, Y_r$ such that*

$$Y = Y_1 \cup \ldots \cup Y_r.$$

We may clearly assume that the decomposition is minimal, *in the sense that $Y_i \not\subseteq Y_j$ for $i \neq j$. In this case $Y_1, \ldots, Y_r$ are unique up to reordering.*

The closed subsets $Y_1, \ldots, Y_r$ in the proposition are the *irreducible components* of Y and the decomposition in the proposition is the *irreducible decomposition* of Y.

Proof of Proposition 1.3.12. Suppose first that there are nonempty closed subsets Y of X that do not have such a decomposition. Since X is Noetherian, we may choose a minimal such Y. In particular, Y is not irreducible, hence we may write $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 closed and strictly contained in Y. Note that Y_1 and Y_2 are nonempty, hence by the minimality of Y, we may write both Y_1 and Y_2 as finite unions of irreducible subsets. In this case, Y is also a finite union of irreducible subsets, a contradiction.

Suppose now that we have two minimal decompositions

$$Y = Y_1 \cup \ldots \cup Y_r = Y_1' \cup \ldots \cup Y_s',$$

with the Y_i and Y_j' irreducible. For every $i \leq r$, we get an induced decomposition

$$Y_i = \bigcup_{j=1}^{s} (Y_i \cap Y_j'),$$

with the $Y_i \cap Y_j'$ closed for all j. Since Y_i is irreducible, it follows that there is $j \leq s$ such that $Y_i = Y_i \cap Y_j' \subseteq Y_j'$. Arguing in the same way, we see that there is $\ell \leq r$ such that $Y_j' \subseteq Y_\ell$. In particular, we have $Y_i \subseteq Y_\ell$, hence by the minimality assumption, we have $i = \ell$, and therefore $Y_i = Y_j'$. By iterating this argument and by reversing the roles of the Y_α and the Y_β', we see that $r = s$ and the Y_α and the Y_β' are the same up to relabeling. $\square$

REMARK 1.3.13. It is clear that if X is a Noetherian topological space, W is a closed subset of X, and Z is a closed subset of W, then the irreducible decomposition of Z is the same whether considered in W or in X.

Recall that by a theorem due to Gauss, if R is a UFD, then the polynomial ring $R[x]$ is a UFD. A repeated application of this result gives that every polynomial ring $k[x_1,\ldots,x_n]$ is a UFD. In particular, a nonzero polynomial $f\in k[x_1,\ldots,x_n]$ is irreducible if and only if the ideal (f) is prime.

EXAMPLE 1.3.14. Given a polynomial $f\in k[x_1,\ldots,x_n]\smallsetminus k$, the subset $V(f)$ is irreducible if and only if f is a power of an irreducible polynomial. In fact, if the irreducible decomposition of f is $f=cf_1^{m_1}\cdots f_r^{m_r}$, for some $c\in k^*$, then the irreducible components of $V(f)$ are $V(f_1),\ldots,V(f_r)$.

EXERCISE 1.3.15. Let Y be the algebraic subset of $\mathbf{A}^3$ defined by the two polynomials x^2-yz and $xz-x$. Show that Y is a union of three irreducible components. Describe them and find the corresponding prime ideals.

EXERCISE 1.3.16. Show that if X and Y are topological spaces, with X irreducible, and $f\colon X\to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible.

EXERCISE 1.3.17. Let X be a topological space, and consider a finite open cover

$$X=U_1\cup\ldots\cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i\cap U_j\neq\emptyset$.

EXERCISE 1.3.18. Let X be a Noetherian topological space and Y a subset X. Show that if $Y=Y_1\cup\ldots\cup Y_r$ is the irreducible decomposition of Y, then $\overline{Y}=\overline{Y_1}\cup\ldots\cup\overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

EXERCISE 1.3.19. Let X be a Noetherian topological space and Y a nonempty closed subset of X, with irreducible decomposition

$$Y=Y_1\cup\ldots\cup Y_r.$$

Show that if U is an open subset of X, then the irreducible decomposition of $U\cap Y$ is given by

$$U\cap Y=\bigcup_{i,U\cap Y_i\neq\emptyset}(U\cap Y_i).$$

We end these general topological considerations by discussing the notion of locally closed subsets.

DEFINITION 1.3.20. Let X be a topological space. A subset V of X is *locally closed* if for every $x\in V$, there is an open neighborhood U_x of x in X such that $U_x\cap V$ is closed in U_x.

REMARK 1.3.21. One should contrast the above definition with the local characterization of closed subsets: V is closed in X if and only if for every $x\in X$, there is an open neighborhood U_x of x in X such that $U_x\cap V$ is closed in U_x.

PROPOSITION 1.3.22. *If V is a subset of a topological space X, then the following are equivalent:*

i) *V is a locally closed subset.*
ii) *V is open in $\overline{V}$.*
iii) *We can write $V = U \cap F$, with U open and F closed.*

PROOF. If V is locally closed, let us choose for every $x \in V$ an open neighborhood U_x of x as in the definition. In this case V is closed in U by Remark 1.3.21, hence $V = U \cap F$ for some F closed in X, proving i)⇒iii). In order to see iii)⇒ii), note that if if $V = U \cap F$, with U open and F closed, then $\overline{V} \subseteq F$, hence $V = U \cap \overline{V}$ is open in $\overline{V}$. Finally, the implication ii)⇒i) is clear: if $V = W \cap \overline{V}$ for some W open in X, then for every $x \in V$, if we take $U_x = W$, we have $U_x \cap V$ closed in U_x. $\square$

Let $X \subseteq \mathbf{A}^n$ be a closed subset. We always consider on X the subspace topology. We now introduce a basis of open subsets on X.

DEFINITION 1.3.23. A *principal affine open subset* of X is an open subset of the form
$$D_X(f) := X \smallsetminus V(f) = \{x \in X \mid f(x) \neq 0\},$$
for some $f \in k[x_1, \ldots, x_n]$.

Note that $D_X(f)$ is nonempty if and only if $f \not\in I(X)$. It is clear that $D_X(f) \cap D_X(g) = D_X(fg)$. Every open subset of X can be written as $X \smallsetminus V(J)$ for some ideal J in R. Since J is finitely generated, we can write $J = (f_1, \ldots, f_r)$, in which case
$$X \smallsetminus V(J) = D_X(f_1) \cup \ldots \cup D_X(f_r).$$
Therefore every open subset of X is a finite union of principal affine open subsets of X. We thus see that the principal affine open subsets give a basis for the topology of X.

EXERCISE 1.3.24. Let X be a topological space and Y a locally closed subset of X. Show that a subset Z of Y is locally closed in X if and only if it is locally closed in Y.

1.4. Regular functions and morphisms

DEFINITION 1.4.1. An *affine algebraic variety* (or affine variety, for short) is a a closed subset of some affine space $\mathbf{A}^n$. A *quasi-affine variety* is a locally closed subset of some affine space $\mathbf{A}^n$, or equivalently, an open subset of an affine algebraic variety. A quasi-affine variety is always endowed with the subspace topology.

The above is only a temporary definition: a (quasi)affine variety is not just a topological space, but it comes with more information that distinguishes which maps between such objects are allowed. We will later formalize this as a ringed space. We now proceed describing the "allowable" maps.

DEFINITION 1.4.2. Let $Y \subseteq \mathbf{A}^n$ be a locally closed subset. A *regular function* on Y is a map $\phi\colon Y \to k$ that can locally be given by a quotient of polynomial functions, that is, for every $y \in Y$, there is an open neighborhood U_y of y in Y, and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that
$$g(u) \neq 0 \quad \text{and} \quad \phi(u) = \frac{f(u)}{g(u)} \quad \text{for all} \quad u \in U_y.$$

We write $\mathcal{O}(Y)$ for the set of regular functions on Y. If Y is an affine variety, then $\mathcal{O}(Y)$ is also called the *coordinate ring* of Y. By convention, we put $\mathcal{O}(Y) = 0$ if $Y = \emptyset$.

REMARK 1.4.3. It is easy to see that $\mathcal{O}(Y)$ is a subalgebra of the k-algebra of functions $Y \to k$, with respect to point-wise operations. For example, suppose that ϕ_1 and ϕ_2 are regular functions, $y \in Y$ and U_1 and U_2 are open neighborhoods of y, and $f_1, f_2, g_1, g_2 \in k[x_1, \ldots, x_n]$ are such that for all $u \in U_y$ we have

$$g_i(u) \neq 0 \quad \text{and} \quad \phi_i(u) = \frac{f_i(u)}{g_i(u)} \quad \text{for} \quad i = 1, 2.$$

If we take $U = U_1 \cap U_2$ and $f = f_1 g_2 + f_2 g_1$, $g = g_1 g_2$, then for all $u \in U$, we have

$$g(u) \neq 0 \quad \text{and} \quad (\phi_1 + \phi_2)(u) = \frac{f(u)}{g(u)}.$$

REMARK 1.4.4. It follows from definition that if $\phi\colon Y \to k$ is a regular function such that $\phi(y) \neq 0$ for every $y \in Y$, then the function $\frac{1}{\phi}$ is a regular function, too.

EXAMPLE 1.4.5. If X is a locally closed subset of $\mathbf{A}^n$, then the projection π_i on the i^{th} component, given by

$$\pi_i(a_1, \ldots, a_n) = a_i$$

induces a regular function $X \to k$. Indeed, if $f_i = x_i \in k[x_1, \ldots, x_n]$, then $\pi_i(a) = f_i(a)$ for all $a \in X$.

When Y is closed in $\mathbf{A}^n$, one can describe more precisely $\mathcal{O}(Y)$. It follows by definition that we have a k-algebra homomorphism

$$k[x_1, \ldots, x_n] \to \mathcal{O}(Y)$$

that maps a polynomial f to the function $\big(u \to f(u)\big)$. By definition, the kernel of this map is the ideal $I(Y)$. With this notation, we have the following

PROPOSITION 1.4.6. *The induced k-algebra homomorphism*

$$k[x_1, \ldots, x_n]/I(Y) \to \mathcal{O}(Y)$$

is an isomorphism.

A similar description holds for principal affine open subsets of affine varieties. Suppose that Y is closed in $\mathbf{A}^n$ and $U = D_Y(h)$, for some $h \in k[x_1, \ldots, x_n]$. We have a k-algebra homomorphism

$$\Phi\colon k[x_1, \ldots, x_n]_h \to \mathcal{O}(U),$$

that maps $\frac{f}{h^m}$ to the map $\big(u \to f(u)/h(u)^m\big)$. With this notation, we have the following generalization of the previous proposition.

PROPOSITION 1.4.7. *The above k-algebra homomorphism induces an isomorphism*

$$k[x_1, \ldots, x_n]_h/I(Y)_h \to \mathcal{O}\big(D_Y(h)\big).$$

Of course it is enough to prove this more general version.

Proof of Proposition 1.4.7. The kernel of Φ consists of those fractions $\frac{f}{h^m}$ such that $\frac{f(u)}{h(u)}=0$ for every $u\in D_Y(h)$. It is clear that this condition is satisfied if $f\in I(Y)$. Conversely, if this condition holds, then $f(u)h(u)=0$ for every $u\in Y$. Therefore $fh\in I(Y)$, hence $\frac{f}{h^m}=\frac{fh}{h^{m+1}}\in I(Y)_h$. This shows that Φ is injective.

We now show that Φ is surjective. Consider $\phi\in\mathcal{O}\big(D_Y(h)\big)$. Using the hypothesis and the fact that $D_Y(h)$ is quasi-compact (being a Noetherian topological space), we can write

$$D_Y(h)=V_1\cup\ldots\cup V_r$$

and we have $f_i,g_i\in k[x_1,\ldots,x_n]$ for $1\leq i\leq r$ such that $g_i(u)\neq 0$ and $\phi(u)=\frac{f_i(u)}{g_i(u)}$ for all $u\in V_i$ and all i. Since the principal affine open subsets form a basis for the topology on Y, we may assume that $V_i=D_Y(h_i)$ for all i, for some $h_i\in k[x_1,\ldots,x_n]\smallsetminus I(Y)$. Since $g_i(u)\neq 0$ for all $u\in Y\smallsetminus V(h_i)$, it follows from Theorem 1.1.9 that

$$h_i\in \operatorname{rad}\big(I(Y)+(g_i)\big).$$

After possibly replacing each h_i by a suitable power, and then by a suitable element with the same class mod $I(Y)$, we may and will assume that $h_i\in(g_i)$. Finally, after multiplying both f_i and g_i by a suitable polynomial, we may assume that $g_i=h_i$ for all i.

We know that on $D_Y(g_i)\cap D_Y(g_j)=D_Y(g_ig_j)$ we have

$$\frac{f_i(u)}{g_i(u)}=\frac{f_j(u)}{g_j(u)}.$$

Applying the injectivity statement for $D_Y(g_ig_j)$, we conclude that

$$\frac{f_i}{g_i}=\frac{f_j}{g_j}\quad\text{in}\quad k[x_1,\ldots,x_n]_{g_ig_j}/I(Y)_{g_ig_j}.$$

Therefore there is a positive integer N such that

$$(g_ig_j)^N(f_ig_j-f_jg_i)\in I(Y)\quad\text{for all}\quad i,j.$$

After replacing each f_i and g_i by $f_ig_i^N$ and g_i^{N+1}, respectively, we may assume that

$$f_ig_j-f_jg_i\in I(Y)\quad\text{for all}\quad i,j.$$

On the other hand, we have

$$D_Y(h)=\bigcup_{i=1}^r D_Y(g_i),$$

hence $Y\cap V(h)=Y\cap V(g_1,\ldots,g_r)$, and by Theorem 1.1.9, we have

$$\operatorname{rad}\big(I(Y)+(h)\big)=\operatorname{rad}\big(I(Y)+(g_1,\ldots,g_r)\big).$$

In particular, we can write

$$h^m-\sum_{i=1}^r a_ig_i\in I(Y)\quad\text{for some}\quad m\geq 1\quad\text{and}\quad a_1,\ldots,a_r\in k[x_1,\ldots,x_n].$$

We claim that

$$\phi=\Phi\left(\frac{a_1f_1+\ldots+a_rf_r}{h^m}\right).$$

Indeed, for $u \in D_Y(g_j)$, we have

$$\frac{f_j(u)}{g_j(u)} = \frac{a_1(u)f_1(u) + \ldots + a_r(u)f_r(u)}{h(u)^m}$$

since

$$h(u)^m f_j(u) = \sum_{i=1}^r a_i(u)g_i(u)f_j(u) = \left(\sum_{i=1}^r a_i(u)f_i(u)\right) g_j(u).$$

This completes the proof of the claim and thus that of the proposition. □

Example 1.4.8. In general, it is not the case that a regular function admits a global description as the quotient of two polynomial functions. Consider, for example the closed subset W of $\mathbf{A}^4$ defined by $x_1x_2 = x_3x_4$. Inside W we have the plane L given by $x_2 = x_3 = 0$. We define the regular function $\phi\colon W \smallsetminus L \to k$ given by

$$\phi(u_1, u_2, u_3, u_4) = \begin{cases} \frac{u_1}{u_3}, & \text{if } u_3 \neq 0; \\ \frac{u_4}{u_2}, & \text{if } u_2 \neq 0. \end{cases}$$

It is an easy exercise to check that there are no polynomials $P, Q \in k[x_1, x_2, x_3, x_4]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad \phi(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in W \smallsetminus L.$$

We now turn to maps between quasi-affine varieties. If Y is a subset of $\mathbf{A}^m$ and $f\colon X \to Y$ is a map, then the composition $X \to Y \hookrightarrow \mathbf{A}^m$ is written as $(f_1, \ldots, f_m)$, with $f_i\colon X \to k$. We often abuse notation writing $f = (f_1, \ldots, f_m)$.

Definition 1.4.9. If $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ are locally closed subsets, a map $f = (f_1, \ldots, f_m)\colon X \to Y$ is a *morphism* if $f_i \in \mathcal{O}(X)$ for all i.

Remark 1.4.10. It follows from definition that $f\colon X \to Y$ is a morphism if and only if the composition

$$X \to Y \hookrightarrow \mathbf{A}^m$$

is a morphism

Remark 1.4.11. If $X \subseteq \mathbf{A}^n$ is a locally closed subset, then a morphism $X \to \mathbf{A}^1$ is the same as a regular function $X \to k$.

Example 1.4.12. If X is a locally closed of $\mathbf{A}^n$, then the inclusion map $\iota\colon X \to \mathbf{A}^n$ is a morphism (this follows from Example 1.4.5). This implies that the identity map $1_X\colon X \to X$ is a morphism.

Proposition 1.4.13. *If X and Y are quasi-affine varieties, then every morphism $f\colon X \to Y$ is continuous.*

Proof. Suppose that X and Y are locally closed in $\mathbf{A}^n$ and $\mathbf{A}^m$, respectively, and write $f = (f_1, \ldots, f_m)$. We will show that if $V \subseteq Y$ is a closed subset, then $f^{-1}(V)$ is a closed subset of X. By assumption, we can write

$$V = Y \cap V(I) \quad \text{for some ideal} \quad I \subseteq k[x_1, \ldots, x_n].$$

In order to check that $f^{-1}(V)$ is closed, it is enough to find for every $x \in X$ an open neighborhood U_x of x in X such that $U_x \cap f^{-1}(V)$ is closed in U_x (see Remark 1.3.21). Since each f_i is a regular function, after replacing X by a suitable open neighborhood of x, we may assume that there are $P_i, Q_i \in k[x_1, \ldots, x_n]$ such that

$$Q_i(u) \neq 0 \quad \text{and} \quad f_i(u) = \frac{P_i(u)}{Q_i(u)} \quad \text{for all} \quad u \in X.$$

For every $h \in I$, there are polynomials $A_h, B_h \in k[x_1, \ldots, x_n]$ such that

$$B_h(u) \neq 0 \quad \text{and} \quad h\left(\frac{P_1(u)}{Q_1(u)}, \ldots, \frac{P_m(u)}{Q_m(u)}\right) = \frac{A_h(u)}{B_h(u)} \quad \text{for all} \quad u \in X.$$

It is then clear that for $u \in X$ we have $u \in f^{-1}(V)$ if and only if $A_h(u) = 0$ for all $h \in I$. Therefore $f^{-1}(V)$ is closed. □

PROPOSITION 1.4.14. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms between quasi-affine varieties, the composition $g \circ f$ is a morphism.*

PROOF. Suppose that $X \subseteq \mathbf{A}^m$, $Y \subseteq \mathbf{A}^n$ and $Z \subseteq \mathbf{A}^q$ are locally closed subsets and let us write $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_q)$. We need to show that $g_i \circ f \in \mathcal{O}(X)$ for $1 \leq i \leq q$. Let us fix such i, a point $x \in X$, and let $y = f(x)$. Since $g_i \in \mathcal{O}(Y)$ is a morphism, there is an open neighborhood V_y of y and $P, Q \in k[x_1, \ldots, x_n]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad g_i(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in V_y.$$

Similarly, since f is a morphism, we can find an open neighborhood U_x of x and $A_j, B_j \in k[x_1, \ldots, x_m]$ for $1 \leq j \leq n$ such that

$$B_j(u) \neq 0 \quad \text{and} \quad f_j(u) = \frac{A_j(u)}{B_j(u)} \quad \text{for all} \quad u \in U_x.$$

It follows from Proposition 1.4.13 that $U_x \cap f^{-1}(V_y)$ is open and we have

$$g_i \circ f(u) = \frac{P\left(\frac{A_1(u)}{B_1(u)}, \ldots, \frac{A_n(u)}{B_n(u)}\right)}{Q\left(\frac{A_1(u)}{B_1(u)}, \ldots, \frac{A_n(u)}{B_n(u)}\right)}.$$

After clearing the denominators, we see that indeed, $g_i \circ f$ is a regular function in the neighborhood of x. □

It follows from Proposition 1.4.14 (and Example 1.4.12) that we may consider the *category of quasi-affine varieties* over k, whose objects are locally closed subsets of affine spaces over k, and whose arrows are the morphisms as defined above. Moreover, since a regular function on X is the same as a morphism $X \to \mathbf{A}^1$, we see that if $f\colon X \to Y$ is a morphism of quasi-affine varieties, we get an induced map

$$f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X), \quad f^{\#}(\phi) = \phi \circ f.$$

This is clearly a morphism of k-algebras. By mapping every quasi-affine variety X to $\mathcal{O}(X)$ and every morphism $f\colon X \to Y$ to $f^{\#}$, we obtain a contravariant functor from the category of quasi-affine varieties over k to the category of k-algebras.

DEFINITION 1.4.15. A morphism $f\colon X \to Y$ is an *isomorphism* if it is an isomorphism in the above category. It is clear that this is the case if and only if f is bijective and f^{-1} is a morphism.

The following result shows that for affine varieties, this functor induces an anti-equivalence of categories. Let $\mathcal{A}fVar_k$ be the full subcategory of the category of quasi-affine varieties whose objects consist of the closed subsets of affine spaces over k and let $\mathcal{C}_k$ denote the category whose objects are reduced, finitely generated k-algebras and whose arrows are the morphisms of k-algebras.

THEOREM 1.4.16. *The contravariant functor*

$$\mathcal{A}fVar_k \to \mathcal{C}_k$$

that maps X to $\mathcal{O}(X)$ and $f\colon X \to Y$ to $f^\#\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is an anti-equivalence of categories.

PROOF. Note first that if X is an affine variety, then $\mathcal{O}(X)$ is indeed a reduced, finitely generated k-algebra. Indeed, if X is a closed subset of $\mathbf{A}^n$, then it follows from Proposition 1.4.6 that we have an isomorphism $\mathcal{O}(X) \simeq k[x_1, \ldots, x_n]/I(X)$, which gives the assertion.

In order to show that the functor is an anti-equivalence of categories, it is enough to check two things:

i) For every affine varieties X and Y, the map

$$\mathrm{Hom}_{\mathcal{A}fVar_k}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_k}\big(\mathcal{O}(Y), \mathcal{O}(X)\big), \quad f \to f^\#$$

is a bijection.

ii) For every reduced, finitely generated k-algebra A, there is an affine variety X with $\mathcal{O}(X) \simeq A$.

The assertion in ii) is clear: since A is finitely generated, we can find an isomorphism $A \simeq k[x_1, \ldots, x_m]/J$, for some positive integer m and some ideal J. Moreover, since A is reduced, J is a radical ideal. If $X = V(J) \subseteq \mathbf{A}^m$, then it follows from Theorem 1.1.9 that $J = I(X)$ and therefore $\mathcal{O}(X) \simeq A$ by Proposition 1.4.6.

In order to prove the assertion in i), suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. By Proposition 1.4.6, we have canonical isomorphisms

$$\mathcal{O}(X) \simeq k[x_1, \ldots, x_m]/I(X) \quad \text{and} \quad \mathcal{O}(Y) \simeq k[y_1, \ldots, y_n]/I(Y).$$

If $f\colon X \to Y$ is a morphism and we write $f = (f_1, \ldots, f_n)$, then $f^\#(\overline{y_i}) = \overline{f}_i$. Since f is determined by the classes $\overline{f_1}, \ldots, \overline{f_n}$ mod $I(X)$, it is clear that the map in i) is injective.

Suppose now that $\alpha\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a morphism of k-algebras and let $f_i \in k[x_1, \ldots, x_m]$ be such that $\overline{f_i} = \alpha(\overline{y_i}) \in \mathcal{O}(X)$. It is then clear that $f = (f_1, \ldots, f_n)$ gives a morphism $X \to \mathbf{A}^n$. Its image lies inside Y since for every $g \in I(Y)$ we have $g(f_1, \ldots, f_n) \in I(X)$, hence $g\big(f(u)\big) = 0$ for all $u \in X$. Therefore f gives a morphism $X \to Y$ such that $f^\# = \alpha$. $\square$

DEFINITION 1.4.17. We extend somewhat the notion of affine variety by saying that a quasi-affine variety is affine if it is isomorphic (in the category of quasi-affine varieties) to a closed subset of some affine space.

An important example that does not come directly as a closed subset of an affine space is provided by the following proposition.

PROPOSITION 1.4.18. *Let X be a closed subset of $\mathbf{A}^n$ and $U = D_X(g)$, for some $g \in k[x_1, \ldots, x_n]$. If J is the ideal in $k[x_1, \ldots, x_n, y]$ generated by $I(X)$ and $1 - g(x)y$, then U is isomorphic to $V(J)$. In particular, U is an affine variety*[2].

PROOF. Define $\phi\colon U \to V(J)$ by $\phi(u) = \big(u, 1/g(u)\big)$. It is clear that $\phi(u)$ lies indeed in $V(J)$ and that ϕ is a morphism. Moreover, we also have a morphism $\psi\colon V(J) \to U$ induced by the projection onto the first n components. It is straightforward to check that ϕ and ψ are inverse to each other. $\square$

NOTATION 1.4.19. If X is a quasi-affine variety and $f \in \mathcal{O}(X)$, then we put

$$D_X(f) = \{u \in X \mid f(u) \neq 0\}.$$

If X is affine, say it is isomorphic to the closed subset Y of $\mathbf{A}^n$, then f corresponds to the restriction to Y of some $g \in k[x_1, \ldots, x_n]$. In this case, it is clear that $D_X(f)$ is isomorphic to $D_Y(g)$, hence it is an affine variety.

REMARK 1.4.20. If X is a locally closed subset of $\mathbf{A}^n$, then X is open in $\overline{X}$. Since the principal affine open subsets of $\overline{X}$ give a basis of open subsets for the topology of X, it follows from Proposition 1.4.18 that the open subsets of X that are themselves affine varieties give a basis for the topology of X.

EXERCISE 1.4.21. Suppose that $f\colon X \to Y$ is a morphism of affine algebraic varieties, and consider the induced homomorphism $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D_Y(u)) = D_X(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D_Y(u)) \to \mathcal{O}(D_X(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

EXERCISE 1.4.22. Let X be an affine algebraic variety, and let $\mathcal{O}(X)$ be the ring of regular functions on X. For every ideal J of $\mathcal{O}(X)$, let

$$V(J) := \{p \in X \mid f(p) = 0 \text{ for all } f \in J\}.$$

For $S \subseteq X$, consider the following ideal of $\mathcal{O}(X)$

$$I_X(S) := \{f \in \mathcal{O}(X) \mid f(p) = 0 \text{ for all } p \in S\}.$$

Show that for every subset S of X and every ideal J in $\mathcal{O}(X)$, we have

$$V\big(I_X(S)\big) = \overline{S} \quad \text{and} \quad I_X\big(V(J)\big) = \mathrm{rad}(J).$$

In particular, the maps $V(-)$ and $I_X(-)$ define order-reversing inverse bijections between the closed subsets of X and the radical ideals in $\mathcal{O}(X)$. Via this correspondence, the irreducible closed subsets correspond to the prime ideals in $\mathcal{O}(X)$ and the points of X correspond to the maximal ideals in $\mathcal{O}(X)$. This generalizes the case $X = \mathbf{A}^n$ that was discussed in Section 1.1.

[2]This justifies calling these subsets principal affine open subsets.

We have seen that a morphism $f\colon X \to Y$ between affine varieties is determined by the corresponding k-algebra homomorphism $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. For such a morphism, it follows from the above exercise that the closed subsets in X and Y are in bijection with the radical ideals in $\mathcal{O}(X)$ and, respectively, $\mathcal{O}(Y)$. In the next proposition we translate the operations of taking the image and inverse image as operations on ideals.

PROPOSITION 1.4.23. *Let $f\colon X \to Y$ be a morphism of affine varieties and $\phi = f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the corresponding k-algebra homomorphism. For a point x in X or Y, we denote by $\mathfrak{m}_x$ the corresponding maximal ideal.*

i) *If $x \in X$ and $y = f(x)$, then $\mathfrak{m}_y = \phi^{-1}(\mathfrak{m}_x)$.*
ii) *More generally, if $\mathfrak{a}$ is an ideal in $\mathcal{O}(X)$ and $W = V(\mathfrak{a})$, then $I_Y\big(\overline{f(W)}\big) = \phi^{-1}\big(I_X(W)\big)$.*
iii) *In particular, we have $I_Y\big(\overline{f(X)}\big) = \ker(\phi)$. Therefore $\overline{f(X)} = Y$ if and only if ϕ is injective.*
iv) *If $\mathfrak{b}$ is an ideal in $\mathcal{O}(Y)$ and $Z = V(\mathfrak{b})$, then $f^{-1}(Z) = V\big(\mathfrak{b}\cdot\mathcal{O}(X)\big)$.*

PROOF. The assertion in i) is a special case of that in ii), hence we begin by showing ii). We have

$$I_Y\big(\overline{f(W)}\big) = I_Y\big(f(W)\big) = \{g \in \mathcal{O}(Y) \mid g\big(f(x)\big) = 0 \text{ for all } x \in W\}$$
$$= \{g \in \mathcal{O}(Y) \mid \phi(g) \in I_X(W)\} = \phi^{-1}\big(I_X(W)\big).$$

By taking $W = X$, we obtain the assertion in iii)

Finally, if $\mathfrak{b}$ and Z are as in iv), we see that

$$f^{-1}(Z) = \{x \in X \mid g\big(f(x)\big) = 0 \text{ for all } g \in \mathfrak{b}\} = V\big(\mathfrak{b}\cdot\mathcal{O}(X)\big).$$

□

REMARK 1.4.24. If $f\colon X \to Y$ is a morphism of affine varieties, then $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective if and only if f factors as $X \xrightarrow{g} Z \xrightarrow{\iota} Y$, where Z is a closed subset of Y, ι is the inclusion map, and g is an isomorphism.

EXERCISE 1.4.25. Let $Y \subseteq \mathbf{A}^2$ be the *cuspidal curve* defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbf{A}^1 \to Y$. Is it an isomorphism ?

EXERCISE 1.4.26. Suppose that $\mathrm{char}(k) = p > 0$, and consider the map $f\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $f(a_1, \ldots, a_n) = (a_1^p, \ldots, a_n^p)$. Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism.

EXERCISE 1.4.27. Use Exercise 1.3.16 to show that the affine variety

$$M_{m,n}^r(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \le r\}$$

is irreducible.

EXERCISE 1.4.28. Let $n \ge 2$ be an integer.

i) Show that the set

$$B_n = \left\{(a_0, a_1, \ldots, a_n) \in \mathbf{A}^{n+1} \mid \mathrm{rank}\begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \le 1\right\}$$

is a closed subset of $\mathbf{A}^{n+1}$.

ii) Show that

$$B_n = \{(s^n, s^{n-1}t, \ldots, t^n) \mid s, t \in k\}.$$

Deduce that B_n is irreducible.

EXERCISE 1.4.29. In order to get an example of a quasi-affine variety which is not affine, consider $U = \mathbf{A}^2 \smallsetminus \{0\}$. Show that the canonical homomorphism $\mathcal{O}(\mathbf{A}^2) \to \mathcal{O}(U)$ is an isomorphism and deduce that U is not affine.

EXERCISE 1.4.30. Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself.

EXERCISE 1.4.31. Show that if X is a quasi-affine variety such that $\mathcal{O}(X) = k$, then X consists of only one point.

1.5. Local rings and rational functions

Let X be a quasi-affine variety and W an irreducible closed subset of X.

DEFINITION 1.5.1. The *local ring* of X at W is the k-algebra

$$\mathcal{O}_{X,W} := \varinjlim_{U \cap W \neq \emptyset} \mathcal{O}(U).$$

Here the direct limit is over the open subsets of X with $U \cap W \neq \emptyset$, ordered by reverse inclusion, and where for $U_1 \subseteq U_2$, the map $\mathcal{O}(U_2) \to \mathcal{O}(U_1)$ is given by restriction of functions.

REMARK 1.5.2. Note that the poset indexing the above direct limit is filtering: given any two open subsets U_1 and U_2 that intersect W nontrivially, we have $U_1 \cap U_2 \cap W \neq \emptyset$ (we use here the fact that W is irreducible). Because of this, the elements of $\mathcal{O}_{X,W}$ can be described as pairs (U, ϕ), where U is open with $W \cap U \neq \emptyset$ and $\phi \in \mathcal{O}(U)$, modulo the following equivalence relation:

$$(U_1, \phi_1) \sim (U_2, \phi_2)$$

if there is an open subset $U \subseteq U_1 \cap U_2$, with $U \cap W \neq \emptyset$, such that $\phi_1|_U = \phi_2|_U$. Operations are defined by restricting to the intersection: for example, we have

$$(U_1, \phi_1) + (U_2, \phi_2) = (U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2}).$$

In order to describe $\mathcal{O}_{X,W}$, we begin with the following lemma.

LEMMA 1.5.3. *If W is an irreducible closed subset of X and V is an open subset of X with $V \cap W \neq \emptyset$, we have a canonical k-algebra isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}_{V,W \cap V}.$$

PROOF. The assertion follows from the fact that the following subset

$$\{U \subseteq V \mid U \text{ open}, U \cap W \neq \emptyset\} \subseteq \{U \subseteq X \mid U \text{ open}, U \cap W \neq \emptyset\}$$

is final. Explicitly, we have the morphism

$$\mathcal{O}_{V,W \cap V} \to \mathcal{O}_{X,W}, \quad (U, \phi) \to (U, \phi),$$

with inverse

$$\mathcal{O}_{X,W} \to \mathcal{O}_{V,W \cap V}, \quad (U, \phi) \to (U \cap V, \phi|_{U \cap V}).$$

$\square$

Given a quasi-affine variety X, the open subsets of X that are affine varieties give a basis for the topology of X (see Remark 1.4.20). By Lemma 1.5.3, we see that it is enough to compute $\mathcal{O}_{X,W}$ when X is an affine variety. This is the content of the next result.

PROPOSITION 1.5.4. *Let X be an affine variety and W an irreducible closed subset of X. If $\mathfrak{p} \subseteq \mathcal{O}(X)$ is the prime ideal corresponding to W, then we have a canonical isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

In particular, $\mathcal{O}_{X,W}$ is a local ring, with maximal ideal consisting of classes of pairs (U, ϕ), with $\phi_{U \cap W} = 0$.

PROOF. Since the principal affine open subsets of X form a basis for the topology of X, we obtain using Proposition 1.4.7 a canonical isomorphism

$$\mathcal{O}_{X,W} \simeq \varinjlim_{f} \mathcal{O}(X)_f,$$

where the direct limit on the right-hand side is over those $f \in \mathcal{O}(X)$ such that $D_X(f) \cap W \neq \emptyset$. This condition is equivalent to $f \not\in \mathfrak{p}$ and it is straightforward to check that the maps $\mathcal{O}(X)_f \to \mathcal{O}(X)_{\mathfrak{p}}$ induce an isomorphism

$$\varinjlim_{f} \mathcal{O}(X)_f \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

The last assertion in the proposition follows easily from the fact that $\mathcal{O}(X)_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}\mathcal{O}(X)_{\mathfrak{p}}$ □

There are two particularly interesting cases of this definition. First, if we take $W = \{x\}$, for a point $x \in X$, we obtain the local ring $\mathcal{O}_{X,x}$ of X *at* x. Its elements are *germs of regular functions* at x. This is a local ring, whose maximal ideal consists of germs of functions vanishing at x. As we will see, this local ring is responsible for the properties of X in a neighborhood of x. If X is an affine variety and $\mathfrak{m}$ is the maximal ideal corresponding to x, then Proposition 1.5.4 gives an isomorphism

$$\mathcal{O}_{X,x} \simeq \mathcal{O}(X)_{\mathfrak{m}}.$$

EXERCISE 1.5.5. Let $f\colon X \to Y$ be a morphism of quasi-affine varieties, and let $Z \subseteq X$ be a closed irreducible subset. Recall that by Exercise 1.3.16, we know that $W := \overline{f(Z)}$ is irreducible. Show that we have an induced morphism of k-algebras

$$g\colon \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z}$$

and that g is a local homomorphism of local rings (that is, it maps the maximal ideal of $\mathcal{O}_{Y,W}$ inside the maximal ideal of $\mathcal{O}_{X,Z}$). If X and Y are affine varieties, and

$$\mathfrak{p} = I_X(Z) \quad \text{and} \quad \mathfrak{q} = I_Y(W) = (f^{\#})^{-1}(\mathfrak{p}),$$

then via the isomorphisms given by Proposition 1.5.4, g gets identified to the homomorphism

$$\mathcal{O}(Y)_{\mathfrak{q}} \to \mathcal{O}(X)_{\mathfrak{p}}$$

induced by $f^{\#}$ via localization.

EXERCISE 1.5.6. Let X and Y be quasi-affine varieties. By the previous exercise, if $f\colon X \to Y$ is a morphism, $p \in X$ is a point, and $f(p) = q$, then f induces a local ring homomorphism $\phi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$.

i) Show that if $f'\colon X \to Y$ is another morphism with $f'(p) = q$, and induced homomorphism $\phi'\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then $\phi = \phi'$ if and only if there is an open neighborhood U of p such that $f|_U = g|_U$.

ii) Show that given any local morphism of local k-algebras $\psi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, there is an open neighborhood W of p, and a morphism $g\colon W \to Y$ with $g(p) = q$, and inducing ψ.

iii) Deduce that $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as k-algebras if and only if there are open neighborhoods W of p and V of q, and an isomorphism $h\colon W \to V$, with $h(p) = q$.

Another important example of local ring of X occurs when X is an irreducible variety and we take $W = X$. The resulting local ring is, in fact, a field, the *field of rational functions* $k(X)$ of X. Indeed, if $U \subseteq X$ is an affine open subset, then it follows from Lemma 1.5.3 and Proposition 1.5.4 that $k(X)$ is isomorphic to the field of fractions of the domain $\mathcal{O}(X)$. The elements of $k(X)$ are *rational functions* on X, that is, pairs (U, ϕ), where U is a nonempty open subset of X and $\phi\colon U \to k$ is a regular function, where we identify two such pairs if the two functions agree on some nonempty open subset of their domains (in fact, as we will see shortly, in this case they agree on the intersection of their domains). We now discuss in more detail rational functions and, more generally, rational maps.

LEMMA 1.5.7. *If X and Y are quasi-affine varieties and f_1 and f_2 are two morphisms $X \to Y$, then the subset*

$$\{a \in X \mid f_1(a) = f_2(a)\} \subseteq X$$

is closed.

PROOF. If Y is a locally closed subset in $\mathbf{A}^n$, then we write $f_i = (f_{i,1}, \ldots, f_{i,n})$ for $i = 1, 2$. With this notation, we have

$$\{a \in X \mid f_1(a) = f_2(a)\} = \bigcap_{j=1}^{n} \{a \in X \mid (f_{1,j} - f_{2,j})(a) = 0\},$$

hence this set is closed in X, since each function $f_{1,j} - f_{2,j}$ is regular, hence continuous. $\square$

DEFINITION 1.5.8. Let X and Y be quasi-affine varieties. A *rational map* $f\colon X \dashrightarrow Y$ is given by a pair (U, ϕ), where U is a dense, open subset of X and $\phi\colon U \to Y$ is a morphism, and where we identify (U_1, ϕ_1) with (U_2, ϕ_2) if there is an open dense subset $V \subseteq U_1 \cap U_2$ such that $\phi_1|_V = \phi_2|_V$. In fact, in this case we have $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$ by Lemma 1.5.7. We also note that since U_1 and U_2 are dense open subsets of X, then also $U_1 \cap U_2$ is a dense subset of X.

REMARK 1.5.9. If $f\colon X \dashrightarrow Y$ is a rational map and (U_i, ϕ_i) are the representatives of f, then we can define a map $\phi\colon U = \bigcup_i U_i \to Y$ by $\phi(u) = \phi_i(u)$ if $u \in U_i$. This is well-defined and it is a morphism, since its restriction to each of the U_i is a morphism. Moreover, (U, ϕ) is a representative of f. The open subset U, the largest one on which a representative of f is defined, is the *domain of definition* of f.

DEFINITION 1.5.10. Given a quasi-affine variety X, the set of rational functions $X \dashrightarrow k$ is denoted by $k(X)$. Since the intersection of two dense open sets is again open and dense, we may define the sum and product of two rational functions. For example, given two rational functions with representatives (U_1, ϕ_1) and (U_2, ϕ_2), we define their sum by the representative

$$(U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2}),$$

and similarly for the product. It is straightforward to see that using also scalar multiplication, $k(X)$ is a k-algebra. Note that when X is irreducible, we recover our previous definition.

EXERCISE 1.5.11. Let X be a quasi-affine variety, and let $X_1, \ldots, X_r$ be its irreducible components. Show that there is a canonical isomorphism

$$k(X) \simeq k(X_1) \times \cdots \times k(X_r).$$

EXERCISE 1.5.12. Let W be the closed subset in $\mathbf{A}^2$, defined by $x^2 + y^2 = 1$. What is the domain of definition of the rational function on W given by $\frac{1-y}{x}$?

Our next goal is to define a category in which the arrows are given by rational function. For simplicity, we only consider irreducible varieties.

DEFINITION 1.5.13. A morphism $f\colon X \to Y$ is *dominant* if $Y = \overline{f(X)}$. Equivalently, for every nonempty open subset $V \subseteq Y$, we have $f^{-1}(V) \neq \emptyset$. Note that if U is open and dense in X, then f is dominant if and only if the composition $U \hookrightarrow X \xrightarrow{f} Y$ is dominant. We can thus define the same notion for rational maps: if $f\colon X \dashrightarrow Y$ is a rational map with representative (U, ϕ), we say that f is *dominant* if $\phi\colon U \to Y$ is dominant.

Suppose that X, Y, and Z are irreducible quasi-affine varieties and $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ are rational maps, with f dominant. In this case we may define the composition $g \circ f$, which is a rational map; moreover, if g is dominant, too, then $g \circ f$ is dominant. Indeed, choose a representative (U, ϕ) for f and a representative (V, ψ) for g. Since the morphism $\phi\colon U \to Y$ is dominant, it follows that $W := \phi^{-1}(V)$ is nonempty. We then take $g \circ f$ to be the rational function defined by the composition

$$W \xrightarrow{f|_W} V \longrightarrow Z.$$

It is straightforward to see that this independent of the representatives for f and g. Moreover, if g is dominant, then $g \circ f$ is dominant: if Z' is a nonempty open subset of Z, then $\psi^{-1}(Z')$ is nonempty and open since g is dominant and therefore $\phi^{-1}\big(\psi^{-1}(Z')\big)$ is nonempty, since f is dominant.

It is clear that the identity map is dominant. Moreover, composition of dominant rational map is associative. We thus obtain a category in which the objects are the irreducible quasi-affine varieties over k and the set $\mathrm{Hom}_{\mathrm{rat}}(X, Y)$ of arrows from X to Y consists of the dominant rational maps $X \dashrightarrow Y$, with the composition defined above. We are then led to the following important concept.

DEFINITION 1.5.14. A rational dominant map $f\colon X \dashrightarrow Y$ between irreducible quasi-affine varieties is *birational* if it is an isomorphism in the above category. More precisely, this is the case if there is a dominant rational map $g\colon Y \dashrightarrow X$ such that

$$g \circ f = 1_X \quad \text{and} \quad f \circ g = 1_Y.$$

A *birational morphism* is a morphism which is birational as a rational map. Two irreducible quasi-affine varieties X and Y are *birational* if there is a birational map $X \dashrightarrow Y$.

This notion plays a fundamental role in the classification of algebraic varieties. On one hand, birational varieties share interesting geometric properties. On the other hand, classifying algebraic varieties up to birational equivalence turns out to be a more reasonable endeavor than classifying varieties up to isomorphism.

EXAMPLE 1.5.15. If U is an open subset of the irreducible quias-affine variety X, then the inclusion map $i\colon U \hookrightarrow X$ is a birational morphism. Its inverse is given by the rational map represented by the identity morphism of U.

EXAMPLE 1.5.16. An interesting example, which we will come back to later, is given by the morphism

$$f\colon \mathbf{A}^n \to \mathbf{A}^n, \quad f(x_1,\ldots,x_n) = (x_1, x_1x_2, \ldots, x_1x_n).$$

Note that the linear subspace given $L = (x_1 = 0)$ is mapped to 0, but f induces an isomorphism

$$\mathbf{A}^n \smallsetminus L = f^{-1}(\mathbf{A}^n \smallsetminus L) \to \mathbf{A}^n \smallsetminus L,$$

with inverse given by $g(y_1,\ldots,y_n) = (y_1, y_2/y_1, \ldots, y_n/y_1)$.

EXAMPLE 1.5.17. Let X be the closed subset of $\mathbf{A}^2$ (on which we denote the coordinates by x and y), defined by $x^2 - y^3 = 0$. Let $f\colon \mathbf{A}^1 \to X$ be the morphism given by $f(t) = (t^3, t^2)$. Note that f is birational: if $g\colon X \smallsetminus \{(0,0)\} \to \mathbf{A}^1$ is the morphism given by $g(u,v) = \frac{u}{v}$, then g gives a rational map $X \dashrightarrow \mathbf{A}^1$ that is an inverse of f. Note that since $f^{-1}(0,0) = \{0\}$, it follows that the morphism f is bijective, However, f is not an isomorphism: otherwise, by Theorem 1.4.16 the induced homomorphism

$$f^{\#}\colon \mathcal{O}(X) = k[x,y]/(x^2 - y^3) \to k[t], \quad f^{\#}(x) = t^3, f^{\#}(y) = t^2$$

would be an isomorphism. However, it is clear that t is not in the image.

If $f\colon X \dashrightarrow Y$ is a rational, dominant map, then by taking $Z = \mathbf{A}^1$, we see that by precomposing with f we obtain a map

$$f^{\#}\colon k(Y) \to k(X).$$

It is straightforward to see that this is a field homomorphism.

THEOREM 1.5.18. *We have an anti-equivalence of categories between the category of irreducible quasi-affine varieties and dominant rational maps and the category of finite type field extensions of k and k-algebra homomorphisms, that maps a variety X to $k(X)$ and a rational dominant map $f\colon X \dashrightarrow Y$ to $f^{\#}\colon k(Y) \to k(X)$.*

PROOF. It is clear that we have a contravariant functor as described in the theorem. Note that if X is an irreducible quasi-affine variety, then $k(X)$ is a finite type extension of k: indeed, if U is an affine open subset of X, then we have $k(X) \simeq k(U) \simeq \mathrm{Frac}\big(\mathcal{O}(U)\big)$.

In order to show that this functor is an anti-equivalence, it is enough to prove the following two statements:

i) Given any two quasi-affine varieties X and Y, the map

$$\mathrm{Hom}_{\mathrm{rat}}(X,Y) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big)$$

is bijective.

ii) Given any finite type field extension K/k, there is an irreducible quasi-affine variety X such that $k(X) \simeq K$.

The assertion in ii) is easy to see: if $K = k(a_1, \ldots, a_n)$, let $A = k[a_1, \ldots, a_n]$. We can thus write $A \simeq k[x_1, \ldots, x_n]/P$ for some (prime) ideal P and if $X = V(P) \subseteq \mathbf{A}^n$, then X is irreducible and $k(X) \simeq K$.

In order to prove i), suppose that X and Y are locally closed in $\mathbf{A}^m$ and, respectively, $\mathbf{A}^n$. Since X and Y are open in $\overline{X}$ and $\overline{Y}$, respectively, by Proposition 1.3.22, and since inclusions of open subsets are birational, it follows that the inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ induce an isomorphism

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{rat}}(\overline{X}, \overline{Y}),$$

and also isomorphisms

$$k(X) \simeq k(\overline{X}) \quad \text{and} \quad k(Y) \simeq k(\overline{Y}).$$

We may thus replace X and Y by $\overline{X}$ and $\overline{Y}$, respectively, in order to assume that X and Y are closed subsets of the respective affine spaces.

It is clear that

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big),$$

where each set on the right-hand side consists of the dominant morphisms $D_X(g) \to Y$. Moreover, since $\mathcal{O}(Y)$ is a finitely generated k-algebra, we have

$$\mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big),$$

where each set on the right-hand side consists of the injective k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)_g$. Since the map $f \to f^{\#}$ gives a bijection

$$\mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big) \simeq \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big)$$

by Theorem 1.4.16 and Proposition 1.4.23, this completes the proof. □

Corollary 1.5.19. *A dominant rational map $f\colon X \to Y$ between irreducible quasi-affine varieties X and Y is birational if and only if the induced homomorphism $f^{\#}\colon k(Y) \to k(X)$ is an isomorphism.*

Remark 1.5.20. A rational map $f\colon X \dashrightarrow Y$ between the irreducible quasi-affine varieties X and Y is birational if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \simeq V$. The "if" assertion is clear, so we only need to prove the converse. Suppose that f is defined by the morphism $\phi\colon X' \to Y$ and its inverse g is defined by the morphism $\psi\colon Y' \to X$, where $X' \subseteq X$ and $Y' \subseteq Y$ are open subsets. The equality $f \circ g = 1_Y$ as rational functions implies by Lemma 1.5.7 that the composition

$$\psi^{-1}(X') \xrightarrow{\psi} X' \xrightarrow{\phi} Y$$

is the inclusion. In particular, we deduce that

$$\psi\big(\psi^{-1}(X')\big) \subseteq \phi^{-1}\big(\psi^{-1}(X')\big) \subseteq \phi^{-1}(Y').$$

Similarly, the equality of rational functions $g \circ f = 1_X$ shows that the composition

$$\phi^{-1}(Y') \xrightarrow{\phi} Y' \xrightarrow{\psi} X$$

is the inclusion; in particular, we obtain

$$\phi\big(\phi^{-1}(Y')\big) \subseteq \psi^{-1}(X').$$

It is now clear that ϕ and ψ induce inverse morphisms between $\phi^{-1}(Y')$ and $\psi^{-1}(X')$.

EXERCISE 1.5.21. Let $X \subset \mathbf{A}^n$ be a hypersurface defined by an equation $f(x_1, \ldots, x_n) = 0$, where $f = f_{d-1} + f_d$, with f_{d-1} and f_d nonzero, homogeneous of degrees $d-1$ and d, respectively. Show that if X is irreducible, then X is birational to $\mathbf{A}^{n-1}$.

1.6. Products of (quasi-)affine varieties

We begin by showing that for positive integers m and n, the Zariski topology on $\mathbf{A}^m \times \mathbf{A}^n = \mathbf{A}^{m+n}$ is finer than the product topology.

PROPOSITION 1.6.1. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets, then $X \times Y$ is a closed subset of $\mathbf{A}^{m+n}$.*

PROOF. The assertion follows from the fact that if $X = V(I)$ and $Y = V(J)$, for ideals $I \subseteq k[x_1, \ldots, x_m]$ and $J \subseteq k[y_1, \ldots, y_n]$, then

$$X \times Y = V(I \cdot R + J \cdot R),$$

where $R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. ☐

COROLLARY 1.6.2. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are open (respectively, locally closed) subsets, then $X \times Y$ is an open (respectively, locally closed) subset of $\mathbf{A}^{m+n} = \mathbf{A}^m \times \mathbf{A}^n$. In particular, the topology on $\mathbf{A}^m \times \mathbf{A}^n$ is finer than the product topology.*

PROOF. The assertion for open subsets follows from Proposition 1.6.1 and the fact that

$$\mathbf{A}^{m+n} \smallsetminus X \times Y = \big(\mathbf{A}^m \times (\mathbf{A}^n \smallsetminus Y)\big) \cup \big((\mathbf{A}^m \smallsetminus X) \times \mathbf{A}^n\big).$$

The assertion for locally closed subsets follows immediately from the assertions for open and closed subsets. ☐

COROLLARY 1.6.3. *Given any quasi-affine varieties X and Y, the topology on $X \times Y$ is finer than the product topology.*

PROOF. If X and Y are locally closed subsets of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, then $X \times Y$ is a locally closed subset of $\mathbf{A}^{m+n}$. Since the topology on $\mathbf{A}^{m+n}$ is finer than the product topology by the previous corollary, we are done. ☐

EXAMPLE 1.6.4. The topology on $\mathbf{A}^m \times \mathbf{A}^n$ is strictly finer than the product topology. For example, the diagonal in $\mathbf{A}^1 \times \mathbf{A}^1$ is closed (defined by $x - y \in k[x, y]$), but it is not closed in the product topology.

REMARK 1.6.5. If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then $X \times Y \subseteq \mathbf{A}^{m+n}$ is a locally closed subset, and the two projections induce morphisms $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$. These make $X \times Y$ the product of X and Y in the category of quasi-affine varieties over k. Indeed, given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, it is clear that there is a unique morphism $\phi\colon Z \to X \times Y$ such that $p \circ \phi = f$ and $q \circ \phi = g$, namely $\phi = (f, g)$.

This implies, in particular, that if $f\colon X \to X'$ and $g\colon Y \to Y'$ are isomorphisms, then the induced map $X \times Y \to X' \times Y'$ is an isomorphism.

Proposition 1.6.6. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then the two projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are open*[3].

Proof. We show that p is open, the argument for q being entirely similar. Note first that by Remark 1.6.5, we may replace X and Y by isomorphic quasi-affine varieties. Moreover, if we write $X = \bigcup_i X_i$ and $Y = \bigcup_j Y_j$, then for any open subset W of $X \times Y$, we have

$$p(W) = p\left(\bigcup_{i,j} W \cap (X_i \times Y_j)\right),$$

hence if order to show that p is open, it is enough to show that each projection $X_i \times Y_j \to X_i$ is open. By Remark 1.4.20, both X and Y can be covered by open subsets that are affine varieties. We may thus assume that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. Let $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$ be the rings corresponding to $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively. Using again the fact that every open subset of $X \times Y$ is a union of principal affine open subsets, we see that it is enough to show that $p(W)$ is open in $\mathbf{A}^m$ for a nonempty subest $W = D_{X \times Y}(h)$, where $h \in k[x, y]$.

Let us write

$$h = \sum_{i=1}^{r} f_i(x) g_i(y). \tag{1.6.1}$$

We may and will assume that for the given set W, h and the expression (1.6.1) are chosen such that r is minimal. Note that in this case, the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. Indeed, if this is not the case and $\sum_{i=1}^{r} \lambda_i g_i = P(y) \in I(Y)$, such that $\lambda_j \neq 0$ for some j, then we may take $h' = h - \lambda_j^{-1} f_j(x) P(y)$; we then have $D_{X \times Y}(h') = D_{X \times Y}(h)$ and we can write

$$h' = \sum_{i, i \neq j} \big(f_i(x) - \lambda_i \lambda_j^{-1} f_j(x)\big) g_i(y),$$

contradicting the minimality of r.

Suppose now that $u \in p(W)$. This implies that $u \in X$ such that there is $v \in Y$, with $h(u, v) \neq 0$. In particular, there is j such that $f_j(u) \neq 0$. It is enough to show that in this case $D_X(f_j)$, which contains u, is contained in $p(W)$. Suppose, arguing by contradiction, that there is $u' \in D_X(f_j) \smallsetminus p(W)$. This implies that for every $v \in Y$, we have $\sum_{i=1}^{r} f_i(u') g_i(v) = 0$, hence $\sum_{i=1}^{r} f_i(u') g_i \in I(Y)$. Since $f_j(u') \neq 0$, this contradicts the fact that the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. $\square$

Corollary 1.6.7. *If X and Y are irreducible quasi-affine varieties, then $X \times Y$ is irreducible.*

Proof. We need to show that if U and V are nonempty, open subsets of $X \times Y$, then $U \cap V$ is nonempty. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projections. By the proposition, the nonempty subsets $p(U)$ and $p(V)$ of X are open. Since X is irreducible, we can find $a \in p(U) \cap p(V)$. In this case, the subsets $\{b \in Y \mid (a, b) \in U\}$ and $\{b \in Y \mid (a, b) \in V\}$ of Y are nonempty. They are also open: this follows from the fact that the map $Y \to X \times Y$, $y \to (a, y)$ is

[3] Recall that a continuous map $\phi\colon Z_1 \to Z_2$ is *open* if for every open subset U of Z_1, its image $\phi(U)$ is open in Z_2.

a morphism, hence it is continuous. Since Y is irreducible, these two subsets must intersect, hence there is a point $(a, b) \in U \cap V$. $\square$

Our next goal is to describe the ideal defining the product of two affine varieties. Suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. We have seen in the proof of Proposition 1.6.1 that if $I(X) \subseteq \mathcal{O}(\mathbf{A}^m)$ and $I(Y) \subseteq \mathcal{O}(\mathbf{A}^n)$ are the ideals defining X and Y, respectively, then $X \times Y$ is the algebraic subset of $\mathbf{A}^{m+n}$ defined by

$$J := I(X) \cdot \mathcal{O}(\mathbf{A}^{m+n}) + I(Y) \cdot \mathcal{O}(\mathbf{A}^{m+n}).$$

We claim that, in fact, J is the ideal defining $X \times Y$, that is, J is a radical ideal. Note that $\mathcal{O}(\mathbf{A}^{m+n})$ is canonically isomorphic to $\mathcal{O}(\mathbf{A}^m) \otimes_k \mathcal{O}(\mathbf{A}^n)$ and by the right-exactness of the tensor product, we have

$$\mathcal{O}(\mathbf{A}^{m+n})/J \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

The assertion that J is a radical ideal (or equivalently, that $\mathcal{O}(\mathbf{A}^{m+n})/J$ is a reduced ring is the content of the following

PROPOSITION 1.6.8. *If X and Y are affine varieties, then the ring $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced.*

Before giving the proof of the proposition, we need some algebraic preparations concerning separable extensions.

LEMMA 1.6.9. *If k is any field and K/k is a finite, separable field extension, then for every field extension k'/k, the ring $K \otimes_k k'$ is reduced.*

PROOF. Since K/k is finite and separable, it follows from the Primitive Element theorem that there is an element $u \in K$ such that $K = k(u)$. Moreover, separability implies that if $f \in k[x]$ is the minimal polynomial of u, then all roots of f in some algebraic closure of k are distinct. The isomorphism $K \simeq k[x]/(f)$ induces an isomorphism

$$K \otimes_k k' \simeq k'[x]/(f).$$

If $g_1, \ldots, g_r$ are the irreducible factors of f in $k'[x]$, any two of them are relatively prime (otherwise f would have multiple roots in some algebraic closure of k). It then follows from the Chinese Remainder theorem that we have an isomorphism

$$K \otimes_k k' \simeq \prod_{i=1}^{r} k'[x]/(g_i).$$

Since each factor on the right-hand side is a field (the polynomial g_i being irreducible), the product is a reduced ring. $\square$

LEMMA 1.6.10. *If k is a perfect*[4] *field and K/k is a finitely generated field extension, then there is a transcendence basis $x_1, \ldots, x_n$ of K over k such that K is separable over $k(x_1, \ldots, x_n)$.*

PROOF. Of course, the assertion is trivial if $\mathrm{char}(k) = 0$, hence we may assume that $\mathrm{char}(k) = p > 0$. Let us write $K = k(x_1, \ldots, x_m)$. We may assume that $x_1, \ldots, x_n$ give a transcendence basis of K/k, and suppose that $x_{n+1}, \ldots, x_{n+r}$ are not separable over $K' := k(x_1, \ldots, x_n)$, while $x_{n+r+1}, \ldots, x_m$ are separable over K'. If $r = 0$, then we are done. Otherwise, since x_{n+1} is not separable over K',

[4]Recall that a field k is perfect if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and $k = k^p$. Equivalently, a field is perfect if every finite extension K/k is separable.

it follows that there is an irreducible polynomial $f \in K'[T]$ such that $f \in K'[T^p]$ and such that $f(x_{n+1}) = 0$. We can find a nonzero $u \in k[x_1, \ldots, x_n]$ such that $g = uf \in k[x_1, \ldots, x_n, T^p]$.

We claim that there is $i \leq n$ such that $\frac{\partial g}{\partial x_i} \neq 0$. Indeed, otherwise we have $g \in k[x_1^p, \ldots, x_n^p, T^p]$, and since k is perfect, we have $k = k^p$, hence $g = h^p$ for some $h \in k[x_1, \ldots, x_n, T]$; this contradicts the fact that f is irreducible.

After relabeling the variables, we may assume that $i = n$. The assumption on i says that x_n is (algebraic and) separable over $K'' := k(x_1, \ldots, x_{n-1}, x_{n+1})$. Note that since x_n is algebraic over K'' and K is algebraic over $k(x_1, \ldots, x_{n-1}, x_n)$, it follows that K is algebraic over K'', and since all transcendence bases of K over k have the same number of elements, we conclude that $x_1, \ldots, x_{n-1}, x_{n+1}$ is a transcendence basis of K over k. We may thus switch x_n and x_{n+1} to lower r. After finitely many steps, we obtain the conclusion of the lemma. □

PROPOSITION 1.6.11. *If k is a perfect field, then for every field extensions K/k and k'/k, the ring $K \otimes_k k'$ is reduced.*

PROOF. We may assume that K is finitely generated over k. Indeed, we can write

$$K = \varinjlim_i K_i,$$

where the direct limit is over all $k \subseteq K_i \subseteq K$, with K_i/k finitely generated. Since we have an induced isomorphism

$$K \otimes_k k' \simeq \varinjlim_i K_i \otimes_k k',$$

and a direct limit of reduced rings is reduced, we see that it is enough to prove the proposition when K/k is finitely generated.

In this case we apply Lemma 1.6.10 to find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $K_1 := k(x_1, \ldots, x_n)$. We have

$$K \otimes_k k' = K \otimes_{K_1} K_1 \otimes_k k'.$$

Since $K_1 \otimes_k k'$ is a ring of fractions of $k'[x_1, \ldots, x_n]$, we have an injective homomorphism

$$K_1 \otimes_k k' \hookrightarrow K_2 := k'(x_1, \ldots, x_n).$$

By tensoring with K, we get an injective homomorphism

$$K \otimes_k k' \hookrightarrow K \otimes_{K_1} K_2.$$

Since K/K_1 is a finite separable extension, we deduce from Lemma 1.6.9 that $K \otimes_{K_1} K_2$ is reduced, hence $K \otimes_k k'$ is reduced. □

We can now prove our result about the coordinate ring of the product of two affine varieties.

PROOF OF PROPOSITION 1.6.8. We will keep using the fact that the tensor product over k is an exact functor. Note first that we may assume that X and Y are irreducible. Indeed, let $X_1, \ldots, X_r$ be the irreducible components of X and $Y_1, \ldots, Y_s$ the irreducible components of Y. Since $X = X_1 \cup \ldots \cup X_r$, it is clear that the canonical homomorphism

$$\mathcal{O}(X) \to \prod_{i=1}^{r} \mathcal{O}(X_i)$$

is injective. Similarly, we have an injective homomorphism

$$\mathcal{O}(Y) \to \prod_{j=1}^{s} \mathcal{O}(Y_i)$$

and we thus obtain an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow \prod_{i,j} \mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j).$$

The right-hand side is a reduced ring if each $\mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j)$ is reduced, in which case $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. We thus may and will assume that both X and Y are irreducible.

We know that in this case $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains and let $k(X)$ and $k(Y)$ be the respective fraction fields. Since k is algebraically closed, it is perfect, hence $k(X) \otimes_k k(Y)$ is a reduced ring by Proposition 1.6.11. The inclusions

$$\mathcal{O}(X) \hookrightarrow k(X) \quad \text{and} \quad \mathcal{O}(Y) \hookrightarrow k(Y)$$

induce an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow k(X) \otimes_k k(Y),$$

which implies that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. □

We now give another application of Lemma 1.6.10. We first make a definition.

DEFINITION 1.6.12. A hypersurface in $\mathbf{A}^n$ is a closed subset of the form

$$\{u \in \mathbf{A}^n \mid f(u) = 0\} \quad \text{for some} \quad f \in k[x_1, \ldots, x_n] \smallsetminus k.$$

PROPOSITION 1.6.13. *Every irreducible variety is birational to an (irreducible) hypersurface in an affine space* $\mathbf{A}^n$.

PROOF. Let X be an irreducible variety, with function field $K = k(X)$. By Lemma 1.6.10, we can find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $k(x_1, \ldots, x_n)$. In this case, it follows from the Primitive Element theorem that there is $u \in K$ such that $K = k(x_1, \ldots, x_n, u)$. If $f \in k(x_1, \ldots, x_n)[t]$ is the minimal polynomial of u, then

$$K \simeq k(x_1, \ldots, x_n)[t]/(f).$$

It is easy to see that after multiplying u by a suitable nonzero element of $k[x_1, \ldots, x_n]$, we may assume that $f \in k[x_1, \ldots, x_n, t]$ and f is irreducible. In this case, we see by Theorem 1.5.18 that X is birational to the affine variety $V(f) \subseteq \mathbf{A}^{n+1}$. □

We end this section with some exercises about linear algebraic groups. We begin with a definition.

DEFINITION 1.6.14. A *linear algebraic group* over k is an affine variety G over k that is also a group, and such that the multiplication $\mu\colon G \times G \to G$, $\mu(g, h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. If G_1 and G_2 are linear algebraic groups, a *morphism of algebraic groups* is a morphism of affine varieties $f\colon G_1 \to G_2$ that is also a group homomorphism.

Linear algebraic groups over k form a category. In particular, we have a notion of isomorphism between linear algebraic groups: this is an isomorphism of affine algebraic varieties that is also a group isomorphism.

EXERCISE 1.6.15.
i) Show that $(k, +)$ and $(k^*, \cdot)$ are linear algebraic groups.
ii) Show that the set $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k has a structure of linear algebraic group.
iii) Show that the set $\mathrm{SL}_n(k)$ of $n \times n$ matrices with coefficients in k and with determinant 1 has a structure of linear algebraic group.
iv) Show that if G and H are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) *torus* $(k^*)^n$ is a linear algebraic group with respect to componentwise multiplication.

DEFINITION 1.6.16. Let G be a linear algebraic group and X a quasi-affine variety. An *algebraic group action* of G on X is a (say, left) action of G on X such that the map $G \times X \to X$ giving the action is a morphism of algebraic varieties.

EXERCISE 1.6.17. Show that $\mathrm{GL}_n(k)$ has an algebraic action on $\mathbf{A}^n$.

EXERCISE 1.6.18. Let G be a linear algebraic group acting algebraically on an affine variety X. Show that in this case G has an induced linear action on $\mathcal{O}(X)$ given by
$$(g \cdot \phi)(u) = \phi\big(g^{-1}(u)\big).$$
While $\mathcal{O}(X)$ has in general infinite dimension over k, show that the action of G on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace V of $\mathcal{O}(X)$ that is preserved by the G-action (Hint: consider the image of f by the corresponding k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

EXERCISE 1.6.19. Let G and X be as in the previous problem. Consider a system of k-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \mathrm{GL}_N(k)$, and an isomorphism of X with a closed subset of $\mathbf{A}^N$, such that the action of G on X is induced by the standard action of $\mathrm{GL}_N(k)$ on $\mathbf{A}^N$. Use a similar argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_N(k)$.

EXERCISE 1.6.20. Show that the linear algebraic group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbf{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action ? Note that the orbits are locally closed subsets of $M_{m,n}(k)$ (as we will see later, this is a general fact about orbits of algebraic group actions).

1.7. Affine toric varieties

In this section we discuss a class of examples of affine varieties that are associated to semigroups.

DEFINITION 1.7.1. A *semigroup* is a set S endowed with an operation $+$ (we will use in general the additive notation) which is commutative, associative and has a unit element 0. If S is a semigroup, a *subsemigroup* of S is a subset $S' \subseteq S$ closed under the operation in S and such that $0_S \in S'$ (in which case, S' becomes a semigroup with the induced operation). A map $\phi\colon S \to S'$ between two semigroups is a *semigroup morphism* if $\phi(u_1 + u_2) = \phi(u_1) + \phi(u_2)$ for all u_1 and u_2, and if $\phi(0) = 0$.

EXAMPLE 1.7.2. i) Every Abelian group is a semigroup.

ii) The field k, endowed with the multiplication, is a semigroup.

iii) The set $\mathbf{N}$ of non-negative integers, with the addition, is a semigroup.

iv) The set $\{m \in \mathbf{N} \mid m \neq 1\}$ is a subsemigroup of $\mathbf{N}$.

v) If S_1 and S_2 are semigroups, then $S_1 \times S_2$ is a semigroup, with componentwise addition.

Given a semigroup S, we consider the semigroup algebra $k[S]$. This has a basis over k indexed by the elements of S. We denote the elements of this basis by χ^u, for $u \in S$. The multiplication is defined by $\chi^{u_1} \cdot \chi^{u_2} = \chi^{u_1+u_2}$ (hence $1 = \chi^0$). This is a k-algebra. Note that if $\phi\colon S_1 \to S_2$ is a morphism of semigroups, then we get a morphism of k-algebras $k[S_1] \to k[S_2]$ that maps χ^u to $\chi^{\phi(u)}$.

EXAMPLE 1.7.3. We have an isomorphism

$$k[\mathbf{N}^r] \simeq k[x_1, \ldots, x_r], \quad \chi^{e_i} \to x_i,$$

where e_i is the tuple that has 1 on the i^{th} component and 0 on all the others. We similarly have an isomorphism

$$k[\mathbf{Z}^r] \simeq k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}].$$

EXAMPLE 1.7.4. In general, if S_1 and S_2 are semigroups, we have a canonical isomorphism

$$k[S_1 \times S_2] \simeq k[S_1] \otimes_k k[S_2].$$

We will assume that our semigroups satisfy two extra conditions. First, we will assume that they are *finitely generated*: a semigroup S satisfies this property if it has finitely many generators $u_1, \ldots, u_r \in S$ (this means that every element in S can be written as $\sum_{i=1}^r a_i u_i$, for some $a_1, \ldots, a_r \in \mathbf{N}$). In other words, the unique morphism of semigroups $\mathbf{N}^r \to S$ that maps e_i to u_i for all i is surjective. Note that in this case, the induced k-algebra homomorphism

$$k[x_1, \ldots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is onto, hence $k[S]$ is finitely generated.

We will also assume that S is *integral*, that is, it is isomorphic to a subsemigroup of a finitely generated, free Abelian group. Since we have an injective morphism of semigroups $S \hookrightarrow \mathbf{Z}^r$, we obtain an injective k-algebra homomorphism $k[S] \hookrightarrow k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$. In particular, $k[S]$ is a domain.

EXERCISE 1.7.5. Suppose that S is the image of a morphism of semigroups $\phi\colon \mathbf{N}^r \to \mathbf{Z}^m$ (this is how semigroups are usually described). Show that the kernel of the induced surjective k-algebra homomorphism

$$k[x_1, \ldots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is the ideal

$$\big(x^a - x^b \mid a, b \in \mathbf{N}^r, \phi(a) = \phi(b)\big).$$

We have seen that if S is an integral, finitely generated semigroup, then $k[S]$ is a finitely generated k-algebra, which is also a domain. Therefore it corresponds to an irreducible affine variety over k, uniquely defined up to canonical isomorphism. We will denote this variety[5] by $\mathrm{TV}(S)$. Its points are in bijection with the maximal ideals in $k[S]$, or equivalently, with the k-algebra homomorphisms $k[S] \to k$. Such

[5]This is not standard notation in the literature.

homomorphisms in turn are in bijection with the semigroup morphisms $S \to (k, \cdot)$. Via this bijection, if we consider $\phi\colon S \to (k, \cdot)$ as a point in $\mathrm{TV}(S)$ and $\chi^u \in k[S]$, then

$$\chi^u(\phi) = \phi(u) \in k.$$

Given a morphism of finitely generated, integral semigroups $S \to S'$, the k-algebra homomorphism $k[S] \to k[S']$ corresponds to a morphism $\mathrm{TV}(S') \to \mathrm{TV}(S)$.

The affine variety $TV(S)$ carries more structure, induced by the semigroup S, which we now describe. First, we have a morphism

$$TV(S) \times TV(S) \to TV(S)$$

corresponding to the k-algebra homomorphism

$$k[S] \to k[S] \otimes_k k[S], \quad \chi^u \to \chi^u \otimes \chi^u.$$

At the level of points (identified, as above, to semigroup morphisms to k), this is given by

$$(\phi, \psi) \to \phi \cdot \psi, \quad \text{where} \quad (\phi \cdot \psi)(u) = \phi(u) \cdot \psi(u).$$

It is clear that the operation is commutative, associative, and has an identity element, given by the morphism $S \to k$ that takes constant value 1.

Remark 1.7.6. If $S \to S'$ is a morphism between integral, finitely generated semigroups, it is clear that the induced morphism of affine varieties $\mathrm{TV}(S') \to \mathrm{TV}(S)$ is compatible with the operation defined above.

Example 1.7.7. If $S = \mathbf{N}^r$, then the operation that we get on $\mathrm{TV}(S) = \mathbf{A}^r$ is given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n).$$

In particular, note that $\mathrm{TV}(S)$ is *not* a group.

Example 1.7.8. With the operation defined above, $\mathrm{TV}(\mathbf{Z})$ is a linear algebraic group isomorphic to $(k^*, \cdot)$. In general, if M is a finitely generated, free Abelian group, then the above operation makes $\mathrm{TV}(M)$ a linear algebraic group. In fact, we have $M \simeq \mathbf{Z}^r$, for some r, and therefore $\mathrm{TV}(M)$ is isomorphic, as an algebraic group, to the torus $(k^*)^r$ (see Exercise 1.6.15 for the definition of the algebraic tori). It follows from the lemma below that we can recover M from $\mathrm{TV}(M)$, together with the group structure, as

$$M \simeq \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M), k^*\big).$$

Lemma 1.7.9. *For every finitely generated, free Abelian groups M and M', the canonical map*

$$\mathrm{Hom}_{\mathbf{Z}}(M, M') \to \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M'), \mathrm{TV}(M)\big)$$

is a bijection.

Proof. A morphism of algebraic groups $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is given by a k-algebra homomorphism $f\colon k[M] \to k[M']$ such that the induced diagram

$$\begin{tikzcd} k[M] \arrow[r, "f"] \arrow[d, "\alpha_M"] & k[M'] \arrow[d, "\alpha_{M'}"] \\ k[M] \otimes k[M] \arrow[r, "f \otimes f"] & k[M'] \otimes k[M'], \end{tikzcd}$$

is commutative, where α_M and $\alpha_{M'}$ are the k-algebra homomorphisms inducing the group structure. Given $u \in M$, we see that if $f(\chi^u) = \sum_{u' \in M'} a_{u,u'} \chi^{u'}$, then

$$\sum_{u' \in M'} a_{u,u'} \chi^{u'} \otimes \chi^{u'} = \sum_{u',v' \in M'} a_{u,u'} a_{u,v'} \chi^{u'} \otimes \chi^{v'}.$$

First, this implies that if u', $v' \in M'$ are distinct, then $a_{u,u'} \cdot a_{u,v'} = 0$. Therefore there is a unique $u' \in M'$ such that $a_{u,u'} \neq 0$ (note that $\chi^u \in k[M]$ is invertible, hence $f(\chi^u) \neq 0$). Moreover, for this u' we have $a_{u,u'}^2 = a_{u,u'}$, hence $a_{u,u'} = 1$. This implies that we have a (unique) map $\phi\colon M \to M'$ such that f is given by $f(\chi^u) = \chi^{\phi(u)}$. Since f is a ring homomorphism, we see that ϕ is a semigroup morphism. This shows that the map in the lemma is bijective. $\square$

Exercise 1.7.10. Given an integral semigroup S, show that there is an injective semigroup morphism $\iota\colon S \hookrightarrow S^{\mathrm{gp}}$, where S^{gp} is a finitely generated Abelian group, that satisfies the following universal property: given any semigroup morphism $h\colon S \to A$, where A is an Abelian group, there is a unique group morphism $g\colon S^{\mathrm{gp}} \to A$ such that $g \circ \iota = h$. Hint: if $S \hookrightarrow M$ is an injective semigroup morphism, where M is a finitely generated, free Abelian group, then show that one can take S^{gp} to be the subgroup of M generated by S. Note that it follows from this description that S^{gp} is finitely generated (since M is) and S^{gp} is generated as a group by S.

Suppose now that S is an arbitrary integral, finitely generated semigroup. The semigroup morphism $\iota\colon S \to S^{\mathrm{gp}}$ induces a k-algebra homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ and correspondingly a morphism of affine algebraic varieties $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$.

Lemma 1.7.11. *With the above notation, the morphism $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is an isomorphism onto a principal affine open subset of* $\mathrm{TV}(S)$*.*

Proof. Suppose that $u_1, \ldots, u_r$ is a finite system of generators of S. In this case S^{gp} is generated as a semigroup by $u_1, \ldots, u_r$, and $-(u_1 + \ldots + u_r)$. This shows that we can identify the homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ with the localization homomorphism of $k[S]$ at $\chi^{u_1+\ldots+u_s}$. $\square$

Since the morphism $\mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is compatible with the operations on the two varieties, we conclude that in particular, the action of the torus $\mathrm{TV}(S^{\mathrm{gp}})$, considered as an open subset of $\mathrm{TV}(S)$, extends to an action of $\mathrm{TV}(S^{\mathrm{gp}})$ on $\mathrm{TV}(S)$. We are thus led to the following

Definition 1.7.12. An affine toric variety is an irreducible affine variety X, together with an open subset T that is (isomorphic to) a torus, such that the action of the torus on itself extends to an action of T of X.

We note that in the literature, it is common to require an affine toric variety to be normal, but we do not follow this convention. For the definition of normality and for the description in the context of toric varieties, see Definition 1.7.26 and Proposition 1.7.30 below.

We have seen that for every (integral, finitely generated) semigroup S, we obtain a toric variety $\mathrm{TV}(S)$. The following proposition shows that, in fact, every affine toric variety arises in this way.

Proposition 1.7.13. *Let X be an irreducible affine variety, $T \subseteq X$ an open subset which is a torus such that the action of T on itself extends to an action on*

X. Then there is a finitely generated, integral semigroup S and an isomorphism $X \simeq \mathrm{TV}(S)$ which induces an isomorphism of algebraic groups $T \simeq \mathrm{TV}(S^{\mathrm{gp}})$, and which is compatible with the action.

Proof. Let $M = \mathrm{Hom}_{\mathrm{alg-gp}}(T, k^*)$, so that we have a canonical isomorphism $T \simeq \mathrm{TV}(M)$. The dominant inclusion morphism $T \to X$ induces an injective k-algebra homomorphism $f\colon \mathcal{O}(X) \to \mathcal{O}(T) = k[M]$, hence we may assume that $\mathcal{O}(X)$ is a subalgebra of $k[M]$. The fact that the action of T on itself extends to an action of T on X is equivalent to the fact that the k-algebra homomorphism

$$k[M] \to k[M] \otimes_k k[M], \quad \chi^u \to \chi^u \otimes \chi^u$$

induces a homomorphism $\mathcal{O}(X) \to k[M] \otimes_k \mathcal{O}(X)$. In other words, if $f = \sum_{u \in M} a_u \chi^u$ lies in $\mathcal{O}(X)$, then $\sum_{u \in M} a_u \chi^u \otimes \chi^u$ lies in $k[M] \otimes_k \mathcal{O}(X)$. This implies that for every $u \in M$ such that $a_u \neq 0$, we have $\chi^u \in \mathcal{O}(X)$. It follows that if $S = \{u \in M \mid \chi^u \in \mathcal{O}(X)\}$, then $\mathcal{O}(X) = k[S]$. It is clear that S is integral and since $k[S]$ is a finitely generated k-algebra, it follows easily that S is a finitely generated semigroup. In order to complete the proof of the proposition, it is enough to show that $M = S^{\mathrm{gp}}$.

It follows from Exercise 1.7.10 that we may take S^{gp} to be the subgroup of M generated by S. By hypothesis, the composition

$$TV(M) \xrightarrow{g} \mathrm{TV}(S^{\mathrm{gp}}) \xrightarrow{h} X = \mathrm{TV}(S)$$

is an isomorphism onto an open subset of X. Since we also know that h is an isomorphism onto an open subset of X, it follows that g gives is an isomorphism onto an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$. In particular, this implies that g is injective. We now show that $M = S^{\mathrm{gp}}$.

Since M is a finitely generated, free Abelian group, we can find a basis $e_1, \ldots, e_n$ of M such that S^{gp} has a basis given by $a_1 e_1, \ldots, a_r e_r$, for some $r \leq n$ and some positive integers $a_1, \ldots, a_r$. In this case g gets identified to the morphism

$$(k^*)^n \to (k^*)^r, \quad (t_1, \ldots, t_n) \to (t_1^{a_1}, \ldots, t_r^{a_r}).$$

Since g is injective, we see that $r = n$. Moreover, if $a_j > 1$ for some j, then $\mathrm{char}(k) = p > 0$ and for every i we have $a_i = p^{e_i}$ for some nonnegative integer e_i. It is easy to see that in this case g is surjective (cf. Exercise 1.4.26). Since we know that it gives an isomorphism of $\mathrm{TV}(M)$ with an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$, it follows that g is an isomorphism. However, this implies $a_i = 1$ for all i. Therefore we have $S^{\mathrm{gp}} = M$. $\square$

We now turn to the description of toric morphisms. Suppose that X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$.

Definition 1.7.14. With the above notation, a *toric morphism* $X \to Y$ is a morphism of algebraic varieties $f\colon X \to Y$ that induces a morphism of algebraic groups $g\colon T_X \to T_Y$.

Remark 1.7.15. Note that if $f\colon X \to Y$ is a toric morphism as above, then f is a morphism of varieties with torus action, in the sense that

$$f(t \cdot x) = g(t) \cdot f(x) \quad \text{for every } t \in T_X, x \in X.$$

Indeed, this follows by Lemma 1.5.7 from the fact that we have this equality for $(t, x) \in T_X \times T_X$.

If $\phi\colon S_1 \to S_2$ is a semigroup morphism between two integral, finitely generated semigroups, we get an induced group morphism $S_1^{\rm gp} \to S_2^{\rm gp}$. We then obtain an induced morphism $f\colon \operatorname{TV}(S_2) \to \operatorname{TV}(S_1)$ that restricts to a morphism of algebraic groups $\operatorname{TV}(S_2^{\rm gp}) \to \operatorname{TV}(S_1^{\rm gp})$; therefore f is a toric morphism. The next proposition shows that all toric morphisms arise in this way, from a unique semigroup homomorphism.

PROPOSITION 1.7.16. *If S_1 and S_2 are finitely generated, integral semigroups, then the canonical map*

$$\operatorname{Hom}_{\rm semigp}(S_1, S_2) \to \operatorname{Hom}_{\rm toric}\big(\operatorname{TV}(S_2), \operatorname{TV}(S_1)\big)$$

is a bijection.

PROOF. By definition, a toric morphism $\operatorname{TV}(S_2) \to \operatorname{TV}(S_1)$ is given by a k-algebra homomorphism $k[S_1] \to k[S_2]$ such that the induced homomorphism $f\colon k[S_1^{\rm gp}] \to k[S_2^{\rm gp}]$ gives a morphism of algebraic groups $\operatorname{TV}(S_2^{\rm gp}) \to \operatorname{TV}(S_1^{\rm gp})$. It follows from Lemma 1.7.9 that we have a group morphism $\phi\colon S_1^{\rm gp} \to S_2^{\rm gp}$ such that $f(\chi^u) = \chi^{\phi(u)}$ for every $u \in S_1^{\rm gp}$. Since f induces a homomorphism $k[S_1] \to k[S_2]$, we have $\phi(S_1) \subseteq S_2$, hence ϕ is induces a semigroup morphism $S_1 \to S_2$. This shows that the map in the proposition is surjective and the injectivity is straightforward. $\square$

REMARK 1.7.17. We can combine the assertions in Proposition 1.7.13 and 1.7.16 as saying that the functor from the category of integral, finitely generated semigroups to the category of affine toric varieties, that maps S to $\operatorname{TV}(S)$, is an anti-equivalence of categories.

EXAMPLE 1.7.18. If $S = \mathbf{N}^r$, then $\operatorname{TV}(S) = \mathbf{A}^r$, with the torus $(k^*)^r \subseteq \mathbf{A}^r$ acting by component-wise multiplication.

EXAMPLE 1.7.19. If $S = \{m \in \mathbb{N} \mid m \neq 1\}$, then $S^{\rm gp} = \mathbb{Z}$. If we embed X in $\mathbb{A}^2$ as the curve with equation $u^3 - v^2 = 0$, then the embedding $T \simeq k^* \hookrightarrow X$ is given by $\lambda \to (\lambda^2, \lambda^3)$. The action of T on X is described by $\lambda \cdot (u, v) = (\lambda^2 u, \lambda^3 v)$.

EXERCISE 1.7.20. Show that if X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$, then $X \times Y$ has a natural structure of toric variety, with torus $T_X \times T_Y$. Describe the semigroup corresponding to $X \times Y$ in terms of the semigroups of X and Y.

EXERCISE 1.7.21. Let S be the sub-semigroup of $\mathbb{Z}^3$ generated by e_1, e_2, e_3 and $e_1 + e_2 - e_3$. These generators induce a surjective morphism $f\colon k[\mathbb{N}^4] = k[t_1, \ldots, t_4] \to k[S]$. Show that the kernel of f is generated by $t_1t_2 - t_3t_4$. We have $S^{\rm gp} = \mathbb{Z}^3$, the embedding of $T = (k^*)^3 \hookrightarrow X$ is given by $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2/\lambda_3)$, and the action of T on X is induced via this embedding by component-wise multiplication.

The following lemma provides a useful tool for dealing with torus-invariant objects. Consider $X = \operatorname{TV}(S)$ and let $T = \operatorname{TV}(S^{\rm gp})$ be the corresponding torus. As in the case of any algebraic group action, the action of T on X induces an action of T on $\mathcal{O}(X)$ (see Exercise 1.6.18). Explicitly, in our setting this is given by

$$\phi \cdot \chi^u = \phi(u)^{-1}\chi^u \quad \text{for all} \quad u \in S, \phi \in \operatorname{Hom}_{\rm gp}(S^{\rm gp}, k^*).$$

LEMMA 1.7.22. *With the above notation, a subspace $V \subseteq k[S]$ is T-invariant (that is, $t \cdot g \in V$ for every $g \in V$) if and only if it is S-homogenous, in the sense that for every $g = \sum_{u \in S} a_u \chi^u \in V$, we have $\chi^u \in V$ whenever $a_u \neq 0$.*

PROOF. We only need to prove the "only if" part, the other direction being straightforward. By definition, V is T-invariant if and only if for every group morphism $\phi\colon S^{\mathrm{gp}} \to k^*$ and every $g = \sum_{u \in S} a_u \chi^u \in V$, we have

$$\sum_{u \in S} a_u \phi(u)^{-1} \chi^u \in V.$$

Iterating, we obtain

$$\sum_{u \in S} a_u \phi(u)^{-m} \chi^u \in V \quad \text{for all} \quad m \geq 1. \tag{1.7.1}$$

Claim. Given pairwise distinct $u_1, \ldots, u_d \in S$, we can find $\phi \in T$ such that $\phi(u_i) \neq \phi(u_{i'})$ for $i \neq i'$. Indeed, let us choose an isomorphism $S^{\mathrm{gp}} \simeq \mathbf{Z}^n$, so that each u_i corresponds to $(a_{i,1}, \ldots, a_{i,n})$. After adding to each $(a_{i,1}, \ldots, a_{i,n})$ the element $(m, \ldots, m)$ for $m \gg 0$, we may assume that $a_{i,j} \geq 0$ for all i and j. Since each polynomial

$$Q_{i,i'} = \prod_{j=1}^{n} x_j^{a_{i,j}} - \prod_{j=1}^{n} x_j^{a_{i',j}}, \quad \text{for} \quad i \neq i'$$

is nonzero, it follows that the open subset $U_{i,i'}$ defined by $Q_{i,i'} \neq 0$ is a nonempty subset of $\mathbf{A}^n$. Since $\mathbf{A}^n$ is irreducible, it follows that the intersection

$$(k^*)^n \cap \bigcap_{i \neq i'} U_{i,i'}$$

is nonempty, giving the claim.

By applying the claim to those $u \in S$ such that $a_u \neq 0$, we deduce from (1.7.1) and from the formula for the Vandermonde determinant that $\chi^u \in V$ for all u such that $a_u \neq 0$. $\square$

In the next two exercises we describe the torus-invariant subvarieties of $\mathrm{TV}(S)$ and the orbits of the torus action. We begin by defining the corresponding concept at the level of the semigroup.

DEFINITION 1.7.23. A face F of a semigroup S is a subsemigroup such that whenever $u_1, u_2 \in S$ have $u_1 + u_2 \in F$, we have $u_1 \in F$ and $u_2 \in F$.

Note that if F is a face of S, then $S \smallsetminus F$ is a subsemigroup of S. Moreover, if S is generated by $u_1, \ldots, u_n$. then a face F of S is generated by those u_i that lie in F. In particular, if S is an integral, finitely generated semigroup, then S has only finitely many faces, and each of these is an integral, finitely generated semigroup.

EXERCISE 1.7.24. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T \subset X$. A subset Y of X is *torus-invariant* if $t \cdot Y \subseteq Y$ for every $t \in T$.

i) Show that a closed subset Y of X is torus-invariant if and only if each irreducible component of Y is torus-invariant.
ii) Show that the torus-invariant irreducible closed subsets of X are precisely the closed subsets defined by ideals of the form

$$\bigoplus_{u \in S \smallsetminus F} k\chi^u,$$

where F is a face of S.

iii) Show that if Y is the closed subset defined by the ideal in ii), then we have $\mathcal{O}(Y) \simeq k[F]$, hence Y has a natural structure of affine toric variety.

EXERCISE 1.7.25. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T_X \subseteq X$.

i) Show that if $M \hookrightarrow M'$ is an injective morphism of finitely generated, free Abelian groups, then the induced morphism of tori $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is surjective.
ii) Show that if F is a face of S with corresponding closed invariant subset Y, then the inclusion of semigroups $F \subseteq S$ induces a morphism of toric varieties $f_Y : X \to Y$, which is a retract of the inclusion $Y \hookrightarrow X$. Show that the torus O_F in Y is an orbit for the action of T_X on X.
iii) Show that the map $F \to O_F$ gives a bijection between the faces of S and the orbits for the T_X-action on X.

We now discuss normality for the varieties we defined. Recall that if $R \to S$ is a ring homomorphism, then the set of elements of S that are integral over R form a subring of S, the *integral closure* of R in S (see Proposition A.2.2).

DEFINITION 1.7.26. An integral domain A is *integrally closed* if it is equal to its integral closure in its field of fractions. It is *normal* if, in addition, it is Noetherian. An irreducible, affine variety X is *normal* if $\mathcal{O}(X)$ is a normal ring.

REMARK 1.7.27. If A is an integral domain and B is the integral closure of A in its fraction field, then B is integrally closed. Indeed, the integral closure of B in K is integral over A (see Proposition A.2.3), hence it is contained in B.

EXAMPLE 1.7.28. Every UFD is integrally closed. Indeed, suppose that A is a UFD and $u = \frac{a}{b}$ lies in the fraction field of A and it is integral over A. We may assume that a and b are relatively prime. Consider a monic polynomial $f = x^m + c_1x^{m-1} + \ldots c_m \in A[x]$ such that $f(u) = 0$. Since

$$a^m = -b \cdot (c_1a^{m-1} + \ldots c_mb^{m-1}),$$

it follows that b divides a^m. Since b and a are relatively prime, it follows that b is invertible, hence $u \in A$.

In particular, we see that every polynomial ring $k[x_1, \ldots, x_n]$ is integrally closed.

DEFINITION 1.7.29. An integral, finitely generated semigroup S is *saturated* if whenever $mu \in S$ for some $u \in S^{\mathrm{gp}}$ and some positive integer m, we have $u \in S$.

PROPOSITION 1.7.30. *If S is an integral, finitely generated semigroup, the variety* $\mathrm{TV}(S)$ *is normal if and only if S is saturated.*

PROOF. The rings $k[S] \subseteq k[S^{\mathrm{gp}}]$ have the same fraction field, and $k[S^{\mathrm{gp}}] \simeq k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ for some n, so $k[S^{\mathrm{gp}}]$ is normal, being a UFD. Therefore $k[S]$ is normal if and only if it is integrally closed in $k[S^{\mathrm{gp}}]$.

Suppose first that $k[S]$ is normal. If $u \in S^{\mathrm{gp}}$ and if $mu \in S$, then $(\chi^u)^m \in k[S]$ and $\chi^u \in k[S^{\mathrm{gp}}]$. As $k[S]$ is integrally closed in $k[S^{\mathrm{gp}}]$, it follows that $\chi^u \in k[S]$, so $u \in S$.

Conversely, let us assume that S is saturated, and let R be the integral closure of $k[S]$ in $k[S^{\mathrm{gp}}]$. It is clear that R is a torus-invariant linear subspace of $k[S^{\mathrm{gp}}]$, hence it follows from Lemma 1.7.22 that it is S^{gp}-homogeneous. In order to show that $R = k[S]$ it is thus enough to check that for every $\chi^u \in R$, we have $u \in S$. By assumption, χ^u satisfies an equation of the form

$$(\chi^u)^m + a_1(\chi^u)^{m-1} + \ldots + a_m\chi^{v_m} = 0,$$

for a positive integer m and $a_1, \ldots, a_m \in k[S]$. By only considering the scalar multiples of χ^{mu}, we may assume that in fact $a_i = c_i\chi^{v_i}$ for some $c_i \in k$ and $v_i \in S$. It follows that $v_i + (m-i)u = mu$ if $a_i \neq 0$, hence $iu = v_i$. Since some a_i must be nonzero, we have $iu \in S$ for some $i \geq 1$, and because S is saturated we deduce $u \in S$. $\square$

Exercise 1.7.31. We have seen in Exercise 1.7.24 that if X is an affine toric variety and Y is a torus-invariant irreducible subset, then Y has a natural structure of toric variety. Show that if X is normal, then every such Y is normal.

CHAPTER 2

General algebraic varieties

In this chapter we introduce general algebraic varieties. Roughly speaking, these are objects obtained by gluing finitely many affine algebraic varieties and by imposing an analogue of the Hausdorff condition. The gluing could be expressed in terms of atlases (as in differential geometry), but the usual language for handling this is that of ringed spaces and we take this approach, following [**Mum88**]. We thus begin with a brief discussion of sheaves that is needed for the definition of algebraic varieties. A more detailed treatment of sheaves will be given in Chapter 8.

2.1. Presheaves and sheaves

Let X be a topological space. Recall that associated to X we have a category $\mathcal{C}at(X)$, whose objects consist of the open subsets of X and such that for every open subsets U and V of X, the set of arrows $U \to V$ contains precisely one element if $U \subseteq V$ and it is empty, otherwise.

Definition 2.1.1. Given a topological space X and a category $\mathcal{C}$, a *presheaf* on X of objects in $\mathcal{C}$ is a contravariant functor $\mathcal{F} \colon \mathcal{C}at(X) \to \mathcal{C}$. Explicitly, this means that for every open subset U of X, we have an object $\mathcal{F}(U)$ in $\mathcal{C}$, and for every inclusion of open sets $U \subseteq V$, we have a *restriction map*

$$\rho_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

that satisfies:

i) $\rho_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$ for every open subset $U \subseteq X$, and
ii) $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$ for every open subsets $U \subseteq V \subseteq W$ of X.

It is common to denote $\rho_{V,U}(s)$ by $s|_U$. The elements of $\mathcal{F}(U)$ are *the sections of* $\mathcal{F}$ *over* U. A common notation for $\mathcal{F}(U)$ is $\Gamma(U, \mathcal{F})$.

The important examples for us are when $\mathcal{C}$ is the category of R-modules or the category of commutative R-algebras (where R is a fixed commutative ring). In particular, when $R = \mathbf{Z}$, we have the category of Abelian groups and the category of rings.

We now introduce *sheaves*: these are presheaves in which the sections can be described locally. For the sake of concreteness, whenever dealing with sheaves, we assume that $\mathcal{C}$ is a subcategory of the category of sets and that a morphism in $\mathcal{C}$ is an isomorphism if and only if it is bijective (note that this is the case for the categories mentioned above).

Definition 2.1.2. Let X be a topological space. A presheaf $\mathcal{F}$ on X of objects in $\mathcal{C}$ is a *sheaf* if for every family of open subsets $(U_i)_{i\in I}$ of X, with $U = \bigcup_{i\in I} U_i$, given $s_i \in \mathcal{F}(U_i)$ for every i such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \text{for every} \quad i, j \in I,$$

there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

REMARK 2.1.3. A special case of the condition in the definition of a sheaf is that when $I = \emptyset$: in this case it says that $\mathcal{F}(\emptyset)$ has exactly one element.

EXAMPLE 2.1.4. If X is a topological space, then we have a presheaf $\mathcal{C}_{X,\mathbf{R}}$ of $\mathbf{R}$-algebras on X, where $\mathcal{C}_{X,\mathbf{R}}(U)$ is the $\mathbf{R}$-algebra of continuous functions $U \to \mathbf{R}$, with the restriction maps given by restriction of functions. It is clear that this is a sheaf, the *sheaf of continuous functions* on X.

EXAMPLE 2.1.5. If X is a $\mathcal{C}^\infty$-manifold, then we have a sheaf of $\mathbf{R}$-algebras $\mathcal{C}^\infty_{X,\mathbf{R}}$ on X, where $\mathcal{C}^\infty_{X,\mathbf{R}}(U)$ is the $\mathbf{R}$-algebra of $\mathcal{C}^\infty$ functions $U \to \mathbf{R}$, with the restriction maps being given by restriction of functions.

EXAMPLE 2.1.6. If X is a quasi-affine variety over an algebraically closed field k, then we have a sheaf $\mathcal{O}_X$ of k-algebras, such that $\mathcal{O}_X(U)$ is the k-algebra of regular functions $U \to k$, with the restriction maps given by restriction of functions. This is the *sheaf of regular functions* on X.

EXAMPLE 2.1.7. Given a continuous map $f \colon X \to Y$ of topological spaces, we have a sheaf of sets $\mathcal{F}$ on Y such that $\mathcal{F}(U)$ is the set of *sections of f over U*, that is, of continuous maps $s \colon U \to X$ such that $f\big(s(y)\big) = y$ for all $y \in U$; the restriction maps given by restriction of functions.

REMARK 2.1.8. If $\mathcal{C}$ is the category of R-modules, for a ring R, it is sometimes convenient to rewrite the sheaf condition for $\mathcal{F}$ as follows: given an open cover $U = \bigcup_i U_i$, we have an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \mathcal{F}(U_i \cap U_j),$$

where

$$\alpha(s) = (s|_{U_i})_i \quad \text{and} \quad \beta((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j} - s_j|_{U_i \cap U_j})_{i,j\in I}.$$

DEFINITION 2.1.9. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X of objects in $\mathcal{C}$, a morphism of presheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ is given by a functorial transformation between the two contravariant functors. Explicitly, for every open subset $U \subseteq X$, we have a morphism $\phi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ such that if $U \subseteq V$ are open subsets of X, then

$$\phi_U(s|_U) = \phi_V(s)|_U \quad \text{for every} \quad s \in \mathcal{F}(V).$$

The same definition applies for sheaves to give the notion of *morphism of sheaves.*

It is clear that morphisms of presheaves can be composed and in this way the presheaves on X of objects in $\mathcal{C}$ form a category. We also have the category of sheaves on X of objects in $\mathcal{C}$, that forms a full subcategory of the category of presheaves. In particular, we may consider isomorphisms of presheaves or of sheaves.

DEFINITION 2.1.10. Given a presheaf $\mathcal{F}$ on X (of objects in some category $\mathcal{C}$) and an open subset W of X, we obtain a presheaf $\mathcal{F}|_W$ on W such that for every open subset U of W, we take $\mathcal{F}|_W(U) = \mathcal{F}(U)$, with the restriction maps given by those for $\mathcal{F}$. This presheaf is the *restriction* of $\mathcal{F}$ to W. It is clear that if $\mathcal{F}$ is a sheaf, then $\mathcal{F}|_W$ is a sheaf. If $\phi \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, then we obtain in the obvious way an induced morphism $\phi|_W \colon \mathcal{F}|_W \to \mathcal{G}|_W$. We thus get a functor from the category of presheaves on X of objects in $\mathcal{C}$ to the category of presheaves on U of objects in $\mathcal{C}$ and a similar functor between the corresponding categories of sheaves.

From now on we assume, for simplicity, that the category $\mathcal{C}$ is either the category of R-modules or the category of R-algebras, where R is a commutative ring.

DEFINITION 2.1.11. If $\mathcal{F}$ is a presheaf on X (of R-modules or of R-algebras), then the *stalk* of $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the direct limit is over all open neighborhoods of x, ordered by reverse inclusion. Note that both categories we consider have direct limits. More generally, if W is an irreducible, closed subset of X, then the *stalk* of $\mathcal{F}$ at W is

$$\mathcal{F}_W := \varinjlim_{U \cap W \neq \emptyset} \mathcal{F}(U),$$

where the direct limit is over all open subsets U of X, with $U \cap W \neq \emptyset$.

EXAMPLE 2.1.12. If $\mathcal{O}_X$ is the sheaf of regular functions on a quasi-affine variety X and W is an irreducible closed subset of X, then the stalk of $\mathcal{O}_X$ at W is the local ring $\mathcal{O}_{X,W}$ of X at W. On general topological spaces, we typically only consider the stalks at the points of X, but in the case of algebraic varieties, it is sometimes natural to also consider the more general stalks.

REMARK 2.1.13. As in the case of a quasi-affine variety, we see that in general, the poset in the definition of $\mathcal{F}_W$ is filtering: given two open subsets U and V with $U \cap W \neq \emptyset$ and $V \cap W \neq \emptyset$, we have $(U \cap V) \cap W \neq \emptyset$, by the irreducibility of W. As a result, we may describe $\mathcal{F}_W$ as the set of all pairs (U, s), for some open subset U with $U \cap W \neq \emptyset$ and some $s \in \mathcal{F}(U)$, modulo the equivalence relation given by $(U, s) \sim (U', s')$ if there is an open subset $V \subseteq U \cap U'$, with $V \cap W \neq \emptyset$ and such that $s|_V = s'|_V$. If $s \in \mathcal{F}(U)$, for some open subset U with $U \cap W \neq \emptyset$, we write s_W for the image of s in $\mathcal{F}_W$.

REMARK 2.1.14. Note that if $\phi \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, then for every irreducible closed subset $W \subseteq X$, we have an induced morphism $\phi_W \colon \mathcal{F}_W \to \mathcal{G}_W$, that maps (U, s) to $\big(U, \phi(s)\big)$. We thus obtain a functor from the category of sheaves on X with values in $\mathcal{C}$ to $\mathcal{C}$.

REMARK 2.1.15. If $\mathcal{F}$ is a sheaf on X and $s, t \in \mathcal{F}(U)$ are such that $s_x = t_x$ for every $x \in U$, then $s = t$.

DEFINITION 2.1.16. Let $\mathcal{F}$ be a presheaf of R-modules or R-algebras on a topological space X. A *subpresheaf* of $\mathcal{F}$ is a presheaf $\mathcal{G}$ such that for every open subset U of X, $\mathcal{G}(U)$ is a submodule (respectively, an R-subalgebra) of $\mathcal{F}(U)$ and such that the restriction maps for $\mathcal{G}$ are induced by those for $\mathcal{F}$. In this case we write $\mathcal{F} \subseteq \mathcal{G}$. It is clear that in this case the inclusion maps define a morphism of presheaves $\mathcal{G} \to \mathcal{F}$. If both $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we say that $\mathcal{G}$ is a *subsheaf* of $\mathcal{F}$.

EXAMPLE 2.1.17. If X is a $\mathcal{C}^\infty$-manifold, then $\mathcal{C}^\infty_{X,\mathbf{R}}$ is a subsheaf of $\mathcal{C}_{X,\mathbf{R}}$.

DEFINITION 2.1.18. Let $\mathcal{C}$ be a category. If $f \colon X \to Y$ is a continuous map between two topological spaces and $\mathcal{F}$ is a presheaf on X of objects in $\mathcal{C}$, then we define the presheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}\big(f^{-1}(U)\big),$$

with the restriction maps being induced by those of $\mathcal{F}$. Moreover, if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, we clearly get a morphism $f_*\mathcal{F} \to f_*\mathcal{G}$ of presheaves on Y, so that we have the *push-forward* functor from the category of presheaves on X to the category of presheaves on Y. It is easy to see that if $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is a sheaf on Y.

EXAMPLE 2.1.19. If $f\colon X \to Y$ is a continuous map between topological spaces, then we have a morphism of sheaves

$$\mathcal{C}_{Y,\mathbf{R}} \to f_*\mathcal{C}_{X,\mathbf{R}}, \quad \mathcal{C}_{Y,\mathbf{R}}(U) \ni \phi \to \phi \circ f \in \mathcal{C}_{X,\mathbf{R}}\big(f^{-1}(U)\big).$$

The following exercises illustrate the advantages of working with sheaves, as opposed to presheaves.

EXERCISE 2.1.20. Show that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent:

i) The morphism ϕ is an isomorphism.
ii) There is an open cover $X = \bigcup_i U_i$ such that $\phi|_{U_i}$ is an isomorphism for all i.
iii) For every $x \in X$, the induced morphism ϕ_x is an isomorphism.

EXERCISE 2.1.21. Let $\mathcal{F}$ be a sheaf and $\mathcal{F}_1$ and $\mathcal{F}_2$ be subsheaves of $\mathcal{F}$.

i) Show that if there is an open cover $X = \bigcup_{i\in I} U_i$ such that $\mathcal{F}_1|_{U_i} \subseteq \mathcal{F}_2|_{U_i}$ for every i, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.
ii) Show that if $\mathcal{F}_{1,x} \subseteq \mathcal{F}_{2,x}$ for every $x \in X$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

EXERCISE 2.1.22. (Gluing morphisms of sheaves) Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X (of objects in some subcategory $\mathcal{C}$ that satisfies our usual requirements). If we have an open cover $X = \bigcup_{i\in I} U_i$ and for every $i \in I$ we have a morphism of sheaves $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ such that for every $i, j \in I$ we have $\phi_i|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j}$, then there is a unique morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$ for all $i \in I$.

EXERCISE 2.1.23. (Gluing sheaves). Let X be a topological space and suppose that $X = \bigcup_{i\in I} U_i$ is an open cover. Suppose that for every $i \in I$ we have a sheaf $\mathcal{F}_i$ on U_i (of objects in some subcategory $\mathcal{C}$ of the category of sets) and for every $i, j \in I$ we have isomorphisms

$$\phi_{j,i}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}, \quad \text{where} \quad U_{ij} = U_i \cap U_j$$

that satisfy the following compatibility conditions:

i) We have $\phi_{i,i} = \mathrm{Id}_{\mathcal{F}|_{U_i}}$ for every $i \in I$, and
ii) We have

$$\phi_{k,j}|_{U_{ijk}} \circ \phi_{j,i}|_{U_{ijk}} = \phi_{k,i}|_{U_{ijk}} \quad \text{for all} \quad i, j, k \in I,$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. In this case there is a sheaf $\mathcal{F}$ on X with isomorphisms $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ for all $i \in I$, such that

$$\phi_{j,i} \circ \phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \quad \text{for all} \quad i, j \in I. \tag{2.1.1}$$

Moreover, if $\mathcal{G}$ is another such sheaf, with isomorphisms $\psi_i\colon \mathcal{G} \to \mathcal{F}|_{U_i}$ for every $i \in I$ that satisfy the compatibility conditions (2.1.1), then there is a unique morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ such that $\psi_i \circ \alpha|_{U_i} = \phi_i$ for all $i \in I$.

2.2. Prevarieties

Let k be a fixed algebraically closed field. Given a topological space X and an open subset U of X, we consider the k-algebra $\mathcal{F}un_X(U)$ of functions $U \to k$, with point-wise operations. It is clear that this gives a sheaf $\mathcal{F}un_X$ of k-algebras on X, with the restriction maps being induced by restriction of functions. Note that if $f\colon X \to Y$ is a continuous map of topological spaces, then we have a canonical morphism of sheaves

$$\mathcal{F}un_Y \to f_*\mathcal{F}un_X, \quad \text{mapping} \quad \mathcal{F}un_Y(U) \ni \phi \to \phi \circ f \in \mathcal{F}un_X\big(f^{-1}(U)\big).$$

We begin by defining a category $\mathcal{T}op_k$ of topological spaces endowed with a sheaf of k-algebras, whose sections are functions on the given topological space. More precisely, the objects of this category are pairs $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ a sheaf of k-algebras on X which is a subsheaf of $\mathcal{F}un_X$. The sheaf $\mathcal{O}_X$ is the *structure sheaf*. A morphism in this category $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f\colon X \to Y$ such that the morphism of sheaves $\mathcal{F}un_Y \to f_*\mathcal{F}un_X$ induces a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$; in other words, for every open subset U of Y and every $\phi \in \mathcal{O}_Y(U)$, we have $\phi \circ f \in \mathcal{O}_X\big(\phi^{-1}(U)\big)$. It is clear that composition of continuous maps induces a composition of morphisms that makes $\mathcal{T}op_k$ a category.

EXAMPLE 2.2.1. Let $(X, \mathcal{O}_X)$ be an object in $\mathcal{T}op_k$. If U is an open subset of X, then we obtain another object $(U, \mathcal{O}_U)$ in $\mathcal{T}op_k$, where $\mathcal{O}_U = \mathcal{O}_X|_U$. Note that the inclusion map induces a morphism $(U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ in $\mathcal{T}op_k$.

REMARK 2.2.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two objects in $\mathcal{T}op_k$. If $X = \bigcup_{i\in I} U_i$ is an open cover and $\alpha_i\colon U_i \to X$ is the inclusion map, then a map $f\colon X \to Y$ is a morphism if and only if each $f \circ \alpha_i$ is a morphism. Indeed, this follows from the fact that continuity is a local property and the fact that $\mathcal{O}_X$ is a sheaf.

EXAMPLE 2.2.3. An isomorphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathcal{T}op_k$ is a homeomorphism $f\colon X \to Y$ such that for every open subset U of Y and every $\phi\colon U \to k$, we have $\phi \in \mathcal{O}_Y(U)$ if and only if $\phi \circ f \in \mathcal{O}_X\big(f^{-1}(U)\big)$.

EXAMPLE 2.2.4. If X is a locally closed subset of some $\mathbf{A}^n$, then $(X, \mathcal{O}_X)$ is an object in $\mathcal{T}op_k$. Note that if U is an open subset of X, then $\mathcal{O}_U = \mathcal{O}_X|_U$.

EXAMPLE 2.2.5. If X and Y are locally closed subsets of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, then a morphism $f\colon X \to Y$ as defined in Chapter 1 is the same as a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathcal{T}op_k$. Indeed, we know that if $f\colon X \to Y$ is a morphism, then f is continuous and for every open subset U in Y and every regular function $\phi\colon U \to k$, the composition $\phi \circ f$ is regular (see Propositions 1.4.13 and 1.4.14). Conversely, if $f\colon X \to Y$ gives a morphism in $\mathcal{T}op_k$ and if $p_i\colon Y \to k$ is induced by the i^{th} projection $\mathbf{A}^n \to k$, then it follows from definition that $p_i \circ f$ is a regular function on X for every i; therefore f is a morphism as defined in Chapter 1.

We enlarge one more time our notion of affine variety, as follows.

DEFINITION 2.2.6. We say that an object $(X, \mathcal{O}_X)$ in $\mathcal{T}op_k$ is an *affine variety* if it is isomorphic to $(V, \mathcal{O}_V)$ for some closed subset of an affine space $\mathbf{A}^n$. We say that $(X, \mathcal{O}_X)$ is a *quasi-affine* variety if it is isomorphic to $(V, \mathcal{O}_V)$ for some locally closed subspace of an affine space $\mathbf{A}^n$.

DEFINITION 2.2.7. An *algebraic prevariety* over k (or simply *prevariety*) is a pair $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ of subsheaf of k-algebras of $\mathcal{F}un_X$, such that there is a finite open covering $X = \bigcup_{i=1}^r U_i$, with each $(U_i, \mathcal{O}_{U_i})$ being an affine variety.

EXAMPLE 2.2.8. A quasi-affine variety $(V, \mathcal{O}_V)$ is a prevariety. Indeed, we may assume that V is a locally closed subset of some $\mathbf{A}^n$ and we know that there is a finite cover by open subsets $V = V_1, \ldots, V_r$ such that each $(V_i, \mathcal{O}_{V_i})$ is isomorphic to an affine variety (see Remark 1.4.20).

NOTATION 2.2.9. By an abuse of notation, we often denote a prevariety $(X, \mathcal{O}_X)$ simply by X.

DEFINITION 2.2.10. The category of algebraic prevarieties over k is a full subcategory of $\mathcal{T}op_k$. In other words, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are prevarieties, then a morphism of prevarieties $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ such that for every open subset U of Y and every $\phi \in \mathcal{O}_Y(U)$, we have $\phi \circ f \in \mathcal{O}_X\big(f^{-1}(U)\big)$.

REMARK 2.2.11. While strictly speaking we have enlarged our notion of quasi-affine varieties, in fact our old category of quasi-affine varieties and the new one are equivalent.

PROPOSITION 2.2.12. *Every prevariety X is a Noetherian topological space. In particular, it is quasi-compact.*

PROOF. By assumption, we have a finite open cover $X = U_1 \cup \ldots \cup U_r$, such that each U_i is Noetherian. Given a sequence

$$F_1 \supseteq F_2 \supseteq \ldots$$

of closed subsets of X, for every i, we can find n_i such that $F_n \cap U_i = F_{n+1} \cap U_i$ for all $n \geq n_i$. Therefore we have $F_n = F_{n+1}$ for every $n \geq \max_i n_i$, and we thus see that X is Noetherian. $\square$

REMARK 2.2.13. For every prevariety $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ is a subsheaf of $\mathcal{C}_X$, where $\mathcal{C}_X(U)$ is the k-algebra of continuous functions $U \to k$. Indeed, this assertion can be checked locally, and thus follows from the fact that it holds on affine varieties.

REMARK 2.2.14. For every prevariety X, the affine open subsets of X give a basis for the topology of X. Indeed, this follows from the definition of a prevariety and the fact that the assertions holds if X is affine.

REMARK 2.2.15. If $(X, \mathcal{O}_X)$ is a prevariety and $\phi \in \mathcal{O}_X(U)$, for some open subset U of X, then the set

$$V := \{x \in U \mid \phi(x) \neq 0\}$$

is an open subset of X and the function $\frac{1}{\phi}$ lies in $\mathcal{O}_X(V)$. Indeed, this follows from the fact that the assertion holds on affine varieties.

REMARK 2.2.16. If X is a prevariety and W is an irreducible closed subset of X, then we can define $\mathcal{O}_{X,W}$ as in Chapter 1. This is, in fact, equal to the stalk of $\mathcal{O}_X$ at W. If U is an affine open subset with $U \cap W \neq \emptyset$ and $\mathfrak{p} \subseteq \mathcal{O}_X(U)$ is the corresponding ideal, then we have canonical isomorphisms

$$\mathcal{O}_{X,W} \simeq \mathcal{O}_{U,U\cap W} \simeq \mathcal{O}_X(U)_{\mathfrak{p}}.$$

We know that the functor mapping X to $\mathcal{O}(X)$ gives an equivalence of categories between the category of affine varieties over k and the category of reduced, finite type k-algebras. The following exercise gives an explicit construction of the inverse functor. This point of view is useful in several instances, for example when discussing the Proj construction.

Exercise 2.2.17. Recall that if R is any commutative ring, then we have the maximal spectrum $\mathrm{MaxSpec}(R)$, a topological space with the underlying space consisting of all maximal ideals in R (see Exercise 1.1.16). Suppose now that R is an algebra of finite type over an algebraically closed field k. Recall that in this case, for every $\mathfrak{m} \in \mathrm{MaxSpec}(R)$, the canonical homomorphism $k \to R/\mathfrak{m}$ is an isomorphism. For every open subset U of $\mathrm{MaxSpec}(R)$, let $\mathcal{O}(U)$ be the set of functions $s\colon U \to k$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and $a, b \in R$ such that for every $\mathfrak{m} \in U_x$, we have

$$b \not\in \mathfrak{m} \quad \text{and} \quad s(\mathfrak{m}) = \overline{a} \cdot \overline{b}^{-1},$$

where we denote by $\overline{u} \in k \simeq R/\mathfrak{m}$ the class of $u \in R$.

1) Show that $\mathcal{O}$ is a sheaf such that the pair $\big(\mathrm{MaxSpec}(R), \mathcal{O}\big)$ defines an element in $\mathcal{T}op_k$ that, by an abuse of notation, we denote by $\mathrm{MaxSpec}(R)$, too.
2) Show that given a homomorphism of reduced, finite type k-algebras $R \to S$, we have an induced morphism $\mathrm{MaxSpec}(S) \to \mathrm{MaxSpec}(R)$ in $\mathcal{T}op_k$, so that we get a functor from the category of reduced, finite type k-algebras to $\mathcal{T}op_k$.
3) Show that for every R as above, $\mathrm{MaxSpec}(R)$ is an affine variety. Moreover, the functor MaxSpec is an inverse of the functor from the category of affine varieties to the category of reduced, finite type k-algebras, that maps X to $\mathcal{O}(X)$.

2.3. Open and closed immersions

Definition 2.3.1. Let $(X, \mathcal{O}_X)$ be an object in $\mathcal{T}op_k$. If Z is a locally closed subset of X, then we define a subsheaf $\mathcal{O}_Z$ of $\mathcal{C}_Z$, as follows. Given an open subset U of Z, a function $\phi\colon U \to k$ lies in $\mathcal{O}_Z(U)$ if for every $x \in U$, there is an open neighborhood V of x in X and $\psi \in \mathcal{O}_X(V)$ such that $\phi(u) = \psi(u)$ for $u \in V \cap X \subseteq U$. It is clear that restriction of functions makes $\mathcal{O}_Z$ a presheaf of k-algebras. Moreover, since the condition in the definition is local, $\mathcal{O}_Z$ is a sheaf, hence $(Z, \mathcal{O}_Z)$ is an object in $\mathcal{T}op_k$.

Remark 2.3.2. If X and Z are as in the above definition and Y is a locally closed subset of Z, then it follows from the definition that the sheaves on Y defined from $(X, \mathcal{O}_X)$ and from $(Z, \mathcal{O}_Z)$ are equal.

Example 2.3.3. If Z is open in X, then the sheaf $\mathcal{O}_Z$ defined above is just $\mathcal{O}_X|_Z$.

Example 2.3.4. If X is a locally closed subset in $\mathbf{A}^n$, then the sheaf $\mathcal{O}_X$ on X defined from $(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n})$ is the sheaf of regular functions on X. This is an immediate consequence of the definition of regular functions on locally closed subsets of $\mathbf{A}^n$.

Proposition 2.3.5. *For every prevariety $(X, \mathcal{O}_X)$ and every locally closed subset Z of X, the pair $(Z, \mathcal{O}_Z)$ is a prevariety.*

Proof. Note that by assumption, we have an open cover $X = V_1 \cup \ldots \cup V_r$ such that each $(V_i, \mathcal{O}_{V_i})$ is an affine variety. Since it is enough to show that each $(V_i \cap Z, \mathcal{O}_Z|_{V_i \cap Z})$ is a prevariety and $\mathcal{O}_Z|_{V_i \cap Z}$ is the sheaf defined on $Z \cap V_i$ as a locally closed subset of V_i (see Remark 2.3.2), it follows that we may and will assume that X is a closed subset of $\mathbf{A}^n$ and $\mathcal{O}_X$ is the sheaf of regular functions on X. In this case, it follows from Example 2.3.4 that Z is a quasi-affine variety, hence a prevariety by Example 2.2.8. $\square$

Definition 2.3.6. A locally closed *subvariety* of a prevariety $(X, \mathcal{O}_X)$ is a prevariety $(Z, \mathcal{O}_Z)$, where Z is a locally closed subset of X and $\mathcal{O}_Z$ is the sheaf defined in Definition 2.3.1. By the above proposition, this is indeed a prevariety. If Z is in fact open or closed in X, we say that we have an *open subvariety*, respectively, *closed subvariety* of X.

Definition 2.3.7. Note that if Z is a locally closed subvariety of X, then the inclusion map $i\colon Z \to X$ is a morphism of prevarieties. A morphism of prevarieties $f\colon X \to Y$ is a *locally closed* (*open*, *closed*) *immersion* (or *embedding*) if it factors as

$$X \xrightarrow{g} Z \xrightarrow{i} Y,$$

where g is an isomorphism and i is the inclusion of a locally closed (respectively, open, closed) subvariety.

Proposition 2.3.8. *If $f\colon X \to Y$ is a locally closed immersion, then for every map $g\colon W \to Y$, there is a morphism $h\colon W \to X$ such that $g = f \circ h$ if and only if $g(W) \subseteq f(X)$. Moreover, in this case h is unique.*

Proof. It is clear that if we have such h, then $g(W) = f\big(h(W)\big) \subseteq f(X)$, hence it is enough to prove the converse. Moreover, since we may replace X by an isomorphic variety, we may assume that f is the inclusion of a locally closed subvariety. Since f is injective, it is clear that if $g(W) \subseteq f(X)$, then there is a unique map $h\colon W \to X$ such that $f \circ h = g$. We need to prove that h is a morphism. Note first that since X is a subspace of Y, the map h is continuous. Furthermore, if $Y = V_1 \cup \ldots \cup V_r$ is an open cover such that each V_i is affine, in order to show that h is a morphism it is enough to show that each induced map $h^{-1}\big(f^{-1}(V_i)\big) \to f^{-1}(V)$ is a morphism (see Remark 2.2.2). Therefore we may assume that Y is an affine variety, in which case the assertion is clear. $\square$

Exercise 2.3.9. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of algebraic prevarieties.

i) Show that if both f and g are locally closed (respectively open, closed) immersions, then $g \circ f$ is a locally closed (respectively open, closed) immersion.
ii) Show that if g is a locally closed immersion and $g \circ f$ is a locally closed (respectively open, closed) immersion, then f is a locally closed (respectively open, closed) immersion.

Proposition 2.3.10. *If $f\colon X \to Y$ is a morphism of prevarieties, then the following are equivalent:*

i) *The morphism f is a closed immersion.*

ii) *For every affine open subset U of Y, its inverse image $f^{-1}(U)$ is affine, and the induced k-algebra homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X\big(f^{-1}(U)\big)$ is surjective.*

iii) *There is a finite cover $Y = U_1 \cup \ldots \cup U_r$ by affine open subsets such that for every i, the inverse image $f^{-1}(U_i)$ is affine, and the induced k-algebra homomorphism $\mathcal{O}_Y(U_i) \to \mathcal{O}_X\big(f^{-1}(U_i)\big)$ is surjective.*

PROOF. We first prove the implication i)⇒ii). Suppose that f factors as

$$X \xrightarrow{g} Z \xrightarrow{i} Y,$$

with g an isomorphism and i the inclusion map of a closed subvariety. If $U \subseteq Y$ is an affine open subset, then $U \cap Z$ is a closed subvariety of an affine variety, hence it is affine, and the restriction map induces a surjection $\mathcal{O}(U) \to \mathcal{O}(U \cap Z)$. Since the induced morphism $f^{-1}(U) \to U \cap Z$ is an isomorphism, we obtain the assertion in i).

Since the implication ii)⇒iii) is trivial, in order to complete the proof it is enough to show iii)⇒i). With the notation in iii), we see that each induced morphism $f^{-1}(U_i) \to U_i$ is a closed immersion. In particular, it is a homeomorphism onto its image, which is a closed subset of Y. This easily implies that f is a homeomorphism onto its image, which is a closed subset of Y. Let Z be the closed subvariety of Y with underlying set $f(X)$. We need to show that the inverse map $\phi\colon Z \to X$ is a morphism. Since $X = \bigcup_i f^{-1}(U_i)$, it follows from Remark 2.2.2 that it is enough to check that each $\phi^{-1}\big(f^{-1}(U_i)\big) \to f^{-1}(U_i)$ is a morphism. This is clear, since $f^{-1}(U_i) \to U_i$ is a closed immersion. □

REMARK 2.3.11. A morphism $f\colon X \to Y$ is a locally closed immersion if and only if there is an open subset U of Y such that f factors as $X \xrightarrow{g} U \xrightarrow{j} Y$, with g a closed immersion and j the inclusion morphism.

One way to construct algebraic prevarieties is by glueing. This is the content of the next exercise.

EXERCISE 2.3.12. Let $X_1, \ldots, X_r$ be prevarieties and for every i and j, suppose that we have open subvarieties $U_{i,j} \subseteq X_i$ and isomorphisms $\phi_{i,j}\colon U_{i,j} \to U_{j,i}$ such that

i) We have $U_{i,i} = X_i$ and $\phi_{i,i} = \mathrm{Id}_{X_i}$ for every i, and

ii) $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ on $U_{i,j} \cap \phi_{i,j}^{-1}(U_{j,k}) \subseteq U_{i,k}$.

In this case, there is a prevariety X and an open cover $X = U_1 \cup \ldots \cup U_r$ and isomorphisms $f_i\colon U_i \to X_i$ such that for every i and j, we have

$$U_i \cap U_j = f_i^{-1}(U_{i,j}) \quad \text{and} \quad \phi_{i,j} \circ f_i = f_j \quad \text{on} \quad U_i \cap U_j.$$

Moreover, if Y is another such prevariety with an open cover $Y = V_1 \cup \ldots \cup V_r$ and isomorphisms $g_i\colon V_i \to X_i$ that satisfy the same compatibility condition, then there is a unique isomorphism $h\colon X \to Y$ such that $h(U_i) = V_i$ and $g_i \circ h = f_i$ for $1 \leq i \leq r$.

EXAMPLE 2.3.13. Let X and Y be two copies of $\mathbf{A}^1$ and let $U \subseteq X$ and $V \subseteq Y$ be the complement of the origin. We can apply the previous exercise to construct a prevariety W_1 by glueing X and Y along the isomorphism $U \to V$ given by the identity. This prevariety is *the affine line with the origin doubled.* On the other hand, we can glue X and Y along the isomorphism $U \to V$ corresponding to the k-algebra isomorphism

$$k[x, x^{-1}] \to k[x, x^{-1}], \quad x \to x^{-1}.$$

As we will see in Chapter 4, the resulting prevariety is the projective line $\mathbf{P}^1$.

We end this section by extending to arbitrary prevarieties some properties that we proved for affine varieties. We then apply these properties to prove a sufficient criterion for a variety to be affine.

PROPOSITION 2.3.14. *For every prevarieties X and Y, with X affine, the map*

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(\mathcal{O}_X(X), \mathcal{O}_Y(Y)\big)$$

that maps f to the homomorphism taking ϕ to $\phi \circ f$ is a bijection.

PROOF. Recall that we know this result if Y is affine, too (see Theorem 1.4.16). We denote the map in the proposition by α_Y. We first show that α_Y is injective for all Y. Suppose that $f, g \colon Y \to X$ are morphisms such that $\alpha_Y(f) = \alpha_Y(g)$. Consider an affine open cover $Y = \bigcup_{i=1}^r U_i$. For every i, the composition

$$\mathcal{O}_X(X) \xrightarrow{\alpha_Y(f)} \mathcal{O}_Y(Y) \xrightarrow{\beta_i} \mathcal{O}_Y(U_i),$$

where β_i is given by restriction of functions, is equal to $\alpha_{U_i}(f|_{U_i})$. A similar assertion holds for g. Our assumption of f and g thus gives

$$\alpha_{U_i}(f|_{U_i}) = \alpha_{U_i}(g|_{U_i})$$

for all i, and since the U_i are affine, we conclude that $f|_{U_i} = g|_{U_i}$. This implies that $f = g$, completing the proof of injectivity.

We now prove the surjectivity of α_Y for every Y. Let $\phi \colon \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ be a k-algebra homomorphism. We consider again the affine open cover $Y = \bigcup_{i=1}^r U_i$ and consider $\phi_i = \beta_i \circ \phi$. Since each U_i is affine, there are morphisms $f_i \colon U_i \to X$ such that $\alpha_{U_i}(f_i) = \phi_i$ for all i.

Claim. For every i and j, we have $f_i|_{U_{i,j}} = f_j|_{U_{i,j}}$, where $U_{i,j} = U_i \cap U_j$. Indeed, $\alpha_{U_{i,j}}(f_i|_{U_{i,j}})$ is equal to the composition

$$\mathcal{O}_X(X) \xrightarrow{\phi} \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_Y(U_{i,j}),$$

where the second map is given by restriction of functions, and the same holds for $\alpha_{U_{i,j}}(f_j|_{U_{i,j}})$. Since we already know that $\alpha_{U_{i,j}}$ is injective, we obtain the assertion in the claim.

We deduce from the claim that we have a morphism $f \colon Y \to X$ such that $f|_{U_i} = f_i$ for all i. This implies that $\alpha_Y(f) = \phi$: indeed, since the morphism

$$\mathcal{O}_Y(Y) \to \prod_{i=1}^r \mathcal{O}_Y(U_i)$$

is injective, it is enough to note that

$$\beta_i \circ \phi = \phi_i = \alpha_Y(f_i) = \beta_i \circ \alpha_Y(f)$$

for all i. This completes the proof of the proposition. $\square$

PROPOSITION 2.3.15. *Let X be a prevariety and $f \in \Gamma(X, \mathcal{O}_X)$. If*

$$D_X(f) = \{x \in X \mid f(x) \neq 0\},$$

then the restriction map

$$\Gamma(X, \mathcal{O}_X) \to \Gamma\big(D_X(f), \mathcal{O}_X\big)$$

induces a k-algebra isomorphism

$$\Gamma(X, \mathcal{O}_X)_f \simeq \Gamma(D_X(f), \mathcal{O}_X).$$

PROOF. Since $f(x) \neq 0$ for every $x \in D_X(f)$, it follows that $f|_{D_X(f)}$ is invertible (see Remark 2.2.15). By the universal property of localization, we see that the restriction map induces a k-algebra homomorphism

$$\tau_{X,f} \colon \Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma\big(D_X(f), \mathcal{O}_X\big).$$

We will show that this is an isomorphism. Recall that we know this when X is affine (see Proposition 1.4.7).

Consider an affine open cover $X = U_1 \cup \ldots \cup U_r$. Since $\mathcal{O}_X$ is a sheaf, we have exact sequences of $\Gamma(X, \mathcal{O}_X)$-modules

$$0 \to \Gamma(X, \mathcal{O}_X) \to \bigoplus_i \Gamma(U_i, \mathcal{O}_X) \to \bigoplus_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X)$$

and

$$0 \to \Gamma\big(D_X(f), \mathcal{O}_X\big) \to \bigoplus_i \Gamma\big(U_i \cap D_X(f), \mathcal{O}_X\big) \to \bigoplus_{i,j} \Gamma\big(U_i \cap U_j \cap D_X(f), \mathcal{O}_X\big).$$

By localizing the first sequence at f, we obtain again an exact sequence, and we thus get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{O}_X)_f \arrow[r] \arrow[d, "\tau_{X,f}"] & \prod_i \Gamma(U_i, \mathcal{O}_X)_f \arrow[r] \arrow[d, "\gamma"] & \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X)_f \arrow[d, "\delta"] \\
0 \arrow[r] & \Gamma\big(D_X(f), \mathcal{O}_X\big) \arrow[r] & \prod_i \Gamma\big(D_X(f) \cap U_i, \mathcal{O}_X\big) \arrow[r] & \prod_{i,j} \Gamma\big(D_X(f) \cap U_i \cap U_j, \mathcal{O}_X\big)
\end{tikzcd}$$

with exact rows, where

$$\gamma = (\tau_{U_i, f|_{U_i}})_i \quad \text{and} \quad \delta = (\tau_{U_i \cap U_j}, f|_{U_i \cap U_j})_{i,j}.$$

Note that since each U_i is affine, we know that γ is an isomorphism. This implies that $\tau_{X,f}$ is injective. Since this holds for all (X, f), applying the assertion for $(U_i \cap U_j, f|_{U_i \cap U_j})$, we conclude that δ is injective. An easy diagram chase then implies that $\tau_{X,f}$ is surjective. This completes the proof of the proposition. $\square$

PROPOSITION 2.3.16. *Let X be a prevariety and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that the ideal they generate is $\Gamma(X, \mathcal{O}_X)$. If $D_X(f_i)$ is an affine variety for every i, then X is an affine variety.*

PROOF. We put $R = \Gamma(X, \mathcal{O}_X)$. This is clearly a reduced k-algebra. By assumption, we can write

$$\sum_{i=1}^{r} f_i g_i = 1 \quad \text{for some} \quad g_1, \ldots, g_r \in R.$$

We begin by showing that R is a finitely generated k-algebra. Since each $D_X(f_i)$ is affine, we know that $\Gamma\big(D_X(f_i),\mathcal{O}_X\big)$ is a finitely generated k-algebra. By Proposition 2.3.15, we have a canonical isomorphism

$$R_{f_i}\simeq\Gamma\big(D_X(f_i),\mathcal{O}_X\big),$$

hence each R_{f_i} is a finitely generated k-algebra. For each R_{f_i}, we choose finitely many generators of the form $\frac{a_{i,j}}{f_i^{m_{i,j}}}$, for suitable $a_{i,j}\in R$ and $m_{i,j}\in\mathbf{Z}_{\geq 0}$. Let $S\subseteq R$ be the k-algebra generated by the $a_{i,j}$, by the f_i, and by the g_i. It follows that S is a finitely generated k-algebra, with $f_1,\ldots,f_r\in S$, such that they generate the unit ideal in S. Moreover, we have $S_{f_i}=R_{f_i}$ for all i. This implies that if M is the S-module R/S, we have $M_{f_i}=0$ for all i, and therefore $M=0$ (see Proposition C.3.1). Therefore $R=S$, hence R is a finitely generated k-algebra.

Recall that we have the functor MaxSpec on the category of reduced, finitely generated k-algebras, with values in the category of affine varieties that is the inverse of the functor that maps Y to $\Gamma(Y,\mathcal{O}_Y)$ (for what follows, the choice of an inverse functor does not actually play a role). Since R is finitely generated, it follows from Proposition 2.3.14 that we have a canonical morphism $p_X\colon X\to\operatorname{MaxSpec}(R)$ such that the induced k-algebra homomorphism

$$R\simeq\Gamma\big(\operatorname{MaxSpec}(R),\mathcal{O}_{\operatorname{MaxSpec}(R)}\big)\to\Gamma(X,\mathcal{O}_X)$$

is the identity. We show that p_X is an isomorphism.

In fact, it is easy to see explicitly what the map p_X: for every $x\in X$, we have

$$p_X(x)=\{\phi\in R\mid\phi(x)=0\}.$$

This follows from the fact that the bijection in Proposition 2.3.14 is functorial, applied to the inclusion $\{x\}\hookrightarrow X$. The elements $f_i\in R$ define open subsets

$$U_i=\{\mathfrak{m}\in\operatorname{MaxSpec}(R)\mid f_i\not\in\mathfrak{m}\}$$

and since $f_1,\ldots,f_r$ generate the unit ideal in R, it follows that $\operatorname{MaxSpec}(R)=\bigcup_i U_i$. On the other hand, it follows from the description of p_X that $p_X^{-1}(U_i)=D_X(f_i)$ and via the isomorphism $R_{f_i}\simeq\Gamma\big(D_X(f_i),\mathcal{O}_X\big)$ provided by Proposition 2.3.15, the induced map $p_X^{-1}(U_i)\to U_i$ gets identified to

$$p_{D_X(f_i)}\colon D_X(f_i)\to\operatorname{MaxSpec}\big(\Gamma(D_X(f_i),\mathcal{O}_{D_X(f_i)})\big),$$

which is an isomorphism since $D_X(f_i)$ is affine. Since each induced morphism $p_X^{-1}(U_i)\to U_i$ is an isomorphism, it follows that p_X is an isomorphism, hence X is affine. $\square$

2.4. Products of prevarieties

We now show that the category of prevarieties has fibered products. We begin with the case of direct products.

Proposition 2.4.1. *The category of prevarieties over k has direct products.*

Proof. We show that given two prevarieties X and Y, there is a topology on the set $X\times Y$ and a subsheaf of k-algebras $\mathcal{O}_{X\times Y}\subseteq\mathcal{F}un_{X\times Y}$ that make $X\times Y$, together with the two projections, the direct product in the category of prevarieties.

Let us consider open covers $X = U_1 \cup \ldots \cup U_r$ and $Y = V_1 \cup \ldots \cup V_s$, with all U_i and V_j affine varieties. We can thus write

$$X \times Y = \bigcup_{i,j} U_i \times V_j.$$

Note that each $U_i \times V_j$ has the structure of an affine variety; in particular, it is a topological space, with a topology that is finer than the product topology (see Corollary 1.6.2). Note that for every two pairs (i_1, j_1) and (i_2, j_2), we have a priori two structures of algebraic prevariety on

$$(U_{i_1} \times V_{j_1}) \cap (U_{i_2} \times V_{j_2}), \tag{2.4.1}$$

one coming from $U_{i_1} \times V_{j_1}$ and the other one from $U_{i_2} \times V_{j_2}$. However, they are the same, both being equal to the structure of prevariety on the quasi-affine variety $(U_{i_1} \cap U_{i_2}) \times (V_{j_1} \cap V_{j_2})$. This follows from the fact that if A and B are affine (or, more generally, quasi-affine) varieties and if $U_A \subseteq A$ and $U_B \subseteq B$ are open subsets, then the open subvariety $U_A \times U_B$ of $A \times B$ is the product of U_A and U_B in the category of quasi-affine varieties, which characterizes it uniquely, up to a canonical isomorphism.

It is then easy to see that if we declare that a subset W of $X \times Y$ is open if and only if $W \cap (U_i \times V_j)$ is open for all i and j, then this gives a topology on $X \times Y$ such that the topology on each $U_i \times V_j$ is the subspace topology. Note that the topology on $X \times Y$ is finer than the product topology. Moreover, if given an open subset $W \subseteq X \times Y$ and a function $\phi\colon W \to k$, we put $\phi \in \mathcal{O}_{X \times Y}(W)$ when

$$\phi|_{W \cap (U_i \times V_j)} \in \mathcal{O}_{U_i \times V_j}\big(W \cap (U_i \times V_j)\big) \quad \text{for all} \quad i, j,$$

then $\mathcal{O}_{X \times Y}$ is a subsheaf of $\mathcal{F}un_{X \times Y}$ such that $\mathcal{O}_{X \times Y}|_{U_i \times V_j} = \mathcal{O}_{U_i \times V_j}$ for all i and j.

We now show that with this structure, the two projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ make $X \times Y$ the direct product of X and Y in the category of prevarieties. Note first that since $X \times Y$ is covered by the affine open subsets $U_i \times V_j$, it follows that $X \times Y$ is a prevariety. Second, both projections p and q are morphisms: for example, for p this follows from the fact that each projection $U_i \times V_j \to U_i$ is a morphism (see Remark 2.2.2). Given a prevariety Z and morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, there is a unique map $h\colon Z \to X \times Y$ such that $p \circ h = f$ and $q \circ h = g$, namely $h(z) = \big(f(z), g(z)\big)$ for every $z \in Z$. In order to check that this is a morphism, note first that for every i and j, the subset

$$h^{-1}(U_i \times V_j) = f^{-1}(U_i) \cap g^{-1}(V_j)$$

is open in Z. Moreover, the restriction of h to this subset is a morphism: by Remark 2.2.2, in order to check this, it is enough to show that the restriction of h to the subsets in an affine open cover of $h^{-1}(U_i \times V_j)$ is a morphism; this follows from the fact that $U_i \times V_j$ is the direct product of U_i and V_j in the category of affine varieties. This completes the proof of the proposition. $\square$

Remark 2.4.2. It follows from the proof of the proposition that the product of two prevarieties X and Y has as underlying set the Cartesian product $X \times Y$ and the topology is finer than the product topology.

Exercise 2.4.3. Show that if $f\colon Z \to X$ and $g\colon W \to Y$ are locally closed (open, closed) immersions, then we have an induced locally closed (respectively, open, closed) immersion

$$Z \times W \to X \times Y, \quad (z,w) \to \big(f(z), g(w)\big).$$

REMARK 2.4.4. If X and Y are irreducible prevarieties, then $X \times Y$ is irreducible.

PROOF. Consider affine open covers

$$X = U_1 \cup \ldots \cup U_r \quad \text{and} \quad Y = V_1 \cup \ldots \cup V_s.$$

Since each $U_i \times V_j$ is irreducible by Corollary 1.6.7, it is enough to note that each intersection

$$(U_i \times V_j) \cap (U_{i'} \cap V_{j'}) = (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

is nonempty (see Exercise 1.3.17). $\square$

DEFINITION 2.4.5. Given a morphism of prevarieties $f\colon X \to Y$, the *graph morphism* of f is the morphism $j_f\colon X \to X \times Y$ given by $j_f(x) = \big(x, f(x)\big)$. Note that this is indeed a morphism by the universal property of the product. The *graph* of f is the image Γ_f of j_f. When $f = \mathrm{id}_X$, the graph of f is the *diagonal* Δ_X of $X \times X$.

PROPOSITION 2.4.6. *For every morphism $f\colon X \to Y$, the graph morphism $j_f\colon X \to X \times Y$ is a locally closed embedding.*

PROOF. For every $x \in X$, let $V_x \subseteq Y$ be an affine open neighborhood of $f(x)$ and $U_x \subseteq f^{-1}(V_x)$ an affine open neighborhood of x. If $U = \bigcup_{x \in X} U_x \times V_x$, then it is clear that the image of j_f is contained in U. Therefore it is enough to show that the induced morphism $j'_f\colon X \to U$ is a closed immersion. We also note that since U is quasi-compact, the union in the definition of U can be taken over a finite subset of X. Since $(j'_f)^{-1}(U_x \times V_x) = U_x$ is affine, in order to complete the proof of the proposition, it is enough to show that when X and Y are affine, the morphism $j_f^{\#}\colon \mathcal{O}(X \times Y) \to \mathcal{O}(X)$ is surjective. We may assume that X is a closed subset of $\mathbf{A}^m$ and Y is a closed subset of $\mathbf{A}^n$. We denote by $x_1, \ldots, x_m$ the coordinates on $\mathbf{A}^m$ and by $y_1, \ldots, y_n$ the coordinates on $\mathbf{A}^n$. Let us write $f = (f_1, \ldots, f_n)$, with $f_i \in \mathcal{O}(X)$ for $1 \leq i \leq n$. In this case, $j_f^{\#}$ is given by

$$j_f^{\#}(x_i) = x_i \quad \text{and} \quad j_f^{\#}(y_j) = f_j \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

and it is clear that this is surjective. $\square$

We now prove the existence of fibered products in the category of prevarieties.

PROPOSITION 2.4.7. *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be morphisms of prevarieties. If*

$$W = \{(x,y) \in X \times Y \mid f(x) = g(y)\},$$

then W is a locally closed subset of $X \times Y$ and $(W, \mathcal{O}_W)$, with the restrictions of the two projections is the fiber product $X \times_Z Y$ in the category of prevarieties.

PROOF. Consider the morphism $h\colon X \times Y \to Z \times Z$ given by $h(x,y) = \big(f(x), g(y)\big)$. It follows from Proposition 2.4.6 that the diagonal $\Delta_Z \subseteq Z \times Z$ is locally closed in $Z \times Z$, hence $W = h^{-1}(\Delta_Z)$ is locally closed in $X \times Y$. We now consider on W the structure of locally closed subvariety of $X \times Y$. Let $p\colon W \to X$ and $q\colon W \to Y$ be the restrictions of the two projections to W. We need to show that given a prevariety T and morphisms $\alpha\colon T \to X$ and $\beta\colon T \to Y$ such that

$f \circ \alpha = g \circ \beta$, there is a unique morphism $\gamma \colon T \to W$ such that $p \circ \gamma = \alpha$ and $q \circ \gamma = \beta$. Uniqueness of γ as a map is clear: in fact, we need to have $\gamma(t) = \big(\alpha(t), \beta(t)\big)$ for all $t \in T$. In order to check that this is a morphism, note that the composition $T \to W \hookrightarrow X \times Y$ is a morphism since $X \times Y$ is the direct product of X and Y, and thus γ is a morphism by Proposition 2.3.8. ☐

Example 2.4.8. If $f \colon X \to Y$ is a morphism of prevarieties and Z is a locally closed subset of Y, then we have a Cartesian diagram[1]

$$\begin{tikzcd} f^{-1}(W) \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ W \arrow[r, "i"] & Y, \end{tikzcd}$$

in which i and j are the inclusion morphisms. Indeed, the assertion is an immediate application of Proposition 2.3.8.

Remark 2.4.9. Given a Cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y, \end{tikzcd}$$

with X, Y, and Z are affine varieties, it follows that $X \times_Y Z$ is affine too: this follows from the fact that it is a closed subvariety of $X \times Y$. Moreover, the canonical homomorphism

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \to \mathcal{O}(X \times_Y Z)$$

is surjective, with the kernel being the nil-radical of $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z)$. This follows from the anti-equivalence of categories between affine varieties over k and reduced, finitely generated k-algebras, by noting that the tensor product gives the push-out in the category of k-algebras, hence the reduced tensor product gives the push-out in the category of reduced k-algebras.

2.5. Algebraic varieties

Algebraic varieties are prevarieties that satisfy an analogue of the Hausdorff condition. Note that the Zariski topology is almost never Hausdorff: if X is an irreducible prevariety, then any two nonempty open subsets intersect nontrivially. The right condition is suggested by the following observation: if X is an arbitrary topological space and if we consider on $X \times X$ the product topology, then X is Hausdorff if and only if the diagonal Δ_X is closed in $X \times X$.

Definition 2.5.1. An algebraic prevariety X is *separated* if the diagonal Δ_X is a closed subset of the prevariety $X \times X$. An *algebraic variety* over k (or simply, a *variety*) is a separated algebraic prevariety.

Remark 2.5.2. It follows from Proposition 2.3.8 that the diagonal map $X \to X \times X$, given by $x \to (x, x)$ is always a locally closed immersion for every prevariety X. Hence X is separated if and only if this map is a closed immersion.

---

[1]Recall that this means that it is a commutative diagram such that the induced morphism $f^{-1}(W) \to X \times_Y W$ given by the universal property of the fiber product is an isomorphism.

Remark 2.5.3. If $f, g\colon X \to Y$ are morphisms of prevarieties and Y is separated, then the set

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in X. Indeed, this is just the inverse image of the diagonal $\Delta_Y \subseteq Y \times Y$ by the morphism $X \to Y \times Y$, that maps x to $\big(f(x), g(x)\big)$. Because of this, the considerations in Section 1.5 about the domain of rational maps extend to the case of arbitrary algebraic varieties.

Proposition 2.5.4. *The following hold:*

i) *If Z is a subvariety of the prevariety X and X is separated, then Z is separated. In particular, quasi-affine varieties are separated.*
ii) *If $f\colon X \to Y$ is a morphism of prevarieties and Y is separated, then the graph morphism $j_f\colon X \to X \times Y$, given by $j_f(x) = \big(x, f(x)\big)$ is a closed immersion.*
iii) *If X and Y are algebraic varieties, so is $X \times Y$. More generally, if $f\colon X \to Z$ and $g\colon Y \to Z$ are morphisms of varieties, then $X \times_Z Y$ is a closed subvariety of $X \times Y$, and therefore it is a variety.*
iv) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of algebraic varieties, then the morphism $\alpha\colon X \to X \times_Z Y$, given by $\alpha(x) = \big(x, f(x)\big)$, is a closed immersion.*

Proof. If Z is a locally closed subvariety of X, then $Z \times Z$ is a locally closed subvariety of $X \times X$ and $\Delta_Z = (Z \times Z) \cap \Delta_X$. It follows that if Δ_X is closed in $X \times X$, then Δ_Z is closed in $Z \times Z$. Note now that if $X = \mathbf{A}^n$, with coordinates $x_1, \ldots, x_n$, then Δ_X is the closed subset of $\mathbf{A}^n \times \mathbf{A}^n$ defined by $x_1 - y_1, \ldots, x_n - y_n$. We thus conclude that every quasi-affine variety is separated.

Under the assumptions in ii), we know that j_f is a locally closed embedding by Proposition 2.4.6. Its image is the inverse image of Δ_Y by the morphism $h\colon X \times Y \to Y \times Y$ given by $h(x, y) = \big(f(x), y\big)$, hence it is closed in $X \times Y$. Therefore i_f is a closed immersion.

Let us prove iii). Suppose that X and Y are varieties. If

$$p_{1,3}\colon (X \times Y) \times (X \times Y) \to X \times X \quad \text{and} \quad p_{2,4}\colon (X \times Y) \times (X \times Y) \to Y \times Y$$

are the projections given by

$$p_{1,3}(x_1, y_1, x_2, y_2) = (x_1, x_2) \quad \text{and} \quad p_{2,4}(x_1, y_1, x_2, y_2) = (y_1, y_2),$$

then $\Delta_{X \times Y} = p_{1,3}^{-1}(\Delta_X) \cap p_{2,4}^{-1}(\Delta_Y)$ and it is thus a closed subset of $(X \times Y) \times (X \times Y)$. This shows that $X \times Y$ is a variety. Moreover, it follows from Proposition 2.4.7 that the fiber product $X \times_Z Y$ is a locally closed subvariety of $X \times Y$, hence it is a variety by i). In fact it is a closed subvariety, since its underlying subset is the inverse image of Δ_Z via the morphism

$$X \times Y \to Z \times Z, \quad (x, y) \to \big(f(x), g(y)\big).$$

Finally, the assertion in iv) follows from the fact that $X \times_Z Y$ is a closed subvariety of $X \times Y$ and the composition of α with the inclusion map $X \times_Z Y \hookrightarrow X \times Y$ is the graph morphism j_f, which is a closed immersion (see assertion ii) in Exercise 2.3.9). $\square$

The following property is sometimes useful:

PROPOSITION 2.5.5. *If X is an algebraic variety and U, V are affine open subvarieties of X, then $U \cap V$ is affine, too.*

PROOF. Consider the closed immersion $i\colon X \to X \times X$ given by the diagonal map. If U and V are affine variety, then $U \times V$ is affine. Since $U \cap V = i^{-1}(U \times V)$, we see that $U \cap V$ is affine by Proposition 2.3.10. □

PROPOSITION 2.5.6. *Let X be a prevariety and suppose that we have an open cover $X = U_1 \cup \ldots \cup U_r$ by affine open subsets. Then X is separated if and only if for every i and j, the intersection $U_i \cap U_j$ is an affine variety and the homomorphism $\mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j)$ induced by the restriction homomorphisms is surjective.*

PROOF. We know that X is separated if and only if the diagonal morphism $i\colon X \to X \times X$ is a closed immersion. The assertion in the proposition is now an immediate consequence of the description of closed immersions in Proposition 2.3.10, using the fact that the canonical homomorphism

$$\mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \times_k U_j)$$

is an isomorphism (see Section 1.6). □

EXAMPLE 2.5.7. Let us consider the two examples in Example 2.3.13. If X is obtained by glueing two copies of $\mathbf{A}^1$ along the identity automorphism of $\mathbf{A}^1 \smallsetminus \{0\}$, then X is covered by two affine open subsets U and V such that $U \simeq \mathbf{A}^1 \simeq V$, $U \cap V \simeq \mathbf{A}^1 \smallsetminus \{0\}$, and the morphism

$$k[x,y] = \mathcal{O}(U \times V) \to \mathcal{O}(U \cap V) = k[t,t^{-1}]$$

maps both x and y to t. This is clearly not surjective, hence X is not separated. On the other hand, if Y is obtained by glueing two copies of $\mathbf{A}^1$ along the automorphism of $\mathbf{A}^1 \smallsetminus \{0\}$ given by $t \to t^{-1}$, then Y is also covered by two affine open subsets U and V such that $U \simeq \mathbf{A}^1 \simeq V$, $U \cap V \simeq \mathbf{A}^1 \smallsetminus \{0\}$, but now the morphism

$$k[x,y] = \mathcal{O}(U \times V) \to \mathcal{O}(U \cap V) = k[t,t^{-1}]$$

maps x to t and y to t^{-1}. This is surjective, hence Y is separated.

EXERCISE 2.5.8.
i) Show that if $X_1, \ldots, X_n$ are algebraic varieties, then on the disjoint union $X = \bigsqcup_{i=1}^n X_i$ there is a unique structure of algebraic variety such that each inclusion map $X_i \hookrightarrow X$ is an open immersion .
ii) Show that every variety X is a disjoint union of connected open subvarieties; each of these is a union of irreducible components of X.
iii) Show that if X is an affine variety and $R = \mathcal{O}(X)$, then X is disconnected if and only if there is an isomorphism $R \simeq R_1 \times R_2$ for suitable nonzero k-algebras R_1 and R_2.

EXERCISE 2.5.9. Let $f\colon X \dashrightarrow Y$ be a rational map between the irreducible varieties X and Y. The *graph* Γ_f of f is defined as follows. If U is an open subset of X such that f is defined on U, then the graph of $f|_U$ is well-defined, and it is a closed subset of $U \times Y$. By definition, Γ_f is the closure of the graph of $f|_U$ in $X \times Y$.

i) Show that the definition is independent of the choice of U.
ii) Let $p\colon \Gamma_f \to X$ and $q\colon \Gamma_f \to Y$ be the morphisms induced by the two projections. Show that p is a birational morphism, and that q is birational if and only if f is.

iii) Show that if the fiber $p^{-1}(x)$ does not consist of only one point, then f is not defined at $x \in X$.

A useful criterion for proving that a prevariety is separated involves the notion of *separated morphisms*. We will not need this notion until Chapter 8, so the reader can skip this for now, and return to it later.

Definition 2.5.10. Given a morphism of prevarieties $f\colon X \to Y$, note that the diagonal $\Delta_X \subseteq X \times X$ lies inside the locally closed subset $X \times_Y X$. The morphism f is *separated* if Δ_X is closed in $X \times_Y X$.

Example 2.5.11. It is clear that the morphism from X to a point is separated if and only if X is separated.

Example 2.5.12. If $f\colon X \to Y$ is a morphism of prevarieties and X is separated, then f is separated. Indeed, Δ_X is closed in $X \times X$, and thus it is closed in $X \times_Y X$, too.

Example 2.5.13. Let $f\colon X \to Y$ be a morphism of prevarieties and consider an open cover $Y = \bigcup_{i\in I} V_i$. If all $f^{-1}(V_i)$ are separated, then f is a separated morphism. More generally, if each restriction $f^{-1}(V_i) \to V_i$ is separated, then f is separated. Indeed, we can write

$$X \times_Y X = \bigcup_{i\in I} \big(f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)\big)$$

and

$$\Delta_X \cap \big(f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)\big) = \Delta_{f^{-1}(V_i)},$$

which implies our assertion.

The following proposition provides an useful criterion for showing that certain prevarieties are separated.

Proposition 2.5.14. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are separated morphisms of prevarieties, then $g \circ f$ is separated. In particular, if f is a separated morphism and Y is separated, then X is separated, too.*

Proof. We have the following inclusions:

$$\Delta_X \subseteq X \times_Y X \subseteq X \times_Z X.$$

Note that Δ_X is closed in $X \times_Y X$, since f is separated. If $\phi\colon X \times_Z X \to Y \times_Z Y$ is given by $\phi(x_1, x_2) = \big(f(x_1), f(x_2)\big)$, then $X \times_Y X = \phi^{-1}(\Delta_Y)$, hence using the fact that g is separated, we deduce that $X \times_Y X$ is closed in $X \times_Z X$. We thus conclude that Δ_X is closed in $X \times_Z X$. $\square$

CHAPTER 3

Dimension theory

In this chapter we prove the main results concerning the dimension of algebraic varieties. We begin with some general considerations about Krull dimension in topological spaces. We then discuss finite morphisms between affine varieties and show that they are closed maps and preserve the dimension of closed subsets. We then give a proof of the Principal Ideal theorem that relies on Noether normalization and use this to deduce the main properties of dimension for algebraic varieties. The last two sections are devoted to the behavior of the dimension of the fibers of morphisms and to the Chevalley constructibility theorem.

3.1. The dimension of a topological space

Definition 3.1.1. Let X be a nonempty topological space. The *dimension* (also called *Krull dimension*) of X, denoted $\dim(X)$, is the supremum over the non-negative integers r such that there is a sequence

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r,$$

with all Z_i closed, irreducible subsets of X. We make the convention that if X is empty, then $\dim(X) = -1$.

In particular, we may consider the dimension of quasi-affine varieties, endowed with the Zariski topology. Note that in general we could have $\dim(X) = \infty$, even when X is Noetherian, but this will not happen in our setting.

Definition 3.1.2. Let $R \neq 0$ be a commutative ring. The *dimension* (also called *Krull dimension*) of R, denoted $\dim(R)$, is the supremum over the non-negative integers r such that there is a sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r,$$

with all $\mathfrak{p}_i$ prime ideals in R. We make the convention that $\dim(R) = -1$ when $R = 0$.

Remark 3.1.3. It follows from Corollary 1.1.10 and Proposition 1.3.8 that if X is an affine variety, we have $\dim(X) = \dim\big(\mathcal{O}(X)\big)$. More generally, for every commutative ring R one can interpret the dimension of R as the dimension of a topological space, as shown in the following exercise.

Exercise 3.1.4. Let R be a commutative ring and consider the *spectrum* of R:

$$\operatorname{Spec}(R) := \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal in } R\}.$$

For every ideal J in R, consider

$$V(J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid J \subseteq \mathfrak{p}\}.$$

Show that the following hold:

i) For every ideals J_1, J_2 in R, we have
$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2) = V(J_1 \cdot J_2).$$
ii) For every family $(J_\alpha)_\alpha$ of ideals in R, we have
$$\bigcap_\alpha V(J_\alpha) = V\left(\sum_\alpha J_\alpha\right).$$
iii) We have
$$V(0) = \mathrm{Spec}(R) \quad \text{and} \quad V(R) = \emptyset.$$
iv) Deduce that $\mathrm{Spec}(R)$ has a topology (the *Zariski topology*) whose closed subsets are the $V(J)$, with J an ideal in R.
v) Show that $V(J) \subseteq V(J')$ if and only if $\mathrm{rad}(J') \subseteq \mathrm{rad}(J)$. In particular, $V(J') = V(J)$ if and only if $\mathrm{rad}(J') = \mathrm{rad}(J)$.
vi) Show that the closed irreducible subsets in $\mathrm{Spec}(R)$ are those of the form $V(P)$, where P is a prime ideal in R. Deduce that
$$\dim(R) = \dim\big(\mathrm{Spec}(R)\big).$$

The following easy two lemmas show that the notion of dimension behaves as expected when it comes to some basic operations.

Lemma 3.1.5. *If Y is a subspace of X, then*
$$\dim(Y) \leq \dim(X).$$

Proof. Given a sequence of irreducible closed subsets in Y
$$Z_0 \supsetneq Z_2 \supsetneq \ldots \supsetneq Z_r,$$
by taking closures we obtain a sequence of closed subsets in X
$$\overline{Z_0} \supsetneq \overline{Z_2} \supsetneq \ldots \supsetneq \overline{Z_r}$$
(the fact that the inclusions are strict follows from $Z_i = \overline{Z_i} \cap Y$ for all i). This gives the inequality in the lemma. □

Lemma 3.1.6. *If X is a topological space, $Y_1, \ldots, Y_r$ are closed subsets of X, and $Y = Y_1 \cup \ldots \cup Y_r$, then*
$$\dim(Y) = \max_{i=1}^{r} \dim(Y_i).$$
This applies, in particular, if X is Noetherian, and $Y_1, \ldots, Y_r$ are the irreducible components of Y.

Proof. After replacing X by Y, we may assume that $X = Y$. The inequality "$\geq$" follows from Lemma 3.1.5. The opposite inequality follows from the fact that given any sequence
$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$
of irreducible, closed subsets of X, there is i such that $Z_0 \subseteq Y_i$, in which case $\dim(Y_i) \geq r$. □

The next lemma will allow us to reduce understanding the dimension of quasi-affine varieties to the case of affine varieties.

Lemma 3.1.7. *If X is a topological space and $X = U_1 \cup \ldots \cup U_r$, with U_i open subsets of X, then*
$$\dim(X) = \max_{i=1}^{r} \dim(U_i).$$

PROOF. Again, the inequality "$\geq$" follows from Lemma 3.1.5. In order to prove the opposite inequality, consider a sequence

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$

of irreducible, closed subsets of X. Let i be such that $Z_r \cap U_i \neq \emptyset$. Since each $Z_j \cap U_i$ is irreducible and dense in Z_j (see Remarks 1.3.7), we obtain the following sequence of irreducible closed subsets of U_i:

$$Z_0 \cap U_i \supsetneq Z_1 \cap U_i \ldots \supsetneq Z_r \cap U_i,$$

hence $\dim(U_i) \geq r$. This completes the proof of the lemma. $\square$

DEFINITION 3.1.8. If X is a topological space and Y is a closed, irreducible subset of X, then the *codimension* of Y in X, denoted $\mathrm{codim}_X(Y)$, is the supremum over the non-negative integers r for which there is a sequence

$$Z_0 \supsetneq Z_1 \supsetneq \ldots \supsetneq Z_r = Y,$$

with all Z_i closed and irreducible in X.

DEFINITION 3.1.9. Given a prime $\mathfrak{p}$ in a commutative ring R, the *codimension* (also called *height*) of $\mathfrak{p}$, denoted $\mathrm{codim}(\mathfrak{p})$, is the supremum over the non-negative integers r such that there is a sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r = \mathfrak{p},$$

with all $\mathfrak{p}_i$ prime ideals in R.

REMARK 3.1.10. It follows from Exercise 1.4.22 that if X is an affine variety and Y is an irreducible closed subset, defined by the prime ideal $\mathfrak{p} \subset \mathcal{O}(X)$, we have

$$\mathrm{codim}(\mathfrak{p}) = \mathrm{codim}_X(Y).$$

Note also that if $\mathfrak{q}$ is a prime ideal in the commutative ring R and $Z = V(\mathfrak{q}) \subseteq W = \mathrm{Spec}(R)$ is the corresponding irreducible closed subset, then

$$\mathrm{codim}(\mathfrak{q}) = \mathrm{codim}_W(Z).$$

REMARK 3.1.11. Using arguments similar to the ones in the proofs of Lemma 3.1.5 and Proposition 3.1.7, we see that if Y is an irreducible closed subset of a topological space X and U is an open subset of X such that $U \cap Y \neq \emptyset$, then

$$\mathrm{codim}_U(U \cap Y) = \mathrm{codim}_X(Y).$$

REMARK 3.1.12. If X is a Noetherian topological space, with irreducible components $X_1, \ldots, X_r$, and Y is an irreducible, closed subset of X, then

$$\mathrm{codim}_X(Y) = \max\{\mathrm{codim}_{X_i}(Y) \mid Y \subseteq X_i\}.$$

Indeed, given any chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r \subseteq X$$

of irreducible, closed subsets of X, by irreducibility of Y_r, there is i such that $Y_r \subseteq X_i$. This gives the inequality "$\leq$" and the opposite inequality is obvious.

It is sometimes convenient to have a notion of codimension for arbitrary closed subsets. We thus make the following

DEFINITION 3.1.13. If Y is a non-empty closed subset of a Noetherian topological space X and $Y_1, \ldots, Y_r$ are the irreducible components of Y, we put

$$\mathrm{codim}_X(Y) := \min\{\mathrm{codim}_X(Y_i) \mid 1 \leq i \leq r\}.$$

3.2. Properties of finite morphisms

In order to prove the basic results concerning the dimension of affine algebraic varieties, we will make use of Noether's Normalization lemma. In order to exploit this, we will need some basic properties of finite morphisms. In this chapter we only discuss such morphisms between affine varieties; we will consider the general notion in Chapter 5.

DEFINITION 3.2.1. A morphism of affine varieties $f\colon X\to Y$ is *finite* if the corresponding ring homomorphism $f^{\#}\colon \mathcal{O}(Y)\to\mathcal{O}(X)$ is finite.

EXAMPLE 3.2.2. Let Y be an affine variety and $a_1,\ldots,a_n\in\mathcal{O}(Y)$. If

$$X=\{(u,t)\in Y\times\mathbf{A}^1\mid t^n+a_1(u)t^{n-1}+\ldots+a_n(u)=0\},$$

then X is a closed subset of $Y\times\mathbf{A}^1$, and the composition

$$X\overset{i}{\hookrightarrow}Y\times\mathbf{A}^1\overset{p}{\longrightarrow}Y,$$

where i is the inclusion and p is the projection onto the first component, is finite. In fact, $\mathcal{O}(X)$ is free over $\mathcal{O}(Y)$, with a basis given by the classes of $1,t,\ldots,t^{n-1}$.

EXAMPLE 3.2.3. Given an irreducible closed subset $X\subseteq\mathbf{A}^N$, with $\mathrm{trdeg}\big(k(X)/k\big)=n$, it follows from Theorem 1.2.2 (and its proof) that after a linear change of coordinates $y_i=\sum_{j=1}^n a_{i,j}x_j$, with $\det(a_{i,j})\neq 0$, the inclusion homomorphism $k[y_1,\ldots,y_n]\hookrightarrow\mathcal{O}(X)$ is finite. In other words, there is a linear automorphism $\phi\colon\mathbf{A}^N\to\mathbf{A}^N$, such that if $i\colon X\hookrightarrow\mathbf{A}^N$ is the inclusion, and $p\colon\mathbf{A}^N\to\mathbf{A}^n$ is the projection $p(u_1,\ldots,u_N)=(u_1,\ldots,u_n)$, the composition

$$X\overset{i}{\hookrightarrow}\mathbf{A}^N\overset{\phi}{\longrightarrow}\mathbf{A}^N\overset{p}{\longrightarrow}\mathbf{A}^n$$

is a finite morphism.

EXAMPLE 3.2.4. If X is an affine variety and Y is a closed subset of X, then Y is an affine variety and the inclusion map $Y\hookrightarrow X$ is finite. Indeed, the morphism $\mathcal{O}(X)\to\mathcal{O}(Y)$ is surjective, hence finite.

REMARK 3.2.5. It is straightforward to see that if $f\colon X\to Y$ and $g\colon Y\to Z$ are finite morphisms between affine varieties, then the composition $g\circ f$ is finite.

EXAMPLE 3.2.6. If X is an affine variety and Y consists of one point, then the unique morphism $f\colon X\to Y$ is finite if and only if X is a finite set. Indeed, note first that if X consists of r points, then $\mathcal{O}(X)=k^{\times r}$, hence $\mathcal{O}(X)$ is clearly a finitely generated k-vector space. For the converse, if $X_1,\ldots,X_n$ are the irreducible components of X, then for every i, the composition $X_i\hookrightarrow X\to Y$ is finite by Remark 3.2.5 and Example 3.2.4. Since it is enough to show that each X_i consists of one point, we may assume that X is irreducible. In this case, the canonical injective homomorphism $k\to\mathcal{O}(X)$ is finite, and since k is a field and $\mathcal{O}(X)$ is an integral domain, we conclude that $\mathcal{O}(X)$ is a field. The finite field extension $k\to\mathcal{O}(X)$ must be an isomorphism, since k is algebraically closed.

REMARK 3.2.7. If $f\colon X\to Y$ is a finite morphism of affine varieties and $Z\subseteq X$ and $W\subseteq Y$ are closed subsets such that $f(Z)\subseteq W$, then the induced morphism

$g\colon Z \to W$ is finite. Indeed, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}(Y) \arrow[r, "f^\#"] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ \mathcal{O}(W) \arrow[r, "g^\#"] & \mathcal{O}(Z). \end{tikzcd}$$

Since $f^\#$ is a finite homomorphism and the vertical homomorphisms in the diagram are surjective, it follows that $g^\#$ is finite as well.

In particular, using also Example 3.2.6, we see that if $f\colon X \to Y$ is finite, then for every $y \in Y$, the fiber $f^{-1}(y)$ is finite.

We collect in the next proposition some basic properties of finite ring homomorphisms (in fact, the same properties hold for integral homomorphisms).

PROPOSITION 3.2.8. *Let $\phi\colon A \to B$ be a finite ring homomorphism.*

i) *If $\mathfrak{q}$ is a prime ideal in B and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$, then $\mathfrak{q}$ is a maximal ideal if and only if $\mathfrak{p}$ is a maximal ideal.*
ii) *If $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$ are prime ideals in B, then $\phi^{-1}(\mathfrak{q}_1) \neq \phi^{-1}(\mathfrak{q}_2)$.*
iii) *If ϕ is injective, then for every prime ideal $\mathfrak{p}$ in A, there is a prime ideal $\mathfrak{q}$ in B such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*
iv) *Given prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A and a prime ideal $\mathfrak{q}_1$ in B such that $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$, there is a prime ideal $\mathfrak{q}_2$ in B such that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$.*

PROOF. Under the assumption in i), note that we have a finite, injective homomorphism of integral domains

$$A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}.$$

In this case, $A/\mathfrak{p}$ is a field if and only if $B/\mathfrak{q}$ is a field (see Proposition A.2.1). This gives i).

In order to prove ii), we first recall that the map $\mathfrak{q} \to \mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ gives a bijection between the primes $\mathfrak{q}$ in B with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and the primes in the ring $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. Since ϕ is finite, the induced homomorphism

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$$

is again finite. Given $\mathfrak{q}_1$ and $\mathfrak{q}_2$ as in ii), suppose that $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p} = \phi^{-1}(\mathfrak{q}_2)$. In this case, it follows from i) that both $\mathfrak{q}_1B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ and $\mathfrak{q}_2B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ are maximal ideals. Since the first one is strictly contained in the second one, we obtain a contradiction.

We now prove iii). Since B is a finitely generated A-module, we see that $B_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module, and it is nonzero since it contains $A_{\mathfrak{p}}$. It thus follows from Nakayama's lemma (see Proposition C.1.1) that $B_{\mathfrak{p}} \neq \mathfrak{p}B_{\mathfrak{p}}$. Since the ring $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is nonzero, it contains a prime ideal and every such prime ideal is of the form $\mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, for some prime ideal $\mathfrak{q}$ in B, with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

Finally, suppose that $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{q}_1$ are as in iv). The induced homomorphism

$$\overline{\phi}\colon A/\mathfrak{p}_1 \longrightarrow B/\mathfrak{q}_1$$

is finite and injective. We may thus apply iii) to find a prime ideal in $B/\mathfrak{q}_1$ whose inverse image via $\overline{\phi}$ is $\mathfrak{p}_2/\mathfrak{p}_1$. This ideal is of the form $\mathfrak{q}_2/\mathfrak{q}_1$, for some prime ideal $\mathfrak{q}_2$ containing $\mathfrak{q}_1$ and it is clear that $\phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$. $\square$

We now reformulate geometrically the properties of finite homomorphisms in the above proposition.

COROLLARY 3.2.9. *Let $f\colon X \to Y$ be a finite morphism of affine varieties and $\phi = f^{\#}$ the corresponding homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$.*

1) *The map f is closed, that is, $f(Z)$ is closed in Y for every closed subset Z of X. In particular, the map f is surjective if and only if ϕ is injective.*
2) *If $Z_1 \subsetneq Z_2$ are irreducible closed subsets of X, then $f(Z_1) \subsetneq f(Z_2)$ are irreducible closed subsets of Y.*
3) *If f is surjective, then given any irreducible, closed subset W of Y, there is an irreducible, closed subset Z in X such that $f(Z) = W$.*
4) *If Z_1 is an irreducible, closed subset of X and $W_1 \supseteq W_2$ are irreducible, closed subsets of Y, with $W_1 = f(Z_1)$, then there is $Z_2 \subseteq Z_1$ irreducible and closed such that $f(Z_2) = W_2$.*

PROOF. Let Z be a closed subset in X. In order to show that $f(Z)$ is closed, after writing Z as the union of its irreducible components, we see that it is enough to prove the assertion when Z is irreducible. Let $\mathfrak{q} \subseteq \mathcal{O}(X)$ be the prime ideal corresponding to Z. Recall that by Proposition 1.4.23, we have

$$\overline{f(Z)} = V\big(\phi^{-1}(\mathfrak{q})\big).$$

If $\mathfrak{m}$ is a maximal ideal in $\mathcal{O}(Y)$ containing $\phi^{-1}(\mathfrak{q})$, we deduce from assertions iv) and i) in the proposition that there is a maximal ideal $\mathfrak{n}$ in $\mathcal{O}(X)$ such that $\mathfrak{q} \subseteq \mathfrak{n}$ and $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. Therefore

$$V\big(\phi^{-1}(\mathfrak{q})\big) = f(Z)$$

and therefore $f(Z)$ is closed. In order to prove the second assertion in 1), recall that by Proposition 1.4.23, we know that ϕ is injective if and only if $\overline{f(X)} = Y$. Since $f(X)$ is closed, we obtain the assertion.

The assertions in 2), 3), and 4) now follow from assertions ii), iii), and respectively iv) in the proposition using the above description of the images of closed subsets of X. $\square$

COROLLARY 3.2.10. *If $f\colon X \to Y$ is a finite, surjective morphism of affine varieties, then*

$$\dim(X) = \dim(Y).$$

Moreover, if Z is a closed, irreducible subset of X, then

$$\operatorname{codim}_X(Z) = \operatorname{codim}_Y\big(f(Z)\big).$$

PROOF. If

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$

is a sequence of irreducible closed subsets in X, then it follows from assertions 1) and 2) in Corollary 3.2.9 that we have the following sequence of irreducible closed subsets in Y:

$$f(Z_0) \supsetneq f(Z_1) \supseteq \ldots \supsetneq f(Z_r).$$

This gives $\dim(Y) \geq \dim(X)$.

Suppose now that

$$W_0 \supsetneq W_1 \supsetneq \ldots \supsetneq W_s$$

is a sequence of irreducible closed subsets in Y. Assertion 3) in Corollary 3.2.9 gives an irreducible closed subset $T_0 \subseteq X$ such that $f(T_0) = W_0$. Using repeatedly assertion 4) in Corollary 3.2.9, we obtain a sequence of irreducible closed subsets in X

$$T_0 \supsetneq T_1 \supsetneq \ldots \supsetneq T_s$$

such that $f(T_i) = W_i$ for all i. We thus have $\dim(X) \geq \dim(Y)$ and by combining the two inequalities we get $\dim(X) = \dim(Y)$. The proof of the second assertion is entirely analogous, so we leave it as an exercise. $\square$

3.3. Main results of dimension theory

The following result, the *Principal Ideal* theorem, is the starting point of dimension theory. A similar statement holds for prime ideals in an arbitrary Noetherian ring; a proof in the general setting, relying on some basic commutative algebra, is given in Appendix J. We give a proof in our geometric setting, following closely [**Mum88**].

THEOREM 3.3.1. *(Krull) If X is an algebraic variety, $f \in \mathcal{O}(X)$, and Y is an irreducible component of*

$$V(f) = \{u \in X \mid f(u) = 0\},$$

then $\operatorname{codim}_X(Y) \leq 1$.

We begin with some comments about the statement.

REMARK 3.3.2. If $X_1, \ldots, X_r$ are the irreducible components of X and $f|_{X_i} \neq 0$ for all i, then $\operatorname{codim}_X(Y) \geq 1$. Indeed, since Y is irreducible, there is i such that $Y \subseteq X_i$, and our assumption on f implies that this inclusion is strict.

REMARK 3.3.3. With notation as in the theorem, if U is an open subset of X with $U \cap Y \neq \emptyset$, it is enough to prove the assertion in the theorem for U, $f|_U$, and $Y \cap U$. Indeed, it follows from Remark 3.1.11 that

$$\operatorname{codim}_X(Y) = \operatorname{codim}_U(U \cap Y),$$

while Exercise 1.3.19 implies that $U \cap Y$ is an irreducible component of $V(f|_U) = V(f) \cap U \subseteq U$.

REMARK 3.3.4. It is enough to prove the theorem when X is affine and irreducible and $Y = V(f)$. First, note that if we have a sequence

$$Z_0 \supsetneq Z_1 \supsetneq Z_2 = Y,$$

with all Z_i irreducible closed subsets of X, then $\operatorname{codim}_{Z_0}(Y) \geq 2$ and Y is an irreducible component also for $V(f|_{Z_0}) = V(f) \cap Z_0 \subseteq Z_0$. This shows that we may assume that X is irreducible. Second, let us choose an affine open subset $U \subseteq X$ that meets Y, but does not meet the other irreducible components of $V(f)$. By the previous remark, it is enough to prove the theorem for U, $f|_U$, and $Y \cap U$, and by our choice of U, we have $U \cap Y = V(f|_U)$.

REMARK 3.3.5. The theorem is easy to prove when X is affine and $\mathcal{O}(X)$ is a UFD. Indeed, the assertion is clearly true when $f = 0$ (in which case $Y = X$ and $\operatorname{codim}_X(Y) = 0$). Suppose now that $f \neq 0$. In this case, it follows from Example 1.3.14 that if the prime decomposition of f is $f = uf_1^{m_1} \cdots f_r^{m_r}$, with u invertible, then there is i such that $Y = V(f_i)$. If there is an irreducible closed subset Z with $Y \subsetneq Z \subsetneq X$ and $I_X(Z) = \mathfrak{p}$, then $\mathfrak{p} \subsetneq (f_i)$. Let $h \in \mathfrak{p}$ be any nonzero element and let m be the exponent of f_i in the prime decomposition of h

is minimal. If we write $h = f_i^m h'$, since $\mathfrak{p}$ is prime and $f_i \not\in \mathfrak{p}$, we have $h' \in (f_i)$, contradicting the definition of m.

The proof of the theorem makes use of Noether's Normalization lemma to reduce the general case to that treated in Remark 3.3.5. We will also need some basic facts about norm maps for finite field extensions, for which we refer to Appendix D.

PROOF OF THEOREM 3.3.1. As we have seen in Remark 3.3.4, we may assume that X is affine and irreducible and $Y = V(f)$. Let $A = \mathcal{O}(X)$ and put $K = k(X)$. By Noether's Normalization lemma, if $n = \mathrm{trdeg}_k(K)$, we can find a k-subalgebra $B \simeq k[x_1, \ldots, x_n]$ of A such that the inclusion map $B \hookrightarrow A$ is finite (hence integral, see Proposition A.1.3). We denote by L the fraction field of B, so that the field extension K/L is finite (see Remark 1.2.1). We denote by $\mathfrak{p} \subseteq A$ the prime ideal corresponding to Y and let $\mathfrak{q} = \mathfrak{p} \cap B$.

Let $h = N_{K/L}(f)$. Note that $h \neq 0$. Moreover, since A is an integral extension of B, $f \in A$, and B is integrally closed (see Example 1.7.28), we have $h \in \mathfrak{q}$ by Proposition D.2.1.

In fact, we have $\mathfrak{q} = \mathrm{rad}(h)$. Indeed, suppose that $u \in \mathfrak{q}$. Since $\mathfrak{p} = \mathrm{rad}(f)$, it follows that we can find a positive integer m and $w \in A$ such that $u^m = fw$. By the multiplicative property of the norm and the behavior of $N_{K/L}$ on elements in L (for both properties, see Proposition D.1.1), we deduce

$$u^{m \cdot [K:L]} = N_{K/L}(u)^m = h \cdot N_{K/L}(w) \in (h).$$

Since B is a UFD, we deduce from Remark 3.3.5 that $\mathrm{codim}(\mathfrak{q}) \leq 1$. On the other hand, since the morphism $B \hookrightarrow A$ is finite and injective, it follows from Proposition 3.2.10 that $\mathrm{codim}(\mathfrak{p}) = \mathrm{codim}(\mathfrak{q})$. This completes the proof of the theorem. $\square$

REMARK 3.3.6. If X is an affine variety with irreducible components $X_1, \ldots, X_r$ and $f \in \mathcal{O}(X)$ is a non-zero-divisor, then $f|_{X_i} \neq 0$ for every i. Indeed, let $\mathfrak{p}_i = I_X(X_i)$ and suppose that we have $f \in \mathfrak{p}_1$. Let us choose $g_j \in \mathfrak{p}_j \smallsetminus \mathfrak{p}_1$ for $j \geq 2$. Since $\mathfrak{p}_1$ is prime, if $g = \prod_{j \geq 2} g_j$, then $g \not\in \mathfrak{p}_1$. In particular, $g \neq 0$. However, $fg \in \bigcap_{j \geq 1} \mathfrak{p}_j$, hence $fg = 0$, contradicting the fact that f is a non-zero-divisor. For a more general assertion, valid in arbitrary Noetherian rings, see Proposition E.2.1.

We thus see, by combining Theorem 3.3.1 and Remark 3.3.2, that if f is a non-zero-divisor in $\mathcal{O}(X)$, for an affine variety X, then every irreducible component of $V(f)$ has codimension 1 in X.

We now deduce from Theorem 3.3.1 the basic properties of dimension of algebraic varieties. We begin with a generalization of the theorem to the case of several functions.

COROLLARY 3.3.7. *If X is an algebraic variety and $f_1, \ldots, f_r$ are regular functions on X, then for every irreducible component Y of*

$$V(f_1, \ldots, f_r) = \{u \in X \mid f_1(u) = \ldots = f_r(u) = 0\},$$

we have $\mathrm{codim}_X(Y) \leq r$.

PROOF. We do induction on r, the case $r = 1$ being a consequence of the theorem. Arguing as in Remarks 3.3.3 and 3.3.4, we see that we may assume that X is affine and $Y = V(f_1, \ldots, f_r)$. We need to show that for every sequence

$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_m$$

of irreducible closed subsets of X, we have $m \leq r$. By Noetherianity, we may assume that there is no irreducible closed subset Z, with $Y \subsetneq Z \subsetneq Y_1$.

By assumption, there is i (say, $i = 1$) such that $Y_1 \not\subseteq V(f_i)$. Since there are no irreducible closed subsets strictly between Y and Y_1, it follows that Y is an irreducible component of $Y_1 \cap V(f_1)$. After replacing X by an affine open subset meeting Y, but disjoint from the other components of $Y_1 \cap V(f_1)$, we may assume that in fact $Y = Y_1 \cap V(f_1)$, hence $I_X(Y) = \mathrm{rad}\big(I_X(Y_1) + (f_1)\big)$. It follows that for $2 \leq i \leq r$, we can find positive integers q_i and $g_i \in I_X(Y_1)$ such that

$$f_i^{q_i} - g_i \in (f_1). \tag{3.3.1}$$

We will show that Y_1 is an irreducible component of $V(g_2, \ldots, g_r)$. If this is the case, then we conclude by induction that $m - 1 \leq r - 1$, hence we are done. Note first that (3.3.1) gives

$$Y = V(f_1, \ldots, f_r) = V(f_1, g_2, \ldots, g_r).$$

If there is an irreducible closed subset Z such that

$$Y_1 \subsetneq Z \subseteq V(g_2, \ldots, g_r),$$

then $Y = Z \cap V(f_1)$, and the theorem implies $\mathrm{codim}_Z(Y) \leq 1$, contradicting the fact that we have $Y \subsetneq Y_1 \subsetneq Z$. Therefore Y_1 is an irreducible component of $V(g_2, \ldots, g_r)$, completing the proof of the corollary. □

Corollary 3.3.8. *For every positive integer n, we have $\dim(\mathbf{A}^n) = n$.*

Proof. It is clear that $\dim(\mathbf{A}^n) \geq n$, since we have the following sequence of irreducible closed subsets in $\mathbf{A}^n$:

$$V(x_1, \ldots, x_n) \subsetneq V(x_1, \ldots, x_{n-1}) \subsetneq \ldots \subsetneq V(x_1) \subsetneq \mathbf{A}^n.$$

In order to prove the reverse inequality, it is enough to show that for every point $p = (a_1, \ldots, a_n) \in \mathbf{A}^n$, we have $\mathrm{codim}_X(\{p\}) \leq n$. This follows from Corollary 3.3.7, since $Y = V(x_1 - a_1, \ldots, x_n - a_n)$. □

Corollary 3.3.9. *If X is an irreducible variety, then*

$$\dim(X) = \mathrm{trdeg}_k k(X).$$

In particular, we have $\dim(X) < \infty$.

Proof. By taking a finite cover by affine open subsets and using Lemma 3.1.7, we see that it is enough to prove the assertion when X is affine. It follows from Noether's Normalization lemma that if $n = \mathrm{trdeg}_k k(X)$, then there is a finite, surjective morphism $f\colon X \to \mathbf{A}^n$. The assertion then follows from the previous corollary via Corollary 3.2.10. □

Remark 3.3.10. It follows from the previous corollary and Lemma 3.1.6 that for every algebraic variety X, we have $\dim(X) < \infty$.

Remark 3.3.11. Another consequence of Corollary 3.3.9 is that if X is an irreducible quasi-affine variety, then for every nonempty open subset U of X, we have $\dim(U) = \dim(X)$.

DEFINITION 3.3.12. If X is a Noetherian topological space, we say that X has *pure dimension* if all its irreducible components have the same dimension. A *curve* is an algebraic variety of pure dimension 1 and a *surface* is an algebraic variety of pure dimension 2; moreover, in both these cases, unless we explicitly mention otherwise, the variety is assumed to be irreducible.

COROLLARY 3.3.13. *If X is an algebraic variety, then the following hold:*

i) *If $Y \subseteq Z$ are closed irreducible subsets, then every saturated*[1] *chain*
$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r = Z$$
of irreducible closed subsets has length $r = \mathrm{codim}_Z(Y)$.

ii) *If X has pure dimension, then for every irreducible closed subset $Y \subseteq X$, we have*
$$\dim(Y) + \mathrm{codim}_X(Y) = \dim(X).$$

PROOF. We begin by showing the following statement: given irreducible, closed subsets $Y \subsetneq Z$, with $\mathrm{codim}_Z(Y) = 1$, we have $\dim(Y) = \dim(Z) - 1$. For this, we may of course assume that $X = Z$. Note also that in light of Remark 3.3.11, we may replace Z by any open subset U with $U \cap Y \neq \emptyset$, since $\dim(U) = \dim(Z)$ and $\dim(U \cap Y) = \dim(Y)$. In particular, after replacing Z by an affine open subset U with $U \cap Y \neq \emptyset$, we may assume that Z is affine.

Let $f \in I_Z(Y) \smallsetminus \{0\}$. Since $\mathrm{codim}_Z(Y) = 1$, we see that Y is an irreducible component of $V(f)$. After replacing Z by an affine open subset that intersects Y, but does not intersect the other components of $V(f)$, we may assume that $Y = V(f)$. We now make use of the argument in the proof of Theorem 3.3.1. Noether's Normalization lemma gives a finite, surjective morphism $p \colon Z \to \mathbf{A}^n$ and we have seen that $p\big(V(f)\big) = V(h)$, for some nonzero $h \in \mathcal{O}(\mathbf{A}^n)$, hence the ideal $I\big(p(Y)\big) \subseteq k[x_1, \ldots, x_n]$ is principal, say generated by a polynomial g. This implies that $\dim\big(p(Y)\big) = n-1$: indeed, arguing as in the proof of Noether's Normalization lemma, we see that after a suitable linear change of coordinates, we may assume that g is a monic polynomial in x_n, with coefficients in $k[x_1, \ldots, x_{n-1}]$, in which case the morphism
$$k[x_1, \ldots, x_{n-1}] \hookrightarrow k[x_1, \ldots, x_n]/(g)$$
is finite and injective, hence we get the assertion via Corollaries 3.2.10 and 3.3.8. Since Corollary 3.2.10 gives $\dim(Z) = n$ and $\dim(Y) = \dim\big(p(Y)\big) = n - 1$, this completes the proof of our initial statement.

This assertion implies that given any saturated chain
$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r = Z$$
of irreducible, closed subsets, we have $\dim(Y_i) = \dim(Y_{i-1}) + 1$ for $1 \leq i \leq r$, hence $\dim(Z) = \dim(Y) + r$. In particular, all such chains have the same length. Since there is such a chain of length $\mathrm{codim}_Z(Y)$, we obtain the assertion in i), as well as the assertion in ii) when X is irreducible.

Suppose now that we are in the setting of ii). Using Remark 3.1.12, the assertion when X is irreducible, and the fact that X is pure dimensional, we obtain
$$\begin{aligned}\mathrm{codim}_X(Y) &= \max\{\mathrm{codim}_{X_i}(Y) \mid Y \subseteq X_i\} \\ &= \max\{\dim(X_i) - \dim(Y) \mid Y \subseteq X_i\} = \dim(X) - \dim(Y),\end{aligned}$$

[1] This means that for every i, with $1 \leq i \leq r$, there is no closed, irreducible subset Z, with $Y_{i-1} \subsetneq Z \subsetneq Y_i$; equivalently, we have $\mathrm{codim}_{Y_i}(Y_{i-1}) = 1$.

completing the proof of the proposition. $\square$

REMARK 3.3.14. if X is an algebraic variety, and p is a point on X, then $\dim_p(X) := \dim(\mathcal{O}_{X,p})$ is equal to the largest dimension of an irreducible component of X that contains p. Indeed, it follows from definition that $\dim_p(X) = \operatorname{codim}_X(\{p\})$ and we deduce from Corollary 3.3.13 that if $X_1, \ldots, X_r$ are the irreducible components of X that contain p, then

$$\dim_p(X) = \max_{i=1}^{r} \operatorname{codim}_{X_i}(\{p\}) = \max_{i=1}^{r} \dim(X_i).$$

REMARK 3.3.15. Suppose that X is an algebraic variety, $f \in \mathcal{O}(X)$ is a non-zero-divisor, and

$$Y = \{x \in X \mid f(x) = 0\}.$$

In this case, for every $x \in Y$, we have

$$\dim(\mathcal{O}_{Y,x}) = \dim(\mathcal{O}_{X,x}) - 1. \tag{3.3.2}$$

In order to see this, we use the interpretation of the two dimensions given by the previous remark. Note first that it follows from Remark 3.3.6 that f does not vanish on any irreducible component of X. If Y' is an irreducible component of Y that contains x and if X' is an irreducible component of X that contains Y', then it follows from Theorem 3.3.1 that $\operatorname{codim}_{X'}(Y') = 1$ and Corollary 3.3.13 implies $\dim(Y') = \dim(X') - 1$. This gives the inequality "$\leq$" in (3.3.2). On the other hand, given any irreducible component Z of X that contains x, then every irreducible component W of $Y \cap Z$ that contains x satisfies $\operatorname{codim}_W(Z) = 1$ by Theorem 3.3.1. Using again Corollary 3.3.13, we obtain

$$\dim(\mathcal{O}_{Y,x}) \geq \dim W = \dim(Z) - 1,$$

hence we get the inequality "$\geq$" in (3.3.2).

We end this section with the following partial converse to Corollary 3.3.7.

PROPOSITION 3.3.16. *Let X be an algebraic variety. If Y is an irreducible closed subset with $\operatorname{codim}_X(Y) = r \geq 1$, then there are $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that Y is an irreducible component of $V(f_1, \ldots, f_r)$.*

PROOF. Let $X_1, \ldots, X_N$ be the irreducible components of X. Note that there is $f_1 \in I_X(Y)$ such that $X_i \not\subseteq V(f_1)$ for all i. Indeed, otherwise

$$I_X(Y) \subseteq \bigcup_{i=1}^{N} I_X(X_i).$$

Since all $I_X(X_i)$ are prime ideals and $I_X(Y) \not\subseteq I_X(X_i)$ (recall that $r \geq 1$), this contradicts the Prime Avoidance lemma (see Lemma E.1.1).

For such f_1, we have $\operatorname{codim}_{V(f_1)} Y \leq r - 1$. Iterating, we find $f_1, \ldots, f_r \in I_X(Y)$ such that $\operatorname{codim}_{V(f_1,\ldots,f_r)}(Y) = 0$, that is, Y is an irreducible component of $V(f_1, \ldots, f_r)$. $\square$

REMARK 3.3.17. In general, if X and Y are as in the proposition, it might not be possible to find $f_1, \ldots, f_r$ such that $Y = V(f_1, \ldots, f_r)$ (not even if we are willing to restrict to affine open neighborhoods of a given point). Consider, for example

$$X = V(x_1x_2 - x_3x_4) \subseteq \mathbf{A}^4 \quad \text{and} \quad Y = V(x_1, x_3).$$

In this case we have $\dim(X) = 3$ and $\dim(Y) = 2$, hence $\operatorname{codim}_X(Y) = 1$ by Corollary 3.3.13. However, for every affine open neighborhood U of the origin, there is no $f \in \mathcal{O}(U)$ such that $V(f) = Y$. Can you prove this?

EXERCISE 3.3.18. Show that if X and Y are algebraic varieties, then

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

EXERCISE 3.3.19. Show that if X is an algebraic variety and Z is a locally closed subset of X, then

$$\dim(Z) = \dim(\overline{Z}) > \dim(\overline{Z} \smallsetminus Z).$$

EXERCISE 3.3.20. Show that if X is an affine variety such that $\mathcal{O}(X)$ is a UFD, then for every closed subset $Y \subseteq X$, having all components of codimension 1, the ideal $I_X(Y)$ defining Y is principal.

EXERCISE 3.3.21. Show that if X and Y are irreducible closed subsets of $\mathbf{A}^n$, then every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$ (Hint: describe $X \cap Y$ as the intersection of $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^n$ with the diagonal $\Delta = \{(x, x) \mid x \in \mathbf{A}^n\}$).

3.4. Dimension of fibers of morphisms

We now discuss the main results concerning the dimensions of fibers of a morphism between algebraic varieties. More generally, we will be interested in the dimension of $f^{-1}(Z)$, where Z is a closed subset of Y.

We fix a dominant morphism $f\colon X \to Y$ between irreducible algebraic varieties and let $k(Y) \hookrightarrow k(X)$ be the induced extension of function fields. We put

$$r = \operatorname{trdeg}_{k(Y)} k(X) = \dim(X) - \dim(Y).$$

THEOREM 3.4.1. *With the above notation, if W is an irreducible component of $f^{-1}(Z)$ that dominates Z, then*

$$\operatorname{codim}_X(W) \leq \operatorname{codim}_Y(Z), \quad \textit{or equivalently}, \quad \dim(W) \geq \dim(Z) + r.$$

In particular, for every point y in the image of f, all irreducible components of $f^{-1}(y)$ have dimension $\geq r$.

PROOF. Note that if U is an open subset such that $Z \cap U \neq \emptyset$, since $\overline{f(W)} = Z$, we have $W \cap f^{-1}(U) \neq \emptyset$. By Corollary 3.3.11, we may thus replace f by $f^{-1}(U) \to U$. In particular, we may and will assume that Y is affine. In this case, if $s = \operatorname{codim}_Y(Z)$, it follows from Proposition 3.3.16 that there are $g_1, \ldots, g_s \in \mathcal{O}(Y)$ such that Z is an irreducible component of $V(g_1, \ldots, g_s)$. Since $W \subseteq f^{-1}(Z)$, we have $W \subseteq W' = V\big(f^{\#}(g_1), \ldots, f^{\#}(g_s)\big)$.

In fact, W is an irreducible component of W': if $W \subseteq W'' \subseteq W'$, with W'' closed and irreducible, we have

$$Z = \overline{f(W)} \subseteq \overline{f(W'')} \subseteq V(g_1, \ldots, g_s).$$

Since Z is an irreducible component of $V(g_1, \ldots, g_s)$, we deduce that $Z = \overline{f(W'')}$. In particular, we have $W'' \subseteq f^{-1}(Z)$, and since W is an irreducible component of $f^{-1}(Z)$, we conclude that $W = W''$. Therefore W is an irreducible component of W'. Corollary 3.3.7 then implies that $\operatorname{codim}_X(W) \leq s$. $\square$

THEOREM 3.4.2. *With the above notation, there is a nonempty open subset V of Y such that $V \subseteq f(X)$ and for every irreducible, closed subset $Z \subseteq Y$ with $Z \cap V \neq \emptyset$, and every irreducible component W of $f^{-1}(Z)$ that dominates Z, we have*

$$\operatorname{codim}_X(W) = \operatorname{codim}_Y(Z), \quad \textit{or equivalently,} \quad \dim(W) = \dim(Z) + r.$$

In particular, for every $y \in V$, every irreducible component of $f^{-1}(y)$ has dimension r.

PROOF. It is clear that we may replace f by $f^{-1}(U) \to U$ for any nonempty open subset, hence we may and will assume that Y is affine. We show that we may further assume that X is affine, too. Indeed, if we know the theorem in this case, we consider an open cover by affine open subsets

$$X = U_1 \cup \ldots \cup U_m$$

and let $V_i \subseteq Y$ be the nonempty open subset constructed for the morphism $U_i \to Y$. In this case it is straightforward to check that $V = \bigcap_i V_i$ satisfies the conditions in the theorem.

Suppose now that X and Y are irreducible affine varieties and let $f^{\#} \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be the induced homomorphism. This is injective, since f is dominant. We consider the $k(Y)$-algebra $S = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} k(Y)$. This is a domain with fraction field $k(X)$. By Noether's Normalization lemma, we can find $y_1, \ldots, y_r \in S$ that are algebraically independent over $k(Y)$ and such that the inclusion

$$\alpha \colon k(Y)[y_1, \ldots, y_r] \hookrightarrow S$$

is finite. After replacing each y_i by some $a_i y_i$, for a suitable nonzero $a_i \in \mathcal{O}(Y)$, we may assume that $y_i \in \mathcal{O}(X)$ for all i.

Claim. There is a nonzero $s \in \mathcal{O}(Y)$ such that the inclusion

$$\mathcal{O}(Y)_s[y_1, \ldots, y_r] \hookrightarrow \mathcal{O}(X)_s$$

is finite. In order to see this, let us choose generators $x_1, \ldots, x_N$ of $\mathcal{O}(X)$ as a k-algebra. Since α is finite, it follows that each x_i satisfies a monic equation of the form:

$$x_i^{m_i} + a_{i,1} x_i^{m_i - 1} + \ldots + a_{i,m_i} = 0 \quad \text{for some} \quad a_{i,j} \in k(Y)[y_1, \ldots, y_r].$$

If $s \in \mathcal{O}(Y) \smallsetminus \{0\}$ is such that $s a_{i,j} \in \mathcal{O}(Y)[y_1, \ldots, y_r]$ for all i and j, then it follows that each x_i is integral over $\mathcal{O}(Y)_s[y_1, \ldots, y_r]$, hence $\mathcal{O}(X)_s$ is finite over $\mathcal{O}(Y)_s[y_1, \ldots, y_r]$, proving the claim.

After replacing f by $D_X\big(f^{\#}(s)\big) = f^{-1}\big(D_Y(s)\big) \to D_Y(s)$, we may thus assume that f factors as

$$X \xrightarrow{g} Y \times \mathbf{A}^r \xrightarrow{p} Y,$$

where p is the first projection and g is finite and surjective. It is clear that in this case f is surjective. Moreover, if Z and W are as in the statement of the theorem, then $g(W) \subseteq Z \times \mathbf{A}^r$, and using Corollary 3.2.9, as well as Exercise 3.3.18, we have

$$\dim(W) = \dim\big(g(W)\big) \leq \dim(Z \times \mathbf{A}^r) = \dim(Z) + r.$$

Since the opposite inequality follows by Theorem 3.4.1, we have in fact equality. $\square$

COROLLARY 3.4.3. *If $f \colon X \to Y$ is a morphism of algebraic varieties such that all fibers of f have dimension r (in particular, f is surjective), then $\dim(X) = \dim(Y) + r$.*

Proof. If $Y_1, \ldots, Y_m$ are the irreducible components of Y, each morphism $f^{-1}(Y_i) \to Y_i$ has all fibers of dimension r. Since

$$\dim(X) = \max_{i=1}^{m} \dim\left(f^{-1}(Y_i)\right) \quad \text{and} \quad \dim(Y) = \max_{i=1}^{m} \dim(Y_i),$$

we see that it is enough to prove the assertion in the corollary when Y is irreducible.

Suppose now that $X_1, \ldots, X_s$ are the irreducible components of X. It follows from Theorem 3.4.2 that for every i, if we put

$$d_i := \dim(X_i) - \dim\left(\overline{f(X_i)}\right),$$

then there is an open subset U_i of $\overline{f(X_i)}$ such that every fiber of $X_i \to \overline{f(X_i)}$ over a point in U_i has dimension d_i. The hypothesis implies that $d_i \leq r$ for every i; moreover, there is i_0 such that $d_{i_0} = r$ and $\overline{f(X_i)} = Y$. The former fact implies that for every i, we have

$$\dim(X_i) \leq \dim\left(\overline{f(X_i)}\right) + r \leq \dim(Y) + r,$$

hence $\dim(X) \leq \dim(Y) + r$. On the other hand, the latter fact implies that $\dim(X_{i_0}) = \dim(Y) + r$, hence $\dim(X) \geq \dim(Y) + r$, completing the proof of the corollary. $\square$

Example 3.4.4. Let a, b, and c be positive integers and let

$$f\colon \mathbf{A}^3 \to \mathbf{A}^3, \quad \text{given by} \quad f(u,v,w) = (u^a v^b w, u^c v, u).$$

This is birational, with inverse

$$g\colon V = \{(x,y,z) \in \mathbf{A}^3 \mid yz \neq 0\} \to \mathbf{A}^3$$

given by

$$g(x,y,z) = (z, yz^{-c}, xy^{-b}z^{-a+bc}).$$

Therefore f induces an isomorphism $f^{-1}(V) \to V$. In particular, for $P \in V$, the fiber $f^{-1}(P)$ is a point.

On the other hand, if $P = (x_0, y_0, 0)$, then

$$f^{-1}(P) = \begin{cases} V(u) \simeq \mathbf{A}^2, & \text{if} \quad x_0 = y_0 = 0; \\ \emptyset, & \text{otherwise} \end{cases}$$

If $P = (x_0, 0, z_0)$, with $z_0 \neq 0$, then

$$f^{-1}(P) = \begin{cases} V(v, u - z_0) \simeq \mathbf{A}^1, & \text{if} \quad x_0 = 0; \\ \emptyset, & \text{otherwise} \end{cases}$$

3.5. Constructible subsets and Chevalley's theorem

Definition 3.5.1. A subset of a topological space X is *constructible* if it is a finite union of locally closed subsets.

Proposition 3.5.2. *If X is a topological space, the set of constructible subsets of X is the smallest set that contains the open subsets of X and is closed under finite unions, finite intersections, and complements.*

Proof. The fact that a finite union of constructible sets is constructible is clear. Suppose now that A and B are constructible and let us show that $A \cap B$ is constructible. We can write

$$A = A_1 \cup \ldots \cup A_r \quad \text{and} \quad B = B_1 \cup \ldots \cup B_s,$$

with the A_i and B_j locally closed. In this case we have

$$A \cap B = \bigcup_{i,j} (A_i \cap B_j).$$

Since the intersection of two locally closed sets is locally closed, we see that $A \cap B$ is constructible.

If A is constructible and we write $A = A_1 \cup \ldots \cup A_r$, with the A_i locally closed, we have

$$X \smallsetminus A = \bigcap_{i=1}^{r} (X \smallsetminus A_i).$$

Since each A_i is locally closed, we can write it as $U_i \cap F_i$, with F_i closed and U_i open. In this case

$$X \smallsetminus A_i = (X \smallsetminus U_i) \cup (X \smallsetminus F_i)$$

is the union of a closed set with an open set, hence it is constructible. Since we have already seen that a finite intersection of constructible sets is constructible, we conclude that $X \smallsetminus A$ is constructible.

The minimality statement in the proposition is straightforward: given a set $\mathcal{C}$ of subsets of X as in the statement, this contains the open subsets by assumption, hence it also contains the closed sets, since we assume that $\mathcal{C}$ is closed under complements. Therefore $\mathcal{C}$ also contains the locally closed subsets (since it is closed under finite intersections) and therefore contains all constructible subsets (since it is closed under finite unions). $\square$

This notion is important because of the following result, due to Chevalley.

Theorem 3.5.3. *If $f: X \to Y$ is a morphism between algebraic varieties, the image $f(X)$ is constructible. More generally, for every constructible subset A of X, its image $f(A)$ is constructible.*

Proof. If A is constructible in X, we write $A = A_1 \cup \ldots \cup A_r$, with all A_i locally closed in X. Since $f(A) = f(A_1) \cup \ldots \cup f(A_r)$, it is enough to show that the image of each composition $A_i \hookrightarrow X \to Y$ is constructible. Therefore it is enough to consider the case $A = X$.

We prove that $f(X)$ is constructible by induction on $\dim(X)$. If $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X in irreducible components, we have

$$f(X) = f(X_1) \cup \ldots \cup f(X_r),$$

hence it is enough to show that each $f(X_i)$ is irreducible. We may thus assume that X is irreducible and after replacing Y by $\overline{f(X)}$, we may assume that Y is irreducible, too, and f is dominant (note that a constructible subset of $\overline{f(X)}$ is constructible also as a subset of Y). By Theorem 3.4.2, there is an open subset V of Y such that $V \subseteq f(X)$. We can thus write

$$f(X) = V \cup g(X'), \tag{3.5.1}$$

where $X' = X \smallsetminus g^{-1}(V)$ is a closed subset of X, with $\dim(X') < \dim(X)$. By induction, we know that $g(X')$ is constructible, and we deduce from (3.5.1) that $f(X)$ is constructible. $\square$

EXERCISE 3.5.4. i) Show that if Y is a topological space and A is a constructible subset of Y, then there is a subset V of A that is open and dense in $\overline{A}$ (in particular, V is locally closed in Y).

ii) Use part i) and Chevalley's theorem to show that if G is an algebraic group[2] having an algebraic action on the algebraic variety X, then every orbit is a locally closed subset of X. Deduce that X contains closed orbits.

[2]An algebraic group is defined like a *linear* algebraic group, but the variety is not necessarily affine.

CHAPTER 4

Projective varieties

In this chapter we introduce a very important class of algebraic varieties, the *projective varieties*.

4.1. The Zariski topology on the projective space

In this section we discuss the Zariski topology on the projective space, by building an analogue of the correspondence between closed subsets in affine space and radical ideals in the polynomial ring. As usual, we work over a fixed algebraically closed field k.

DEFINITION 4.1.1. For a non-negative integer n, the projective space $\mathbf{P}^n = \mathbf{P}^n_k$ is the set of all 1-dimensional linear subspaces in k^{n+1}.

For now, this is just a set. We proceed to endow it with a topology and in the next section we will put on it a structure of algebraic variety. Note that a 1-dimensional linear subspace in k^{n+1} is described by a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$, with two points $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ giving the same subspace if and only if there is $\lambda \in k^*$ such that $\lambda a_i = b_i$ for all i. In this way, we identify $\mathbf{P}^n$ with the quotient of the set $\mathbf{A}^{n+1} \smallsetminus \{0\}$ by the action of k^* given by

$$\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n).$$

Let $\pi \colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ be the quotient map. We denote the image in $\mathbf{P}^n$ of a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$ by $[a_0, \ldots, a_n]$.

Let $S = k[x_0, \ldots, x_n]$. The relevant structure on S, for the study of $\mathbf{P}^n$, is that of a graded k-algebra. Recall that a graded (commutative) ring R is a commutative ring that has a decomposition as an Abelian group

$$R = \bigoplus_{m \in \mathbf{Z}} R_m$$

such that $R_i \cdot R_j \subseteq R_{i+j}$ for all i and j. We say that R is $\mathbf{N}$-graded if $R_m = 0$ for $m < 0$.

Note that the definition implies that if R is a graded ring, then R_0 is a subring of R and each R_m is an R_0-module, making R an R_0-algebra. We say that R is a graded A-algebra, for a commutative ring A, if R is a graded ring such that R_0 is an A-algebra (in which case R becomes an A-algebra, too). If R and S are graded rings, a *graded homomorphism* $\phi \colon R \to S$ is a ring homomorphism such that $\phi(R_m) \subseteq S_m$ for all $m \in \mathbf{Z}$.

The polynomial ring S is an $\mathbf{N}$-graded k-algebra, with S_m being the set of homogeneous polynomials of degree m. In general, if R is a graded ring, a nonzero element of R_m is *homogeneous of degree m*. By convention, 0 is homogeneous of degree m for every m. Given an arbitrary element $f \in R$, if we write

$$f = \sum_i f_i, \quad \text{with} \quad f_i \in R_i,$$

then the f_i are the *homogeneous componenets* of f.

REMARK 4.1.2. Note that the action of k^* on $\mathbf{A}^{n+1} \smallsetminus \{0\}$ is an algebraic action: in fact, it is induced by the algebraic action of k^* on $\mathbf{A}^{n+1}$ corresponding to the homomorphism

$$S \to k[t, t^{-1}] \otimes_k S, \quad f \to f(tx_1, \ldots, tx_n).$$

EXERCISE 4.1.3. For an ideal I in a graded ring R, the following are equivalent:
i) The ideal I can be generated by homogeneous elements of R.
ii) For every $f \in I$, all homogeneous components of f lie in I.
iii) The decomposition of R induces a decomposition $I = \bigoplus_{m \in \mathbf{Z}} (I \cap R_m)$.

An ideal that satisfies the equivalent conditions in the above exercise is a *homogeneous* (or *graded*) ideal. Note that if I is a homogeneous ideal in a graded ring R, then the quotient ring R/I becomes a graded ring in a natural way:

$$R/I = \bigoplus_{m \in \mathbf{Z}} R_m/(I \cap R_m).$$

We now return to the study of $\mathbf{P}^n$. The starting observation is that while it does not make sense to evaluate a polynomial in S at a point $p \in \mathbf{P}^n$, it makes sense to say that a *homogeneous* polynomial vanishes at p: indeed, if f is homogeneous of degree d and $\lambda \in k^*$, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d \cdot f(a_0, \ldots, a_n),$$

hence $f(\lambda a_0, \ldots, \lambda a_n) = 0$ if and only if $f(a_0, \ldots, a_n) = 0$. More generally, given any $f \in S$, we say that f vanishes at p if every homogeneous component of f vanishes at p.

Given any homogeneous ideal I of S, we define the *zero-locus* $V(I)$ of I to be the subset of $\mathbf{P}^n$ consisting of all points $p \in \mathbf{P}^n$ such that every polynomial f in I vanishes at p. Like the corresponding notion in the affine space, this notion satisfies the following basic properties. The proof is straightforward, hence we leave it as an exercise.

PROPOSITION 4.1.4. *The following hold:*
1) $V(S) = \emptyset$.
2) $V(0) = \mathbf{P}^n$.
3) *If I and J are ideals in S with $I \subseteq J$, then $V(J) \subseteq V(I)$.*
4) *If $(I_\alpha)_\alpha$ is a family of ideals in S, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

5) *If I and J are ideals in S, then*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

It follows from the proposition that we can put a topology on $\mathbf{P}^n$ (the *Zariski topology*) in which the closed subsets of $\mathbf{P}^n$ are the subsets of the form $V(I)$, where I is a homogeneous ideal in S.

REMARK 4.1.5. A closed subset $Y \subseteq \mathbf{A}^{n+1}$ is invariant by the k^*-action (that is, $\lambda \cdot Y = Y$ for every $\lambda \in k^*$) if and only if the ideal $I_{\mathbf{A}^n}(Y) \subseteq S$ is homogeneous (cf. Lemma 1.7.22). Indeed, if f is homogeneous, then for every $\lambda \in k^*$ and every $u \in \mathbf{A}^{n+1}$, we have $f(\lambda u) = 0$ if and only if $f(u) = 0$. We thus see that if I is a homogeneous ideal, then its zero-locus in $\mathbf{A}^{n+1}$ is k^*-invariant. In particular, if $I_{\mathbf{A}^n}(Y)$ is homogeneous, then Y is k^*-invariant. Conversely, if Y is k^*-invariant and $f \in I_{\mathbf{A}^n}(Y)$, let us write $f = \sum_i f_i$, with $f_i \in S_i$. For every $u \in Y$ and every $\lambda \in k^*$, we have $\lambda u \in Y$, hence

$$0 = f(\lambda u) = \sum_{i \geq 0} \lambda^i \cdot f_i(u).$$

It is easy to see that since this property holds for infinitely many λ, we have $f_i(u) = 0$ for all i, hence $I_{\mathbf{A}^n}(Y)$ is homogeneous.

REMARK 4.1.6. The topology on $\mathbf{P}^n$ is the quotient topology with respect to the k^*-action on $\mathbf{A}^{n+1} \smallsetminus \{0\}$. In other words, if $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ is the quotient map, then a subset Z of $\mathbf{P}^n$ is closed if and only if its inverse image $\pi^{-1}(Z)$ is closed. For this, we may assume that Z is nonempty. If $\pi^{-1}(Z)$ is closed, then it is clear that $\pi^{-1}(Z) \cup \{0\}$ is closed, hence by the previous remark, there is a homogeneous ideal $I \subseteq S$ such that $\pi^{-1}(Z) \cup \{0\}$ is the zero-locus of I. In this case, it is clear that Z is the zero-locus of I in $\mathbf{P}^n$. The converse is clear.

We now construct a map in the opposite direction. Given any subset $S \subseteq \mathbf{P}^n$, let $I(S)$ be the set of polynomials in S that vanish at all points in S. Note that $I(S)$ is a homogeneous radical ideal of S (the fact that it is homogeneous follows from the fact that if $f \in I(S)$, then all homogeneous components of f lie in $I(S)$). This definition satisfies the following properties, that are straightforward to check.

PROPOSITION 4.1.7. *The following hold:*

1) $I(\emptyset) = S$.
2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$, *then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$.
3) *If* $W_1 \subseteq W_2$, *then* $I(W_2) \subseteq I(W_1)$.

We now turn to the compositions of the two maps. The first property is tautological.

PROPOSITION 4.1.8. *For every subset* S *of* $\mathbf{P}^n$, *we have* $V\big(I(S)\big) = \overline{S}$.

PROOF. The proof follows verbatim the proof in the case of affine space (see Proposition 1.1.8). ☐

The more interesting statement concerns the other composition. This is the content of the next proposition, a graded version of the Nullstellensatz.

PROPOSITION 4.1.9. *If* $J \subseteq S$ *is a radical ideal different from* $S_+ = (x_0, \ldots, x_n)$, *then* $I\big(V(J)\big) = J$.

Note that $V(S_+) = \emptyset$, hence $I\big(V(S_+)\big) = S$. The ideal S_+, which behaves differently in this correspondence, is the *irrelevant ideal*.

PROOF OF PROPOSITION 4.1.9. The inclusion "$\supseteq$" is trivial, hence we only need to prove the reverse inclusion. It is enough to show that every homogeneous polynomial $f \in I\big(V(J)\big)$ lies in J.

We make use of the map $\pi\colon \mathbf{A}^{n+1}\smallsetminus\{0\}\to\mathbf{P}^n$. Let Z be the closed subset of $\mathbf{A}^{n+1}$ defined by J, so that $Z\smallsetminus\{0\}=\pi^{-1}\big(V(J)\big)$. Our assumption on f says that f vanishes on $Z\smallsetminus\{0\}$. If $\deg(f)>0$, then $f(0)=0$, and we conclude by Hilbert's Nullstellensatz that $f\in J$. On the other hand, if $\deg(f)=0$ and $f\neq 0$, then it follows that $V(J)=\emptyset$. This implies that $Z\subseteq\{0\}$ and another application of Hilbert's Nullstellensatz gives $S_+\subseteq J$. Since $J\neq S_+$ by assumption, we have $J=S$, in which case $f\in J$. □

Corollary 4.1.10. *The two maps $V(-)$ and $I(-)$ give inclusion reversing inverse bijections between the set of homogeneous radical ideals in S different from S_+ and the closed subsets of $\mathbf{P}^n$.*

Proof. Note that for every closed subset Z of $\mathbf{P}^n$, we have $I(Z)\neq S_+$. Indeed, if $I(Z)=S_+$, then it follows from Proposition 4.1.8 that

$$Z=V\big(I(Z)\big)=V(S_+)=\emptyset.$$

However, in this case $I(Z)=I(\emptyset)=S$. The assertion in the corollary follows directly from Propositions 4.1.8 and 4.1.9. □

Exercise 4.1.11. Show that if I is a homogeneous ideal in a graded ring S, then the following hold:

i) The ideal I is radical if and only if for every *homogeneous* element $f\in S$, with $f^m\in I$ for some $m\geq 1$, we have $f\in I$.
ii) The radical $\mathrm{rad}(I)$ of I is a homogeneous ideal.

Exercise 4.1.12. Show that if I is a homogeneous ideal in a graded ring S, then I is a prime ideal if and only if for every homogeneous elements $f,g\in S$ with $fg\in I$, we have $f\in I$ or $g\in I$. Deduce that a closed subset Z of $\mathbf{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal. In particular, $\mathbf{P}^n$ is irreducible.

Definition 4.1.13. If X is a closed subset of $\mathbf{P}^n$ and I_X is the corresponding homogeneous radical ideal, then $S_X:=S/I_X$ is the *homogeneous coordinate ring* of X. Note that this is an $\mathbf{N}$-graded k-algebra. In particular, S is the *homogeneous coordinate ring* of $\mathbf{P}^n$.

Suppose that X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X. For every homogeneous $g\in S_X$ of positive degree, we consider the following open subset of X:

$$D_X^+(g)=X\smallsetminus V(\widetilde{g}),$$

where $\widetilde{g}\in S$ is any homogeneous polynomial which maps to $g\in S_X$. Note that if h is another homogeneous polynomial of positive degree, we have

$$D_X^+(gh)=D_X^+(g)\cap D_X^+(h).$$

Remark 4.1.14. Every open subset of X is of the form $X\smallsetminus V(J)$, where J is a homogeneous ideal in S. By choosing a system of homogeneous generators for J, we see that this is the union of finitely many open subsets of the form $D_X^+(g)$. Therefore the open subsets $D_X^+(g)$ give a basis of open subsets for the topology of X.

Definition 4.1.15. For every closed subset X of $\mathbf{P}^n$, we define the *affine cone* $C(X)$ over X to be the union in $\mathbf{A}^{n+1}$ of the corresponding lines in X. Note that if X is nonempty, then

$$C(X)=\pi^{-1}(Z)\cup\{0\}.$$

If $X = V(I)$ is nonempty, for a homogeneous ideal $I \subseteq S$, it is clear that $C(X)$ is the zero-locus of I in $\mathbf{A}^{n+1}$. Therefore $C(X)$ is a closed subset of $\mathbf{A}^n$ for every X. Moreover, we see that $\mathcal{O}\big(C(X)\big) = S_X$.

EXERCISE 4.1.16. Show that if G is an irreducible algebraic group acting on a variety X, then every irreducible component of X is invariant under the G-action.

REMARK 4.1.17. Let X be a closed subset of $\mathbf{P}^n$, with corresponding homogeneous radical ideal $I_X \subseteq S$, and let $C(X)$ be the affine cone over X. Since $C(X)$ is k^*-invariant, it follows from the previous exercise that the irreducible components of $C(X)$ are k^*-invariant, as well. By Remark 4.1.5, this means that the minimal prime ideals containing I_X are homogeneous. They correspond to the irreducible components $X_1, \ldots, X_r$ of X, so that the irreducible components of $C(X)$ are $C(X_1), \ldots, C(X_r)$.

4.2. Regular functions on quasi-projective varieties

Our goal in this section is to define a structure sheaf on $\mathbf{P}^n$. The main observation is that if F and G are homogeneous polynomials of the same degree, then we may define a function $\frac{F}{G}$ on the open subset $\mathbf{P}^n \smallsetminus V(G)$ by

$$[a_0, \ldots, a_n] \to \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Indeed, if $\deg(F) = d = \deg(G)$, then

$$\frac{F(\lambda a_0, \ldots, \lambda a_n)}{G(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d \cdot F(a_0, \ldots, a_n)}{\lambda^d \cdot G(a_0, \ldots, a_n)} = \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Let W be a locally closed subset in $\mathbf{P}^n$. A *regular function* on W is a function $f\colon W \to k$ such that for every $p \in W$, there is an open neighborhood $U \subseteq W$ of p and homogeneous polynomials of the same degree F and G such that $G(q) \neq 0$ for every $q \in U$ and

$$f(q) = \frac{F(q)}{G(q)} \quad \text{for all} \quad q \in U.$$

The set of regular functions on W is denoted by $\mathcal{O}(W)$. Note that $\mathcal{O}(W)$ is a k-algebra with respect to the usual operations on functions. For example, if $f_1(q) = \frac{F_1(q)}{G_1(q)}$ for $q \in U_1$ and $f_2(q) = \frac{F_2(q)}{G_2(q)}$ for $q \in U_2$, where U_1 and U_2 are open neighborhoods of p, then $F_1G_2 + F_2G_1$ and G_1G_2 are homogeneous polynomials of the same degree and

$$f_1(q) + f_2(q) = \frac{(F_1G_2 + F_2G_1)(q)}{(G_1G_2)(q)} \quad \text{for} \quad q \in U_1 \cap U_2.$$

Moreover, it is clear that if V is an open subset of W, the restriction to V of a regular function on W is a regular function of V. We thus obtain in this way a subpresheaf $\mathcal{O}_W$ of k-algebras of $\mathcal{F}un_W$. In fact, this is a sheaf, as follows immediately from the fact that regular functions are defined in terms of a local property.

REMARK 4.2.1. Note that if W is a locally closed subset of $\mathbf{P}^n$, then the sheaf $\mathcal{O}_W$ we defined is the one induced from $\mathcal{O}_{\mathbf{P}^n}$ as in Section 2.3.

Our first goal is to show that all spaces defined in this way are algebraic varieties. Let U_i be the open subset defined by $x_i \neq 0$. Note that we have

$$\mathbf{P}^n = \bigcup_{i=0}^{n} U_i.$$

The key fact is the following assertion:

PROPOSITION 4.2.2. *For every i, with $0 \leq i \leq n$, the map*

$$\psi_i \colon \mathbf{A}^n \to U_i, \quad \psi(v_1, \ldots, v_n) = [v_1, \ldots, v_i, 1, v_{i+1}, \ldots, v_n]$$

is an isomorphism in $\mathcal{T}op_k$.

PROOF. It is clear that ψ_i is a bijection, with inverse

$$\phi_i \colon U_i \to \mathbf{A}^n, \quad [u_0, \ldots, u_n] \to (u_0/u_i, \ldots, u_{i-1}/u_i, u_{i+1}/u_i, \ldots, u_n/u_i).$$

In order to simplify the notation, we give the argument for $i = 0$, the other cases being analogous. Consider first a principal affine open subset of $\mathbf{A}^n$, of the form $D(f)$, for some $f \in k[x_1, \ldots, x_n]$. Note that if $\deg(f) = d$, then we can write

$$f(x_1/x_0, \ldots, x_n/x_0) = \frac{g(x_0, \ldots, x_n)}{x_0^d}$$

for a homogeneous polynomial $g \in S$ of degree d. It is then clear that $\phi_0^{-1}\big(D(f)\big) = D^+_{\mathbf{P}^n}(x_0 g)$, hence this is open in U_0. Since the principal affine open subsets in $\mathbf{A}^n$ give a basis for the topology of $\mathbf{A}^n$, we see that ϕ_0 is continuous.

Consider now an open subset of U_0 of the form $D^+_{\mathbf{P}^n}(h)$, for some homogeneous $h \in S$, of positive degree. If we put $h_0 = h(1, x_1, \ldots, x_n)$, we see that $\phi_0\big(D^+_{\mathbf{P}^n}(h)\big) = D(h_0)$ is open in $\mathbf{A}^n$. Since the open subsets of the form $D^+_{\mathbf{P}^n}(h)$ give a basis for the topology of $\mathbf{P}^n$, we conclude that ϕ_0 is a homeomorphism.

We now need to show that if U is open in $\mathbf{A}^n$ and $\alpha \colon U \to k$, then $\alpha \in \mathcal{O}_{\mathbf{A}^n}(U)$ if and only if $\alpha \circ \phi_0 \in \mathcal{O}_{\mathbf{P}^n}\big(\phi_0^{-1}(U)\big)$. If $\alpha \in \mathcal{O}_{\mathbf{A}^n}(U)$, then for every point $p \in U$, we have an open neighborhood $U_p \subseteq U$ of p and $f_1, f_2 \in k[x_1, \ldots, x_n]$ such that

$$f_2(u) \neq 0 \quad \text{and} \quad \alpha(u) = \frac{f_1(u)}{f_2(u)} \quad \text{for all} \quad u \in U_p.$$

As above, we can write

$$f_1(x_1/x_0, \ldots, x_n/x_0) = \frac{g_1(x_0, \ldots, x_n)}{x_0^d} \quad \text{and} \quad f_2(x_1/x_0, \ldots, x_n/x_0) = \frac{g_2(x_0, \ldots, x_n)}{x_0^d}$$

for some homogeneous polynomials $g_1, g_2 \in S$ of the same degree, in which case we see that

$$g_2(v) \neq 0 \quad \text{and} \quad \alpha\big(\phi_0(v)\big) = \frac{g_1(v)}{g_2(v)} \quad \text{for all} \quad v \in \phi_0^{-1}(U_p).$$

Since this holds for every $p \in U$, we see that $\alpha \circ \phi_0$ is a regular function on $\phi_0^{-1}(U)$.

Conversely, suppose that $\alpha \circ \phi_0$ is a regular function on $\phi_0^{-1}(U)$. This means that for every $q \in \phi_0^{-1}(U)$, there is an open neighborhood $V_q \subseteq \phi_0^{-1}(U)$ of q and homogeneous polynomials $h_1, h_2 \in S$ of the same degree such that

$$h_2(v) \neq 0 \quad \text{and} \quad \alpha\big(\phi_0(v)\big) = \frac{h_1(v)}{h_2(v)} \quad \text{for all} \quad v \in V_q.$$

In this case, we have

$$h_2(1, u_1, \ldots, u_n) \neq 0 \quad \text{and} \quad \alpha(u_1, \ldots, u_n) = \frac{h_1(1, u_1, \ldots, u_n)}{h_2(1, u_1, \ldots, u_n)}$$

for all $u = (u_1, \ldots, u_n) \in \phi_0(V_q)$. Since this holds for every $q \in \phi_0^{-1}(U)$, it follows that α is a regular function on U. This completes the proof of the fact that ϕ_0 is an isomorphism. $\square$

COROLLARY 4.2.3. *For every locally closed subset W of $\mathbf{P}^n$, the space $(W, \mathcal{O}_W)$ is an algebraic variety.*

PROOF. It is enough to show the assertion for $W = \mathbf{P}^n$: the general case is then a consequence of Propositions 2.3.5 and 2.5.4. We have already seen that $\mathbf{P}^n$ is a prevariety. In order to show that it is separated, using Proposition 2.5.6, it is enough to show that each $U_i \cap U_j$ is affine and that the canonical morphism

$$\tau_{i,j} \colon \mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j) \tag{4.2.1}$$

is surjective. Suppose that $i < j$ and let us denote by $x_1, \ldots, x_n$ the coordinates on the image of ϕ_i and by $y_1, \ldots, y_n$ the coordinates on the image of ϕ_j. Note that via the isomorphism ϕ_i, the open subvariety $U_i \cap U_j$ is mapped to the open subset

$$\{(u_1, \ldots, u_n) \in \mathbf{A}^n \mid u_j \neq 0\},$$

which is affine, being a principal affine open subset of $\mathbf{A}^n$. Similarly, ϕ_j maps $U_i \cap U_j$ to the open subset

$$\{(u_1, \ldots, u_n) \in \mathbf{A}^n \mid u_{i+1} \neq 0\}.$$

Furthermore, since we have

$$\phi_j \circ \phi_i^{-1}(u_1, \ldots, u_n) = \left(\frac{u_1}{u_j}, \ldots, \frac{u_i}{u_j}, \frac{1}{u_j}, \frac{u_{i+1}}{u_j}, \ldots, \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, \ldots, \frac{u_n}{u_j}\right)$$

for all $(u_1, \ldots, u_n) \in \phi_i(U_i \cap U_j)$, we see that the morphism

$$\tau_{i,j} \colon k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n, x_j^{-1}]$$

satisfies $\tau(x_\ell) = x_\ell$ for all ℓ and $\tau(y_{i+1}) = x_j^{-1}$. Therefore $\tau_{i,j}$ is surjective for all i and j, proving that $\mathbf{P}^n$ is separated. $\square$

EXAMPLE 4.2.4. The map

$$\pi \colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n, \quad \pi(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$$

is a morphism. Indeed, with the notation in the proof of Proposition 4.2.2, it is enough to show that for every i, the induced map $\pi^{-1}(U_i) \to U_i$ is a morphism. However, via the isomorphism $U_i \simeq \mathbf{A}^n$, this map becomes

$$\mathbf{A}^{n+1} \smallsetminus V(x_i) \to \mathbf{A}^n, \quad (x_0, \ldots, x_n) \to (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i),$$

which is clearly a morphism.

DEFINITION 4.2.5. A *projective variety* is an algebraic variety that is isomorphic to a closed subvariety of some $\mathbf{P}^n$. A *quasi-projective* variety is an algebraic variety that is isomorphic to a locally closed subvariety ofsome $\mathbf{P}^n$.

Remark 4.2.6. It follows from definition that if X is a projective variety and Y is a closed subvariety of X, then Y is a projective variety as well. Similarly, if X is a quasi-projective variety and Z is a locally closed subvariety of X, then Z is a quasi-projective variety.

Remark 4.2.7. Every quasi-affine variety is quasi-projective: this follows from the fact that $\mathbf{A}^n$ is isomorphic to an open subvariety of $\mathbf{P}^n$.

Remark 4.2.8. Note that unlike the coordinate ring of an affine variety, the homogeneous coordinate ring of a projective variety $X \subseteq \mathbf{P}^n$ is not an intrinsic invariant: it depends on the embedding in the projective space.

We next show that the distinguished open subsets $D_X^+(g)$ are all affine varieties[1].

Proposition 4.2.9. *For every closed subvariety X of $\mathbf{P}^n$ and every homogeneous element $g \in S_X$ of positive degree, the variety $D_X^+(g)$ is affine.*

Proof. Since X is a closed subvariety of $\mathbf{P}^n$ and $D_X^+(g) = D_{\mathbf{P}^n}^+(\widetilde{g}) \cap X$, where $\widetilde{g} \in S$ is any lift of g, it is enough to prove the assertion when $X = \mathbf{P}^n$. Let $U = D_{\mathbf{P}^n}^+(g)$ and put $d = \deg(g)$.

Consider the regular functions $f_0, \ldots, f_n$ on U given by $f_i(u_0, \ldots, u_n) = \frac{u_i^d}{g(u)}$. Note that they generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. Indeed, since $g \in S_+ = \mathrm{rad}(x_0^d, \ldots, x_n^d)$, it follows that there is m such that $g^m \in (x_0^d, \ldots, x_n^d)$. If we write $g^m = \sum_{i=1}^n h_i x_i^d$ and if we consider the regular functions $\alpha_i \colon U \to k$ given by

$$\alpha_i(u_1, \ldots, u_n) = \frac{h_i(u)}{g(u)^{m-1}},$$

then $\sum_{i=0}^n f_i \cdot \alpha_i = 1$, hence $f_0, \ldots, f_n$ generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. By Proposition 2.3.16, we see that it is enough to show that each subset $U \cap U_i$ is affine, where U_i is the open subset of $\mathbf{P}^n$ defined by $x_i \neq 0$. However, by the isomorphism $U_i \simeq \mathbf{A}^n$ given in Proposition 4.2.2, the open subset $U \cap U_i$ becomes isomorphic to the subset

$$\{u = (u_1, \ldots, u_n) \in \mathbf{A}^n \mid g(u_1, \ldots, u_i, 1, u_{i+1}, \ldots, u_n) \neq 0\},$$

which is affine by Proposition 1.4.18. This completes the proof. $\square$

Since the open subsets $D_X^+(g)$ are affine, they are determined by their rings of regular functions. Our next goal is to describe these rings.

We begin with some general considerations regarding localization in graded rings. If S is a graded ring and $T \subseteq S$ is a multiplicative system consisting of homogeneous elements of S, then the ring of fractions $T^{-1}S$ has an induced grading, in which

$$(T^{-1}S)_m = \left\{ \frac{f}{t} \mid t \in T, f \in S_{\deg(t)+m} \right\}.$$

Note that even if S is $\mathbf{N}$-graded, $T^{-1}S$ is not, in general, $\mathbf{N}$-graded. We will use two special cases. If $g \in S$ is a homogeneous element, then S_g is a graded ring, and we denote by $S_{(g)}$ its degree 0 part. Similarly, if $\mathfrak{p}$ is a homogeneous prime ideal in S and if we take T to be the set of homogeneous elements in $S \smallsetminus \mathfrak{p}$, then $T^{-1}S$ is a graded ring and we denote its degree 0 part by $S_{(\mathfrak{p})}$. Therefore $S_{(g)}$ is the subring

[1] For another proof of this proposition, making use of the Veronese embedding, see Exercise 4.2.23 below.

of S_g consisting of fractions $\frac{h}{g^m}$, where h is a homogeneous element of S, of degree $m \cdot \deg(g)$. Similarly, $S_{(\mathfrak{p})}$ is the subring of $S_{(\mathfrak{p})}$ consisting of all fractions of the form $\frac{f}{h}$, where $f, g \in S$ are homogeneous, of the same degree, with $g \not\in \mathfrak{p}$. Note that $S_{(\mathfrak{p})}$ is a local ring, with maximal ideal

$$\{f/h \in S_{(\mathfrak{p})} \mid f \in \mathfrak{p}\}.$$

Let X be a closed subset of $\mathbf{P}^n$, with corresponding radical ideal I_X and homogeneous coordinate ring S_X. Note that if $h \in S_X$ is homogeneous, of positive degree, we have a morphism of k-algebras

$$\Phi\colon (S_X)_{(h)} \to \mathcal{O}\big(D_X^+(h)\big),$$

such that $\Phi(f/h^m)$ is given by the function $p \to \frac{\widetilde{f}(p)}{\widetilde{h}^m(p)}$, where $\widetilde{f}, \widetilde{h} \in S$ are elements mapping to $f, h \in S_X$, respectively (it is clear that $\Phi(f/h^m)$ is independent of the choice of $\widetilde{f}$ and $\widetilde{h}$).

Proposition 4.2.10. *For every X and h as above, the morphism Φ is an isomorphism.*

Proof. We will prove a more general version in Proposition 4.3.16 below. $\square$

We end this section with the description of the dimension of a closed subset of $\mathbf{P}^n$ in terms of the homogeneous coordinate ring.

Proposition 4.2.11. *If $X \subseteq \mathbf{P}^n$ is a nonempty closed subset, with homogeneous coordinate ring S_X, then $\dim(X) = \dim(S_X) - 1$.*

Proof. Note that the morphism $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ induces a surjective morphism $f\colon C(X) \smallsetminus \{0\} \to X$ whose fibers are 1-dimensional (in fact, they are all isomorphic to $\mathbf{A}^1 \smallsetminus \{0\}$). It follows from Corollary 3.4.3 that

$$\dim\big(C(X)\big) = 1 + \dim(X).$$

Since S_X is the coordinate ring of the affine variety $C(X)$, we obtain the assertion in the proposition. $\square$

Corollary 4.2.12. *If X and Y are nonempty closed subsets of $\mathbf{P}^n$, with $\dim(X) + \dim(Y) \geq n$, then $X \cap Y$ is nonempty and every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$.*

Proof. Note that $\big(C(X) \cap C(Y)\big) \smallsetminus \{0\} = C(X \cap Y) \smallsetminus \{0\}$. It is clear $C(X) \cap C(Y)$ is nonempty, since it contains 0. In this case, it follows from Exercise 3.3.21 that every irreducible component of $C(X) \cap C(Y)$ has dimension

$$\geq \dim\big(C(X)\big) + \dim\big(C(Y)\big) - (n+1) = \dim(X) + \dim(Y) - n + 1.$$

This implies that $C(X) \cap C(Y)$ is not contained in $\{0\}$, hence $X \cap Y$ is non-empty. Moreover, the irreducible components of $C(X) \cap C(Y)$ are of the form $C(Z)$, where Z is an irreducible component of $X \cap Y$, hence

$$\dim(Z) = \dim\big(C(Z)\big) - 1 \geq \dim(X) + \dim(Y) - n.$$

$\square$

EXERCISE 4.2.13. A *hypersurface* in $\mathbf{P}^n$ is a closed subset defined by

$$\{[x_0, \ldots, x_n] \in \mathbf{P}^n \mid F(x_0, \ldots, x_n) = 0\},$$

for some homogeneous polynomial F, of positive degree. Given a closed subset $X \subseteq \mathbf{P}^n$, show that the following are equivalent:

i) X is a hypersurface.
ii) The ideal $I(X)$ is a principal ideal.
iii) All irreducible component of X have codimension 1 in $\mathbf{P}^n$.

Note that if X is any irreducible variety and U is a nonempty open subset of X, then the map taking $Z \subseteq U$ to $\overline{Z}$ and the map taking $W \subseteq X$ to $W \cap U$ give inverse bijections (preserving the irreducible decompositions) between the nonempty closed subsets of U and the nonempty closed subsets of X that have no irreducible component contained in the $X \smallsetminus U$. This applies, in particular, to the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \quad (x_1, \ldots, x_n) \to [1, x_1, \ldots, x_n].$$

The next exercise describes this correspondence at the level of ideals.

EXERCISE 4.2.14. Let $S = k[x_0, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n]$. For an ideal J in R, we put

$$J^{\text{hom}} := \left(f^{\text{hom}} \mid 0 \neq f \in J\right),$$

where $f^{\text{hom}} = x_0^{\deg(f)} \cdot f(x_1/x_0, \ldots, x_n/x_0) \in S$. On the other hand, if $\mathfrak{a}$ is a homogeneous ideal in S, then we put $\overline{\mathfrak{a}} := \{h(1, x_1, \ldots, x_n) \mid h \in \mathfrak{a}\} \subseteq R$.

An ideal $\mathfrak{a}$ in S is called x_0-saturated if $(\mathfrak{a} \colon x_0) = \mathfrak{a}$ (recall that $(\mathfrak{a} \colon x_0) := \{u \in S \mid x_0 u \in \mathfrak{a}\}$).

i) Show that the above maps give inverse bijections between the ideals in R and the x_0-saturated homogeneous ideals in S.
ii) Show that we get induced bijections between the radical ideals in R and the homogeneous x_0-saturated radical ideals in S. Moreover, a homogeneous radical ideal $\mathfrak{a}$ is x_0-saturated if and only if either no irreducible component of $V(\mathfrak{a})$ is contained in the hyperplane $(x_0 = 0)$, or if $\mathfrak{a} = S$.
iii) The above correspondence induces a bijection between the prime ideals in R and the prime ideals in S that do not contain x_0.
iv) Consider the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \ (u_1, \ldots, u_n) \to (1 \colon u_1 \colon \ldots \colon u_n),$$

which allows us to identify $\mathbf{A}^n$ with the complement of the hyperplane $(x_0 = 0)$ in $\mathbf{P}^n$. Show that for every ideal J in R we have $\overline{V_{\mathbf{A}^n}(J)} = V_{\mathbf{P}^n}(J^{\text{hom}})$.
v) Show that for every homogeneous ideal $\mathfrak{a}$ in S, we have $V_{\mathbf{P}^n}(\mathfrak{a}) \cap \mathbf{A}^n = V_{\mathbf{A}^n}(\overline{\mathfrak{a}})$.

EXERCISE 4.2.15. Recall that $GL_{n+1}(k)$ denotes the set of invertible $(n+1) \times (n+1)$ matrices with entries in k. Let $PGL_{n+1}(k)$ denote the quotient $GL_{n+1}(k)/k^*$, where k^* acts on $GL_{n+1}(k)$ by

$$\lambda \cdot (a_{i,j})_{i,j} = (\lambda a_{i,j})_{i,j}.$$

i) Show that $PGL_{n+1}(k)$ has a natural structure of linear algebraic group, and that it is irreducible.

ii) Prove that $PGL_{n+1}(k)$ acts algebraically on $\mathbf{P}^n$.

DEFINITION 4.2.16. Two subsets of $\mathbf{P}^n$ are *projectively equivalent* if they differ by an automorphism in $PGL_{n+1}(k)$ (we will see later that these are, indeed, all automorphisms of $\mathbf{P}^n$).

DEFINITION 4.2.17. A *linear subspace* of $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ defined by an ideal generated by homogeneous polynomials of degree one. A *hyperplane* is a linear subspace of codimension one.

EXERCISE 4.2.18. Consider the projective space $\mathbf{P}^n$.

i) Show that a closed subset Y of $\mathbf{P}^n$ is a linear subspace if and only if the affine cone $C(Y) \subseteq \mathbf{A}^{n+1}$ is a linear subspace.
ii) Show that if L is a linear subspace in $\mathbf{P}^n$ of dimension r, then there is an isomorphism $L \simeq \mathbf{P}^r$.
iii) Show that the hyperplanes in $\mathbf{P}^n$ are in bijection with the points of "another" projective space $\mathbf{P}^n$, called the *dual* of $\mathbf{P}^n$, and usually denoted by $(\mathbf{P}^n)^*$. We denote the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H by $[H]$.
iv) Show that the subset
$$\big\{\big(p,[H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H\big\}$$
is closed in $\mathbf{P}^n \times (\mathbf{P}^n)^*$.
v) Show that given two sets of points in $\mathbf{P}^n$
$$\Gamma = \{P_0, \ldots, P_{n+1}\} \text{ and } \Gamma' = \{Q_0, \ldots, Q_{n+1}\},$$
such that no $(n+1)$ points in the same set lie in a hyperplane, there is a unique $A \in PGL_{n+1}(k)$ such that $A \cdot P_i = Q_i$ for every i.

EXERCISE 4.2.19. Let $X \subseteq \mathbf{P}^n$ be an irreducible closed subset of codimension r. Show that if $H \subseteq \mathbf{P}^n$ is a hypersurface such that X is not contained in H, then every irreducible component of $X \cap H$ has codimension $r+1$ in $\mathbf{P}^n$.

EXERCISE 4.2.20. Let $X \subseteq \mathbf{P}^n$ be a closed subset of dimension r. Show that there is a linear space $L \subseteq \mathbf{P}^n$ of dimension $(n-r-1)$ such that $L \cap X = \emptyset$.

EXERCISE 4.2.21. (The Segre embedding). Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1) - 1$, and let us denote the coordinates on $\mathbf{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\phi_{m,n} \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism
$$f_{m,n} \colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n], \quad f_{m,n}(z_{i,j}) = x_i y_j.$$
Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if X and Y are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.

EXERCISE 4.2.22. (The Veronese embedding). Let n and d be positive integers, and let $M_0, \ldots, M_N$ be all monomials in $k[x_0, \ldots, x_n]$ of degree d (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism $\nu_{n,d} \colon \mathbf{P}^n \to \mathbf{P}^N$ that takes the point $[a_0, \ldots, a_n]$ to the point $\big[M_0(a), \ldots, M_N(a)\big]$.
2) Consider the ring homomorphism $f_d \colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_d)$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\nu_{n,d}$ (in particular, this image is closed).
3) Show that $\nu_{n,d}$ is a closed immersion.
4) Show that if Z is a hypersurface of degree d in $\mathbf{P}^n$ (this means that $I(Z) = (F)$, where F is a homogeneous polyomial of degree d), then there is a hyperplane H in $\mathbf{P}^N$ such that for every projective variety $X \subseteq \mathbf{P}^n$, the morphism $\nu_{n,d}$ induces an isomorphism between $X \cap Z$ and $\nu_{n,d}(X) \cap H$. This shows that the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane.
5) The *rational normal curve* in $\mathbf{P}^d$ is the image of the Veronese embedding $\nu_{1,d} \colon \mathbf{P}^1 \to \mathbf{P}^d$, mapping $[a, b]$ to $[a^d, a^{d-1}b, \ldots, b^d]$ (more generally, this term applies to any subvariety of $\mathbf{P}^d$ projectively equivalent to this one). Show that the rational normal curve is the zero-locus of the 2×2-minors of the matrix
$$\begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$

EXERCISE 4.2.23. Use the Veronese embedding to deduce the assertion in Proposition 4.2.9 from the case when h is a linear form (which follows from Proposition 4.2.2).

EXERCISE 4.2.24. A *plane Cremona transformation* is a birational map of $\mathbf{P}^2$ into itself. Consider the following example of *quadratic* Cremona transformation: $\phi \colon \mathbf{P}^2 \to \mathbf{P}^2$, given by $\phi(x \colon y \colon z) = (yz \colon xz \colon xy)$, when no two of x, y, or z are zero.

1) Show that ϕ is birational, and its own inverse.
2) Find open subsets $U, V \subset \mathbf{P}^2$ such that ϕ induces an isomorphism $U \simeq V$.
3) Describe the open sets on which ϕ and ϕ^{-1} are defined.

4.3. A generalization: the MaxProj construction

We now describe a generalization of the constructions in the previous two sections. A key idea introduced by Grothendieck in algebraic geometry is that it is often better to study morphisms $f \colon X \to Y$, instead of varieties X (the case of a variety being recovered as the special case when Y is a point). More precisely, instead of studying varieties with a certain property, one should extend this property to morphisms and study it in this context. We begin with one piece of terminology.

DEFINITION 4.3.1. Given a variety Y, a *variety over* Y is a morphism $f \colon X \to Y$, where X is another variety. A morphism between varieties $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$ is a morphism of varieties $g \colon X_1 \to X_2$ such that $f_2 \circ g = f_1$. It is clear that we can compose morphisms of varieties over Y and we get, in this way, a category that we denote by Var/X.

Following the above philosophy, we introduce in this section the Proj construction, that allows us to study projective varieties over Y, when Y is affine (as we will see, these are simply closed subvarieties of a product $Y \times \mathbf{P}^n$). We will return later to the case when Y is an arbitrary variety, after discussing quasi-coherent sheaves.

The setting is the following: we fix an $\mathbf{N}$-graded, reduced, finitely generated k-algebra $S = \bigoplus_{m\in\mathbf{N}} S_m$. This implies that S_0 is a finitely generated k-algebra and it is also easy to see that each S_m is a finitely generated S_0-module. We put $S_+ = \bigoplus_{m>0} S_m$.

Exercise 4.3.2. Given homogeneous elements $t_0, \ldots, t_n \in S_+$, show that they generate S as an S_0-algebra if and only if they generate S_+ as an ideal.

For the sake of simplicity, we always assume that S is generated as an S_0-algebra by S_1. This condition is equivalent with the fact that S is isomorphic, as a graded ring, to the quotient of $S_0[x_0, \ldots, x_n]$ by a homogeneous ideal, where the grading on this polynomial ring is given by the total degree of the monomials. Note that by the above exercise, our assumption implies that S_1 generates S_+ as an ideal.

Consider the affine varieties $W = \mathrm{MaxSpec}(S)$ and $W_0 = \mathrm{MaxSpec}(S_0)$ (see Exercise 2.2.17 for the notation). The inclusion $S_0 \hookrightarrow S$ corresponds to a morphism $f\colon W \to W_0$. The grading on S translates into an algebraic action of the torus k^* on W, as follows. We have a morphism

$$\alpha\colon k^* \times W \to W$$

corresponding to the k-algebra homomorphism $S \to k[t, t^{-1}] \otimes_k S$ mapping $\sum_i f_i$ to $\sum_i t^i f_i$, where $f_i \in S_i$ for all i. One can check directly that this gives an action of k^* on W, but we prefer to argue as follows: let us choose a surjective graded homomorphism of S_0-algebras $\phi\colon S_0[x_0, \ldots, x_n] \to S$, corresponding to a closed immersion $j\colon W \hookrightarrow W_0 \times \mathbf{A}^{n+1}$ such that if $p\colon W_0 \times \mathbf{A}^{n+1} \to W_0$ is the first projection, we have $p \circ j = f$. As before, we have a morphism

$$\beta\colon k^* \times W_0 \times \mathbf{A}^{n+1} \to W_0 \times \mathbf{A}^{n+1}.$$

Since ϕ is a graded homomorphism, we see that the two morphisms are compatible via j, in the sense that

$$j\big(\alpha(\lambda, w)\big) = \beta\big(\lambda, j(w)\big) \quad \text{for all} \quad \lambda \in k^*, w \in W.$$

It is straightforward to check that

$$\beta(\lambda, w_0, x_0, \ldots, x_n) = (w_0, \lambda x_0, \ldots, \lambda x_n) \quad \text{for all} \quad \lambda \in k^*, w_0 \in W_0, (x_0, \ldots, x_n) \in \mathbf{A}^{n+1}.$$

Therefore β gives an algebraic action of k^* on $W_0 \times \mathbf{A}^{n+1}$, and thus α gives an algebraic action of k^* on W. We will keep using this embedding for describing the action of k^* on W. To simplify the notation, we will write $\lambda \cdot w$ for $\alpha(\lambda, w)$.

Lemma 4.3.3. *Given the above action of k^* on W, the following hold:*

i) *An orbit consists either of one point or it is 1-dimensional.*
ii) *A point is fixed by the k^*-action if and only if it lies in $V(S_+)$.*
iii) *If O is a 1-dimensional orbit, then O is a closed subset of $W \smallsetminus V(S_+)$, $\overline{O} \simeq \mathbf{A}^1$, and $\overline{O} \cap V(S_+)$ consists of one point.*

Proof. By embedding W in $W_0 \times \mathbf{A}^{n+1}$ as above, we reduce the assertions in the lemma to the case when $W = W_0 \times \mathbf{A}^{n+1}$, in which case they are all clear. Note that via this embedding, we have $V(S_+) = W_0 \times \{0\}$. $\square$

REMARK 4.3.4. By arguing as in Remark 4.1.5, we see that a closed subset $Z \subseteq W$ is invariant by the k^*-action (that is, $\lambda \cdot Z = Z$ for every $\lambda \in k^*$) if and only if the corresponding ideal $I_W(Z)$ is homogeneous.

DEFINITION 4.3.5. Given S as above, we define $\mathrm{MaxProj}(S)$ to be the set of one-dimensional orbit closures for the action of k^* on W. Since every such orbit is clearly irreducible, being the image of a morphism $k^* \to W$, it follows from Lemma 4.3.3 and Remark 4.3.4 that these orbit closures are in bijection with the homogeneous prime ideals $\mathfrak{q} \subseteq S$ such that $S_+ \not\subseteq \mathfrak{q}$ and $\dim(S/\mathfrak{q}) = 1$.

We put a topology on $X = \mathrm{MaxProj}(S)$ by declaring that a subset is closed if it consists of all 1-dimensional orbit closures contained in some torus-invariant closed subset of W. Equivalently, the closed subsets are those of the form

$$V(I) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid I \subseteq \mathfrak{q}\},$$

for some homogeneous ideal $I \subseteq S$. The assertions in the next lemma, which are straightforward to prove, imply that this gives indeed a topology on $\mathrm{MaxProj}(S)$.

LEMMA 4.3.6. *With the above notation, the following hold:*

i) *We have* $V(0) = \mathrm{MaxProj}(S)$ *and* $V(S) = \emptyset$.

ii) *For every two homogeneous ideals I and J in S, we have*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

iii) *For every family $(I_\alpha)_\alpha$ of homogeneous ideals in S, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

Since every homogeneous ideal is generated by finitely many homogeneous elements, we see that every open set can be written as a finite union of sets of the form

$$D_X^+(f) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid f \not\in \mathfrak{q}\},$$

where $f \in S$ is a homogeneous element. In fact, we may take f of positive degree, since if $t_0, \ldots, t_n \in S_1$ generate S_+, we have

$$D_X^+(f) = \bigcup_{i=0}^n D_X^+(t_i f).$$

As a special case of this equality for $f = 1$, we have

$$\mathrm{MaxProj}(S) = D_X^+(t_0) \cup \ldots \cup D_X^+(t_n).$$

REMARK 4.3.7. It is clear that if I is a homogeneous ideal in S, then $V(I) = V\big(\mathrm{rad}(I)\big)$. Moreover, if

$$I' = \{f \in S \mid f \cdot S_+ \subseteq \mathrm{rad}(I)\},$$

then $V(I) = V(I')$.

For future reference, we give the following variant of graded Nullstellensatz.

PROPOSITION 4.3.8. *Let S be a graded ring as in the proposition. If I is a homogeneous, radical ideal in S, and $f \in S$ is homogeneous, such that $f \in \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{MaxProj}(S)$ with $\mathfrak{q} \supseteq I$, then $f \cdot S_+ \subseteq I$. If $\deg(f) > 0$, then $f \in I$.*

PROOF. We first prove the last assertion, assuming $\deg(f) > 0$. After writing S as a quotient of a polynomial ring over S_0, we see that we may assume that $S = A[x_0, \ldots, x_n]$, with the standard grading. Recall that we take $W_0 = \mathrm{MaxSpec}(S_0)$ and $W = \mathrm{MaxSpec}(S) = W_0 \times \mathbf{A}^{n+1}$. Let $Y \subseteq W$ be the closed subset defined by I. Note that W is k^*-invariant. Our assumption says that f vanishes on $\{w_0\} \times L$, whenever L is a line in $\mathbf{A}^{n+1}$ with $\{w_0\} \times \mathbf{A}^{n+1} \subseteq Y$. On the other hand, since $\deg(f) > 0$, we see that f automatically vanishes along $W_0 \times \{0\}$, hence f vanishes along Y (we use the fact that Y is a union of k^*-orbits). We thus conclude that $f \in I$. The first assertion in the proposition now follows by applying what we know to each product fg, with $g \in S_1$. $\square$

Given an ideal $\mathfrak{q} \in \mathrm{MaxProj}(S)$, let T denote the set of homogeneous elements in $S \smallsetminus \mathfrak{q}$. Recall that the ring of fractions $T^{-1}S$ carries a natural grading, whose degree 0 part is denoted by $S_{(\mathfrak{q})}$. This is a local ring, with maximal ideal $\mathfrak{m}_{\mathfrak{q}} := \mathfrak{q} \cdot T^{-1}S \cap S_{(\mathfrak{q})}$. Similarly, given a homogeneous element $f \in S$, the localization S_f carries a natural grading, whose degree 0 part is denoted $S_{(f)}$.

LEMMA 4.3.9. *For every $t \in S_1$, the following hold:*

i) *We have an isomorphism of graded rings $S_t \simeq S_{(t)}[x, x^{-1}]$.*
ii) *Every homogeneous ideal in S_t is of the form $\bigoplus_{m \in \mathbf{Z}} (I \cap S_{(t)}) t^m$.*
iii) *We have a homeomorphism between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$.*
iv) *For every $\mathfrak{q} \in \mathrm{MaxProj}(S)$, the residue field of $S_{(\mathfrak{q})}$ is equal to k.*

PROOF. Since the element $\frac{t}{1} \in S_t$ has degree 1 and is invertible, it follows easily that the homomorphism of graded $S_{(t)}$-algebras

$$S_{(t)}[x, x^{-1}] \to S_t$$

that maps x to $\frac{t}{1}$ is an isomorphism. This gives i) and the assertion in ii) is straightforward to check.

It is clear that localization induces a bijection between the homogeneous prime ideals in S that do not contain t and the homogeneous prime ideals in S_t. Moreover, it follows from ii) that every such prime ideal in S_t is of the form $\bigoplus_{m \in \mathbf{Z}} \mathfrak{p} t^m$, for a unique prime ideal $\mathfrak{p}$ in $S_{(t)}$. If $\mathfrak{q} \subseteq S$ corresponds to $\mathfrak{p} \subseteq S_{(t)}$, then

$$(S/\mathfrak{q})_t \simeq (S_{(t)}/\mathfrak{p})[x, x^{-1}], \tag{4.3.1}$$

hence

$$\dim(S/\mathfrak{q}) = \dim\big((S/\mathfrak{q})_t\big) = \dim(S_{(t)}/\mathfrak{p}) + 1.$$

Therefore $\mathfrak{q}$ lies in $\mathrm{MaxProj}(S)$ if and only if $\mathfrak{p}$ is a maximal ideal in $S_{(t)}$. This gives the bijection between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$ and it is straightforward to check, using the definitions of the two topologies, that this is a homeomorphism.

Finally, given any $\mathfrak{q} \in \mathrm{MaxProj}(S)$, we can find $t \in S_1$ such that $\mathfrak{q} \in D^+(t)$. If $\mathfrak{p}$ is the corresponding ideal in $S_{(t)}$, then the isomorphism (4.3.1) implies that the residue field of $S_{(\mathfrak{q})}$ is isomorphic as a k-algebra to the residue field of $(S_{(t)})_{\mathfrak{p}}$, hence it is equal to k. $\square$

We now define a sheaf of functions on $X = \mathrm{MaxProj}(S)$, with values in k, as follows. For every open subset U in X, let $\mathcal{O}_X(U)$ be the set of functions $\phi\colon U \to k$ with the following property: for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and homogeneous elements $f, g \in S$ of the same degree such that for every $\mathfrak{q} \in U_x$, we have $g \not\in \mathfrak{q}$ and $\phi(\mathfrak{q})$ is equal to the image of $\frac{f}{g}$ in the residue field of $S_{(\mathfrak{q})}$, which is equal to k by Lemma 4.3.9. It is straightforward to check that $\mathcal{O}_X(U)$ is a k-subalgebra of $\mathcal{F}un_X(U)$ and that, with respect to restriction of functions, $\mathcal{O}_X$ is a sheaf. This is the *sheaf of regular functions* on X. From now on, we denote by $\mathrm{MaxProj}(S)$ the object $(X, \mathcal{O}_X)$ in $\mathcal{T}op_k$.

REMARK 4.3.10. It is clear from the definition that we have a morphism in $\mathcal{T}op_k$

$$\mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$$

that maps $\mathfrak{q}$ to $\mathfrak{q} \cap S_0$.

PROPOSITION 4.3.11. *If we have a surjective, graded homomorphism $\phi\colon S \to T$, then we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{MaxProj}(T) \arrow[r, "j"] \arrow[d] & \mathrm{MaxProj}(S) \arrow[d, "f"] \\
\mathrm{MaxSpec}(T_0) \arrow[r, "i"] & \mathrm{MaxProj}(S_0),
\end{tikzcd}$$

in which i is a closed immersion and j given an isomorphism onto $V(I)$ (with the induced sheaf from the ambient space)[2]*, where $I = \ker(\phi)$.*

PROOF. Note first that since ϕ is surjective, the induced homomorphism $S_0 \to T_0$ is surjective as well, hence the induced morphism $i\colon \mathrm{MaxSpec}(T_0) \to \mathrm{MaxSpec}(S_0)$ is a closed immersion. Since ϕ is graded and surjective, we have $T_+ = \phi(S_+)$ and $S_+ = \phi^{-1}(T_+)$, hence $S_+ \subseteq \phi^{-1}(\mathfrak{p})$ if and only if $T_+ \subseteq \mathfrak{p}$. We can thus define $j\colon \mathrm{MaxProj}(T) \to \mathrm{MaxProj}(S)$ by $j(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. It is straightforward to see that the diagram in the proposition is commutative and that j gives a homeomorphism of $\mathrm{MaxProj}(T)$ onto the closed subset $V(I)$ of $\mathrm{MaxProj}(S)$. Furthermore, it is easy to see, using the definition, that if U is an open subset of $V(I)$, then a function $\phi\colon U \to k$ has the property that $\phi \circ j$ is regular on $j^{-1}(U)$ if and only if it can be locally extended to a regular function on open subsets in $\mathrm{MaxProj}(S)$. This gives the assertion in the proposition. $\square$

We now consider in detail the case when $S = A[x_0, \ldots, x_n]$, with the standard grading. As before, let $W_0 = \mathrm{MaxSpec}(A)$. We have seen that a point $\mathfrak{p}$ in $X = \mathrm{MaxProj}(S)$ corresponds to a subset in $W_0 \times \mathbf{A}^{n+1}$, of the form $\{w_0\} \times L$, where L is a 1-dimensional linear subspace in k^{n+1}, corresponding to a point in $\mathbf{P}^n$. We thus have a bijection between $\mathrm{MaxProj}(S)$ and $W_0 \times \mathbf{P}^n$. Moreover, since $x_0, \ldots, x_n$ span S_1, we see that

$$X = \bigcup_{i=0}^{n} D_X^+(x_i).$$

The above bijection induces for every i a bijection between $D_X^+(x_i)$ and $W_0 \times D_{\mathbf{P}^n}^+(x_i)$. In fact, this is the same as the homeomorphism between $D_X^+(x_i)$ and

$$\mathrm{MaxSpec}\big(A[x_0, \ldots, x_n]_{(x_i)}\big) = \mathrm{MaxSpec}\big(A[x_0/x_i, \ldots, x_n/x_i]\big)$$

given by assertion iii) in Lemma 4.3.9. Furthermore, arguing as in the proof of Proposition 4.2.2, we see that each of these homeomorphisms gives an isomorphism of objects in $\mathcal{T}op_k$. We thus obtain the following

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[2]Once we will show that $\mathrm{MaxProj}(S)$ and $\mathrm{MaxProj}(T)$ are algebraic varieties, this simply says that j is a closed immersion.

PROPOSITION 4.3.12. *If $S = A[x_0, \ldots, x_n]$, with the standard grading, and $W_0 = \mathrm{MaxSpec}(A)$, then we have an isomorphism*

$$\mathrm{MaxProj}(S) \simeq W_0 \times \mathbf{P}^n$$

of varieties over W_0.

COROLLARY 4.3.13. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then $\mathrm{MaxProj}(S)$ is a quasi-projective variety.*

PROOF. By the assumption on S, we have a graded, surjective morphism of S_0-algebras

$$S_0[x_0, \ldots, x_n] \to S.$$

If $W_0 = \mathrm{MaxSpec}(S_0)$, then it follows from Propositions 4.3.11 and 4.3.12 that we have a closed immersion

$$\mathrm{MaxProj}(S) \hookrightarrow \mathrm{MaxProj}\big(S_0[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n,$$

which gives the assertion in the corollary, since a product of quasi-projective varieties is quasi-projective by Exercise 4.2.21. ☐

REMARK 4.3.14. If X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X, then $\mathrm{MaxProj}(S_X) \simeq X$. More generally, suppose that A is a reduced, finitely generated k-algebra, $W_0 = \mathrm{MaxSpec}(A)$, and X is a closed subvariety of $W_0 \times \mathbf{P}^n$. If I is a radical, homogeneous ideal in $A[x_0, \ldots, x_n]$ such that $X = V(I)$, then

$$X \simeq \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big).$$

Indeed, the surjection

$$A[x_0, \ldots, x_n] \to A[x_0, \ldots, x_n]/I$$

induces by Proposition 4.3.11 a closed immersion

$$\iota\colon \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big) \hookrightarrow \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big).$$

It is then clear that, via the isomorphism $\mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n$ provided by Proposition 4.3.12, the image of ι is equal to X.

PROPOSITION 4.3.15. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then for every homogeneous $f \in S$, of positive degree, the open subset $D_X^+(f) \subseteq X = \mathrm{MaxProj}(S)$ is affine.*

PROOF. By Proposition 4.3.11, it is enough to prove this when $S = S_0[x_0, \ldots, x_n]$. The argument in this case follows the one in the proof of Proposition 4.2.9. ☐

We now give a generalization of Proposition 4.2.10 describing the regular functions on the affine open subsets $D_X^+(f)$ in $\mathrm{MaxProj}(S)$.

PROPOSITION 4.3.16. *Let S be a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, and let $X = \mathrm{MaxProj}(S)$. For every homogeneous $f \in S$, of positive degree, consider the homomorphism*

$$\Phi\colon S_{(f)} \to \mathcal{O}\big(D_X^+(f)\big)$$

that maps $\frac{g}{f^m}$ to the function taking $\mathfrak{q} \in D_X^+(f)$ to the image of $\frac{g}{f^m}$ in the residue field of $S_{(\mathfrak{q})}$, which is isomorphic to k. Then Φ is an isomorphism.

PROOF. The proof is similar to that of Proposition 1.4.7. We first show that Φ is injective. Suppose that $\frac{g}{f^m}$ lies in the kernel of Φ. In this case, for every $\mathfrak{q} \in X \smallsetminus V(f)$, we have $g \in \mathfrak{q}$. This implies that $fg \in \mathfrak{q}$ for every $\mathfrak{q} \in X$, hence $fg = 0$ by Proposition 4.3.8, hence $\frac{g}{f^m} = 0$ in $(S_X)_{(f)}$. This proves the injectivity of Φ.

In order to prove the surjectivity of Φ, consider $\phi \in \mathcal{O}\big(D_X^+(f)\big)$. By hypothesis, and using the quasi-compactness of $D_X^+(f)$, we may write

$$D_X^+(f) = V_1 \cup \ldots \cup V_r,$$

for some open subsets V_i such that for every i, there are $g_i, h_i \in S$ homogeneous of the same degree such that for every $\mathfrak{q} \in V_i$, we have $h_i \not\in \mathfrak{q}$ and $\phi(\mathfrak{q})$ is the image of $\frac{g_i}{h_i}$ in the residue field of $S_{(\mathfrak{q})}$. We may assume that $V_i = X \smallsetminus V(f_i)$ for $1 \leq i \leq r$, for some homogeneous $f_i \in S$, of positive degree. Since $h_i \not\in \mathfrak{q}$ for every $\mathfrak{q} \in X \smallsetminus V(f_i)$, it follows from Proposition 4.3.8 that $f_i \in \mathrm{rad}(h_i)$. After possibly replacing f_i by a suitable power, we may assume that $f_i \in (h_i)$ for all i. Finally, after multiplying both g_i and h_i by the same homogeneous element, we may assume that $f_i = h_i$ for all i.

We know that for $u \in X \smallsetminus V(g_i g_j)$ the two fractions $\frac{g_i(u)}{h_i(u)}$ and $\frac{g_j(u)}{h_j(u)}$ have the same image in the residue field of every $S_{(\mathfrak{q})}$. By the injectivity statement we have already proved, this implies that

$$\frac{g_i}{h_i} = \frac{g_j}{h_j} \quad \text{in} \quad S_{h_i h_j}.$$

Therefore there is a positive integer N such that

$$(h_i h_j)^N (g_i h_j - g_j h_i) = 0 \quad \text{for all} \quad i, j.$$

After replacing each g_i and h_i by $g_i h_i^N$ and h_i^{N+1}, respectively, we see that we may assume that

$$g_i h_j - g_j h_i = 0 \quad \text{for all} \quad i, j.$$

On the other hand, since

$$D_X^+(f) = \bigcup_{i=1}^r D_X^+(h_i),$$

we have

$$V(f) = V(h_1, \ldots, h_r),$$

and therefore Proposition 4.3.8 implies that $f \in \mathrm{rad}(h_1, \ldots, h_r)$. We can thus write

$$f^m = \sum_{i=1}^r a_i h_i \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a_1, \ldots, a_r \in S.$$

Moreover, by only considering the terms in $S_{m \cdot \deg(f)}$, we see that we may assume that each a_i is homogeneous, with $\deg(a_i) + \deg(h_i) = m \cdot \deg(f)$.

In order to complete the proof, it is enough to show that

$$\phi = \Phi\left(\frac{a_1 g_1 + \ldots + a_r g_r}{f^m}\right).$$

Note that for $\mathfrak{q} \in D_X^+(h_j)$, we have

$$\frac{g_j}{h_j} = \frac{a_1 g_1 + \ldots + a_r g_r}{f^m} \quad \text{in} \quad S_{(\mathfrak{q})}$$

since

$$h_j \cdot \sum_{i=1}^r a_i g_i = \sum_{i=1}^r a_i h_i g_j = f^m g_j.$$

This completes the proof. $\square$

REMARK 4.3.17. Suppose that S is an $\mathbf{N}$-graded k-algebra as above and

$$f\colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$$

is the corresponding morphism. If $a \in S_0$ and we consider the $\mathbf{N}$-graded k-algebra S_a, then we have a map

$$j\colon \mathrm{MaxProj}(S_a) \to \mathrm{MaxProj}(S)$$

that maps $\mathfrak{q}$ to its inverse image in S. This gives an open immersion, whose image is $f^{-1}\big(D_Y(a)\big)$: this follows by choosing generators $t_1, \ldots, t_r \in S_1$ of S as an S_0-algebra, and by showing that for every i, the induced map

$$\mathrm{MaxSpec}\big((S_a)_{(t_i)}\big) \to \mathrm{MaxSpec}\big(S_{(t_i)}\big)$$

is an open immersion, with image equal to the principal affine open subset corresponding to $\frac{a}{1} \in S_{(t_i)}$.

REMARK 4.3.18. Suppose again that S is an $\mathbf{N}$-graded k-algebra as above and $f\colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$ is the corresponding morphism. If J is an ideal in S_0, then the inverse image $f^{-1}\big(V(J)\big)$ is the closed subset $V(J \cdot S)$. This is the image of the closed immersion

$$\mathrm{MaxProj}\big(S/\mathrm{rad}(J \cdot S)\big) \hookrightarrow \mathrm{MaxProj}(S)$$

(see Proposition 4.3.11).

REMARK 4.3.19. For every S as above, we have a surjective morphism

$$\pi\colon \mathrm{MaxSpec}(S) \smallsetminus V(S_+) \to \mathrm{MaxProj}(S).$$

Since all fibers are of dimension 1 (in fact, they are all isomorphic to $\mathbf{A}^1 \smallsetminus \{0\}$), we conclude that

$$\dim\big(\mathrm{MaxProj}(S)\big) = \dim\big(\mathrm{MaxSpec}(S) \smallsetminus V(S_+)\big) - 1 \le \dim(S) - 1.$$

Moreover, this is an equality, unless every irreducible component of maximal dimension of $\mathrm{MaxSpec}(S)$ is contained in $V(S_+)$, in which case we have $\dim(S) = \dim(S_0)$.

EXERCISE 4.3.20. Show that if S is an $\mathbf{N}$-graded k-algebra as above and $X = \mathrm{MaxProj}(S)$, then for every $\mathfrak{q} \in X$, there is a canonical isomorphism

$$\mathcal{O}_{X,\mathfrak{q}} \simeq S_{(\mathfrak{q})}.$$

EXERCISE 4.3.21. Let $\phi\colon S \to T$ be a graded k-algebra homomorphism, where both S and T are $\mathbf{N}$-graded k-algebras, as above. Show that if $\phi_m\colon S_m \to T_m$ is an isomorphism for $m \gg 0$, then we have an isomorphism $\mathrm{MaxProj}(T) \to \mathrm{MaxProj}(S)$ that maps $\mathfrak{q}$ to $\phi^{-1}(\mathfrak{q})$.

EXERCISE 4.3.22. Let S be an $\mathbf{N}$-graded k-algebra as above. Given $d \ge 1$, we consider the $\mathbf{N}$-graded k-algebra

$$S^{(d)} = \bigoplus_{j \ge 0} S_j^{(d)}, \quad \text{where} \quad S_j^{(d)} = S_{jd}.$$

i) Show that $S^{(d)}$ is generated in degree 1.

ii) Show that the inclusion $S^{(d)} \hookrightarrow S$ induces an isomorphism of algebraic varieties $\mathrm{MaxProj}(S) \to \mathrm{MaxProj}(S^{(d)})$ that maps an ideal $\mathfrak{p}$ to $\mathfrak{p} \cap S^{(d)}$.

CHAPTER 5

Proper, finite, and flat morphisms

In this chapter we discuss an algebraic analogue of compactness for algebraic varieties, *completeness*, and a corresponding relative notion, *properness*. In particular, we prove Chow's lemma, which relates arbitrary complete varieties to projective varieties. As a special case of proper morphisms, we have finite morphisms, which we have already encountered in the case of morphisms of affine varieties. We prove an irreducibility criterion for varieties that admit a proper morphism onto an irreducible variety, such that all fibers are irreducible, of the same dimension; we also prove the semicontinuity of fiber dimension for proper morphisms. Finally we discuss an algebraic property, *flatness*, that is very important in the study of families of algebraic varieties.

5.1. Proper morphisms

We will define a notion that is analogous to that of *compactness* for usual topological spaces. Recall that the Zariski topology on algebraic varieties is quasi-compact, but not Hausdorff. As we have seen, *separatedness* is the algebraic counterpart to the Hausdorff property. A similar point of view allows us to define the algebraic counterpart of compactness. The key observation is the following.

REMARK 5.1.1. Let us work in the category of Hausdorff topological spaces. A topological space X is compact if and only if for every other topological space Z, the projection map $p\colon X \times Z \to Z$ is closed. More generally, a continuous map $f\colon X \to Y$ is proper (recall that this means that for every compact subspace $K \subseteq Y$, its inverse image $f^{-1}(K)$ is compact) if and only if for every continuous map $g\colon Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed.

DEFINITION 5.1.2. A morphism of varieties $f\colon X \to Y$ is *proper* if for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is closed. A variety X is *complete* if the morphism from X to a point is proper, that is, for every variety Z, the projection $X \times Z \to Z$ is closed.

REMARK 5.1.3. Note that if $f\colon X \to Y$ is a proper morphism, then it is closed (simply apply the definition to the identity map $Z = Y \to Y$.

We collect in the next proposition some basic properties of this notion.

PROPOSITION 5.1.4. *In what follows all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper morphisms, then $g \circ f$ is a proper morphism.*
ii) *If $f\colon X \to Y$ is a proper morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is proper.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is proper.*

iv) *If X is a complete variety, then any morphism $f\colon X \to Y$ is proper.*
v) *If $f\colon X \to Y$ is a morphism and Y has an open cover $Y = U_1 \cup \ldots \cup U_r$ such that each induced morphism $f^{-1}(U_i) \to U_i$ is proper, then f is proper.*

Proof. Under the assumption in i), given any morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{tikzcd} X \times_Y Y \times_Z W \arrow[r, "p"] \arrow[d] & Y \times_Z W \arrow[r, "q"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

In this case, the big rectangle is Cartesian. The assumption implies that the morphisms p and q are closed, hence the composition $q \circ p$ is closed. This gives i).

For ii), we argue similarly: given a morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{tikzcd} X \times_Y Z \times_Z W \arrow[r] \arrow[d, "p"'] & X \times_Y Z \arrow[r] \arrow[d, "q"] & X \arrow[d, "f"] \\ W \arrow[r] & Z \arrow[r, "g"] & Y. \end{tikzcd}$$

Since the big rectangle is Cartesian, it follows from the hypothesis that p is closed. This proves that q is proper.

If $i\colon X \hookrightarrow Y$ is a closed immersion, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is a closed immersion, whose image is $g^{-1}\big(i(X)\big)$ (see Example 2.4.8). Since every closed immersion is clearly closed, it follows that i is proper.

Suppose now that X is a complete variety and $f\colon X \to Y$ is an arbitrary morphism. We can factor f as

$$X \overset{i_f}{\hookrightarrow} X \times Y \xrightarrow{p} Y,$$

where i_f is the graph morphism associated to f and p is the projection. The map p is proper, by property ii), since X is complete, and i_f is proper by iii), being a closed immersion, since X and Y are separated. Therefore the composition $f = p \circ i_f$ is proper, proving iv).

Under the assumptions in v), consider a morphism $g\colon Z \to Y$ and let $p\colon X \times_Y Z \to Z$ be the induced morphism. We have an induced open cover $Z = \bigcup_{i=1}^r g^{-1}(U_i)$ and for every i, we have an induced morphism

$$p_i\colon p^{-1}\big(g^{-1}(U_i)\big) = f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \to g^{-1}(U_i).$$

Since $f^{-1}(U_i) \to U_i$ is proper, it follows that p_i is closed, which easily implies that p is closed. $\square$

Remark 5.1.5. It follows from property ii) in the proposition that if $f\colon X \to Y$ is a proper morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is a complete variety (possibly empty).

Exercise 5.1.6. Show that if X is a connected, complete variety, then $\Gamma(X, \mathcal{O}_X) = k$. Deduce that a complete variety is also affine if and only if it is a finite set of points.

We state the following proposition in a way that applies to several classes of morphisms.

PROPOSITION 5.1.7. *Suppose that $\mathcal{P}$ is a class of morphisms between algebraic varieties that satisfies the following conditions:*

i) *Every closed immersion is in $\mathcal{P}$.*
ii) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are in $\mathcal{P}$, then $g \circ f$ is in $\mathcal{P}$.*
iii) *Given a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with f in $\mathcal{P}$, then f' is in $\mathcal{P}$.

Under these assumptions, if we have morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$ such that $g \circ f$ is in $\mathcal{P}$, then f is in $\mathcal{P}$.

PROOF. Consider the Cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X \arrow[r, "g\circ f"] & Z. \end{tikzcd}$$

Since $g \circ f$ is in $\mathcal{P}$, it follows that q is in $\mathcal{P}$. Let $h\colon X \to X \times_Z Y$ be given by $h(x) = \big(x, f(x)\big)$. Note that h is a closed immersion (see assertion iv) in Proposition 2.5.4), hence it is in $\mathcal{P}$. Since $f = q \circ h$, we conclude that f is in $\mathcal{P}$. $\square$

REMARK 5.1.8. By Proposition 5.1.4, we may apply the above the above proposition if we take $\mathcal{P}$ to be the class of proper morphisms. It is clear that we can also apply it by taking $\mathcal{P}$ to consist of all closed immersions or of all locally closed immersions.

The following is the main result of this section.

THEOREM 5.1.9. *The projective space $\mathbf{P}^n$ is a complete variety.*

PROOF. We need to show that given any variety Y, the projection morphism $p\colon \mathbf{P}^n \times Y \to Y$ is closed. If we consider an affine open cover $Y = \bigcup_{i=1}^r U_i$, it is enough to show that each projection $\mathbf{P}^n \times U_i \to U_i$ is closed. Therefore we may and will assume that Y is affine, say $Y = \operatorname{MaxSpec}(A)$ and we need to show that the canonical morphism

$$f\colon X = \operatorname{MaxProj}\big(A[x_0, \ldots, x_n]\big) \to Y$$

is closed.

Let $W = V(I)$ be a closed subset of X. Recall that if

$$I' = \{f \in A[x_0, \ldots, x_n] \mid f \cdot (x_0, \ldots, x_n) \subseteq \operatorname{rad}(I)\},$$

then $V(I') = V(I)$. We need to show that if $\mathfrak{m} \not\in f(W)$, then there is $h \in A$ such that $\mathfrak{m} \in D_Y(h)$ and $D_Y(h) \cap f(W) = \emptyset$. For this, it is enough to find $h \in A$ such that $h \in I'$ and $h \not\in \mathfrak{m}$. Indeed, in this case, for every $\mathfrak{q} \in W = V(I')$, we have $h \in \mathfrak{q} \cap A$, hence $\mathfrak{q} \cap A \not\in D_Y(h)$.

For every i, with $0 \leq i \leq n$, consider the affine open subset $U_i = D_X(x_i)$ of X. Since U_i is affine, with $\mathcal{O}(U_i) = A[x_0, \ldots, x_n]_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i]$, and $W \cap U_i$ is the open subset defined by

$$I_{(x_i)} = \{g/x_i^m \mid m \geq 0, g \in I \cap A[x_0, \ldots, x_n]_m\},$$

the condition that $\mathfrak{m} \not\in f(U_i)$ is equivalent to the fact that

$$\mathfrak{m} \cdot A[x_0/x_i, \ldots, x_n/x_i] + I_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i].$$

By putting the condition that 1 lies on the left-hand side and by clearing the denominators, we conclude that

$$x_i^m \in \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I \quad \text{for some} \quad m \in \mathbf{N}.$$

Since such a condition holds for all i, we conclude that if $N \gg 0$ then

$$(x_0, \ldots, x_n)^N \subseteq \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I.$$

This implies

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq \mathfrak{m} \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N + (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N$$

and we deduce from Nakayama's lemma that

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N.$$

This implies that there is $h \in A \smallsetminus \mathfrak{m}$ such that $h \cdot (x_0, \ldots, x_n)^N \subseteq I$, hence $h \in I'$. This completes the proof of the theorem. □

Corollary 5.1.10. *Every projective variety is complete. Moreover, every morphism of varieties $f\colon X \to Y$, with X projective, is proper; in particular, it is closed.*

Proof. This follows from the theorem, using various assertions in Proposition 5.1.4. Since X is a projective variety, there is a closed immersion $i\colon X \hookrightarrow \mathbf{P}^n$ for some n. Note that i is proper by assertion iii) in the proposition and $\mathbf{P}^n$ is complete by the theorem, hence we conclude that X is complete, using assertion i) in the proposition. The fact that every morphism $X \to Y$ is proper now follows from assertion iv) in the proposition. □

Corollary 5.1.11. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then the canonical morphism $f\colon \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$ is proper.*

Proof. The morphism f factors as

$$\mathrm{MaxProj}(S) \overset{i}{\hookrightarrow} \mathrm{MaxSpec}(S_0) \times \mathbf{P}^n \xrightarrow{p} \mathrm{MaxSpec}(S_0),$$

where i is a closed immersion and p is the projection. Since $\mathbf{P}^n$ is complete, we deduce that p is proper by assertion ii) in Proposition 5.1.4 and i is a closed immersion by assertion iii) in the proposition. We thus conclude that f is proper by assertion i) in the proposition. □

For the sake of completeness, we mention the following embedding theorem. Its proof is more involved (see, for example, [Con07]).

THEOREM 5.1.12 (Nagata, Deligne). *For every algebraic variety X, there is an open immersion $i\colon X \hookrightarrow Y$, where Y is complete. More generally, every morphism of algebraic varieties $f\colon X \to Z$ factors as a composition*

$$X \overset{i}{\hookrightarrow} Y \xrightarrow{p} Z,$$

with i an open immersion and p a proper morphism.

The next exercise deals with an important example of a proper, birational morphism: the *blow-up* of the affine space at the origin.

EXERCISE 5.1.13. Thinking of $\mathbf{P}^{n-1}$ as the set of lines in $\mathbf{A}^n$, define the *blow-up of $\mathbf{A}^n$ at 0* as the set

$$\mathrm{Bl}_0(\mathbf{A}^n) := \big\{\big(P,[\ell]\big) \in \mathbf{A}^n \times \mathbf{P}^{n-1} \mid P \in \ell\big\}.$$

1) Show that $\mathrm{Bl}_0(\mathbf{A}^n)$ is a closed subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$.
2) Show that the restriction of the projection onto the first component gives a morphism $\pi\colon \mathrm{Bl}_0(\mathbf{A}^n) \to \mathbf{A}^n$ that is an isomorphism over $\mathbf{A}^n \smallsetminus \{0\}$.
3) Show that $\pi^{-1}(0) \simeq \mathbf{P}^{n-1}$.
4) Show that π is a proper morphism.

5.2. Chow's lemma

In this section we discuss a result that is very useful in reducing statements about complete varieties to the case of projective varieties. More generally, it allows reducing statements about proper morphisms to a special case of what we will later define as projective morphisms. In order to make things more transparent, we begin with the statement in the absolute case.

THEOREM 5.2.1. *(Chow's lemma) If X is a complete variety, then there is a projective variety Y and a morphism $g\colon Y \to X$ that induces an isomorphism between dense open subsets of Y and X.*

Here is the relative version of the above result:

THEOREM 5.2.2. *(Chow's lemma, relative version) If $f\colon X \to Z$ is a proper morphism of algebraic varieties, then there is a proper morphism $g\colon Y \to X$ that satisfies the following conditions:*

i) *The morphism g induces an isomorphism between dense open subsets of Y and X.*
ii) *The composition $f \circ g$ factors as*

$$Y \overset{i}{\hookrightarrow} Z \times \mathbf{P}^N \xrightarrow{p} Z,$$

where i is a closed immersion, N is a positive integer, and p is the projection onto the first factor.

Of course, it is enough to only prove the relative statement. We give the proof following [Mum88].

PROOF OF THEOREM 5.2.2. If g satisfies ii), then it is automatically proper, since $f \circ g$ is a proper morphism (see Remark 5.1.8). We first note that we may assume that X is irreducible. Indeed, if $X_1, \ldots, X_r$ are the irreducible components of X and if we can construct morphisms $Y_i \to X_i$ as in the theorem, then we have an induced morphism $Y = \bigsqcup_i Y_i \to X$ which satisfies the required conditions (note that if we have closed immersions $Y_i \hookrightarrow Z \times \mathbf{P}^{n_i}$, then we can construct a closed immersion $Y \hookrightarrow Z \times \mathbf{P}^d$, where $d+1 = \sum_{i=1}^r (n_i+1)$, by embedding the $\mathbf{P}^{n_i}$ in $\mathbf{P}^d$ as disjoint linear subspaces).

Suppose now that X is irreducible and consider an affine open cover $X = U_1 \cup \ldots \cup U_n$. Since each U_i is an affine variety, it admits a locally closed immersion in a projective space $\mathbf{P}^{m_i}$. We thus obtain a morphism $U_i \hookrightarrow Z \times \mathbf{P}^{m_i}$ which is again a locally closed immersion (see Remark 5.1.8) and we denote its image by $\overline{U_i}$. Using the Segre embedding we see that we have a closed immersion

$$\overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \hookrightarrow Z \times \mathbf{P}^{m_1} \times \ldots \times \mathbf{P}^{m_n} \hookrightarrow Z \times \mathbf{P}^N,$$

where $N+1 = \prod_i (m_i+1)$.

Let $U^* = U_1 \cap \ldots \cap U_n$. Since X is irreducible, U^* is a nonempty open subset of X. We consider two locally closed immersions. First, we have

$$\alpha\colon U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion map. This is a locally closed immersion since it factors as the composition

$$U^* \to U^* \times_Z \times \ldots \times_Z U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n},$$

with the first map being a diagonal map (hence a closed immersion) and the second being a product of open immersions (hence an open immersion). We denote by W the closure of $\alpha(U^*)$. Since W is a closed subvariety of $\overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$, we see that the canonical morphism $W \to Z$ factors as

$$W \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

where the first morphism is a closed immersion and the second morphism is the projection onto the first component.

We also consider the map

$$\beta\colon U^* \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion. Again, this is a locally closed immersion, and we denote the closure of its image by Y. It is clear that the projection onto the last n components

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

induces a morphism $q\colon Y \to W$, while the projection onto the first component

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X$$

induces a morphism $g\colon Y \to X$. The restriction of g to U^* is the identity, hence g is birational. Note that q is a closed map, since f is proper. In particular, since its image contains the dense open subset U^*, it follows that q is surjective.

The key assertion is that q is an isomorphism. Once we know this, we see that $f \circ g$ factors as

$$Y \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

with the first map being a closed immersion, and therefore g has the required properties.

In order to show that q is an isomorphism, we consider for every i the map

$$\alpha_i\colon U_i \hookrightarrow X \times_Z \overline{U_i},$$

given by the inclusion on each component. This is again a locally closed immersion. Moreover, since the maps

$$U_i \hookrightarrow X \times_Z U_i \quad \text{and} \quad U_i \hookrightarrow U_i \times_Z \overline{U_i}$$

are closed immersions (as the graphs of the inclusion maps $U_i \hookrightarrow X$ and $U_i \hookrightarrow \overline{U_i}$, respectively), it follows that

$$\overline{\alpha_i(U_i)} \cap (X \times_Z U_i) = \{(u,u) \mid u \in U_i\} = \overline{\alpha_i(U_i)} \cap (U_i \times_Z \overline{U_i}).$$

Consider the projection map

$$\pi_{1,i} \colon X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X \times_Z \overline{U_i}.$$

Since $\pi_{1,i}(Y) \subseteq \overline{\alpha_i(U^*)} = \overline{\alpha_i(U_i)}$, we deduce that

$$V_i := Y \cap (X \times_Z \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n})$$
$$= Y \cap (U_i \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}) = Y \cap \{(u_0, u_1, \ldots, u_n) \mid u_0 = u_i \in U_i\}.$$

The first formula for V_i shows that $V_i = q^{-1}(V_i')$, where

$$V_i' = W \cap \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n}$$

is an open subset of W. From the second formula for V_i' we deduce that $Y = V_1 \cup \ldots \cup V_n$ and since q is surjective, it follows that $W = V_1' \cup \ldots \cup V_n'$.

In order to conclude the proof, it is thus enough to show that each induced morphism $V_i \to V_i'$ is an isomorphism. We define the morphism

$$\gamma_i \colon V_i' \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

by

$$\gamma_i(u_1, \ldots, u_n) = (u_i, u_1, \ldots, u_n).$$

This is well-defined, and since it maps U^* to U^*, it follows that its image lies inside Y. Moreover, we clearly have $q \circ \gamma_i(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$; in particular, the image of γ_i lies inside V_i. Finally, if $u = (u_0, u_1, \ldots, u_n) \in V_i$, then $u_0 = u_i$ lies in U_i, hence $u = \gamma_i\big(q(u)\big)$. This shows that γ_i gives an inverse of $q|_{V_i} \colon V_i \to V_i'$ and thus completes the proof of the theorem. $\square$

Remark 5.2.3. Let $\mathcal{P}$ be the class of morphisms of algebraic varieties $f \colon X \to Y$ that factor as $X \xrightarrow{i} Y \times \mathbf{P}^n \xrightarrow{p} Y$, for some $n \geq 0$, where i is a closed immersion and p is the projection onto the first component. We claim that $\mathcal{P}$ satisfies the conditions in Proposition 5.1.7. Properties i) and iii) are straightforward to check, so we only need to show that if $f \colon X \to Y$ and $g \colon Y \to Z$ admit such factorizations, then so does the composition $g \circ f$. Suppose that we have the following factorizations of f and g:

$$X \xrightarrow{i} Y \times \mathbf{P}^m \xrightarrow{p} Y \quad \text{and} \quad Y \xrightarrow{j} Z \times \mathbf{P}^n \xrightarrow{q} Z.$$

Consider the closed immersion i' given by the composition

$$X \xrightarrow{i} Y \times \mathbf{P}^m \xrightarrow{j \times 1_{\mathbf{P}^n}} Z \times \mathbf{P}^n \times \mathbf{P}^m \xrightarrow{1_Z \times \nu} Z \times \mathbf{P}^N,$$

where $\nu \colon \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N$, with $N = (m+1)(n+1) - 1$, is the Segre embedding. Since it is clear that $g \circ f$ factors as

$$X \xrightarrow{i'} Z \times \mathbf{P}^N \xrightarrow{p'} Z,$$

where p' is the projection onto the first component, we conclude that $g \circ f$ lies in $\mathcal{P}$.

We can thus apply Proposition 5.1.7 to conclude that in the setting of Theorem 5.2.2, since $f \circ g$ lies in $\mathcal{P}$, we also have that g lies in $\mathcal{P}$.

5.3. Finite morphisms

We discussed in Chapter 3 finite morphisms between affine varieties. We now consider the general notion.

DEFINITION 5.3.1. The morphism $f\colon X \to Y$ between algebraic varieties is *finite* if for every affine open subset $V \subseteq Y$, its inverse image $f^{-1}(V)$ is an affine variety, and the induced k-algebra homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X\big(f^{-1}(V)\big)$$

is finite.

It is not clear that in the case when X and Y are affine varieties, the above definition coincides with our old one. However, this follows from the following theorem.

PROPOSITION 5.3.2. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. If there is an affine open cover $Y = \bigcup_{i=1}^{r} V_i$ such that each $U_i = f^{-1}(V_i)$ is an affine variety and the induced morphism*

$$\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$$

is finite, then f is a finite morphism.

We begin with a lemma which is useful in several other situations.

LEMMA 5.3.3. *If X is an algebraic prevariety and U, $V \subseteq X$ are affine open subsets, then for every $p \in U \cap V$, there is open neighborhood $W \subseteq U \cap V$ of p that is a principal affine open subset in both U and V.*

PROOF. We first choose an open neighborhood $W_1 \subseteq U \cap V$ of p of the form $W_1 = D_U(f)$ for some $f \in \mathcal{O}(U)$. We next choose another open neighborhood $W \subseteq W_1$ of the form $W = D_V(g)$, for some $g \in \mathcal{O}(V)$. It is enough to show that W is a principal affine open subset also in U.

Since $\mathcal{O}(W_1) \simeq \mathcal{O}(U)_f$, it follows that there is $h \in \mathcal{O}(U)$ such that $g|_{W_1} = \frac{h}{f^m}$ for some non-negative integer m. In this case we have $W = D_U(fh)$, completing the proof. $\square$

PROOF OF PROPOSITION 5.3.2. Note that if W is a principal affine open subset of some of the V_i, then $f^{-1}(W)$ is affine and the induced morphism

$$\mathcal{O}_Y(W) \to \mathcal{O}_X\big(f^{-1}(W)\big) \tag{5.3.1}$$

is finite. Indeed, if $W = D_{V_i}(\phi)$, then $f^{-1}(W) = D_{U_i}(\phi \circ f)$ is affine and the morphism (5.3.1) is identified to

$$\mathcal{O}_Y(V_i)_\phi \to \mathcal{O}(U_i)_{\phi\circ f},$$

which is finite.

Let $V \subseteq Y$ be an arbitrary affine open subset. Since V is covered by the open subsets $V \cap V_i$, applying for each pair (V, V_i) Lemma 5.3.3, and using what we have already seen, we see that we can cover V by finitely many principal affine open subsets $W_1, \ldots, W_s$, such that each $f^{-1}(W_i)$ is affine and the induced morphism

$$\mathcal{O}_Y(W_i) \to \mathcal{O}_X\big(f^{-1}(W_i)\big) \tag{5.3.2}$$

is finite. Let us write $W_i = D_V(\phi_i)$, for some $\phi_i \in \mathcal{O}_Y(V)$. The condition that $V = \bigcup_{i=1}^s W_i$ is equivalent to the fact that $\phi_1, \ldots, \phi_s$ generate the unit ideal in $\mathcal{O}_Y(V)$. This implies that the $f^{\#}(\phi_i) = \phi_i \circ f$ generate the unit ideal in $\mathcal{O}_X\big(f^{-1}(V)\big)$. Since each $D_{f^{-1}(V)}(\phi_i \circ f)$ is affine, it follows from Proposition 2.3.16 that $f^{-1}(V)$ is affine.

Moreover, the $\mathcal{O}_Y(V)$-module $\mathcal{O}_X\big(f^{-1}(V)\big)$ has the property that $\mathcal{O}_X\big(f^{-1}(V)\big)_{\phi_i}$ is a finitely generated module over $\mathcal{O}_Y(V)_{\phi_i}$ for all i. Since the ϕ_i generate the unit ideal in $\mathcal{O}_Y(V)$, we conclude using Corollary C.3.5 that $\mathcal{O}_X\big(f^{-1}(V)\big)$ is a finitely generated $\mathcal{O}_Y(V)$-module. $\square$

Remark 5.3.4. If $f\colon X \to Y$ is a finite morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is finite. Indeed, if V is an affine open neighborhood of y, then $U = f^{-1}(V)$ is affine and the induced morphism $f^{-1}(V) \to V$ is finite. Applying to this morphism Remark 3.2.7, we deduce that $f^{-1}(y)$ is finite.

In the next proposition we collect some general properties of finite morphisms.

Proposition 5.3.5. *In what follows, all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms, then $g \circ f\colon X \to Z$ is a finite morphism.*
ii) *If $f\colon X \to Y$ is a finite morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $h\colon X \times_Y Z \to Z$ is a finite morphism.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is a finite morphism.*
iv) *If $f\colon X \to Y$ is a morphism and $Y = V_1 \cup \ldots \cup V_r$ is an open cover such that each induced morphism $f^{-1}(V_i) \to V_i$ is finite, then f is finite.*

Proof. The assertions in i) and iii) are straightforward to see and the one in iv) follows by covering each V_i by affine open subsets and then using Proposition 5.3.2. We now prove the assertion in ii). Let $V = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. For every i, consider an affine open cover $g^{-1}(V_i) = \bigcup_j U_{i,j}$. Note that we have

$$h^{-1}(U_{i,j}) = f^{-1}(V_i) \times_{V_i} U_{i,j}.$$

Using Proposition 5.3.2, we thus see that it is enough to prove the assertion when X, Y, and Z are affine varieties. In this case, $X \times_Y Z$ is affine, since it is a closed subvariety of $X \times Z$ (see Proposition 2.4.7). Moreover, the morphism

$$h^{\#}\colon \mathcal{O}(Z) \to \mathcal{O}(X \times_Y Z)$$

factors as

$$\mathcal{O}(Z) = \mathcal{O}(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{f^{\#} \otimes 1} \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{p} \mathcal{O}(X \times_Y Z).$$

The homomorphism $f^{\#} \otimes 1$ is finite since $f^{\#}$ is finite and p is surjective (this follows, for example, from the fact that $X \times_Y Z$ is a closed subvariety of $X \times Z$, but see also Remark 2.4.9 for a more precise statement). This completes the proof of ii). $\square$

The next proposition extends to arbitrary morphisms some properties that we have already proved for finite morphisms between affine varieties.

Proposition 5.3.6. *Let $f\colon X \to Y$ be a finite morphism.*

1) *The map f is closed.*
2) *If $Z_1 \subsetneq Z_2$ are irreducible closed subsets of X, then $f(Z_1) \subsetneq f(Z_2)$ are irreducible, closed subsets of Y.*

3) *If f is surjective, then given any irreducible, closed subset W of Y, there is an irreducible, closed subset Z in X such that $f(Z) = W$.*
4) *If Z_1 is an irreducible, closed subset of X and $W_1 \supseteq W_2$ are irreducible, closed subsets of Y, with $W_1 = f(Z_1)$, then there is $Z_2 \subseteq Z_1$ irreducible and closed such that $f(Z_2) = W_2$.*

PROOF. We have already seen these properties when X and Y are affine varieties in Corollary 3.2.9. Let $Y = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. By definition, each $f^{-1}(V_i)$ is affine and the induced morphism $f^{-1}(V_i) \to V_i$ is finite, hence it satisfies the properties in the proposition. Since each map $f^{-1}(V_i) \to V_i$ is closed, it follows that f is closed, hence we have 1). The assertions in 2), 3), and 4) similarly follow from the corresponding ones for the morphisms $f^{-1}(V_i) \to V_i$. □

COROLLARY 5.3.7. *Every finite morphism $f\colon X \to Y$ is proper.*

PROOF. Given any morphism of varieties $g\colon Z \to Y$, assertion ii) in Proposition 5.3.5 implies that the induced morphism $X \times_Y Z \to Z$ is finite. This is thus closed by assertion 1) in Proposition 5.3.6, which shows that f is proper. □

We mention the following converse to Corollary 5.3.7: every proper morphism with finite fibers is finite. This is a deeper result that we will only prove later (see Corollary 14.1.8).

The following proposition gives another property of finite morphisms that we have seen for affine varieties.

PROPOSITION 5.3.8. *If $f\colon X \to Y$ is a finite, surjective morphism of algebraic varieties, then for every closed subset Z of X, we have*

$$\dim\big(f(Z)\big) = \dim(Z).$$

Moreover, if Z is irreducible, then

$$\mathrm{codim}_Y\big(f(Z)\big) = \mathrm{codim}_X(Z).$$

PROOF. This can be deduced from the properties in Proposition 5.3.6 as in the proof of Corollary 3.2.10. □

EXAMPLE 5.3.9. If L_1 and L_2 are disjoint linear subspaces of $\mathbf{P}^n$, with $\dim(L_1)+\dim(L_2) = n-1$, then the *projection of* $\mathbf{P}^n$ *onto* L_2, *with center* L_1 is the morphism $\pi\colon \mathbf{P}^n \smallsetminus L_1 \longrightarrow L_2$ such that $\pi(p)$ is the intersection of L_2 with the linear span $\langle L_1, p\rangle$ of L_1 and p. In order to see that this is indeed a morphism, let's apply an element of $PGL_{n+1}(k)$ to $\mathbf{P}^n$ in order to have

$$L_1 = (x_0 = \ldots = x_r = 0) \quad \text{and} \quad L_2 = (x_{r+1} = \ldots = x_n = 0).$$

We consider the isomorphism $\mathbf{P}^r \simeq L_2$ given by

$$[u_0, \ldots, u_r] \to [u_0, \ldots, u_r, 0, \ldots, 0].$$

Note that if $p = [a_0, \ldots, a_n] \in \mathbf{P}^n \smallsetminus L_1$, then the linear span of L_1 and p is the set

$$\{[\lambda a_0, \ldots, \lambda a_r, b_{r+1}, \ldots, b_n] \mid \lambda \in k^*, b_{r+1}, \ldots, b_n \in k\}.$$

We thus see that the map $\pi\colon \mathbf{P}^n \smallsetminus L_1 \to \mathbf{P}^r$ is given by

$$\pi\big([a_0, \ldots, a_n]\big) = [a_0, \ldots, a_r]$$

and it is now straightforward to check that π is a morphism.

Let us show that if X is a closed subvariety of $\mathbf{P}^n$ such that $X \cap L_1 = \emptyset$, then the induced morphism $\pi_X \colon X \to L_2$ is finite. This is an easy consequence of the fact that proper morphisms with finite fibers are finite, since the hypothesis implies that π_X has finite fibers: the fiber over a point $q \in L_2$ lies in the linear span $\langle L_1, q\rangle$ of L_1 and q, which has dimension equal to $\dim(L_1)+1$; if this is not finite, then its intersection with the hyperplane $L_1 \subseteq \langle L_1, q\rangle$ would be non-empty by Corollary 4.2.12. However, we will give a direct argument for the finiteness of π_X, since we haven't proved yet the fact that proper morphisms with finite fibers are finite.

After a linear change of coordinates as above, we may assume that

$$\pi \colon \mathbf{P}^n \smallsetminus L_1 \to \mathbf{P}^r, \quad \pi_X\big([a_0, \ldots, a_n]\big) = [a_0, \ldots, a_r].$$

Note that π is the composition of $(n-r)$ maps, each of which is the projection from a point onto a hyperplane. Indeed, if

$$\pi_i \colon \mathbf{P}^{r+i} \smallsetminus \{[0, \ldots, 0, 1]\} \to \mathbf{P}^{r+i-1}, \quad \pi_i\big([u_0, \ldots, u_{r+i}]\big) = [u_0, \ldots, u_{r+i-1}]$$

for $1 \leq i \leq n-r$, then it is clear that $\pi = \pi_1 \circ \ldots \circ \pi_{n-r}$. Since a composition of finite morphisms is finite, we see that we only need to prove our assertion when $r = n-1$.

It is enough to show that if $U_i = (x_i \neq 0) \subseteq \mathbf{P}^{n-1}$, then for each i, with $0 \leq i \leq n-1$, the inverse image $\pi_X^{-1}(U_i)$ is affine and the induced homomorphism

$$\mathcal{O}(U_i) \to \mathcal{O}\big(\pi_X^{-1}(U_i)\big) \tag{5.3.3}$$

is a finite homomorphism. The fact that $\pi_X^{-1}(U_i)$ is affine is clear, since this is equal to $D_X^+(x_i)$, hence it is affine by Proposition 4.2.9. Moreover, by Proposition 4.2.10, we can identify the homomorphism (5.3.3) with

$$k[x_0, \ldots, x_{n-1}]_{(x_i)} = k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right] \to (S_X)_{(x_i)}, \tag{5.3.4}$$

where S_X is the homogeneous coordinate ring of X. Since $(S_X)_{(x_i)}$ is generated by $\frac{x_j}{x_i}$, with $0 \leq j \leq n$, in order to show that (5.3.4) is a finite homomorphism, it is enough to show that each $\frac{x_j}{x_i} \in (S_X)_{(x_i)}$ is integral over $k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right]$. This is clear if $j \leq n-1$, hence we only need to consider $\frac{x_n}{x_i}$. By hypothesis, we have $[0, \ldots, 0, 1] \not\in X$. Therefore there is a homogeneous polynomial f, say of degree d, in the ideal I_X corresponding to X such that x_n^d appears in f with nonzero coefficient. If $d = 0$, then X is empty, in which case the assertion to prove is trivial. If $d > 0$, we may assume that $f = x_n^d + \sum_{i=1}^d g_i(x_0, \ldots, x_{n-1})x_n^{d-i}$. Dividing by x_i^d, we thus conclude that

$$\left(\frac{x_n}{x_i}\right)^d + \sum_{i=1}^d g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right) = 0 \quad \text{in} \quad (S_X)_{(x_i)},$$

hence $\frac{x_n}{x_i}$ is integral over $k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{n-1}}{x_i}\right]$. This gives our assertion.

In particular, we see that if X is a projective d-dimensional variety, then there is a finite morphism $X \to \mathbf{P}^d$. Indeed, if X is a closed subvariety of $\mathbf{P}^n$ different from $\mathbf{P}^n$, by projecting from a point not in X we obtain a finite morphism $X \to Y$, where Y is a d-dimensional subvariety of $\mathbf{P}^{n-1}$. By iterating this construction we obtain a finite morphism $X \to \mathbf{P}^d$.

PROPOSITION 5.3.10. *Let $f\colon X\to Y$ be a dominant morphism of irreducible varieties. If* $\dim(X)=\dim(Y)$*, then there is a non-empty open subset V of Y such that the induced morphism $f^{-1}(V)\to V$ is finite (in this case, one says that f is* generically finite*).*

Note that the converse also holds by Proposition 5.3.8.

PROOF OF PROPOSITION 5.3.10. We may clearly replace Y by an affine open subset and X by the inverse image of this subset, in order to assume that Y is an affine variety. In fact, we may assume that X is affine as well. Indeed, let us choose an affine open subset U of X and suppose that we know the assertion in the proposition for the induced morphism $U\to Y$. In other words, we know that there is a non-empty open subset V of Y, such that the induced morphism $g\colon U\cap f^{-1}(V)\to V$ is finite. Note that if $Z=\overline{f(X\smallsetminus U)}$, then

$$\dim(Z)\leq\dim(X\smallsetminus U)<\dim(X)=\dim(Y),$$

hence Z is a proper closed subset of Y. If we take $V'=V\smallsetminus Z$, then V' is non-empty and the induced morphism $g^{-1}(V')=U\cap f^{-1}(V')\to V'$ is finite. However, it follows from the definition of X' that $f^{-1}(V')\subseteq U$, which implies that V' satisfies the requirement in the proposition.

Suppose now that both X and Y are affine varieties, and consider the homomorphism

$$f^{\#}\colon A=\mathcal{O}(Y)\to\mathcal{O}(X)=B$$

corresponding to f. Note that this is injective since f is dominant. Let $k(Y)=\mathrm{Frac}(A)$ be the field of rational functions of Y. The assumption that $\dim(X)=\dim(Y)$ implies that $\mathrm{Frac}(B)$ is algebraic, hence finite, over $\mathrm{Frac}(A)$ by Corollary 3.3.9. Noether's Normalization lemma thus implies that $B\otimes_A k(Y)$ is a finite $k(Y)$-algebra. Let $b_1,\ldots,b_r\in B$ be generators of B as a k-algebra. Since each b_i is algebraic over $k(Y)$, we see that there is $f_i\in A$ such that $\frac{b_i}{1}$ is integral over A_{f_i}. This implies that if $f=\prod_i f_i$, then each $\frac{b_i}{1}$ is integral over A_f, hence $A_f\to B_f$ is a finite homomorphism. Therefore $V=D_Y(f)$ satisfies the assertion in the proposition. $\square$

DEFINITION 5.3.11. If $f\colon X\to Y$ is a dominant, generically finite morphism of irreducible varieties, then the field extension $k(X)/k(Y)$ is algebraic and finitely generated, hence finite. Its degree is the *degree* of f, denoted $\deg(f)$.

We end this section by introducing another class of morphisms.

DEFINITION 5.3.12. A morphism of algebraic varieties $f\colon X\to Y$ is *affine* if for every affine open subset $V\subseteq Y$, its inverse image $f^{-1}(V)$ is affine.

The next proposition shows that, in fact, it is enough to check the property in the definition for an affine open cover of the target. In particular, this implies that every morphism of affine varieties is affine.

PROPOSITION 5.3.13. *Let $f\colon X\to Y$ be a morphism of algebraic varieties. If there is an open cover $Y=V_1\cup\ldots\cup V_r$, with each V_i affine, such that all $f^{-1}(V_i)$ are affine, then f is an affine morphism.*

PROOF. The argument follows as in the proof of Proposition 5.3.2. $\square$

5.4. Semicontinuity of fiber dimension for proper morphisms

Our goal in this section is to prove the following semicontinuity result for the dimensions of the fibers of a proper morphism.

THEOREM 5.4.1. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties, then for every non-negative integer m, the set*

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$

is closed in Y.

This is an immediate consequence of the following more technical statement, but which is valid for an arbitrary morphism.

PROPOSITION 5.4.2. *If $f\colon X \to Y$ is a morphism of algebraic varieties, then for every non-negative integer m, the set X_m consisting of those $x \in X$ such that the fiber $f^{-1}\big(f(x)\big)$ has an irreducible component of dimension $\geq m$ passing through x, is closed.*

PROOF. Arguing by Noetherian induction, we may assume that the assertion in the proposition holds for every $f|_Z$, where Z is a proper closed subset of X. If X is not irreducible and $X^{(1)}, \ldots, X^{(r)}$ are the irreducible components of X, we know that each $X_m^{(j)}$ is closed in $X^{(j)}$, hence in X. Since

$$X_m = \bigcup_{j=1}^{r} X_m^{(j)},$$

we conclude that X_m is closed.

Therefore we may and will assume that X is irreducible. Of course, we may replace Y by $\overline{f(X)}$ and thus assume that Y is irreducible and f is dominant. In this case, if $m \leq \dim(X) - \dim(Y)$, then $X_m = X$ by Theorem 3.4.1, hence we are done. On the other hand, it follows from Theorem 3.4.2 that there is an open subset V of Y such that if $y \in V$, then every irreducible component of $f^{-1}(y)$ has dimension equal to $\dim(X) - \dim(Y)$. We deduce that if $m > \dim(X) - \dim(Y)$ and we put $Z = X \smallsetminus f^{-1}(V)$, then Z is a proper closed subset of X such that $X_m = Z_m$. Since Z_m is closed in Z, hence in X, by the inductive assumption, we are done. $\square$

PROOF OF THEOREM 5.4.1. With the notation in the proposition, we have

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\} = f(X_m).$$

Since X_m is closed and f is proper, it follows that $f(X_m)$ is closed. $\square$

REMARK 5.4.3. If $f\colon X \to Y$ is an arbitrary morphism of algebraic varieties, we can still say that the subset

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$

is constructible in Y. Indeed, with the notation in Proposition 5.4.2, we see that this set is equal to $f(X_m)$. Since X_m is closed in X by the proposition, its image $f(X_m)$ is constructible by Theorem 3.5.3.

Note that also the set

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) = m\}$$

is constructible in Y, being the difference of two constructible subsets.

5.5. An irreducibility criterion

The following result is an useful irreducibility criterion.

PROPOSITION 5.5.1. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that Y is irreducible and that all fibers of f are irreducible, of the same dimension d (in particular, f is surjective). If either one of the following two conditions holds:*

a) *X is pure-dimensional;*
b) *f is closed,*

then X is irreducible, of dimension $d + \dim(Y)$.

We will be using the proposition for proper morphisms f, so that condition b) will be automatically satisfied.

PROOF OF PROPOSITION 5.5.1. We will show that in general–that is, without assuming a) or b)– the following assertions hold:

i) There is a unique irreducible component of X that dominates Y, and
ii) Every irreducible component Z of X is a union of fibers of f. Its dimension is equal to $\dim\big(\overline{f(Z)}\big) + d$.

Let $X = X_1 \cup \ldots \cup X_r$ be the irreducible decomposition of X. For every $y \in Y$, we put $X_y = f^{-1}(y)$, and $(X_j)_y = X_y \cap X_j$. Since $X_y = \bigcup_{j=1}^r (X_j)_y$, and since X_y is irreducible, it follows that for every y, there is j such that $X_y = (X_j)_y$.

For every i, let $U_i := X_i \smallsetminus \bigcup_{j\neq i} X_j$. This is a nonempty open subset of X. Note that if $y \in f(U_i)$, then X_y can't be contained in $(X_j)_y$ for any $j \neq i$. It follows that

$$X_y = (X_i)_y \quad \text{for all} \quad y \in f(U_i). \tag{5.5.1}$$

Note that some X_ℓ has to dominate Y: since f is surjective, we have $Y = \bigcup_j \overline{f(X_j)}$, and since Y is irreducible, we see that there is ℓ such that $Y = \overline{f(X_\ell)}$. In this case we also have $Y = \overline{f(U_\ell)}$, and Theorem 3.4.2 implies that there is an open subset V of Y contained in $f(U_\ell)$. We deduce from (5.5.1) that $X_y = (X_\ell)_y$ for every $y \in V$, hence for all $j \neq \ell$, we have $X_j \smallsetminus X_\ell \subseteq f^{-1}(Y \smallsetminus V)$. Therefore $X_j = \overline{X_j \smallsetminus X_\ell}$ is contained in $f^{-1}(Y \smallsetminus V)$ (which is closed). We conclude that X_j does not dominate Y for any $j \neq \ell$.

On the other hand, it follows from Theorems 3.4.1 and 3.4.2 that for every i, the following hold

α) $\dim(X_i)_y \geq \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$ for every $y \in f(X_i)$ and
β) There is an open subset W_i in $\overline{f(X_i)}$ such that for all $y \in W_i$ we have $\dim(X_i)_y = \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$.

Since $W_i \cap f(U_i) \neq \emptyset$, it follows from β) and (5.5.1) that $d = \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$ for every i. Furthermore, for every $y \in f(X_i)$, we know by α) that $(X_i)_y$ is a closed subset of dimension d of the irreducible variety X_y of dimension d. Therefore $X_y = (X_i)_y$ for all $y \in f(X_i)$, which says that each X_i is a union of fibers of f. Therefore assertions i) and ii) hold.

In particular, it follows from i) and ii) that if $i \neq \ell$, then $\overline{f(X_i)}$ is a proper subset of Y, and

$$\dim(X_i) = d + \dim(\overline{f(X_i)}) < d + \dim(Y) = \dim(X_\ell).$$

If X is pure-dimensional, then we conclude that X is irreducible.

Suppose now that f is a closed map. Since $f(X_\ell)$ is closed, it follows that $f(X_\ell) = Y$. We have seen that X_ℓ is a union of fibers of f, hence $X_\ell = X$. Therefore X is irreducible also in this case. $\square$

EXAMPLE 5.5.2. Consider the incidence correspondence between points and hyperplanes in $\mathbf{P}^n$, defined as follows. Recall that $(\mathbf{P}^n)^*$ is the projective space parametrizing the hyperplanes in $\mathbf{P}^n$. We write $[H]$ for the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H. Consider the following subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$:

$$\mathcal{Z} = \big\{ \big(p, [H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H \big\}.$$

Note that if we take homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbf{P}^n$ and $y_0, \ldots, y_n$ on $(\mathbf{P}^n)^*$, then $\mathcal{Z}$ is defined by the condition $\sum_{i=0}^n x_i y_i = 0$. It is the straightforward to see, by considering the products of the affine charts on $\mathbf{P}^n$ and $(\mathbf{P}^n)^*$, that $\mathcal{Z}$ is a closed subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$. The projections on the two components induce morphisms $\pi_1 \colon \mathcal{Z} \to \mathbf{P}^n$ and $\pi_2 \colon \mathcal{Z} \to (\mathbf{P}^n)^*$. For every $[H] \in (\mathbf{P}^n)^*$, we have $\pi_2^{-1}([H]) \simeq H$, hence all fibers of π_2 are irreducible, of dimension $n-1$. Since $(\mathbf{P}^n)^*$ is irreducible, it follows from Proposition 5.5.1 that $\mathcal{Z}$ is irreducible, of dimension $2n-1$. Note that the picture is symmetric: for every $p \in \mathbf{P}^n$, the fiber $\pi_1^{-1}(p)$ consists of all hyperplanes in $\mathbf{P}^n$ that contain p, which is a hyperplane in $(\mathbf{P}^n)^*$.

5.6. Flat morphisms

We begin by reviewing the concept of a flat module. Recall that if M is a module over a commutative ring A, then the functor $M \otimes_A -$ from the category of A-modules to itself, is right exact. The module M is *flat* if, in fact, this is an exact functor. Given a ring homomorphism $\phi \colon A \to B$, we say that ϕ is flat (or that B is a flat A-algebra) if B is flat as an A-module.

EXAMPLE 5.6.1. The ring A is flat as an A-module, since $A \otimes_A M \simeq M$ for every A-module M.

EXAMPLE 5.6.2. A direct sum of flat A-modules is flat, since tensor product commutes with direct sums and taking a direct sum is an exact functor. It follows from the previous example that every free module is flat. In particular, every vector space over a field is flat.

EXAMPLE 5.6.3. If $(M_i)_{i \in I}$ is a filtered direct system of flat A-modules, then $M = \varinjlim_{i \in I} M_i$ is a flat A-module. Indeed, since the tensor product commutes with direct limits, for every injective morphism of A-modules $N_1 \hookrightarrow N_2$, the induced morphism

$$N_1 \otimes_A M \to N_2 \otimes_A M$$

can be identified with the direct limit of the injective morphisms

$$N_1 \otimes M_i \to N_2 \otimes_A M_i.$$

Since a filtered direct limit of injective morphisms is injective, we obtain our assertion.

EXAMPLE 5.6.4. If M is a flat A-module, then for every non-zero-divisor $a \in A$, multiplication by a is injective on A, and after tensoring with M, we see that multiplication by a is injective also on M. In particular, if A is a domain, then M is torsion-free.

The converse holds if A is a PID: every torsion-free A-module is flat. Indeed, M is the filtered direct limit of its finitely generated submodules, which are free A-modules, being finitely generated and torsion-free over a PID. Since every filtered direct limit of flat modules is flat, we conclude that M is flat.

EXAMPLE 5.6.5. For every ring A and every multiplicative system $S \subseteq A$, the A-algebra $S^{-1}A$ is flat. Indeed, for every A-module N, we have a canonical isomorphism

$$S^{-1}A \otimes_A N \simeq S^{-1}N$$

and the functor taking N to $S^{-1}N$ is exact.

We do not discuss the more subtle aspects of flatness, which we do not need at this point, and whose treatment is better handled using the Tor functors. We only collect in the next proposition some very easy properties that we need in order to define flatness for morphisms of algebraic varieties.

PROPOSITION 5.6.6. *Let M be an A-module.*

i) *If M is flat, then for every ring homomorphism $A \to B$, the B-module $M \otimes_A B$ is flat.*
ii) *If $B \to A$ is a flat homomorphism and M is flat over A, then M is flat over B.*
iii) *If $\mathfrak{p}$ is a prime ideal in A and M is an $A_{\mathfrak{p}}$-module, then M is flat over A if and only if it is flat over $A_{\mathfrak{p}}$.*
iv) *If $B \to A$ is a ring homomorphism, then M is flat over B if and only if for every prime (respectively, maximal) ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat.*

PROOF. The assertion in i) follows from the fact that for every B-module N, we have a canonical isomorphism

$$(M \otimes_A B) \otimes_B N \simeq M \otimes_A N.$$

Similarly, the assertion in ii) follows from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M \simeq (N \otimes_B A) \otimes_A M.$$

With the notation in iii), note that if M is a flat $A_{\mathfrak{p}}$-module, since $A_{\mathfrak{p}}$ is a flat A-algebra, we conclude that M is flat over A by ii). The converse follows from the fact that if N is an $A_{\mathfrak{p}}$-module, then we have a canonical isomorphism

$$N \otimes_{A_{\mathfrak{p}}} M \simeq N \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M) \simeq N \otimes_A M.$$

We now prove iv). Suppose first that M is flat over B and let $\mathfrak{p}$ be a prime ideal in A. We deduce that $M_{\mathfrak{p}}$ is flat over B from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M_{\mathfrak{p}} \simeq (N \otimes_B M) \otimes_A A_{\mathfrak{p}}.$$

Conversely, suppose that for every maximal ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat. Given an injective map of B-modules $N' \hookrightarrow N$, we see that for every maximal ideal $\mathfrak{p}$, the induced homomorphism

$$N' \otimes_B M_{\mathfrak{p}} \simeq (N' \otimes_B M)_{\mathfrak{p}} \to (N \otimes_B M)_{\mathfrak{p}} \simeq N \otimes_B M_{\mathfrak{p}}$$

is injective. This implies the injectivity of

$$N' \otimes_B M \to N \otimes_B M$$

by Corollary C.3.4. $\square$

REMARK 5.6.7. If $\phi\colon A \to B$ is a flat homomorphism of Noetherian rings and $\mathfrak{p}$ is a prime ideal in A, then for every minimal prime ideal $\mathfrak{q}$ containing $\mathfrak{p}B$, we have $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Indeed, it follows from assertion i) in Proposition 5.6.6 that the morphism $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat. It then follows from Example 5.6.4 that if $\overline{a}$ is a nonzero element in $A/\mathfrak{p}$, then its image in $B/\mathfrak{p}B$ is a non-zero-divisor, hence it can't lie in a minimal prime ideal (see Proposition E.2.1). This gives our assertion.

We now define flatness in our geometric context. We say that a morphism of varieties $f\colon X \to Y$ is *flat* if it satisfies the equivalent conditions in the next proposition.

PROPOSITION 5.6.8. *Given a morphism of varieties $f\colon X \to Y$, the following conditions are equivalent:*

i) *For every affine open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \subseteq f^{-1}(V)$, the induced homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is flat.*
ii) *There are affine open covers $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$ such that for all i, we have $U_i \subseteq f^{-1}(V_i)$ and the induced homomorphism $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ is flat.*
iii) *For every point $x \in X$, if $y = f(x)$, then the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat.*

PROOF. We begin by showing that ii)$\Rightarrow$iii). Given x and y as in iii) and covers as in ii), we choose i such that $x \in U_i$, in which case $y \in V_i$. Note that x corresponds to a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_X(U_i)$ and y corresponds to the inverse image $\mathfrak{q}$ of $\mathfrak{p}$ in $\mathcal{O}_Y(V_i)$. Since

$$B = \mathcal{O}_Y(V_i) \to A = \mathcal{O}_X(U_i)$$

is flat, we conclude that $A_{\mathfrak{q}}$ is B-flat by property iv) in Proposition 5.6.6. It follows that $A_{\mathfrak{p}}$ is flat over $B_{\mathfrak{q}}$ by property ii) in the same proposition.

Since the implication i)$\Rightarrow$ii) is trivial, in order to complete the proof it is enough to show iii)$\Rightarrow$i). Let U and V be affine open subsets as in i). Given the induced homomorphism

$$B = \mathcal{O}_Y(V) \to \mathcal{O}_X(U) = A,$$

it follows from iii) that for every maximal ideal $\mathfrak{p}$ in A, if its inverse image in B is $\mathfrak{q}$, then the induced homomorphism $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ is flat. Assertion iii) in Proposition 5.6.6 implies that $A_{\mathfrak{p}}$ is flat over B for every $\mathfrak{p}$, in which case assertion iv) in the proposition implies that A is flat over B. $\square$

REMARK 5.6.9. The argument for the implication ii)$\Rightarrow$iii) in the proof of the above proposition shows that more generally, if $f\colon X \to Y$ is a flat morphism, then for every irreducible closed subset $V \subseteq X$, if $W = \overline{f(V)}$, then the induced ring homomorphism $\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V}$ is flat.

EXAMPLE 5.6.10. Every open immersion $i\colon U \hookrightarrow X$ is flat: indeed, it is clear that property iii) in the above proposition is satisfied.

EXAMPLE 5.6.11. If X and Y are varieties, then the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are flat. Indeed, by choosing affine covers of X and Y, we reduce to the case when both X and Y are affine. In this case, since $\mathcal{O}(Y)$ is a free k-module, it follows from assertion i) in Proposition 5.6.6 that

$\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is flat over $\mathcal{O}(X)$. This shows that p is flat and the assertion about q follows similarly.

REMARK 5.6.12. A composition of flat morphisms is a flat morphism. Indeed, this follows from definition and property ii) in Proposition 5.6.6.

REMARK 5.6.13. If $f\colon X \to Y$ is flat and $W \subseteq Y$ is an irreducible, closed subset such that $f^{-1}(W) \neq \emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have $\overline{f(V)} = W$. Indeed, we may replace X and Y by suitable affine open subsets that intersect V and W, respectively, to reduce to the case when both X and Y are affine. In this case the assertion follows from Remark 5.6.7.

EXAMPLE 5.6.14. A morphism $f\colon X \to \mathbf{A}^1$ is flat if and only if every irreducible component of X dominates $\mathbf{A}^1$. The "only if" part follows from the previous remark. For the converse, note that under the hypothesis, for every affine open subset U of X, the $k[x]$-module $\mathcal{O}_X(U)$ is torsion-free: if a nonzero $u \in k[x]$ annihilates $v \in \mathcal{O}_X(U)$, it follows that every irreducible component of U on which v does not vanish is mapped by f in the zero-locus of u, a contradiction. We then deduce that f is flat using Example 5.6.4.

Our goal is to prove two geometric properties of flat morphisms. We begin with the following generalization of Proposition 1.6.6.

THEOREM 5.6.15. *If $f\colon X \to Y$ is a flat morphism between algebraic varieties, then f is open.*

The proof will make use of the following openness criterion.

LEMMA 5.6.16. *Let W be a subset of a Noetherian topological space Y. The set W is open if and only if whenever $Z \subseteq Y$ is a closed irreducible subset of Y such that $W \cap Z \neq \emptyset$, then W contains nonempty open subset of Z.*

PROOF. The "only if" part is clear, so we only need to prove the converse. Arguing by Noetherian induction, we may assume that the assertion holds for all proper closed subspaces of Y. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. We may assume that W is nonempty, and suppose that W contains a point y in some Y_i. By hypothesis, there is a nonempty open subset $U \subseteq Y_i$ such that $U \subseteq W$. After replacing U by $U \smallsetminus \bigcup_{j\neq i} Y_j$, we may assume that $U \cap Y_j = \emptyset$ for every $j \neq i$, in which case U is open in Y.

Note that $Y \smallsetminus U$ is a proper closed subset of Y. Moreover, $W \smallsetminus U \subseteq Y \smallsetminus U$ satisfies the same hypothesis as W: if $Z \subseteq Y \smallsetminus U$ is an irreducible closed subset such that $(W \smallsetminus U) \cap Z \neq \emptyset$, then W contains a nonempty open subset of Z, hence the same holds for $W \smallsetminus U$. By induction, we conclude that $W \smallsetminus U$ is open in $Y \smallsetminus U$. This implies that W is open, since

$$Y \smallsetminus W = (Y \smallsetminus U) \smallsetminus (W \smallsetminus U)$$

is closed in $Y \smallsetminus U$, hence in Y. $\square$

PROOF OF THEOREM 5.6.15. If U is an open subset of X, we may replace f by its restriction to U, which is still flat. Therefore we only need to show that $f(X)$ is open in Y and it is enough to show that $f(X)$ satisfies the condition in the lemma. Suppose that W is an irreducible closed subset of Y such that $f(X) \cap W \neq \emptyset$. If V is an irreducible component of $f^{-1}(W)$, then V dominates W by Remark 5.6.13.

In this case, the image of V in W contains an open subset of W by Theorem 3.4.2. This completes the proof. $\square$

Our second main property of flat morphisms will follow from the following

PROPOSITION 5.6.17. *(Going Down for flat homomorphisms) If $\phi\colon A \to B$ is a flat ring homomorphism, then given prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A and $\mathfrak{q}_2$ in B such that $\phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$, there is a prime ideal $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ such that $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$.*

PROOF. As we have seen in the proof of Proposition 5.6.8, the fact that ϕ is flat implies that the induced homomorphism $A_{\mathfrak{p}_2} \to B_{\mathfrak{q}_2}$ is flat. After replacing ϕ by this homomorphism, we may thus assume that $(A, \mathfrak{p}_2)$ and $(B, \mathfrak{q}_2)$ are local rings and ϕ is a local homomorphism. In this case every prime ideal in B is contained in $\mathfrak{q}_2$. Since the prime ideals in B lying over $\mathfrak{p}_1$ are in bijection with the prime ideals in $(A_{\mathfrak{p}_1}/\mathfrak{p}_1 A_{\mathfrak{p}_1}) \otimes_A B$, it is enough to show that this ring is not the zero ring. This is a consequence of the following more general lemma below. $\square$

LEMMA 5.6.18. *If $\phi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a flat local homomorphism of local rings, then the following hold:*

i) *For every A-module M, we have $M \otimes_A B = 0$ if and only if $M = 0$.*
ii) *For every morphism of A-modules $u\colon M \to N$, we have $u = 0$ if and only if $u \otimes_A \mathrm{id}_B = 0$. In particular, ϕ is injective.*
iii) *Given two maps of A-modules*

$$M' \xrightarrow{u} M \xrightarrow{v} M''$$

with $v \circ u = 0$, the above sequence is exact if and only if the induced sequence

$$M' \otimes_A B \xrightarrow{u \otimes \mathrm{id}_B} M \otimes_A B \xrightarrow{v \otimes \mathrm{id}_B} M'' \otimes_A B$$

is exact.

PROOF. In order to prove i), note that if $u \in M$ is nonzero and $I = \mathrm{Ann}_A(u)$, then $I \subseteq \mathfrak{m}$ and $Au \simeq A/I$. We thus have an inclusion $A/I \hookrightarrow M$ and the flatness assumption implies that the induced morphism $B/IB = A/I \otimes B \to M \otimes_A B$ is injective. Since $IB \subseteq \mathfrak{n}$, it follows that B/IB is nonzero, hence $M \otimes_A B$ is nonzero.

If $u\colon M \to N$ is a morphism of A-modules, since M is flat, we have

$$\mathrm{Im}(u \otimes_A \mathrm{id}_B) \simeq \mathrm{Im}(u) \otimes_A B,$$

hence by i), $\mathrm{Im}(u \otimes_A \mathrm{id}_B) = 0$ if and only if $\mathrm{Im}(u) = 0$. We thus obtain the first assertion in ii), and the second one follows by taking u to be the multiplication on A with an element $a \in A$.

The assertion in iii) follows from the one in i), using the fact that since B is flat over A, we have an isomorphism

$$\ker(v \otimes \mathrm{id}_B)/\mathrm{Im}(u \otimes \mathrm{id}_B) \simeq \big(\ker(v)/\mathrm{Im}(u)\big) \otimes_A B.$$

$\square$

PROPOSITION 5.6.19. *If $\phi\colon A \to B$ is a ring homomorphism that satisfies the Going-Down property in the previous proposition, then for every prime ideal $\mathfrak{q}$, if we put $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$, then*

$$\dim(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}) \leq \dim(B_{\mathfrak{q}}) - \dim(A_{\mathfrak{p}}).$$

PROOF. Let $r = \dim(B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q})$ and $s = \dim(A_\mathfrak{p})$. We can choose prime ideals $\mathfrak{p}_s \subsetneq \ldots \subsetneq \mathfrak{p}_0 = \mathfrak{p}$ in A and $\mathfrak{q}_r \subsetneq \ldots \subsetneq \mathfrak{q}_0 = \mathfrak{q}$ in B, with $\mathfrak{p}B \subseteq \mathfrak{q}_r$. Applying the Going-Down property successively, we obtain a sequence of prime ideals $\mathfrak{p}'_s \subseteq \ldots \subseteq \mathfrak{p}'_0 \subseteq \mathfrak{q}_r$ such that $\phi^{-1}(\mathfrak{p}'_i) = \mathfrak{p}_i$ for $0 \leq i \leq s$. In particular, we have $\mathfrak{p}'_i \neq \mathfrak{p}'_{i+1}$ for $0 \leq i \leq s-1$ (however, we might have $\mathfrak{p}'_0 = \mathfrak{q}_s$). From the sequence of prime ideals in B

$$\mathfrak{p}'_s \subsetneq \ldots \subsetneq \mathfrak{p}'_1 \subsetneq \mathfrak{q}_r \subsetneq \ldots \subsetneq \mathfrak{q}_0 = \mathfrak{q},$$

we conclude that $\dim(B_\mathfrak{q}) \geq r + s$. □

By combining the above two propositions, we obtain the following consequence in our geometric setting:

THEOREM 5.6.20. *If $f\colon X \to Y$ is a flat morphism between two algebraic varieties, W is an irreducible closed subset of Y such that $f^{-1}(W) \neq \emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have*

$$\operatorname{codim}_X(V) = \operatorname{codim}_Y(W).$$

PROOF. Note first that V dominates W (see Remark 5.6.13). We apply Proposition 5.6.19 for the flat morphism

$$\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V},$$

which satisfies the Going-Down property by Proposition 5.6.17. Since V is an irreducible component of $f^{-1}(W)$, we obtain the inequality

$$\operatorname{codim}_X(V) \geq \operatorname{codim}_Y(W).$$

In order to prove the opposite inequality, let X' be an irreducible component of X containing V and such that $\operatorname{codim}_X(V) = \operatorname{codim}_{X'}(V)$. If Y' is an irreducible component of Y that contains $\overline{f(X')}$, then X' dominates Y' by Remark 5.6.13. We can thus apply Theorem 3.4.1 to deduce

$$\operatorname{codim}_X(V) = \operatorname{codim}_{X'}(V) \leq \operatorname{codim}_{Y'}(W) \leq \operatorname{codim}_Y(W).$$

This completes the proof of the theorem. □

REMARK 5.6.21. Let $f\colon X \to Y$ be a flat morphism. If X' is an irreducible component of X, then there is an irreducible component Y' of Y such that $f(X') \subseteq Y'$. In this case, it is clear that X' is an irreducible component of $f^{-1}(Y')$, and thus X' dominates Y' by Remark 5.6.13. In particular, such Y' is unique. Conversely, given any irreducible component Y' of Y such that $f^{-1}(Y') \neq \emptyset$, then every irreducible component X' of $f^{-1}(Y')$ is an irreducible component of X. Indeed, if $X' \subsetneq X''$, for some irreducible component X'' of X, then there is an irreducible component Y'' of Y such that $f(X'') \subseteq Y''$. Since X' dominates Y' by Remark 5.6.13, it follows that $Y' \subseteq Y''$, hence $Y' = Y''$. This contradicts the fact that X' is an irreducible component of $f^{-1}(Y')$.

DEFINITION 5.6.22. We say that a morphism $f\colon X \to Y$ is *flat, of relative dimension* n, if it is flat and for every irreducible component X' of X, if Y' is the irreducible component of Y such that $f(X') \subseteq Y'$, then $\dim(X') = \dim(Y') + n$.

REMARK 5.6.23. Of course, if $f\colon X \to Y$ is a flat morphism between irreducible varieties, then f is flat, of relative dimension n, where $n = \dim(X) - \dim(Y)$.

COROLLARY 5.6.24. *If $f\colon X \to Y$ is a flat morphism, of relative dimension n, then for every irreducible closed subset W of Y such that $f^{-1}(W) \neq \emptyset$, and every irreducible component V of $f^{-1}(W)$, we have*

$$\dim(V) = \dim(W) + n.$$

In particular, every non-empty fiber of f has pure dimension n.

PROOF. Note that by Theorem 5.6.20, we have

$$\mathrm{codim}_X(V) = \mathrm{codim}_Y(W)$$

Suppose now that X' is an irreducible component of X containing V and such that $\mathrm{codim}_{X'}(V) = \mathrm{codim}_X(V)$. If Y' is the irreducible component of Y such that $f(X') \subseteq Y'$, then we have

$$\dim(W) = \dim(Y') - \mathrm{codim}_{Y'}(W) \geq \dim(Y') - \mathrm{codim}_Y(W)$$
$$= \dim(X') - n - \mathrm{codim}_X(V) = \dim(X') - n - \mathrm{codim}_{X'}(V) = \dim(V) - n.$$

Similarly, if Y'' is an irreducible component of Y containing W and such that $\mathrm{codim}_{Y''}(W) = \mathrm{codim}_Y(W)$, and if X'' is an irreducible component of $f^{-1}(Y'')$ that contains V, then we have

$$\dim(V) = \dim(X'') - \mathrm{codim}_{X''}(V) \geq \dim(X'') - \mathrm{codim}_X(V) = \dim(Y'') + n - \mathrm{codim}_Y(W)$$
$$= \dim(Y'') + n - \mathrm{codim}_{Y''}(W) = \dim(W) + n.$$

By combining the two inequalities, we obtain the equality in the corollary. $\square$

REMARK 5.6.25. Note that if $f\colon X \to Y$ is a flat morphism such that every non-empty fiber of f has pure dimension n, then f is flat, of relative dimension n. Indeed, if X' is an irreducible component of X and Y' is the irreducible component of Y that contains $f(X')$, then there is an open subset U of Y' such that for every $y \in U$, $f^{-1}(y) \cap X'$ is an irreducible component of $f^{-1}(y)$. We thus conclude using Theorem 3.4.2 that $\dim(X') = \dim(Y') + n$.

CHAPTER 6

Smooth varieties

In this chapter we introduce an important local property of points on algebraic varieties: smoothness. We begin by describing a fundamental construction, the blow-up of a variety along an ideal (in the case of an affine variety). We then define the tangent space of a variety at a point and use it to define smooth points. We make use of the blow-up of the variety at a smooth point to show that the local ring of a smooth point is a domain. After discussing some general properties of smooth varieties, we prove Bertini's theorem on general hyperplane sections of smooth projective varieties and end the chapter by introducing smooth morphisms between smooth varieties.

6.1. Blow-ups

In this section we discuss the blow-up of an affine variety along an ideal. We will later globalize this construction, after having at our disposal coherent sheaves of ideals and the global MaxProj construction.

Let X be an affine variety, with $A = \mathcal{O}(X)$, and let $I \subseteq A$ be an ideal.

Definition 6.1.1. The *Rees algebra* $R(A, I)$ is the $\mathbf{N}$-graded k-subalgebra

$$R(A, I) = \bigoplus_{m \in \mathbf{N}} I^m t^m \subseteq A[t].$$

Since A is reduced, it follows that $A[t]$ is reduced, hence so is $R(A, I)$. Similarly, if X is irreducible, then $A[t]$ is a domain, hence so is $R(A, I)$.

Note that $R(A, I)$ is finitely generated and, in fact, it is generated by its degree 1 component: if $I = (a_1, \ldots, a_r)$, then $a_1 t, \ldots, a_r t$ generate $R(A, I)$. We can thus apply to $R(A, I)$ the MaxProj construction discussed in Section 4.3. Note that the degree 0 component is equal to A.

Definition 6.1.2. The blow-up of X along I is the morphism

$$f \colon \mathrm{MaxProj}\big(R(A, I)\big) \to X.$$

We will typically assume that I is nonzero, since otherwise $\mathrm{MaxProj}\big(R(A, I)\big)$ is empty. We collect in the next proposition some basic properties of this construction.

Proposition 6.1.3. *Let X be an affine variety, with $A = \mathcal{O}(X)$, and let $I \subseteq A$ be a nonzero ideal. If Z is the closed subset of X defined by I and $f \colon \widetilde{X} \to X$ is the blow-up of X along I, then the following hold:*

i) *The morphism f is an isomorphism over $X \smallsetminus Z$.*
ii) *The inverse image $f^{-1}(Z)$ is locally defined in $\widetilde{X}$ by one equation, which is a non-zero-divisor. In particular, every irreducible component of $f^{-1}(Z)$ has codimension 1 in $\widetilde{X}$.*

iii) *If X is irreducible, then $\widetilde{X}$ is irreducible and f is a birational morphism.*
iv) *More generally, if Z does not contain any irreducible component of X, by mapping X' to $f(X')$, we get a bijection between the irreducible components of $\widetilde{X}$ and those of X, such that the corresponding varieties are birational.*

Proof. In order to prove the assertion in i), it is enough to show that if $a \in A$ is such that $D_X(a) \cap V(I) = \emptyset$ (which implies $a \in \sqrt{I}$, hence $I_a = A_a$), then the induced morphism $f^{-1}\big(D_X(a)\big) \to D_X(a)$ is an isomorphism. Since $f^{-1}\big(D_X(a)\big) = \operatorname{MaxProj}\big(R(A,I)_a\big)$ (see Remark 4.3.17) and $R(A,I)_a \simeq R(A_a, I_a)$, we see that it is enough to show that if $I = A$, then f is an isomorphism. However, in this case

$$\widetilde{X} = \operatorname{MaxProj}(A[t]) = \operatorname{MaxSpec}(A) \times \mathbf{P}^0$$

by Proposition 4.3.12, with f being the projection on the first component. This is clearly an isomorphism.

In order to prove ii), note that $f^{-1}(Z) = V\big(I \cdot R(A,I)\big)$. Let us choose generators $a_1, \ldots, a_n$ of I and consider the affine open cover

$$\widetilde{X} = \bigcup_{i=1}^{n} D^+_{\widetilde{X}}(a_i t).$$

Note that by Propositions 4.3.15 and 4.3.16, we have

$$D^+_{\widetilde{X}}(a_i t) \simeq \operatorname{MaxSpec}\big(R(A,I)_{(a_i t)}\big).$$

Since the ideal $I \cdot R(A,I)_{(a_i t)}$ is generated by $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ and $\frac{a_j}{1} = \frac{a_i}{1} \cdot \frac{a_j t}{a_i t}$ for $1 \leq j \leq n$, we conclude that $I \cdot R(A,I)_{(a_i t)}$ is generated by $\frac{a_i}{1}$. Finally, note that $\frac{a_i}{1} \in R(A,I)_{(a_i t)}$ is a non-zero divisor: if $\frac{a_i}{1} \cdot \frac{h}{a_i^m t^m} = 0$ for some $h \in R(A,I)_m$, then there is $q \geq 1$ such that $h a_i^q = 0$ in A, hence $\frac{h}{a_i^m t^m} = 0$ in $R(A,I)_{(a_i t)}$. This gives the first assertion in ii) and the second one follows from the Principal Ideal theorem (see also Remark 3.3.6).

The assertion in iii) is clear: we have seen that in this case $\widetilde{X}$ is irreducible and f is an isomorphism over the nonempty closed subset $X \smallsetminus Z$.

Suppose now that $X_1, \ldots, X_r$ are the irreducible components of X and that Z does not contain any of the X_i. It follows from i) that $\widetilde{X}_i := \overline{f^{-1}(X_i \smallsetminus Z)}$ is an irreducible component of $\widetilde{X}$ such that f induces a birational morphism $\widetilde{X}_i \to X_i$. Since f is proper (see Corollary 5.1.10), the image $f(\widetilde{X}_i)$ is closed, hence $f(\widetilde{X}_i) = X_i$.

Moreover, we have

$$\widetilde{X} \subseteq f^{-1}(Z) \cup \bigcup_{i=1}^{r} \widetilde{X}_i.$$

On the other hand, no irreducible component of $f^{-1}(Z)$ can be an irreducible component of $\widetilde{X}$, since we can find, on a suitable affine chart, a non-zero-divisor that vanishes on $f^{-1}(Z)$ (see Remark 3.3.6). We thus conclude that

$$\widetilde{X} = \bigcup_{i=1}^{r} \widetilde{X}_i,$$

completing the proof of iv). ☐

EXAMPLE 6.1.4. Suppose that $I = \mathfrak{m}$ is the maximal ideal defining a non-isolated point $x \in X = \mathrm{MaxSpec}(A)$, hence $Z = \{x\}$. It follows from Remark 4.3.18 that $f^{-1}(Z)$ is the closed subset associated to the ideal $\mathfrak{m} \cdot R(A, \mathfrak{m})$. Note that

$$R(A, \mathfrak{m})/\mathfrak{m} \cdot R(A, \mathfrak{m}) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} =: \mathrm{gr}_{\mathfrak{m}}(A).$$

Note that if $X_1, \ldots, X_r$ are the irreducible components of X that contain x, then the corresponding irreducible components of $\widetilde{X}$ are precisely those that meet $f^{-1}(Z)$. Since $f^{-1}(Z)$ is locally defined by a non-zero-divisor, we conclude that

$$\dim\left(f^{-1}(Z)\right) = \max_i\{\dim(X_i)\} - 1 = \dim(\mathcal{O}_{X,x}) - 1.$$

Since

$$f^{-1}(Z) \simeq \mathrm{MaxProj}\big(\mathrm{gr}_{\mathfrak{m}}(A)/J\big),$$

where J is the nil-radical of $\mathrm{gr}_{\mathfrak{m}}(A)$, we conclude using Proposition 4.2.11 that

$$\dim\left(\mathrm{gr}_{\mathfrak{m}}(A)\right) = \dim(\mathcal{O}_{X,x}).$$

EXAMPLE 6.1.5. With the above notation, suppose that $X = \mathbf{A}^n$, hence $A = k[x_1, \ldots, x_n]$, and $I = (x_1, \ldots, x_n)$. We thus have a surjective homomorphism

$$\phi\colon A[y_1, \ldots, y_n] \to R(A, I), \quad \phi(y_i) = x_i t \quad \text{for} \quad 1 \leq i \leq n,$$

inducing a closed immersion

$$\iota\colon \widetilde{X} \hookrightarrow X \times \mathbf{P}^{n-1}$$

of varieties over X. Note that if J is the ideal in $A[y_1, \ldots, y_n]$ generated by all differences $x_iy_j - x_jy_i$, for $i \neq j$, then $J \subseteq \ker(\phi)$, hence $\iota(\widetilde{X})$ is contained in $V(J)$. On the other hand, we have seen in Exercise 5.1.13 that $V(J)$ is an irreducible variety, of dimension n. We thus conclude that ι gives an isomorphism of $\widetilde{X}$ with $V(J)$. In particular, our old definition for the blow-up of the affine space at the origin agrees with the new one. For a generalization of this example, see Example 6.3.24 below.

DEFINITION 6.1.6. Suppose that X is an irreducible affine variety, Z is a proper closed subset of X, and $f\colon \widetilde{X} \to X$ is the blow-up of X along I. If Y is any closed subvariety of X such that no irreducible component of Y is contained in Z, then the *strict transform* (or *proper transform*) of Y in $\widetilde{X}$ is given by

$$\widetilde{Y} := \overline{f^{-1}(Y \smallsetminus Z)}.$$

REMARK 6.1.7. With the notation in the above definition, we have an induced morphism $\widetilde{Y} \to Y$ that can be identified with the blow-up of Y along the ideal $J = I \cdot \mathcal{O}(Y)$. Indeed, if $B = \mathcal{O}(Y)$, then the surjection $A \to B$ induces a graded, surjective homomorphism of k-algebras:

$$R(A, I) \to R(B, J).$$

This induces by Proposition 4.3.11 a commutative diagram

$$\begin{tikzcd}
\mathrm{MaxProj}\big(R(B, J)\big) \arrow[r, "j"] \arrow[d, "g"] & \mathrm{MaxProj}\big(R(A, I)\big) \arrow[d, "f"] \\
Y \arrow[r, "i"] & X,
\end{tikzcd}$$

where i and j are closed immersions. By Proposition 6.1.3, g maps birationally each irreducible component of $\widetilde{Y}$ onto a corresponding irreducible component of Y, which implies that the image of j is, indeed, the strict transform of Y.

EXAMPLE 6.1.8. In order to give some intuition about the blow-up, we discuss the strict transform of a curve in $\mathbf{A}^2$ under the blow-up at one point. Let us begin, more generally, with the computation of the strict transform of a hypersurface in $\mathbf{A}^n$ under the blow-up at one point, where $n \geq 2$. Suppose that Y is a hypersurface in $\mathbf{A}^n$, with corresponding radical ideal defined by the non-zero polynomial $f \in k[x_1, \ldots, x_n]$. Given a point $p \in Y$, the *multiplicity* $\operatorname{mult}_p(Y)$ of Y (or of f) at p is the largest $j \geq 1$ such that $f \in \mathfrak{m}_p^j$, where $\mathfrak{m}_p$ is the maximal ideal corresponding to p. Let $\pi \colon \widetilde{\mathbf{A}^n} \to \mathbf{A}^n$ be the blow-up of $\mathbf{A}^n$ at p. After a suitable translation, we may assume that $p = 0$. We can thus write

$$f = f_m + f_{m+1} + \ldots + f_d,$$

where each f_i is homogeneous of degree i, and $f_m \neq 0$, so that $\operatorname{mult}_0(Y) = m$. Recall that by Example 6.1.5, we know that $\widetilde{\mathbf{A}^n}$ is the subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$ given by the equations $x_i y_j = x_j y_i$, for $0 \leq i, j \leq n$, where $y_1, \ldots, y_n$ are the homogeneous coordinates on $\mathbf{P}^{n-1}$. Consider the chart U_i on $\widetilde{Y}$ given by $y_i \neq 0$. Note that in this chart we have $x_j = x_i \frac{y_j}{y_i}$ for $j \neq i$, hence $U_i \simeq \mathbf{A}^n$, with coordinates $u_1, \ldots, u_n$ such that $\pi^{\#}(x_i) = u_i$ and $\pi^{\#}(x_j) = u_i u_j$ for $j \neq i$. If $E = \pi^{-1}(0)$, then $E \cap U_i$ is defined by $u_i = 0$.

The inverse image $\pi^{-1}(Y)$ is defined in U_i by

$$\pi^{\#}(f) = f(u_1 u_i, \ldots, u_i, \ldots, u_n u_i)$$

$$= u_i^m \cdot \big(f_m(u_1, \ldots, 1, \ldots, u_n) + u_i \cdot f_{m+1}(u_1, \ldots, 1, \ldots, u_n) + \ldots + u_i^{m-d} f_d(u_1, \ldots, 1, \ldots, u_n)\big).$$

Since the polynomial

$$\widetilde{f} := f_m(u_1, \ldots, 1, \ldots, u_n) + u_i \cdot f_{m+1}(u_1, \ldots, 1, \ldots, u_n) + \ldots + u_i^{m-d} f_d(u_1, \ldots, 1, \ldots, u_n)$$

defines a hypersurface in U_i that does not contain $E \cap U_i$, it follows that its zero-locus defines $\widetilde{Y} \cap U_i$. In fact, since the homomorphism $k[x_1, \ldots, x_n]_{x_i} \to \mathcal{O}(U_i)_{u_i}$ is an isomorphism, it is easy to deduce that $\widetilde{f}$ is square-free, hence it generates the ideal of $\widetilde{Y} \cap U_i$.

Let us specialize now to the case $n = 2$. In this case f_m is a homogeneous polynomial of degree d, which can thus be written as $f_m = \prod_{j=1}^m \ell_j$, where each ℓ_j is a linear form (we use the fact that k is algebraically closed, hence every polynomial in one variable is the product of degree 1 polynomials). The lines through the origin defined by the factors of f_m are the *tangents* to X at 0. Note that the lines through 0 in $\mathbf{A}^2$ are parametrized by $\mathbf{P}^1 = \pi^{-1}(0)$.

We claim that after the blow-up, the points of intersection of the strict transform $\widetilde{Y}$ with E correspond precisely to the tangent lines to X at 0. Indeed, if we consider for example the chart U_1, note that the points of $\widetilde{Y} \cap E \cap U_1$ are defined by $u_1 = 0 = f_m(1, u_2)$. It follows that if $f_m = \prod_{j=1}^m (a_j x_1 + b_j x_2)$, then the points of $\widetilde{Y} \cap E \cap U_1$ are precisely the points $[b_j, -a_j] \in E$ with $b_j \neq 0$. Similarly, the points of $\widetilde{Y} \cap E \cap U_2$ are precisely the points $[b_j, -a_j] \in E$ with $a_j \neq 0$. This proves our claim. In fact, this is not just a set-theoretic correspondence: tangents that appear with multiplicity > 1 in f_m translate to tangency conditions between $\widetilde{Y}$ and E at the corresponding point. We will return to this phenomenon at a later point.

6.2. The tangent space

We begin with the following general observation. If $(R, \mathfrak{m})$ is a local Noetherian ring, then $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional vector space over the residue field $K = R/\mathfrak{m}$. It follows from Nakayama's lemma that $\dim_K \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators for the ideal $\mathfrak{m}$ (see Remark C.1.3).

In this section we are interested in the case when $(X, \mathcal{O}_X)$ is an algebraic variety, $p \in X$ is a point, and we consider the local ring $\mathcal{O}_{X,p}$, with maximal ideal $\mathfrak{m}_p$. Recall that in this case the residue field is the ground field k.

Definition 6.2.1. The *tangent space* of X at p is the k-vector space

$$T_pX := (\mathfrak{m}_p/\mathfrak{m}_p^2)^* = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k).$$

The following proposition explains the terminology in the above definition. Note that T_pX does not change if we replace X by an affine open neighborhood of p. In particular, we may assume that X is affine and choose a closed immersion $X \hookrightarrow \mathbf{A}^n$.

Proposition 6.2.2. *If X is a closed subvariety of $\mathbf{A}^n$ with corresponding radical ideal $I_X \subseteq k[x_1, \ldots, x_n]$, then T_pX is isomorphic to the linear subspace of k^n defined by the equations*

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)x_i = 0 \quad \textit{for all} \quad f \in I_X.$$

Moreover, it is enough to only consider those equations corresponding to a system of generators of I_X.

In the case of a closed subset X of $\mathbf{A}^n$, we will refer to the linear subspace in the proposition as the *embedded tangent space* in the affine space.

Proof of Proposition 6.2.2. Let $f_1, \ldots, f_r$ be a system of generators of I_X. In this case, if $p = (a_1, \ldots, a_n)$, we have

$$\mathcal{O}_{X,p} = \mathcal{O}_{\mathbf{A}^n,p}/(f_1, \ldots, f_r) \quad \text{and} \quad \mathfrak{m}_p = (x_1 - a_1, \ldots, x_n - a_n)\mathcal{O}_{\mathbf{A}^n,p}/(f_1, \ldots, f_r).$$

Therefore we have

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 = k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)^2 + (f_1, \ldots, f_r).$$

On the other hand, for every $f \in k[x_1, \ldots, x_n]$, we have

$$f \equiv f(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot (x_i - a_i) \mod (x_1 - a_1, \ldots, x_n - a_n)^2.$$

We thus see that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the quotient of the vector space over k with basis $e_i = x_i - a_i$ for $1 \leq i \leq n$, by the subspace generated by

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)e_i \quad \text{for} \quad f \in (f_1, \ldots, f_r).$$

This immediately gives the first assertion in the proposition.

Note now that if $g \in (f_1, \ldots, f_r)$ and we write $g = \sum_{j=1}^r h_j f_j$, then it follows from the product rule and the fact that $f_j(p) = 0$ for all j that

$$\sum_{i=1}^n \frac{\partial g}{\partial x_i}(p)x_i = \sum_{j=1}^r h_j(p) \cdot \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)x_i.$$

The last assertion in the proposition follows. □

REMARK 6.2.3. If $f\colon X \to Y$ is a morphism of varieties and $p \in X$, we obtain a local homomorphism of local rings

$$\phi\colon \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}.$$

This induces a k-linear morphism

$$\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$$

and by taking duals, we obtain a k-linear map $df_p\colon T_pX \to T_{f(p)}Y$. It is easy to see that this definition is functorial: if $g\colon Y \to Z$ is another morphism, then

$$dg_{f(p)} \circ df_p = d(g \circ f)_p.$$

REMARK 6.2.4. If Y is a closed subvariety of the variety X and if $i\colon Y \to X$ is the inclusion, then for every $p \in Y$, the linear map $di_p\colon T_pY \to T_pX$ is injective. This follows from the fact that the homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective and therefore the induced map $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \to \mathfrak{m}_{Y,p}/\mathfrak{m}_{Y,p}^2$ is surjective, where $\mathfrak{m}_{X,p} \subseteq \mathcal{O}_{X,p}$ and $\mathfrak{m}_{Y,p} \subseteq \mathcal{O}_{Y,p}$ are the corresponding maximal ideals.

Note that if Y if if a closed subvariety of $\mathbf{A}^n$ and $i\colon Y \hookrightarrow \mathbf{A}^n$ is the inclusion, then the embedded tangent space of Y at p is the image of di_p, where we identify in the obvious way $T_p\mathbf{A}^n$ to k^n.

REMARK 6.2.5. If X and Y are closed subvarieties of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, and if $f = (f_1, \ldots, f_n)\colon X \to Y$, then via the isomorphisms given by Proposition 6.2.2, the linear map df_p is induced by the linear map $k^m \to k^n$ given with respect of the standard bases by the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)$. Indeed, by functoriality, it is enough to check this when $X = \mathbf{A}^m$ and $Y = \mathbf{A}^n$. Let $x_1, \ldots, x_m$ be the coordinate functions on $\mathbf{A}^m$ and $y_1, \ldots, y_n$ the coordinate functions on $\mathbf{A}^n$. If $p = (a_1, \ldots, a_m)$, then the maximal ideals defining p and $f(p)$ are

$$\mathfrak{m}_p = (x_1 - a_1, \ldots, x_m - a_m) \quad \text{and} \quad \mathfrak{m}_{f(p)} = \big(y_1 - f_1(p), \ldots, y_n - f_n(p)\big).$$

Moreover, the map $\mathfrak{m}_{f(p)} \to \mathfrak{m}_p$ maps $y_i - f_i(p)$ to $f_i - f_i(p)$ and Taylor's formula shows that

$$f_i - f_i(p) - \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(p)(x_j - a_j) \in \mathfrak{m}_p^2,$$

which implies, after taking duals, our assertion.

In the case of a closed subvariety of a projective space, we also have an embedded tangent space: this time, it is a linear subpace of the projective space. This is defined as follows. Suppose that X is a closed subset of $\mathbf{P}^n$, with corresponding radical homogeneous ideal I_X. Given a point $p = [u_0, \ldots, u_n] \in X$, the *projective tangent space* of X at p, that we will denote by $\mathbf{T}_pX$, is the linear subspace of $\mathbf{P}^n$ defined by the equations

$$\sum_{i=0}^n \frac{\partial f}{\partial x_i}(u_0, \ldots, u_n)x_i = 0,$$

where f varies over the homogeneous elements in I_X. Note first that since f is homogeneous, if we replace $(u_0, \ldots, u_n)$ by $(\lambda u_0, \ldots, \lambda u_n)$, for some $\lambda \in k^*$, then the equation gets multiplied by λ. Note also that it is enough to consider a system of homogeneous generators of I_X: if $f = \sum_{j=1}^r g_j f_j$, with $f_j \in I_X$, then we get using the product rule and the fact that $f_j(u_0, \ldots, u_n) = 0$ for all j

$$\sum_{i=0}^n \frac{\partial f}{\partial x_i}(u_0, \ldots, u_n)x_i = \sum_{j=1}^r g_j(u_0, \ldots, u_n) \cdot \sum_{i=0}^n \frac{\partial f_j}{\partial x_i}(u_0, \ldots, u_n)x_i.$$

Finally, we note that $\mathbf{T}_pX$ contains the point p: this is a consequence of Euler's identity, which says that if f is homogeneous, of degree d, then

$$\sum_{i=0} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

The terminology is justified by the following

PROPOSITION 6.2.6. *Let X be a closed subvariety of $\mathbf{P}^n$ and $p \in X$. If i is such that $p \in U_i = (x_i \neq 0)$ and if we identify U_i with $\mathbf{A}^n$ is the usual way, then $\mathbf{T}_pX \cap U_i$ is the image of the embedded tangent space in $\mathbf{A}^n$ for $X \cap U_i$ at p by the translation mapping 0 to p.*

PROOF. In order to simplify the notation, let us assume that $i = 0$. In this case, we may assume that $(u_0, \ldots, u_n) = (1, u_1, \ldots, u_n)$. Note that the ideal of $X \cap U_i$ in $k[x_1, \ldots, x_n]$ is generated by $f(1, x_1, \ldots, x_n)$, where f varies over a set of homogeneous generators of I_X (see Exercise 4.2.14). Fix such f and let $g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n)$. Therefore we have $\frac{\partial g}{\partial x_i}(u_1, \ldots, u_n) = \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n)$. On the other hand, it follows from Euler's identity that

$$\frac{\partial f}{\partial x_0}(1, u_1, \ldots, u_n) = -\sum_{i=1}^n u_i \cdot \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n).$$

This implies that

$$\frac{\partial f}{\partial x_0}(1, u_1, \ldots, u_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n)x_i = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(u_1, \ldots, u_n) \cdot (x_i - u_i).$$

This implies the assertion in the proposition. $\square$

EXERCISE 6.2.7. Given varieties X and Y, for every $x \in X$ and $y \in Y$, the projections $X \times Y \to X$ and $X \times Y \to Y$ induce a linear map

$$T_{(x,y)}(X \times Y) \to T_xX \times T_yY.$$

Show that this is an isomorphism.

6.3. Smooth algebraic varieties

Let X be an algebraic variety. Given a point $p \in X$, recall that we put $\dim_p X := \dim(\mathcal{O}_{X,p})$. This is the largest dimension of an irreducible component of X that contains p (see Remark 3.3.14), and also the codimension of $\{p\}$ in X. Our first goal is to show that $\dim_k T_pX \geq \dim_p X$.

More generally, we will get a similar statement for the localization of a finite type k-algebra at a prime ideal. This applies, in particular, for the local ring $(\mathcal{O}_{X,V}, \mathfrak{m}_V)$ of X at an irreducible closed subset V. Note that in this case the residue field is the field of rational functions on V.

PROPOSITION 6.3.1. *For every local ring $(R, \mathfrak{m})$ that is the localization of a k-algebra of finite type at a prime ideal, we have*

$$\dim(R) \leq \dim_K \mathfrak{m}/\mathfrak{m}^2,$$

where $K = R/\mathfrak{m}$.

PROOF. Suppose that $R = A_{\mathfrak{p}}$, where A is a k-algebra of finite type and $\mathfrak{p}$ is a prime ideal in A. Note that if I is the nil-radical of A and $\overline{R} = R/I \cdot R$, then $\overline{R}$ is local, with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/I \cdot R$, and the same residue field. Since $\dim(R) = \dim(\overline{R})$, while

$$\dim_K \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \dim_K \mathfrak{m}/(\mathfrak{m}^2 + I \cdot R) \leq \dim_K \mathfrak{m}/\mathfrak{m}^2,$$

we see that that it is enough to prove the assertion when A is reduced. Let X be an affine variety with $\mathcal{O}(X) = A$ and let V be the irreducible closed subset defined by $\mathfrak{p}$.

Recall that by Nakayama's lemma, if $r = \dim_K \mathfrak{m}/\mathfrak{m}^2$, then $\mathfrak{m}$ is generated by r elements. This implies that there is $f \in A \smallsetminus \mathfrak{p}$ such that $\mathfrak{p}A_f$ is generated by r elements. After replacing A by A_f, we may thus assume that $\mathfrak{p}$ is generated by r elements. In this case, Corollary 3.3.7 implies $\dim(R) = \operatorname{codim}_X(V) \leq r$, giving the assertion in the proposition. $\square$

DEFINITION 6.3.2. A point $p \in X$ is *nonsingular* (or *smooth*) if $\dim_p X = \dim_k T_pX$. Otherwise, it is *singular*. The variety X is *nonsingular* (or *smooth*) if all its points are nonsingular points.

Given an irreducible, closed subset $V \subseteq X$, we say that X is *nonsingular at V* if $\dim(\mathcal{O}_{X,V}) = \dim_{k(V)} \mathfrak{m}_V/\mathfrak{m}_V^2$. We will see later that this is equivalent with the fact that some point $p \in V$ is a nonsingular point.

EXAMPLE 6.3.3. It is clear that every affine space $\mathbf{A}^n$ is a smooth variety. Since a projective space has an open cover by affine spaces, it follows that every projective space is a smooth variety.

EXAMPLE 6.3.4. Let X be a hypersurface in $\mathbf{A}^n$, defined by the radical ideal $(f) \subseteq k[x_1, \ldots, x_n]$. Since $\dim_p(X) = n - 1$ for every $p \in X$, it follows from definition and Proposition 6.2.2 that the set of singular points in X is the zero locus of the ideal

$$(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

In particular, we see that the set of smooth points is open in X. A generalization of this fact will be given in Theorem 6.3.7 below.

REMARK 6.3.5. Since Krull's Principal Ideal theorem holds in every Noetherian ring, the inequality in Proposition 6.3.1 also holds for arbitrary Noetherian local rings. A Noetherian local ring for which the inequality is an equality is a *regular* ring.

DEFINITION 6.3.6. For every regular local ring $(R, \mathfrak{m})$, a *regular system of parameters* is a minimal set of generators of $\mathfrak{m}$. Note that since R is regular, the length of such a system is equal to $\dim(R)$. If X is a variety and $p \in X$ is a smooth point, we say that some regular functions $f_1, \ldots, f_n$ defined in a neighborhood of p give a regular system of parameters at p if their images in $\mathcal{O}_{X,p}$ give a regular system of parameters.

The following is the main result of this section

THEOREM 6.3.7. *For every variety X, the set X_{sm} of smooth points $p \in X$ is a dense open subset.*

The complement of the smooth locus X_{sm} is the *singular locus* X_{sing} of X. We prove the theorem, assuming the following proposition, and then give the proof of the proposition.

PROPOSITION 6.3.8. *If $p \in X$ is a nonsingular point, then the local ring $\mathcal{O}_{X,p}$ is a domain (that is, p lies on a unique irreducible component of X).*

PROOF OF THEOREM 6.3.7. In order to prove the assertion, we may assume that X is irreducible. Indeed, if $X_1, \ldots, X_r$ are the irreducible components of X, then it follows from Proposition 6.3.8 that no point on the intersection of two distinct components is nonsingular. It thus follows that if $X_i' = X_i \smallsetminus \bigcup_{j \neq i} X_j$, then

$$X_{\mathrm{sm}} = \bigcup_{i=1}^{r} (X_i')_{\mathrm{sm}}.$$

Therefore it is enough to know the assertion for irreducible varieties.

Suppose now that X is irreducible and let $r = \dim(X)$. If $X = \bigcup_i U_i$ is an affine open cover, it is enough to show that each set $X_{\mathrm{sm}} \cap U_i = (U_i)_{\mathrm{sm}}$ is open and nonempty. Therefore we may and will assume that X is a closed subset of an affine space $\mathbf{A}^n$. If $f_1, \ldots, f_m$ are generators of the ideal defining X, then it follows from definition and Proposition 6.2.2 that a point $q \in X$ is a nonsingular point if and only if the rank of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(q)\right)$ is $\geq n - r$. This is the case if and only if one of the $(n-r)$-minors of the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ does not vanish at q, condition that defines an open subset of X.

In order to prove that X_{sm} is nonempty, we may replace X by a birational variety. By Proposition 1.6.13, we may thus assume that X is an irreducible hypersurface in $\mathbf{A}^{r+1}$. Let $f \in k[x_1, \ldots, x_{r+1}]$ be the irreducible polynomial that generates the prime ideal corresponding to X. As we have seen, for a point $q \in X$, we have $q \in X_{\mathrm{sm}}$ if and only there is i such that $\frac{\partial f}{\partial x_i}(q) \neq 0$. If $X_{\mathrm{sm}} = \emptyset$, then $\frac{\partial f}{\partial x_i}$ vanishes on X for $1 \leq i \leq r+1$. Therefore $\frac{\partial f}{\partial x_i} \in (f)$ for all i. If $\deg_{x_i}(f) = d_i$, then we clearly have $\deg_{x_i}\left(\frac{\partial f}{\partial x_i}\right) < d_i$, hence $\frac{\partial f}{\partial x_i} \in (f)$ implies that $\frac{\partial f}{\partial x_i} = 0$. Since this holds for every i, we conclude that $\mathrm{char}(k) = p > 0$ and $f \in k[x_1^p, \ldots, x_n^p]$. Since k is perfect, being algebraically closed, we conclude that $f = g^p$ for some $g \in k[x_1, \ldots, x_{r+1}]$, contradicting the fact that f is irreducible. $\square$

We now turn to the proof of Proposition 6.3.8. This will be a consequence of the following useful fact about smooth points. Let X be a variety and $p \in X$ a smooth point. We put $R = \mathcal{O}_{X,p}$ and let $\mathfrak{m}$ be the maximal ideal in R. Since p is a smooth point, if $n = \dim(R)$, then we can choose generators $a_1, \ldots, a_n$ for $\mathfrak{m}$. Note that $R/\mathfrak{m} = k$ and the classes $\overline{a_1}, \ldots, \overline{a_n} \in \mathfrak{m}/\mathfrak{m}^2$ give a k-basis. Consider the graded k-algebra homomorphism

$$\phi \colon k[x_1, \ldots, x_n] \to \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

that maps each x_i to $\overline{a_i}$. Since the right-hand side is generated by $\mathfrak{m}/\mathfrak{m}^2$, it is clear that ϕ is surjective.

PROPOSITION 6.3.9. *If $p \in X$ is a smooth point, then morphism ϕ defined above is an isomorphism.*

PROOF. Let U be an affine open neighborhood of p and let $A = \mathcal{O}(U)$. If $\mathfrak{n} \subseteq A$ is the maximal ideal corresponding to x, then $R = A_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}A_{\mathfrak{n}}$. Note that

$$\bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} = \mathrm{gr}_{\mathfrak{n}}(A),$$

hence this ring has dimension n by Example 6.1.4. Since ϕ is a surjective homomorphism and $k[x_1, \ldots, x_n]$ is a domain of dimension n, it follows that ϕ is an isomorphism. $\square$

PROOF OF PROPOSITION 6.3.8. Let $R = \mathcal{O}_{X,p}$ and $\mathfrak{m}$ the maximal ideal of R. We know, by Proposition 6.3.9, that the ring $S = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a domain. We now show that this implies that R is a domain. Suppose that $a, b \in R \smallsetminus \{0\}$ are such that $ab = 0$. It follows from Krull's Intersection theorem (see Theorem C.4.1) that there are i and j such that $a \in \mathfrak{m}^i \smallsetminus \mathfrak{m}^{i+1}$ and $b \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. In this case, since S is a domain, we conclude that $ab \not\in \mathfrak{m}^{i+j+1}$, a contradiction. Therefore R is a domain. $\square$

REMARK 6.3.10. It follows from Proposition 6.3.8 that every connected component of a smooth variety is irreducible. Because of this, when dealing with smooth varieties, one can easily reduce to the case when the variety is irreducible.

REMARK 6.3.11. The same line of argument can be used to prove a stronger statement: if A is a k-algebra of finite type, but non-necessarily reduced, and $\mathfrak{m}$ is a maximal ideal in A such that $A_{\mathfrak{m}}$ is a regular local ring, then $A_{\mathfrak{m}}$ is a domain. Indeed, let I be the nil-radical of A and $\overline{A} = A/I$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$. After possibly replacing A by the localization at a suitable element not in $\mathfrak{m}$, we may assume that $\mathfrak{m}$ is generated by n elements, where $n = \dim(A_{\mathfrak{m}}) = \dim(A)$. Consider the following two surjective morphisms:

$$A/\mathfrak{m}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathrm{gr}_{\mathfrak{m}}(A) \xrightarrow{\psi} \mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{A}).$$

By Example 6.1.4, we have

$$\dim\left(\mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{A}\right) = n,$$

which implies that $\psi \circ \phi$ is an isomorphism, which implies that ϕ is injective, hence an isomorphism. The argument in the proof of Proposition 6.3.8 now implies that $A_{\mathfrak{m}}$ is a domain.

REMARK 6.3.12. Suppose that $f \in k[x_1, \ldots, x_n]$ is a non-constant polynomial such that there is no point $p \in \mathbf{A}^n$, with

$$f(p) = 0 = \frac{\partial f}{\partial x_i}(p) \quad \text{for} \quad 1 \leq i \leq n.$$

In this case f generates a radical ideal and the corresponding hypersurface in $\mathbf{A}^n$ is smooth. Indeed, note that if g is a non-constant polynomial such that g^2 divides f, then for every $p \in V(g)$, we have $f(p) = 0$ and $\frac{\partial f}{\partial x_i} = 0$ for all i, a contradiction. The fact that the hypersurface defined by f is smooth now follows from Example 6.3.4.

A similar assertion holds in the projective setting, with an analogous argument: if $F \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of positive degree such that there is no point $p \in \mathbf{P}^n$ with

$$F(p) = 0 = \frac{\partial F}{\partial x_i}(p) \quad \text{for} \quad 0 \leq i \leq n,$$

then the ideal (F) is radical and it defines a smooth hypersurface in $\mathbf{P}^n$. Moreover, in this case we see that if $n \geq 2$, then this hypersurface is irreducible: indeed, two irreducible components would have non-empty intersection by Corollary 4.2.12 and any point on the intersection would be a singular point by Proposition 6.3.8.

EXERCISE 6.3.13. Show that if X and Y are algebraic varieties, the points $x \in X$ and $y \in Y$ are smooth if and only if (x, y) is a smooth point of $X \times Y$.

EXERCISE 6.3.14. Suppose that G is an algebraic group which has a transitive algebraic action on the variety X. Show that X is smooth. Deduce that every algebraic group is a smooth variety.

EXAMPLE 6.3.15. If V is an irreducible, closed subset of X, with $\operatorname{codim}_X(V) = 1$, then X is smooth at V if and only if the maximal ideal of $\mathcal{O}_{X,V}$ is principal, that is, $\mathcal{O}_{X,V}$ is a DVR (for an elementary discussion of DVRs, see Section C.5).

EXAMPLE 6.3.16. Let H be a hyperplane in $\mathbf{P}^n$ and X a closed subvariety of H. Given a point $p \in \mathbf{P}^n \smallsetminus H$, the *projective cone over* X *with vertex* p is the union $C_p(X)$ of the lines joining p with the points on X. Note first that this is a closed subvariety of $\mathbf{P}^n$.

In order to see this, after applying a suitable transformation in $PGL_{n+1}(k)$, we may assume that $H = (x_n = 0)$ and $p = [0, \ldots, 0, 1]$, and use the isomorphism $\mathbf{P}^{n-1} \to H$, given by $[u_0, \ldots, u_{n-1}] \to [u_0, \ldots, u_{n-1}, 0]$, to identify $\mathbf{P}^{n-1}$ and H. In this case,

$$C_p(X) = \{p\} \cup \{[u_0, \ldots, u_n] \in \mathbf{P}^n \mid [u_0, \ldots, u_{n-1}] \in X\}.$$

It is now clear that $C_p(X)$ is closed in $\mathbf{P}^n$; in fact, if $I_X \subseteq k[x_0, \ldots, x_{n-1}]$ is the homogeneous ideal corresponding to X, then the ideal of $C_p(X)$ is $I_X \cdot k[x_0, \ldots, x_n]$. Note that if U is the affine chart $U = (x_n \neq 0) \simeq \mathbf{A}^n$, then $C_p(X) \cap U$ is isomorphic to the affine cone over X.

We claim that p is a smooth point of $C_p(X)$ if and only if X is a linear subspace of H. Indeed, p is a smooth point of $C_p(X)$ if and only if 0 is a smooth point of the affine cone $C(X)$ over X. Note that the embedded tangent cone to $C(X)$ at 0 is defined by the linear polynomials in the ideal I_X of X; in other words, this is equal to the smallest vector subspace of k^n containing $C(X)$. This has the same dimension as $C(X)$ if and only if $C(X)$ is a linear space.

In the remainder of this section we give some further properties of smooth points.

PROPOSITION 6.3.17. *Let X be an algebraic variety and Y a closed subvariety, with $x \in Y_{\rm sm}$, such that there is an affine open neighborhood U of x in X, and $f_1, \ldots, f_r \in \mathcal{O}(U)$ satisfying the following conditions:*

i) *We have $I_U(Y \cap U) = (f_1, \ldots, f_r)$, and*
ii) *The subvariety $Y \cap U$ of U is irreducible, of codimension r.*

In this case x is a smooth point on X.

Note that since x is a smooth point of Y, it follows from Proposition 6.3.8 that x lies on a unique irreducible component of Y. Therefore after possibly replacing U by a smaller open subset, we can always assume that $Y\cap U$ is irreducible.

PROOF OF PROPOSITION 6.3.17. Let $R=\mathcal{O}_{X,x}$ and $\overline{R}$ be the local rings at x of X and Y, respectively. If $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ are the maximal ideals in R and $\overline{R}$, then $\overline{\mathfrak{m}}=\mathfrak{m}/(f_{1,x},\ldots,f_{r,x})$, where we denote by $f_{i,x}$ the image of f_i in R. It follows that

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2=\mathfrak{m}/\mathfrak{m}^2+(f_{1,x},\ldots,f_{r,x}),$$

hence $\dim_k T_xY\geq \dim_k T_xX-r$. Since we clearly have

$$\dim(\mathcal{O}_{X,x})\geq \operatorname{codim}_X(Y)+\dim(\mathcal{O}_{Y,x})=r+\dim(\mathcal{O}_{Y,x}),$$

we conclude that $\dim(\mathcal{O}_{X,x})\geq \dim_k T_xX$ and thus x is a smooth point of X. $\square$

REMARK 6.3.18. An important special case of the above proposition is that of a hypersurface: suppose that X is an algebraic variety and Y is a closed subvariety of X such that for some point $x\in Y$ and for some affine open neighborhood $U\subseteq X$ of x, we have $I_U(Y\cap U)=(f)$, for some non-zero divisor $f\in\mathcal{O}(U)$. In this case, if x is a smooth point of Y, then x is a smooth point of X. Indeed, note that in this case the fact that $Y\cap U$ has codimension 1 in U follows from Theorem 3.3.1 and Remark 3.3.6.

COROLLARY 6.3.19. *If X is a variety and V is an irreducible closed subset of X such that X is nonsingular at V, then $V\cap X_{\rm sm}\neq\emptyset$.*

We will see in Corollary 6.3.23 below that the converse also holds. This is a special case of a result due to Auslander-Buchsbaum and Serre, saying that if R is a regular local ring, then for every prime ideal $\mathfrak{p}$ in R, the localization $R_{\mathfrak{p}}$ is regular (see [**Eis95**, Chapter 19]).

PROOF OF COROLLARY 6.3.19. Let $r=\dim(\mathcal{O}_{X,V})$. By assumption, the maximal ideal in $\mathcal{O}_{X,V}$ is generated by r elements. After possibly replacing X by a suitable affine open subset meeting V, we may assume that X is affine and that $I_X(V)$ is generated by r elements $f_1,\ldots,f_r$. By Theorem 6.3.7, we can find a point $x\in Y_{\rm sm}$. We then deduce from Proposition 6.3.17 that x is a smooth point also on X, hence $Y\cap X_{\rm sm}\neq\emptyset$. $\square$

PROPOSITION 6.3.20. *Let p be a smooth point on a variety X. If $f_1,\ldots,f_r$ are regular functions defined in an open neighborhood of p, vanishing at p, and whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{X,p}$, then there is an affine open neighborhood U of x such that the following conditions hold:*

i) *We have $f_1,\ldots,f_r\in\mathcal{O}(U)$.*
ii) *We have a closed subvariety Y of X with $I_U(Y\cap U)=(f_1,\ldots,f_r)$.*
ii) *The subvariety Y is smooth at p and $\dim_p(Y)=\dim_p(X)-r$.*

PROOF. We begin by choosing an affine open neighborhood U of p such that $f_i\in\mathcal{O}(U)$ for all i and let Y be the closure in X of the zero-locus in U of $f_1,\ldots,f_r$. Since p lies on a unique irreducible component of X by Proposition 6.3.8, we may assume, after possibly shrinking U, that X is irreducible, and let $n=\dim(X)$. Let $R=\mathcal{O}_{X,p}$ and $\overline{R}=R/(f_{1,p},\ldots,f_{r,p})$, where $f_{i,p}$ is the image of f_i in R. If we denote by $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ the maximal ideals in R and $\overline{R}$, respectively, then by assumption, the classes of $f_{1,p},\ldots,f_{r,p}$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, hence $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = n-r$. On the other hand, it follows from Corollaries 3.3.7 and 3.3.13 that $\dim(\overline{R}) \geq n-r$. We thus conclude that $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \leq \dim(\overline{R})$ and it follows from Proposition 6.3.1 that this is an equality, hence $\overline{R}$ is a regular local ring. We also see that $\dim(\overline{R}) = n-r$. Since $\overline{R}$ is a regular ring, it follows from Remark 6.3.11 that $\overline{R}$ is a reduced ring, hence after replacing U by a smaller neighborhood of p, we may assume that $f_1,\ldots,f_r$ generate a radical ideal in $\mathcal{O}(U)$, hence $(f_1,\ldots,f_r) = I_U(Y\cap U)$. Since $\overline{R}$ is a regular ring, it follows that Y is smooth at p, with $\dim_p(Y) = \dim_p(X) - r$. □

The next result describes the behavior of smooth closed subvarieties of a smooth variety.

PROPOSITION 6.3.21. *Let X be an algebraic variety and Y a closed subvariety of X. If $p\in Y$ is a point that is smooth on both Y and X, then after replacing X with a suitable affine open neighborhood of p, the ideal $I = I_X(Y)$ is generated by r elements, where $r = \dim_p(X) - \dim_p(Y)$ (in fact, these elements can be chosen such that their images in $\mathcal{O}_{X,p}$ are part of a regular system of parameters). If this holds, then the r generators of I induce an isomorphism*

$$R/I[x_1,\ldots,x_r] \simeq \bigoplus_{j\geq 0} I^j/I^{j+1} =: \mathrm{gr}_I(R).$$

In particular, for every $m\geq 1$, the R/I-module I^m/I^{m+1} is free.

PROOF. Note first that by Proposition 6.3.8, the point p lies on unique irreducible components of X and Y, hence we may assume that both X and Y are irreducible. We may and will assume that X is affine, with $\mathcal{O}(X) = R$, and Y is defined by $I = I_X(Y)$. Let $\mathfrak{m}$ be the maximal ideal in R corresponding to p. By assumption, we can write

$$r = \dim_k T_pX - \dim_k T_pY. \tag{6.3.1}$$

It follows from (6.3.1) that

$$\dim_k(IR_{\mathfrak{m}} + \mathfrak{m}^2R_{\mathfrak{m}})/\mathfrak{m}^2R_{\mathfrak{m}} = r.$$

We can thus find r elements that are part of a regular system of parameters of $R_{\mathfrak{m}}$ and which lie in $IR_{\mathfrak{m}}$. After possibly replacing X by a smaller affine open neighborhood of x, we may assume, in addition, that these elements are the images $f_{1,p},\ldots,f_{r,p}$ in $R_{\mathfrak{m}}$ of $f_1,\ldots,f_r\in I$. It follows from Proposition 6.3.20 that after possibly replacing X by a suitable open neighborhood of p, we may assume that $f_1,\ldots,f_r$ generate the ideal of a closed subvariety Z, smooth, irreducible, of dimension equal to $\dim(X) - r$. Since $Y\subseteq Z$, it follows that $Y = Z$, which gives the first assertion in the proposition.

The second assertion is trivial if $I = 0$, hence we assume $r > 0$. We have a surjective homomorphism

$$\phi\colon R/I[x_1,\ldots,x_r] \longrightarrow \mathrm{gr}_I(R)$$

that maps each x_i to the class of f_i in I/I^2. Note now that

$$\dim\big(\mathrm{gr}_I(R)\big) \geq \dim(R). \tag{6.3.2}$$

Indeed, it follows from Proposition 6.1.3 that the blow-up $g\colon \widetilde{X} \to X$ of X along I is a birational morphism and $g^{-1}(Y)$ has all irreducible components of codimension 1 in $\widetilde{X}$. If J is the nil-radical of $\mathrm{gr}_I(R)$, then

$$g^{-1}(Y) \simeq \mathrm{MaxProj}\big(\mathrm{gr}_I(R)/J\big),$$

which gives by Remark 4.3.19

$$\dim\big(\mathrm{gr}_I(R)\big) = \dim\big(\mathrm{gr}_I(R)/J\big) \geq \dim\Big(\mathrm{MaxProj}\big(\mathrm{gr}_I(R)/J\big)\Big) + 1 = \dim(X).$$

Since $R/I[x_1, \ldots, x_r]$ is a domain of dimension equal to $\dim(X)$, we conclude that ϕ is an isomorphism, completing the proof of the proposition. $\square$

Corollary 6.3.22. *If X is a smooth variety and $Y, Z \subseteq X$ are irreducible closed subsets, then every irreducible component W of $Y \cap Z$ satisfies*

$$\mathrm{codim}_X(W) \leq \mathrm{codim}_X(Y) + \mathrm{codim}_X(Z).$$

Proof. The idea is similar to the one used when $X = \mathbf{A}^n$ (cf. Exercise 3.3.21). We may replace X by its unique irreducible component that contains W, and thus assume that X is irreducible. Let $n = \dim(X)$. Consider the diagonal $\Delta_X \subseteq X \times X$. Note that we have an isomorphism

$$Y \cap Z \simeq (Y \times Z) \cap \Delta_X,$$

hence we may consider W as an irreducible component of the right-hand side. Since X is smooth of dimension n and $X \times X$ is smooth of dimension $2n$, it follows from the proposition that we can find a non-empty affine open subset $U \subseteq X \times X$ such that $U \cap W \neq \emptyset$, and we have $f_1, \ldots, f_n \in \mathcal{O}(U)$ such that

$$\Delta_X \cap U = \{x \in U \mid f_1(x) = \ldots = f_n(x) = 0\}.$$

We deduce that $W \cap U$ is an irreducible component of

$$\{x \in (Y \times Z) \cap U \mid f_1(x) = \ldots = f_n(x) = 0\},$$

and therefore Corollary 3.3.7 implies that $\mathrm{codim}_{(Y\times Z)\cap U}(W \cap U) \leq n$. Using Corollary 3.3.13, this gives $\dim(W) \geq \dim(Y) + \dim(Z) - n$, and further

$$\mathrm{codim}_X(W) \leq \mathrm{codim}_X(Y) + \mathrm{codim}_X(Z).$$

$\square$

Corollary 6.3.23. *If V is an irreducible closed subset of the algebraic variety X such that $V \cap X_{\mathrm{sm}} \neq \emptyset$, then X is smooth at V.*

Proof. After replacing X by an irreducible component of X_{sm} that meets V, we may assume that X is smooth and irreducible. Furthermore, it follows from Theorem 6.3.7 that there is an open subset U of X such that $U \cap V$ is smooth. After replacing X by U, we may assume that V is smooth, too. Furthermore, it follows from Proposition 6.3.21 that we may assume that X is affine, with $R = \mathcal{O}(X)$, and the prime ideal $\mathfrak{p}$ defining V has the property that $\mathfrak{p}/\mathfrak{p}^2$ is a free $R/\mathfrak{p}$-module, of rank $r = \mathrm{codim}_X(V)$. In this case $\dim_{k(V)} \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2R_{\mathfrak{p}} = r$, hence X is smooth at V. $\square$

Example 6.3.24. If $X = \mathrm{MaxSpec}(A)$ is a smooth variety and $f\colon \widetilde{X} \to X$ is the blow-up of X along the radical ideal I, defining the smooth closed subvariety Y of X, then $\widetilde{X}$ is smooth. Indeed, note first that after covering X by suitable affine open subsets, we may assume that X and Y are irreducible and, by Proposition 6.3.21, that I is generated by $r = \operatorname{codim}_X(Y)$ elements $f_1, \ldots, f_r$. In this case, we can explicitly describe $\widetilde{X}$ by equations, as follows.

The surjection

$$\phi\colon A[y_1, \ldots, y_r] \to R(A, I), \quad \phi(y_i) = f_i t \quad \text{for} \quad 1 \leq i \leq r$$

induces a closed immersion

$$\iota\colon \widetilde{X} \hookrightarrow X \times \mathbf{P}^{r-1}$$

of varieties over X. Note that if J is the ideal generated by all differences $f_i y_j - f_j y_i$, for $i \neq j$, then $J \subseteq \ker(\phi)$, hence ι maps $\widetilde{X}$ inside $V(J)$. We will show that in fact $\iota(X) = V(J)$.

Note first that the morphism $g\colon V(J) \to X$ is an isomorphism over $X \smallsetminus Y$. Indeed, we have

$$\big(A[y_1, \ldots, y_r]/J\big)_{f_i} \simeq A_{f_i}[y_i],$$

and therefore the inverse image of $D(f_i)$ in $V(J)$ is isomorphic to

$$\operatorname{MaxProj}\big(A_{f_i}[y_i]\big) \simeq \operatorname{MaxSpec}(A_{f_i}).$$

We now show that $V(J)$ is a smooth subvariety of $X \times \mathbf{P}^{r-1}$, of codimension $r-1$. This is clear at the points lying over $X \smallsetminus Y$, so that we consider a point $q = \big(p, [u_1, \ldots, u_r]\big) \in V(J)$ lying over Y, hence $f_1(p) = \ldots = f_r(p) = 0$. Let i be such that $u_i \neq 0$ and consider the open subset $U_i = X \times D^+_{\mathbf{P}^{r-1}}(x_i) \subseteq X \times \mathbf{P}^{r-1}$. The intersection $V(J) \cap U_i$ is the zero-locus of the ideal generated by $f_j - f_i \frac{y_j}{y_i}$, for $j \neq i$. Let $\mathfrak{m}$ be the ideal defining q. Note that we can write

$$f_j - f_i \frac{y_j}{y_i} = f_j - \frac{u_j}{u_i} f_i + \left(\frac{y_j}{y_i} - \frac{u_j}{u_i}\right) f_i.$$

Since $\left(\frac{y_j}{y_i} - \frac{u_j}{u_i}\right) f_i \in \mathfrak{m}^2$ and the classes of $f_j - \frac{u_j}{u_i} f_i$ in $\mathfrak{m}/\mathfrak{m}^2$, for $j \neq i$, are linearly independent, it follows from Proposition 6.3.20 that q is a smooth point of $V(J)$, and the codimension of $\widetilde{X}$ in $X \times \mathbf{P}^{r-1}$ around q is $r-1$.

We can now see that $V(J)$ is irreducible, and thus it is equal to $\iota(\widetilde{X})$. Indeed, every irreducible component of $V(J)$ different from $\overline{g^{-1}(X \smallsetminus Y)}$ must be contained in $g^{-1}(Y) = Y \times \mathbf{P}^{r-1}$. However, we have seen that every irreducible component of $V(J)$ has dimension equal to $\dim(X) > \dim(Y) + r - 1$, hence it can't be contained in $Y \times \mathbf{P}^{r-1}$.

We thus conclude that $\widetilde{X}$ is smooth and is defined in $X \times \mathbf{P}^{r-1}$ by the ideal J.

Definition 6.3.25. Given a smooth variety X and two smooth closed subvarieties Y and Z of X, recall that for every $p \in Y \cap Z$, we may consider T_pY and T_pZ as linear subspaces of T_pX. We say that Y and Z intersect transversely if, for every $p \in Y \cap Z$, we have

$$\operatorname{codim}_{T_pX}(T_pY \cap T_pZ) = \operatorname{codim}^p_X(Y) + \operatorname{codim}^p_X(Z)$$

(note that p lies on unique irreducible components X' and Y' of X and Y, respectively, and we put $\operatorname{codim}^p_X(Y) = \operatorname{codim}_{X'}(Y')$; a similar definition applies for $\operatorname{codim}^p_X(Z)$). The condition can be equivalently formulated as follows: for every $p \in Y \cap Z$, we have

$$T_pY + T_pZ = T_pX.$$

PROPOSITION 6.3.26. *If X is a smooth variety and Y, Z are smooth closed subvarieties of X that intersect transversely, then $Y\cap Z$ is smooth, and for every $p\in Y\cap Z$, we have*

$$\operatorname{codim}_X^p(Y\cap Z)=\operatorname{codim}_X^p(Y)+\operatorname{codim}_X^p(Z)\quad \textit{and}$$

$$T_p(Y\cap Z)=T_pY\cap T_pZ.$$

Moreover, for every affine open subset U of X, we have

$$I_U(Y\cap Z\cap U)=I_U(Y\cap U)+I_U(Z\cap U).$$

PROOF. Let $r=\operatorname{codim}_X^p(Y)$ and $s=\operatorname{codim}_X^p(Z)$. It follows from Proposition 6.3.21 that if U is a suitable irreducible affine open neighborhood of p, then $I_U(Y\cap U)$ is generated by r elements and $I_U(Z\cap U)$ is generated by s elements. Consider the ideal

$$J=I_U(Y\cap U)+I_U(Z\cap U)$$

that defines the closed subset $Y\cap Z$. Since J is generated by $r+s$ elements, it follows from Corollaries 3.3.7 and 3.3.13 that every irreducible component of $Y\cap Z$ has dimension $\geq \dim_p(X)-(r+s)$. On the other hand, we have $T_p(Y\cap Z)\subseteq T_p(Y)\cap T_p(Z)$, hence by assumption

$$\dim_k T_p(Y\cap Z)\leq \dim_p(X)-(r+s).$$

This implies that p is a smooth point of $Y\cap Z$ and $T_p(Y\cap Z)=T_pY\cap T_pZ$.

In fact we can do better: it is easy to see, by translating the above argument algebraically, that if $\mathfrak{m}\subseteq\mathcal{O}(U)=R$ is the maximal ideal corresponding to p, then

$$\dim_k \mathfrak{m}/\mathfrak{m}^2+J\leq \dim_p(X)-(r+s)\leq \dim\big(R_{\mathfrak{m}}/JR_{\mathfrak{m}}\big).$$

This implies that $R_{\mathfrak{m}}/JR_{\mathfrak{m}}$ is a regular local ring, hence reduced by Remark 6.3.11. Therefore $J_{\mathfrak{m}}=\big(I_U(Y\cap U)+I_U(Z\cap U)\big)_{\mathfrak{m}}$ for every point in $Y\cap Z$, which implies the last assertion in the proposition (see, for example, Corollary C.3.3). $\square$

We end this section by stating one of the most useful results in algebraic geometry. Given an irreducible algebraic variety X, a *resolution of singularities* of X is a proper, birational morphism $f\colon \widetilde{X}\to X$, with $\widetilde{X}$ a smooth, irreducible variety. One can ask for more properties (for example, one can ask that f is *projective*, in a sense that we will define later, which implies in particular that if X is projective or quasi-projective, then $\widetilde{X}$ has the same property; one can also ask for f to be an isomorphism over $X_{\rm sm}$). The following celebrated result is due to Hironaka.

THEOREM 6.3.27. *If* $\operatorname{char}(k)=0$, *then every irreducible variety X over k has a resolution of singularities.*

REMARK 6.3.28. In fact, Hironaka's theorem is more precise: suppose, for simplicity, that X has a closed immersion in a smooth variety Y (for example, any quasi-projective variety satisfies this condition). In this case the theorem says that there is a sequence of morphisms

$$Y_r\xrightarrow{f_r}Y_{r-1}\longrightarrow\ldots\longrightarrow Y_1\xrightarrow{f_1}Y_0=Y$$

with the following properties:

i) Each f_i, with $1\leq i\leq r$, is the blow-up along a smooth variety Z_{i-1} (hence, by induction, all Y_i are smooth).

ii) For every i, with $1 \leq i \leq r$, the strict transform X_{i-1} of X on Y_{i-1} is not contained in Z_{i-1} (so that the next strict transform X_i is defined).
iii) The strict transform X_r of X on Y_r is smooth.

EXERCISE 6.3.29. Consider the following curves in $\mathbf{A}^2$:

$$X = V(x^2 - y^3), \quad Y = V\big(y^2 - x^2(x+1)\big), \quad \text{and} \quad Z = V(x^2 - y^5).$$

Show that if $\pi \colon \widetilde{\mathbf{A}^2} \to \mathbf{A}^2$ is the blow-up of the origin, then the strict transforms $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, are smooth; the strict transform $\widetilde{Z}$ of Z has one singular point and by blowing that up, the resulting strict transform is smooth.

6.4. Bertini's theorem

Recall that the hyperplanes in $\mathbf{P}^n$ are parametrized by a projective space $(\mathbf{P}^n)^*$. We will be using the following terminology: if Z is an irreducible variety, we say that a property holds for a general point $z \in Z$ if there is an open subset U of Z such that the property holds for all $z \in U$. Note that if we have two such properties, then they both hold for a general point in Z: this follows from the fact that the intersection of two nonempty open subsets is again a nonempty open subset. This terminology is particularly convenient when the points of Z parametrize some geometric objects, as is the case with $(\mathbf{P}^n)^*$.

Given a projective variety $X \subseteq \mathbf{P}^n$, one is often interested in the following type of statement: if X has a certain property, then for a general hyperplane H in $\mathbf{P}^n$, the intersection $X \cap H$ still has the same property. In this section we prove such a result for smoothness.

THEOREM 6.4.1 (Bertini). *If $X \subseteq \mathbf{P}^n$ is a smooth variety, then for a general hyperplane H in $\mathbf{P}^n$, the subvarieties X and H of $\mathbf{P}^n$ intersect transversely; in particular, the intersection $X \cap H$ is smooth, and if X has pure dimension d, then $X \cap H$ has pure dimension $d-1$.*

PROOF. We may assume that X is irreducible: indeed, if we know this, then for every connected component of X, we find a corresponding open subset of $(\mathbf{P}^n)^*$. The intersection of these open subsets then satisfies the conclusion in the theorem. From now on we assume that X is irreducible, and let $d = \dim(X)$.

Note that for every hyperplane H in $\mathbf{P}^n$ and every $p \in H$, we have $\mathbf{T}_p H = H$. It follows from Proposition 6.2.6 that H and X do not intersect transversely if and only if there is $p \in X \cap H$ such that $\mathbf{T}_p X \subseteq H$. Consider the set

$$Z := \big\{\big(p, [H]\big) \in X \times (\mathbf{P}^n)^* \mid \mathbf{T}_p(X) \subseteq H\big\}.$$

We claim that Z is closed in $X \times (\mathbf{P}^n)^*$. In order to check this, let $f_1, \ldots, f_r$ be homogeneous generators for the ideal I_X of X in $\mathbf{P}^n$. The linear subspace $\mathbf{T}_p X$ at a point $p \in X$ is defined by the linear equations

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(p) x_j = 0 \quad \text{for} \quad 1 \leq i \leq r.$$

By assumption, for every $p \in X$, the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$ is $n-d$. The hyperplane H defined by $\sum_{j=0}^{n} a_j x_j = 0$ contains $\mathbf{T}_p X$ if and only if the rank of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ \frac{\partial f_1}{\partial x_0}(p) & \frac{\partial f_1}{\partial x_1}(p) & \ldots & \frac{\partial f_1}{\partial x_n}(p) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_r}{\partial x_0}(p) & \frac{\partial f_r}{\partial x_1}(p) & \ldots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}$$

is $\leq n-d$. Equivalently, all $(n-d+1)$-minors of this matrix must be 0, and it is clear that it is enough to only consider those minors involving the first row. Each of these conditions is of the form

$$\sum_{j=0}^{n} a_j g_j(p) = 0$$

for some homogeneous polynomials $g_0, \ldots, g_n$, all of the same degree. It is now straightforward to check (for example, by covering each of X and $(\mathbf{P}^n)^*$ by the standard affine charts) that the subset Z is closed in $X \times (\mathbf{P}^n)^*$. In particular, Z is a projective variety.

The projections onto the two components induce two morphisms $\pi_1 \colon Z \to X$ and $\pi_2 \colon Z \to (\mathbf{P}^n)^*$. For every $p \in X$, consider the fiber $\pi_1^{-1}(p)$. This is identified with the subset of $(\mathbf{P}^n)^*$ consisting of all hyperplanes containing $\mathbf{T}_p(X)$. This is a linear subspace of dimension $n-d-1$. Indeed, since X is smooth, of dimension d, the linear subspace $\mathbf{T}_p(X)$ of $\mathbf{P}^n$ has dimension d. After choosing suitable coordinates, we may assume that this is given by $x_{d+1} = \ldots = x_n = 0$. In this case, the hyperplane with equation $\sum_{i=0}^{n} a_i x_i = 0$ contains $\mathbf{T}_p(X)$ if and only if $a_0 = \ldots = a_d = 0$; this is thus a linear subspace in $(\mathbf{P}^n)^*$ of codimension $d+1$.

Therefore we conclude from Corollary 3.4.3 that

$$\dim(Z) = \dim(X) + (n-d-1) = n-1.$$

In this case, the morphism $\pi_2 \colon Z \to (\mathbf{P}^n)^*$ can't be dominant. Its image is thus a proper closed subset of $(\mathbf{P}^n)^*$ and if U is the complement of this image, we see that for every hyperplane H in $\mathbf{P}^n$ with $[H] \in U$, X and H intersect transversely, and therefore Proposition 6.3.26 implies that $X \cap H$ is a smooth variety of pure dimension $d-1$ (of course, if $d=0$, this simply means that $X \cap H$ is empty). $\square$

Remark 6.4.2. It follows from the above proof that even if $X \subseteq \mathbf{P}^n$ is a subvariety with finitely many singular points, for a general hyperplane H in $\mathbf{P}^n$, the intersection $X \cap H$ is still smooth. Indeed, with the notation in the proof, we still have that the fiber $\pi_1^{-1}(p)$, for $p \in X$, has dimension $\leq n-d-1$ (in fact, one can get a better bound at the singular points). We thus still have the bound $\dim(Z) \leq n-1$, which implies that Z does not dominate $(\mathbf{P}^N)^*$. Since a general hyperplane does not contain any of the singular points of X, we deduce that such a hyperplane intersects X_{sm} transversally, and therefore $X \cap H$ is smooth.

Remark 6.4.3. There are several other versions of Bertini's theorem. One which is often useful says that if $X \subseteq \mathbf{P}^n$ is an irreducible closed subvariety, with $\dim(X) \geq 2$, then for a general hyperplane $H \subseteq \mathbf{P}^n$, the intersection $X \cap H$ is again irreducible (see [**Jou83**] for this and related results). Another useful version, due to Kleiman, concerns smoothness in the case when instead of a closed subvariety of $\mathbf{P}^n$ one deals with an arbitrary morphism $X \to \mathbf{P}^n$ (this, however, works only over a ground field of characteristic 0). We will prove this result in Proposition 13.2.21.

6.5. Smooth morphisms between smooth varieties

In this section we discuss the notion of smooth morphism between smooth varieties. We will later return to this concept, to consider the case of arbitrary varieties.

DEFINITION 6.5.1. A morphism $f\colon X \to Y$ between smooth algebraic varieties is *smooth* at a point $x \in X$ if the linear map

$$df_x\colon T_xX \to T_{f(x)}Y$$

is surjective. The morphism f is *smooth* if it is smooth at every point.

Given a morphism of smooth varieties $f\colon X \to Y$, for every irreducible component X' of X there is a unique irreducible component Y' of Y such that $f(X') \subseteq Y'$. We can thus easily reduce to the case of morphisms between smooth, irreducible varieties.

PROPOSITION 6.5.2. *If $f\colon X \to Y$ is a smooth morphism between the smooth, irreducible varieties X and Y, then f is dominant and for every $y \in f(X)$, the fiber $f^{-1}(y)$ is smooth, of pure dimension* $\dim(X) - \dim(Y)$*. Moreover, for every $x \in f^{-1}(y)$, we have*

$$T_x\big(f^{-1}(y)\big) = \ker(df_x\colon T_xX \to T_{f(x)}Y).$$

PROOF. By Theorem 3.4.1, we know that every irreducible component of $f^{-1}(y)$ has dimension $\geq \dim(X) - \dim(Y)$. Moreover, the inequality is strict if f is not dominant.

On the other hand, the composition

$$f^{-1}(y) \overset{i}{\hookrightarrow} X \overset{f}{\longrightarrow} Y$$

where i is the inclusion map can also be factored as

$$f^{-1}(y) \longrightarrow \{y\} \hookrightarrow Y.$$

This implies that the restriction of df_x to $T_x\big(f^{-1}(y)\big) \subseteq T_xX$ is zero, hence $T_x\big(f^{-1}(y)\big)$ is contained in the kernel of df_x. Since df_x is surjective, it follows that

$$\dim_k T_x\big(f^{-1}(y)\big) \leq \dim_k \ker(df_x) = \dim_k T_xX - \dim_k T_{f(x)}Y = \dim(X) - \dim(Y).$$

Since $\dim_x \big(f^{-1}(y)\big) \leq \dim_k T_x\big(f^{-1}(y)\big)$, we thus conclude that this is, in fact, an equality. This implies that f is dominant, $T_x\big(f^{-1}(y)\big) = \ker(df_x)$, and $f^{-1}(y)$ is smooth at x, of dimension $\dim(X) - \dim(Y)$. $\square$

EXAMPLE 6.5.3. Consider the morphism $f\colon \mathbf{A}^1 \to \mathbf{A}^1$ given by $f(t) = t^2$, where we assume that $\mathrm{char}(k) \neq 2$. For every point $t \in \mathbf{A}^1$, the map

$$T_t\mathbf{A}^1 = k \to k = T_{f(t)}\mathbf{A}^1$$

is given by multiplication by $2t$ (see Remark 6.2.5). It follows that f is smooth at every point $t \neq 0$, but it is not smooth at 0.

DEFINITION 6.5.4. A morphism of smooth varieties $f\colon X \to Y$ is *étale* at $x \in X$ if it is smooth at x and $\dim_x X = \dim_{f(x)} Y$. The morphism is *étale* if it is étale at every point.

An important result in this setting is the Generic Smoothness theorem, saying that if $\mathrm{char}(k) = 0$, then for every dominant morphism of smooth varieties $f\colon X \to Y$, there is a non-empty open subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is smooth (see Theorem 13.2.18).

Remark 6.5.5. The hypothesis on the characteristic in the Generic Smoothness theorem is essential. If $\mathrm{char}(k) = p$, note that the morphism $f\colon \mathbf{A}^1 \to \mathbf{A}^1$ given by $f(t) = t^p$ is not smooth at any point.

Remark 6.5.6. The Generic Smoothness theorem is the analogue of Sard's theorem in differential topology. Note that by combining it with Proposition 6.5.2, we conclude that if $f\colon X \to Y$ is a dominant morphism of smooth, irreducible algebraic varieties over an algebraically closed field of characteristic 0, then there is a non-empty open subset U of Y such that for every $y \in Y$, the fiber $f^{-1}(y)$ is smooth.

CHAPTER 7

The Grassmann variety and other examples

In this chapter we discuss various geometric examples related to the Grassmann variety. In the first section we construct this variety and discuss several related constructions, such as the Plücker embedding and the incidence correspondence. In the second section we discuss flag varieties, while in the third section we give a resolution of singularities for the generic determinantal variety. We next consider the parameter space for projective hypersurfaces and discuss linear subspaces on such hypersurfaces. In the last section we treat the variety of nilpotent matrices.

7.1. The Grassmann variety

Let $V = k^n$ and let r be an integer with $0 \leq r \leq n$. In this section we describe the structure of algebraic variety on the set $G(r, n)$ parametrizing the r-dimensional linear subspaces of V. These are the *Grassmann varieties*. Given an r-dimensional linear subspace W of V, we denote by $[L]$ the corresponding point of $G(r, n)$.

This is trivial for $r = 0$ or $r = n$: in this case $G(r, n)$ is just a point. The first non-trivial case that we have already encountered is for $r = 1$: in this case $G(r, n) = \mathbf{P}^{n-1}$. A similar description holds for $r = n-1$: hyperplanes in k^n are in bijection with lines in $(k^n)^* \simeq k^n$, hence these are again parametrized by a $\mathbf{P}^{n-1}$ (cf. Exercise 4.2.18).

We now proceed with the description in the general case. Given an r-dimensional linear subspace W of k^n, choose a basis $u_1, \ldots, u_r$ of W. By writing $u_i = (a_{i,1}, \ldots, a_{i,n})$ for $1 \leq i \leq r$, we obtain a matrix $A = (a_{i,j}) \in M_{r,n}(k)$. Note that we have an action of $GL_r(k)$ on $M_{r,n}(k)$ given by left multiplication. Choosing a different basis of W corresponds to multiplying the matrix on the left by an element of $GL_r(k)$. Moreover a matrix in $M_{r,n}(k)$ corresponds to some r-dimensional linear subspace in k^n if and only if it has maximal rank r. We can thus identify $G(r, n)$ with the quotient set $U/GL_r(k)$, where U is the open subset of $M_{r,n}(k)$ consisting of matrices of rank r.

For every subset $I \subseteq \{1, \ldots, n\}$ with r elements, let U_I be the open subset of U given by the non-vanishing of the r-minor on the columns indexed by the elements of I. Note that this subset is preserved by the $GL_r(k)$-action and let V_I be the corresponding subset of $G(r, n)$. We now construct a bijection

$$\phi_I : V_I \to M_{r,n-r}(k) = \mathbf{A}^{r(n-r)}.$$

In order to simplify the notation, say $I = \{1, \ldots, r\}$. Given any matrix $A \in U_I$, let us write it as $A = (A', A'')$ for matrices $A' \in M_{r,r}(k)$ and $A'' \in M_{r,n-r}(k)$. Note that by assumption $\det(A') \neq 0$. In this case there is a unique matrix $B \in GL_r(k)$ such that $B \cdot A = (I_r, C)$, for some matrix $C \in M_{r,n-r}(k)$ (namely $B = (A')^{-1}$, in which case $C = (A')^{-1} \cdot A''$). Therefore every matrix class in V_I is the class of a unique matrix of the form (I_r, C), with $C \in M_{r,n-r}(k)$. This gives the desired bijection between $V_{\{1,\ldots,r\}} \to \mathbf{A}^{r(n-r)}$, and a similar argument works for every V_I.

We put on each V_I the topology and the sheaf of functions that make the above bijection an isomorphism in $\mathcal{T}op_k$. We need to show that these glue to give on $G(r,n)$ a structure of a prevariety: we need to show that for every subsets I and J as above, the subset $\phi_I(V_I \cap V_J)$ is an open subset of $\mathbf{A}^{r(n-r)}$ and the map

$$\phi_J \circ \phi_I^{-1} \colon \phi_I(V_I \cap V_J) \to \phi_J(V_I \cap V_J) \tag{7.1.1}$$

is a morphism of algebraic varieties (in which case, by symmetry, it is an isomorphism). In order to simplify the notation, suppose that $I = \{1, \ldots, r\}$. It is then easy to see that if $\#(I \cap J) = \ell$, then $\phi_I(V_I \cap V_J) \subseteq \mathbf{A}^{r(n-r)}$ is the principal affine open subset defined by the non-vanishing of the $(r-\ell)$-minor on the rows indexed by those $i \in I \smallsetminus J$ and on the columns indexed by those $j \in J \smallsetminus I$. Moreover, the map (7.1.1) is given by associating to a matrix C the $r \times n$ matrix $M = (I_r, C)$, multiplying it on the left with the inverse of the $r \times r$-submatrix of M on the columns in J to get M', and then keeping the $r \times (n-r)$ submatrix of M' on the columns in $\{1, \ldots, n\} \smallsetminus J$. It is clear that this is a morphism.

We thus conclude that $G(r,n)$ is an object in $\mathcal{T}op_k$. In fact, it is a prevariety, since it is covered by open subsets isomorphic to affine varieties. In fact, since each V_I is isomorphic to an affine space, it is smooth and irreducible, and since we have seen that any two V_I intersect, we conclude that $G(r,n)$ is irreducible by Exercise 1.3.17. Furthermore, since each V_I has dimension $r(n-r)$, we conclude that $\dim\big(G(r,n)\big) = r(n-r)$. We collect these facts in the following proposition.

PROPOSITION 7.1.1. *The Grassmann variety $G(r,n)$ is a smooth, irreducible prevariety of dimension $r(n-r)$, that has a cover by open subsets isomorphic to $\mathbf{A}^{r(n-r)}$.*

EXAMPLE 7.1.2. If $r = 1$, the algebraic variety $G(1,n)$ is just $\mathbf{P}^{n-1}$, described via the charts $U_i = (x_i \neq 0) \simeq \mathbf{A}^{n-1}$.

EXAMPLE 7.1.3. If $r = n-1$, the algebraic variety $G(n-1,n)$ has an open cover

$$G(n-1,n) = U_1 \cup \ldots \cup U_n.$$

For every i, we have an isomorphism $\mathbf{A}^{n-1} \simeq U_i$ such that $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots \lambda_n)$ is mapped to the hyperplane generated by $\{e_j + \lambda_j e_i \mid j \neq i\}$. This is the hyperplane defined by the equation $e_i^* - \sum_{j \neq i} \lambda_j e_j^* = 0$. We thus see that the variety structure on $G(n-1,n)$ is the same one as on $(\mathbf{P}^{n-1})^*$, which is isomorphic to $\mathbf{P}^{n-1}$ (cf. Exercise 4.2.18).

Our next goal is to show that, in fact, $G(r,n)$ is a projective variety. Note that if W is an r-dimensional linear subspace of $V = k^n$, then $\wedge^r W$ is a 1-dimensional linear subspace of $\wedge^r V \simeq k^d$, where $d = \binom{n}{r}$. If $e_1, \ldots, e_n$ is the standard basis of k^n, then we have a basis of $\wedge^r V$ given by the $e_I = e_{i_1} \wedge \ldots \wedge e_{i_r}$, where $I = \{i_1, \ldots, i_r\}$ is a subset of $\{1, \ldots, n\}$ with r-elements (and where, in order to write e_I, we order the elements $i_1 < \ldots < i_r$). We correspondingly denote the coordinates on the projective space of lines in $\wedge^r V$ by x_I.

PROPOSITION 7.1.4. *The map $f \colon G(r,n) \to \mathbf{P}^{d-1}$ that maps $[W]$ to $[\wedge^r W]$ is a closed immersion. In particular, $G(r,n)$ is a projective variety.*

The embedding in the above proposition is the *Plücker embedding* of the Grassmann variety.

Proof of Proposition 7.1.4. If $W \subseteq V$ is an r-dimensional linear subspace described by the matrix A, then $f\big([W]\big) \in \mathbf{P}^{d-1}$ is given in the above homogeneous coordinates by the r-minors of A. In particular, we see that the inverse image of the affine chart $W_I = (x_I \neq 0)$ is the affine open subset $V_I \subseteq G(r,n)$.

In order to complete the proof, it is enough to show that for every I, the induced map $V_I \to W_I$ is a morphism and the corresponding ring homomorphism

$$\mathcal{O}(W_I) \to \mathcal{O}(V_I) \tag{7.1.2}$$

is surjective. The argument is the same for all I, but in order to simplify the notation, we assume $I = \{1, \ldots, r\}$. Note that the map $V_I \to W_I$ gets identified to $M_{r,n-r}(k) \to \mathbf{A}^{\binom{n}{r}-1}$, than maps a matrix B to all r-minors of (I_r, B), with the exception of the one on the first r columns. In particular, we see that this map is a morphism. By choosing $r-1$ columns of the first r ones and an additional column of the last $(n-r)$ ones, we obtain every entry of B as an r-minor as above. This implies that the homomorphism (7.1.2) is surjective. $\square$

Remark 7.1.5. The algebraic group $GL_n(k)$ acts on k^n and thus acts on $G(r,n)$ by $g \cdot [W] = [g \cdot W]$. Note that if W is described by the matrix $A \in M_{r,n}(k)$, then $g \cdot W$ is described by $A \cdot g^t$. It is straightforward to see that this is an algebraic action. Since any two linear subspaces can by mapped one to the other by a linear automorphism of k^n, we see that the $GL_n(k)$-action on $G(r,n)$ is transitive.

Remark 7.1.6. If W is an r-dimensional linear subspace of $V = k^r$, then we have an induced surjection $V^* \to W^*$, whose kernel is an $(n-r)$-dimensional linear subspace of $(k^n)^* \simeq k^n$. In this way we get a bijection $G(r,n) \to G(n-r,n)$ and it is not hard to check that this is, in fact, an isomorphism of algebraic varieties.

Remark 7.1.7. Given an arbitrary n-dimensional vector space V over k, let $G(r,V)$ be the set of r-dimensional linear subspace of V. By choosing an isomorphism $V \simeq k^n$, we obtain a bijection $G(r,V) \simeq G(r,n)$ and we put on $G(r,V)$ the structure of an algebraic variety that makes this an isomorphism. Note that this is independent of the choice of isomorphism $V \simeq k^r$: for a different isomorphism, we have to compose the map $G(r,V) \to G(r,n)$ with the action on $G(r,n)$ of a suitable element in $GL_n(k)$.

Remark 7.1.8. It is sometimes convenient to identify $G(r,n)$ with the set of $(r-1)$-dimensional linear subspaces in $\mathbf{P}^{n-1}$.

Notation 7.1.9. Given a finite-dimensional k-vector space V, we denote by $\mathbf{P}(V)$ the projective space parametrizing *hyperplanes* in V. Therefore the homogeneous coordinate ring of $\mathbf{P}(V)$ is given by the symmetric algebra $\mathrm{Sym}^\bullet(V)$. With this notation, the projective space parametrizing the lines in V is given by $\mathbf{P}(V^*)$.

We end this section by discussing the incidence correspondence for the Grassmann variety and by giving some applications. More applications will be given in the next sections.

Consider the set of r-dimensional linear subspaces in $\mathbf{P}^n$, parametrized by $G = G(r+1, n+1)$. The *incidence correspondence* is the subset

$$\mathcal{Z} = \big\{\big(q, [V]\big) \in \mathbf{P}^n \times G \mid q \in V\big\}.$$

Note that this is a closed subset of $\mathbf{P}^n \times G$. Indeed, if we represent $[W]$ by the matrix $A = (a_{i,j})_{0\leq i\leq r+1, 0\leq j\leq n}$, then $\big([b_0,\ldots,b_n],[W]\big)$ lies in $\mathcal{Z}$ if and only if the rank of the matrix

$$B = \begin{pmatrix} b_0 & b_1 & \ldots & b_n \\ a_{0,0} & a_{0,1} & \ldots & a_{0,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{r,0} & a_{r,1} & \ldots & a_{r,n} \end{pmatrix}$$

is $\leq r+1$. This is the case if and only if all $(r+2)$-minors of B vanish. By expanding along the first row, we can write each such minor as $\sum_{j\in I} b_j\delta_j$, where $I \subseteq \{0,\ldots,n\}$ is the subset with $r+2$ elements determining the minor and each δ_j is a suitable minor of A. Consider the closed immersion

$$\mathbf{P}^n \times G \overset{i}{\hookrightarrow} \mathbf{P}^n \times \mathbf{P}^N \overset{j}{\hookrightarrow} \mathbf{P}^M,$$

where i is given by $i(u,v) = \big(u,\phi(v)\big)$, with ϕ being the Plücker embedding, and j is the Segre embedding. It follows from the above discussion that via this embedding, $\mathcal{Z}$ is the inverse image of a suitable linear subspace of $\mathbf{P}^M$, and therefore it is closed in $\mathbf{P}^n \times G$. Since both $\mathbf{P}^n$ and G are projective varieties, we conclude that $\mathcal{Z}$ is a projective variety.

The projections onto the two components induce the morphisms $\pi_1 \colon \mathcal{Z} \to \mathbf{P}^n$ and $\pi_2 \colon \mathcal{Z} \to G$. It follows from the definition that for every $[W] \in G$, we have $\pi_2^{-1}\big([W]\big) \simeq W$.

EXERCISE 7.1.10. Show that the morphism $\pi_2 \colon \mathcal{Z} \to G$ is locally trivial, with fiber[1] $\mathbf{P}^r$.

Since all fibers of π_2 are irreducible, of dimension r, we conclude from Proposition 5.5.1 that $\mathcal{Z}$ is irreducible, of dimension

$$\dim(\mathcal{Z}) = r + \dim(G) = r + (r+1)(n-r).$$

(we use here the fact that G is irreducible and $\mathcal{Z}$ is a projective variety).

Given a point $q \in \mathbf{P}^n$, the fiber $\pi^{-1}(q) \subseteq G$ consists of all r-dimensional linear subspaces of $\mathbf{P}^n$ containing q (equivalently, these are the $(r+1)$-dimensional linear subspaces of k^{n+1} containing a given line). These are in bijection with the Grassmann variety $G(r,n)$.

EXERCISE 7.1.11. Show that the morphism $\pi_1 \colon \mathcal{Z} \to \mathbf{P}^n$ is locally trivial, with fiber $G(r,n)$.

We use the incidence correspondence to prove the following

PROPOSITION 7.1.12. *Let $X \subseteq \mathbf{P}^n$ be a closed subvariety of dimension d and let $G = G(r+1,n+1)$. If we put*

$$M_r(X) = \big\{[W] \in G \mid W \cap X \neq \emptyset\big\},$$

then the following hold:

i) *The set $M_r(X)$ is a closed subset of G, which is irreducible if X is irreducible.*

ii) *We have $\dim\big(M_r(X)\big) = \dim(G) - (n-r-d)$ for $0 \leq r \leq n-d$.*

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[1] Given a variety F, we say that a morphism $f \colon X \to Y$ is locally trivial, with fiber F, if there is an open cover $Y = U_1 \cup \ldots \cup U_r$ such that for every i, we have an isomorphism $f^{-1}(U_i) \simeq U_i \times F$ of varieties over U_i.

PROOF. Using the previous notation, note that $M_r(X) = \pi_2\big(\pi_1^{-1}(X)\big)$, hence $M_r(X)$ is closed, since X is closed and π_2 is a closed map (recall that $\mathcal{Z}$ is a projective variety). Consider now the morphism $\pi_1^{-1}(X) \to X$ induced by π_1. Since all fibers are irreducible, of dimension $r(n-r)$ (being isomorphic to $G(r,n)$), and $\pi_1^{-1}(X)$ is a projective variety, we deduce from Proposition 5.5.1 that if X is irreducible, then $\pi_1^{-1}(X)$ is irreducible, with

$$\dim\left(\pi_1^{-1}(X)\right) = \dim(X) + r(n-r).$$

The irreducibility of $\pi^{-1}(X)$ implies the irreducibility of $\pi_2\big(\pi_1^{-1}(X)\big)$.

It is clear that if $X = X \cup \ldots \cup X_s$ is the irreducible decomposition of X, then we have $M_r(X) = M_r(X_1) \cup \ldots \cup M_r(X_s)$. Therefore, in order to prove ii), we may assume that X is irreducible. We claim that the morphism $\pi_1^{-1}(X) \to M_r(X)$ has at least one finite, non-empty fiber. Using Theorems 3.4.1 and 3.4.1, this implies that

$$\dim\left(M_r(X)\right) = \dim\left(\pi_1^{-1}(X)\right) = d + r(n-r).$$

hence

$$\operatorname{codim}_G\left(M_r(X)\right) = (r+1)(n-r) - d - r(n-r) = n - r - d.$$

We thus only need to find an r-dimensional linear subspace that intersects X in a nonempty, finite set. This is easy to see and we leave the argument as an exercise for the reader. $\square$

EXERCISE 7.1.13. Consider the Grassmann variety $G = G(r+1, n+1)$ parametrizing the r-dimensional linear subspaces in $\mathbf{P}^n$. Show that if Z is a closed subset of G, then the set

$$\widetilde{Z} := \bigcup_{[V]\in Z} V \subseteq \mathbf{P}^n$$

is a closed subset of $\mathbf{P}^n$, with $\dim(\widetilde{Z}) \leq \dim(Z) + r$.

EXERCISE 7.1.14. Show that if X and Y are disjoint closed subvarieties of $\mathbf{P}^n$, then the *join* $J(X,Y) \subseteq \mathbf{P}^n$, defined as the union of all lines in $\mathbf{P}^n$ joining a point in X and a point in Y, is a closed subset of $\mathbf{P}^n$, with

$$\dim\left(J(X,Y)\right) \leq \dim(X) + \dim(Y) + 1.$$

7.2. Flag varieties

In this section we define flag varieties and prove some basic properties. Let V be a vector space over k, with $\dim_k V = n$ and let $1 \leq \ell_1 < \ldots < \ell_r \leq n$. A *flag of type* $(\ell_1, \ldots, \ell_r)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq V$, where $\dim_k(V_i) = \ell_i$. A *complete flag* is a flag of type $(1, 2, \ldots, n)$.

The *flag variety* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ parametrizes flags in V. In other words, this is the set

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) := \{(V_1, \ldots, V_r) \in G(\ell_1, V) \times \cdots \times G(\ell_r, V) \mid V_1 \subseteq \cdots \subseteq V_r\}.$$

In particular, the *complete flag variety* $\mathrm{Fl}(V) = \mathrm{Fl}_{1,\ldots,n}(V)$ parametrizes complete flags in V.

PROPOSITION 7.2.1. *The subset* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ *of* $G(\ell_1, V) \times \cdots \times G(\ell_r, V)$ *is closed, hence* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ *is a projective variety.*

Proof. The assertion is trivial for $r = 1$, hence we may assume $r \geq 2$. For i with $1 \leq i \leq r-1$, consider the map

$$\pi_{i,i+1}\colon G(\ell_1, V) \times \cdots \times G(\ell_r, V) \to G(\ell_i, V) \times G(\ell_{i+1}, V)$$

given by the projection on the factors i and $i+1$. It is clear that

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) = \bigcap_{i=1}^{r-1} \pi_{i,i+1}^{-1}\big(\mathrm{Fl}_{\ell_i,\ell_{i+1}}(V)\big),$$

hence it s enough to prove the assertion in the proposition when $r = 2$.

Let us fix a basis $e_1, \ldots, e_n$ on V. Consider now the set $M_1 \times M_2 \subseteq M_{\ell_1,n}(k) \times M_{\ell_2,n}(k)$ consisting of pairs of matrices of maximal rank. Let Z be the subset of $M_1 \times M_2$ consisting of matrices (A, B) with the property that the linear span of the rows of A is contained in the linear span of the rows of B. Recall that we have morphisms

$$M_1 \to G(\ell_1, V) \quad \text{and} \quad M_2 \to G(\ell_2, V)$$

such that the product map $M_1 \times M_2 \to G(\ell_1, V) \times G(\ell_2, V)$ maps Z onto $\mathrm{Fl}_{\ell_1,\ell_2}(V)$.

Note that Z is closed in $M_1 \times M_2$. Indeed, a pair $\big((a_{i,j}, (b_{i,j})\big)$ lies in Z if and only if the rank of the matrix $(c_{i,j})_{1 \leq i \leq \ell_1+\ell_2, 1 \leq j \leq n}$ given by

$$c_{i,j} = a_{i,j} \quad \text{for} \quad i \leq \ell_1 \quad \text{and} \quad c_{i,j} = b_{i-\ell_1,j} \quad \text{for} \quad \ell_1 + 1 \leq i \leq \ell_1$$

has rank $\leq \ell_2$. Using now the description of $G(\ell_1, V)$ and $G(\ell_2, V)$ in terms of charts arising by covering M_1 and M_2 by corresponding open subsets, it is now easy to see that $\mathrm{Fl}_{\ell_1,\ell_2}(V)$ is closed in $G(\ell_1, V) \times G(\ell_2, V)$. $\square$

Recall that the group $GL(V)$ of linear automorphisms of V has an induced action on each $G(\ell, V)$ and it is clear that the product action on $G(\ell_1, V) \times \cdots \times G(\ell_r, V)$ induces an algebraic action of $GL(V)$ on $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$. This action is clearly transitive: given any two flags of type $(\ell_1, \ldots, \ell_r)$, we can find an invertible linear automorphism of $GL(V)$ that maps one to the other (for example, choose for each flag a basis of V such that the i^{th} element of the flag is generated by the first ℓ_i elements of the basis, and then choose a linear transformation that maps one basis to the other). By Exercise 6.3.14, we conclude that $\mathrm{Fl}(\ell_1, \ldots, \ell_r)(V)$ is a smooth variety.

Example 7.2.2. If $e_1, \ldots, e_n$ is a basis of n and V_i is the linear span of $e_1, \ldots, e_i$, then the stabilizer of the point on the complete flag variety corresponding to $V_1 \subseteq \ldots \subseteq V_n$ is the subgroup $B \subseteq GL(V) \simeq GL_n(k)$ of upper-triangular matrices.

It is clear that if $r = 1$, then $\mathrm{Fl}_{\ell_1}(V) = G(\ell_1, V)$. Suppose now that $r \geq 2$. For every $(\ell_1, \ldots, \ell_r)$ as above the projection

$$G(\ell_1, V) \times \cdots \times G(\ell_r, V) \longrightarrow G(\ell_1, V) \times \cdots \times G(\ell_{r-1}, V)$$

onto the first $(r-1)$ components induces a morphism

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) \longrightarrow \mathrm{Fl}_{\ell_1,\ldots,\ell_{r-1}}(V).$$

The fiber over a point corresponding to the flag $(V_1, \ldots, V_{r-1})$ is isomorphic to the Grassmann variety $G(\ell_r - \ell_{r-1}, V/V_{r-1})$, hence it is irreducible, of dimension $(\ell_r - \ell_{r-1})(n - \ell_r)$. Arguing by induction on r and using Proposition 5.5.1, we obtain the following:

PROPOSITION 7.2.3. *If V is an n-dimensional vector space over k, then for every $(\ell_1,\ldots,\ell_r)$, the flag variety $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is an irreducible variety, of dimension $\sum_{i=1}^r \ell_i(\ell_{i+1}-\ell_i)$, where $\ell_{r+1}=n$. In particular, the complete flag variety $\mathrm{Fl}(V)$ is an irreducible variety of dimension $\frac{n(n-1)}{2}$.*

7.3. A resolution of the generic determinantal variety

Fix positive integers m and n and a non-negative integer $r\leq \min\{m,n\}$. Recall that if we identify the space $M_{m,n}(k)$ of $m\times n$ matrices with entries in k with $\mathbf{A}^{mn}$ in the obvious way, we have a closed subset $M_{m,n}^r(k)$ of $\mathbf{A}^{mn}$ consisting of those matrices of rank $\leq r$. Two cases are trivial: if $r=0$, then $M_{m,n}^r(k)=\{0\}$, and if $r=\min\{m,n\}$, then $M_{m,n}^r(k)=M_{m,n}(k)$.

If we denote the coordinates on $\mathbf{A}^{mn}$ by $x_{i,j}$, for $1\leq i\leq m$ and $1\leq j\leq n$, then $M_{m,n}^r(k)$ is defined by the vanishing of all $(r+1)$-minors of the matrix $(x_{i,j})$. We have already seen that $M_{m,n}^r(k)$ is irreducible in Exercise 1.4.27. We will give another argument for this, that allows us to also compute the dimension of this variety. In fact, we will give a resolution of singularities for $M_{m,n}^r(k)$.

As usual, we identify $M_{m,n}(k)$ with $\mathrm{Hom}_k(k^n,k^m)$. Consider the following subset of $\mathbf{A}^{mn}\times G(n-r,n)$:

$$\mathcal{Y}=\{(A,[W])\in \mathbf{A}^{mn}\times G(n-r,n)\mid W\subseteq \ker(A)\}.$$

We first show that $\mathcal{Y}$ is a closed subset of $\mathbf{A}^{mn}\times G(n-r,n)$. Consider the affine open cover $G(n-r,n)$ by subsets $V_I\simeq \mathbf{A}^{(n-r)r}$ described in Section 7.1. Suppose, as usual, that $I=\{1,\ldots,r\}$. If $B\in M_{(n-r)r}(k)$ represents the linear subspace W and if $M=(I_{n-r},B)$, then $(A,[W])\in\mathcal{Y}$ if and only if $A\cdot M^t=0$. We thus see that $\mathcal{Y}\cap(\mathbf{A}^{mn}\times V_I)$ is the zero-locus of the homogeneous degree 2 polynomials given by writing the entries of $A\cdot M^t$ in terms of the entries of A and M. We thus conclude that $\mathcal{Y}$ is a closed subset of $\mathbf{A}^{mn}\times G(n-r,n)$

The projections onto the two components induce maps $\pi_1\colon \mathcal{Y}\to\mathbf{A}^{mn}$ and $\pi_2\colon\mathcal{Y}\to G(n-r,n)$. Note that since $G(n-r,n)$ is a projective variety, π_1 is a proper morphism. Its image consists of that $A\in M_{m,n}(k)$ such that $\dim_k\ker(A)\geq n-r$: this is precisely $M_{m,n}^r(k)$.

Let us consider the fiber of π_2 over a point $[W]\in G(n-r,n)$. This is identified to the set of all $A\in M_{m,n}(k)$ that vanish of W, which is isomorphic to $\mathrm{Hom}(k^n/W,k^m)\simeq\mathbf{A}^{rm}$. In fact we can say more: π_1 is locally trivial, with fiber $\mathbf{A}^{rm}$. Indeed, for every subset with r elements $I\subseteq\{1,\ldots,n\}$, we have an isomorphism of varieties over V_I:

$$\pi_1^{-1}(V_I)\simeq V_I\times\mathbf{A}^{rm}.$$

In order to see this, let us assume that $I=\{1,\ldots,r\}$. Via the identification $V_I\simeq M_{n-r,r}(k)$, the intersection $\mathcal{Y}\cap(M_{m,n}(k)\times V_I)$ consists of pairs of matrices $A=(a_{i,j})$ (of size $m\times n$) and $B=(b_{p,q})$ (of size $(n-r)\times r$) such that

$$a_{i,\ell}+\sum_{j=1}^r a_{i,n-r+j}b_{\ell,j}=0\quad\text{for}\quad 1\leq i\leq m, 1\leq\ell\leq n-r.$$

It is then clear that by mapping the pair

$$\big((a_{i,j})_{1\leq i\leq m,1\leq j\leq n},(b_{p,q})\big)\quad\text{to}\quad\big((a_{i,j})_{1\leq i\leq m,n-r+1\leq j\leq n},(b_{p,q})\big)$$

we obtain the desired isomorphism. Since $G(n-r,n)$ is smooth, this clearly implies that $\mathcal{Y}$ is smooth. We also see that $\mathcal{Y}$ is irreducible via the general lemma below. Finally, it follows from Theorem 3.4.2 that

$$\dim(\mathcal{Y}) = \dim\big(G(n-r,n)\big) + mr = (n-r)r + mr = (m+n)r - r^2.$$

LEMMA 7.3.1. *If F is an irreducible algebraic variety and $f\colon X \to Y$ is a morphism of algebraic varieties that is locally trivial with fiber F, and such that Y is irreducible, then X is irreducible.*

PROOF. Consider a cover $Y = V_1 \cup \ldots \cup V_i$, with each V_i a nonempty open subset of Y such that $f^{-1}(V_i)$ is isomorphic to $V_i \times F$ as a variety over V_i. In particular, since Y is irreducible, each V_i is irreducible, and therefore $V_i \times F$ is irreducible. Moreover, using again the irreducibility of Y we see that $V_i \cap V_j \neq \emptyset$ for every i and j. Therefore

$$f^{-1}(V_i) \cap f^{-1}(V_j) \simeq (V_i \cap V_j) \times F$$

is nonempty, and we conclude that X is irreducible using Exercise 1.3.17. □

Since $M^r_{m,n}(k)$ is the image of $\mathcal{Y}$, we get another proof for the fact that $M^r_{m,n}(k)$ is irreducible. Note that if $U = M^r_{m,n}(k) \smallsetminus M^{r-1}_{m,n}(k)$, then for every $A \in U$, there is a unique point in $\mathcal{Y}$ mapping to A, namely $\big(A, [\ker(A)]\big)$. By Theorem 3.4.2, we conclude that $\dim\big(M^r_{m,n}(k)\big) = \dim(\mathcal{Y})$, hence the codimension of $M^r_{m,n}(k)$ in $M_{m,n}(k)$ is

$$mn - (m+n)r + r^2 = (m-r)(n-r).$$

In fact, we will show that π_2 is an isomorphism over U; in particular, it is birational. We need to show that the inverse map $U \to f^{-1}(U)$ is a morphism. Of course, since $f^{-1}(U)$ is a locally closed subvariety of $\mathbf{A}^{mn} \times G(n-r,n)$ it is enough to show that the map taking $A \in U$ to $\ker(A) \in G(n-r,n)$ is a morphism. We cover U by the subsets $U_{\Lambda,\Gamma}$, where $\Lambda \subseteq \{1,\ldots,m\}$ and $\Gamma \subseteq \{1,\ldots,n\}$ are subsets with r elements, and where $U_{\Lambda,\Gamma}$ is the subset of $M^r_{m,n}(k)$ consisting of those matrices A such that the minor on the rows in Λ and on the columns in Γ is nonzero. We will show that each map $U_{\Lambda,\Gamma} \to G(n-r,n)$ is a morphism.

In order to simplify the notation, let us assume that $\Lambda = \{1,\ldots,r\}$ and $\Gamma = \{n-r+1,\ldots,n\}$. Let $A \in U_{\Lambda,\Gamma}$. Note that in this case, if $e_1,\ldots,e_n$ is the standard basis of k^n, then $A(e_{n-r+1}),\ldots,A(e_n)$ are linearly independent, hence

$$\ker(A) + \langle e_{n-r+1},\ldots,e_n\rangle = k^n.$$

This implies that $\ker(A) \in V_{\{1,\ldots,n-r\}}$. Moreover, if $\ker(A)$ is described by the matrix $(b_{p,q})_{1\leq p\leq n-r, 1\leq q\leq r}$, then the $b_{p,q}$ are determined by the condition

$$A(e_p) = -\sum_{q=n-r+1}^{n} b_{p,q} A(e_q).$$

It thus follows easily from Cramer's rule that if $A = (a_{i,j}) \in U_{\Lambda,\Gamma}$, then we can write each $b_{p,q}$ as

$$b_{p,q} = \frac{R_{p,q}(A)}{\delta(A)},$$

where $R_{p,q}$ is a polynomial in the $a_{i,j}$, while $\delta(A) = \det(a_{i,j})_{1\leq i\leq r, n-r+1\leq j\leq n}$. This completes the proof of the fact that π_2 is birational. We collect the results we proved in this section in the following proposition

PROPOSITION 7.3.2. *The closed subset $M^r_{m,n}(k)$ of $M_{m,n}(k)$ is irreducible, of codimension $(m-r)(n-r)$, and the morphism $\pi_2\colon \mathcal{Y} \to M^r_{m,n}(k)$ is a resolution of singularities.*

7.4. Linear subspaces on projective hypersurfaces

We consider a projective space $\mathbf{P}^n$ and let S be its homogeneous coordinate ring. Recall that a hypersurface in $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ whose corresponding radical homogeneous ideal is of the form (F), for some nonzero homogeneous polynomial of positive degree. If $\deg(F) = d$, then the hypersurface has *degree* d.

We begin by constructing a parameter space for hypersurfaces of degree d. Note that two polynomials F and G define the same hypersurface if and only if there is $\lambda \in k^*$ such that $F = \lambda G$. Let $\mathbf{P}^{N_d}$ be the projective space parametrizing lines in the vector space S_d, hence $N_d = \binom{n+d}{n} - 1$. We consider on $\mathbf{P}^{N_d}$ the coordinates y_α, where $\alpha = (\alpha_0, \ldots, \alpha_n)$ has $|\alpha| := \sum_i \alpha_i = d$; therefore the point $[c_\alpha]_\alpha$ corresponds to the hypersurface defined by $\sum_\alpha c_\alpha x^\alpha$, where $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Therefore the set $\mathcal{H}_d$ is parametrized by a subset of the projective space $\mathbf{P}^{N_d}$ consisting of classes of homogeneous polynomials $F \in S_d$ such that the ideal (F) is radical. We will denote by $[H]$ the point of $\mathcal{H}_d$ corresponding to the hypersurface $H \subseteq \mathbf{P}^n$.

LEMMA 7.4.1. *The subset $\mathcal{H}_d \subseteq \mathbf{P}^{N_d}$ is a non-empty open subset.*

PROOF. Note that given $F \in S_d$, the ideal (F) is not reduced if and only if there is a positive integer e and a homogeneous polynomial $G \in S_e$ such that G^2 divides F. For every e such that $0 < 2e \leq d$, consider the map

$$\alpha_e\colon \mathbf{P}^{N_e} \times \mathbf{P}^{N_{d-2e}} \to \mathbf{P}^{N_d}$$

that maps $\big([G], [H]\big)$ to $[G^2H]$. It is straightforward to see that this is a morphism. Since the source is a projective variety, it follows that the image of α_e is closed. Since $\mathcal{H}_d$ is equal to

$$\mathbf{P}^{N_d} \smallsetminus \bigcup_{1 \leq e \leq \lfloor d/2 \rfloor} \mathrm{Im}(\alpha_e),$$

we see that this set is open in $\mathbf{P}^{N_d}$. In order to see that it is non-empty, it is enough to consider $f \in S_d$ which is the product of d distinct linear forms. $\square$

REMARK 7.4.2. We have seen in Theorem 6.4.1 that if $X \subseteq \mathbf{P}^n$ is a smooth variety of pure dimension r, then for a general hyperplene $H \subseteq \mathbf{P}^n$, the intersection $X \cap H$ is smooth, of pure dimension $r - 1$. The same assertion holds if we take H a general hypersurface in $\mathbf{P}^n$, of degree d. Indeed, if $\nu_d\colon \mathbf{P}^n \hookrightarrow \mathbf{P}^{N_d}$ is the d^{th} Veronese embeddings, then the intersections $X \cap H$ is isomorphic to the intersection $\nu_d(X) \cap L$, where $L \subseteq \mathbf{P}^{N_d}$ is the hyperplane corresponding to H. We thus conclude by applying Bertini's theorem to $\nu_d(X)$.

By applying the above remark to the case $X = \mathbf{P}^n$, we see that a general hypersurface $H \subseteq \mathbf{P}^n$ of degree d is smooth. The following proposition makes this more precise.

PROPOSITION 7.4.3. *The subset $\mathcal{S}ing_d \subseteq \mathcal{H}_d$ consisting of singular hypersurfaces is an irreducible closed subset, of codimension* 1.

PROOF. Let $\mathcal{Y}$ be the subset of $\mathbf{P}^{N_d}$ consisting of pairs $\big(p, [F]\big)$ such that

$$F(p) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x_i}(p) = 0 \quad \text{for} \quad 0 \leq i \leq n. \tag{7.4.1}$$

It is straightforward to see that $\mathcal{Y}$ is a closed subset of $\mathbf{P}^n \times \mathbf{P}^{N_d}$; in particular, it is a projective variety. Let $\alpha \colon \mathcal{Y} \to \mathbf{P}^n$ and $\beta \colon \mathcal{Y} \to \mathbf{P}^{N_d}$ be the maps induced by the two projections.

We claim that for every $p \in \mathbf{P}^n$, the fiber $\alpha^{-1}(p) \hookrightarrow \mathbf{P}^{N_d}$ is a linear subspace, of codimension $n+1$. Indeed, we may choose coordinates on $\mathbf{P}^n$ such that $p = [1, 0, \ldots, 0]$. In this case, the conditions in (7.4.1) are equivalent with the fact that the coefficients of $x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$ are equal to 0, which gives our claim.

In particular, all fibers of α are irreducible, of the same dimension. Since α is proper, we deduce using Proposition 5.5.1 that $\mathcal{Y}$ is irreducible, and Proposition 3.4.2 gives

$$\dim(\mathcal{Y}) = N_d - 1.$$

Since β is a closed map, it follows that its image is a closed, irreducible subset of $\mathbf{P}^{N_d}$. In order to conclude the proof of the proposition, it is enough to find a singular hypersurface, with only finitely many singular points. Indeed, this implies via Theorem 3.4.1 that $\dim\big(\beta(\mathcal{Y})\big) = \dim(\mathcal{Y}) = N_d - 1$. Since

$$\mathcal{S}ing_d = \beta(\mathcal{Y}) \cap \mathcal{H}_d,$$

it follows that $\mathcal{S}ing_d$ is closed in $\mathcal{H}_d$, and being a non-empty open subset of $\beta(\mathcal{Y})$, it is irreducible, of dimension $N_d - 1$.

In order to construct a hypersurface that satisfies the required condition, it is enough to consider $g \in k[x_0, \ldots, x_{n-1}]$ homogeneous, of degree d, defining a smooth hypersurface in $\mathbf{P}^{n-1}$. Such g exists by Remark 7.4.2. For an explicit example, when $\mathrm{char}(k) \not| d$, one can take

$$g = \sum_{i=0}^{n-1} x_i^d.$$

For any such example, if we consider g as a polynomial in $k[x_0, \ldots, x_n]$, it defines a hypersurface in $\mathbf{P}^n$ that has precisely one singular point, namely $[0, \ldots, 0, 1]$. This completes the proof of the proposition. $\square$

EXAMPLE 7.4.4. Let us describe the hypersurfaces of degree 2 (the *quadrics*) in $\mathbf{P}^n$. For simplicity, let us assume that $\mathrm{char}(k) \neq 2$. Any non-zero homogeneous polynomial $F \in k[x_0, \ldots, x_n]$ of degree 2 can be written as

$$F = \sum_{i,j} a_{i,j} x_i x_j, \quad \text{with} \quad a_{i,j} = a_{j,i} \quad \text{for all} \quad i, j.$$

The *rank* of F is the rank of the symmetric matrix $(a_{i,j})$ (note that if we do a linear change of variables, this rank does not change).

Since k is algebraically closed, it follows that after a suitable linear change of variables, we can write

$$F = \sum_{i=0}^{r} x_i^2, \tag{7.4.2}$$

in which case $\mathrm{rank}(F) = r+1 \geq 1$. This can be deduced from the structure theorem for symmetric bilinear forms over a field, but one can also give a direct argument: we leave this as an exercise for the reader.

Given the expression in (7.4.2), note that (F) is radical if and only if $r \geq 1$ and (F) is prime if and only if $r \geq 2$. It follows from the above description that a quadric is either smooth (precisely when $r = n$) or the projective cone over a quadric of lower dimension.

For example, a quadric in $\mathbf{P}^3$ is either a smooth quadric, or a cone over a smooth conic (quadric in $\mathbf{P}^2$) or a union of 2 planes. After a suitable change of variables, a smooth quadric in $\mathbf{P}^3$ has equation $x_0x_3 + x_1x_2 = 0$. This is the image of the Segre embedding

$$\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3, \quad \big([u_0, u_1], [v_0, v_1]\big) \to [u_0u_1, u_0v_1, u_1v_0, u_1v_1].$$

We next construct the *universal hypersurface* over $\mathcal{H}_d$. In fact, for many purposes, it is more convenient to work with the whole space $\mathbf{P}^{N_d}$ instead of restricting to $\mathcal{H}_d$ (this is due to the fact that $\mathbf{P}^{N_d}$ is complete, while $\mathcal{H}_d$ is not). Define

$$\mathcal{Z}_d := \left\{ \big(p, [F]\big) \in \mathbf{P}^n \times \mathbf{P}^{N_d} \mid F(p) = 0 \right\}.$$

It is easy to see that via the composition of closed embeddings

$$\mathbf{P}^n \times \mathbf{P}^{N_d} \overset{\nu_d \times 1}{\hookrightarrow} \mathbf{P}^{N_d} \times \mathbf{P}^{N_d} \overset{\beta}{\hookrightarrow} \mathbf{P}^M,$$

where ν_d is the d^{th} Veronese embedding and β is the Segre embedding, $\mathcal{Z}_d$ is the inverse image of a hyperplane, hence it is a closed subset of $\mathbf{P}^n \times \mathbf{P}^{N_d}$.

Note that the projections onto the two components induce two morphisms

$$\phi\colon \mathcal{Z}_d \to \mathbf{P}^n \quad \text{and} \quad \psi\colon \mathcal{Z}_d \to \mathbf{P}^{N_d}.$$

Since $\mathbf{P}^n$ and $\mathbf{P}^{N_d}$ are projective varieties, we deduce that both ϕ and ψ are proper morphisms. It follows from definition that for every $[H] \in \mathcal{H}_d$, we have $\psi^{-1}\big([H]\big) = H$.

On the other hand, for every $p \in \mathbf{P}^n$, the fiber $\phi^{-1}(p)$ consists of the classes of those $F \in S_d$ such that $F(p) = 0$. This is a hyperplane in $\mathbf{P}^{N_d}$. We deduce from Proposition 5.5.1 that $\mathcal{Z}_d$ is irreducible, of dimension $N_d + n - 1$.

We now turn to linear subspaces on projective hypersurfaces. Given $r < n$, let $G = G(r+1, n+1)$ be the Grassmann variety parametrizing the r-dimensional linear subspaces in $\mathbf{P}^n$. Consider the incidence correspondence $I \subseteq \mathbf{P}^{N_d} \times G$ consisting of pairs $\big([F], [\Lambda]\big)$ such that F vanishes on Λ.

We first show that I is closed in $\mathbf{P}^{N_d} \times G$. Suppose that we are over the open subset $V = V_{\{1,\ldots,r\}} \simeq \mathbf{A}^{(r+1)(n-r)}$ of G, where a subspace Λ is described by the linear span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{0,r+1} & \ldots & a_{0,n} \\ 0 & 1 & \ldots & 0 & a_{1,r+1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a_{r,r+1} & \ldots & a_{r,n} \end{pmatrix}.$$

The hypersurface corresponding to $c = (c_\alpha)$, which is defined by $f_c = \sum_\alpha c_\alpha x^\alpha$ contains the subspace corresponding to the above matrix if and only if

$$f_c\left(x_0, \ldots, x_r, \sum_{0\leq i\leq r} a_{i,r+1}x_i, \ldots, \sum_{0\leq i\leq r} a_{i,n}x_i\right) = 0 \quad \text{in} \quad k[x_0, \ldots, x_r].$$

We can write

$$f_c\left(x_0, \ldots, x_r, \sum_{0\leq i\leq r} a_{i,r+1}x_i, \ldots, \sum_{0\leq i\leq r} a_{i,n}x_i\right) = \sum_\beta F_\beta(a,c)x^\beta, \tag{7.4.3}$$

where the sum is running over those $\beta = (\beta_0, \ldots, \beta_r)$ with $\sum_i \beta_i = d$. Note that each F_β is a polynomial in the $a_{i,j}$ and c_α variables, homogeneous of degree 1 in the c_α's. With this notation, $I \cap (\mathbf{P}^{N_d} \times V)$ is the zero-locus in $\mathbf{P}^{N_d} \times V$ of the ideal generated by all F_β; in particular, it is a closed subset. The equations over the other charts in G are similar.

In particular, we see that I is a projective variety. Let $\pi_1 \colon I \to \mathbf{P}^{N_d}$ and $\pi_2 \colon I \to G$ be the morphisms induced by the projections onto the two factors.

DEFINITION 7.4.5. For every hypersurface H of degree d in $\mathbf{P}^n$, the *Fano variety* of r-planes in H, denoted $F_r(H)$, is the fiber $\pi_1^{-1}\big([H]\big)$ of π_1, parametrizing the r-dimensional linear subspaces contained in H. .

PROPOSITION 7.4.6. *The projective variety I is irreducible, of dimension*

$$(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1.$$

PROOF. Consider the morphism $\pi_2 \colon I \to G$. By Proposition 5.5.1, it is enough to show that every fiber $\pi^{-1}\big([\Lambda]\big)$ is isomorphic to a linear subspace of $\mathbf{P}^{N_d}$, of codimension $\binom{r+d}{d}$. In order to see this, we may assume that Λ is defined by $x_{r+1} = \ldots = x_n = 0$. It is clear that a polynomial f vanishes on Λ if and only if all coefficients of the monomials in $x_0, \ldots, x_r$ in f vanish; this gives a linear subspace of codimension $\binom{r+d}{d}$. □

EXERCISE 7.4.7. Given a smooth quadric X in $\mathbf{P}^3$, we have 2 families of lines on X: choose coordinates such that X is given by $x_0x_3 - x_1x_2 = 0$, hence X is the image of the Segre embedding $\iota \colon \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. One family of lines is given by $\big(\iota(\mathbf{P}^1 \times \{q\})\big)_{q\in\mathbf{P}^1}$ and the other one is given by $\big(\iota(\{p\} \times \mathbf{P}^1)\big)_{p\in\mathbf{P}^1}$. Show that these are all the lines on X; deduce that the Fano variety of lines on X has two connected components, each of them isomorphic to $\mathbf{P}^1$.

EXAMPLE 7.4.8. Consider lines on cubic surfaces: that is, we specialize to the case when $n = 3 = d$ and $r = 1$. Note that in this case I is an irreducible variety of dimension 19, the same as the dimension of the projective space parametrizing homogeneous polynomials of degree 3 in $S = k[x_0, x_1, x_2, x_3]$. We claim that the morphism $\pi_1 \colon I \to \mathbf{P}^{19}$ is surjective; in other words, every hypersurface in $\mathbf{P}^3$ which is the zero-locus of a degree 3 homogeneous polynomial contains at least one line. In order to see this, it is enough to exhibit such a hypersurface that only contains finitely many lines (this follows from Theorem 3.4.1). At least for $\mathrm{char}(k) \neq 3$, such an example is given by the Fermat cubic surface below.

EXAMPLE 7.4.9. Suppose that $\mathrm{char}(k) \neq 3$ and let X be the Fermat surface in $\mathbf{P}^3$ defined by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

Of course, if $\mathrm{char}(k) = 3$, then the zero locus of this polynomial is the hyperplane $x_0 + x_1 + x_2 + x_3 = 0$, which contains infinitely many lines.

Up to reordering the variables, every line $L \subseteq X$ can be given by equations of the form

$$x_0 = \alpha x_2 + \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2 + \delta x_3,$$

for some $\alpha, \beta, \gamma, \delta \in k$. This line lies on X if and only if

$$(\alpha x_2 + \beta x_3)^3 + (\gamma x_2 + \delta x_3)^3 + x_2^3 + x_3^3 = 0 \quad \text{in} \quad k[x_2, x_3].$$

This is equivalent to the following system of equations:

$$\alpha^3 + \gamma^3 = -1, \alpha^2\beta + \gamma^2\delta = 0, \alpha\beta^2 + \gamma\delta^2 = 0, \text{ and } \beta^3 + \delta^3 = -1.$$

If α, β, γ, δ are all nonzero, then it follows from the third equation that

$$\gamma = -\alpha\beta^2\delta^{-2},$$

and plugging in the second equation, we get

$$\alpha^2\beta + \alpha^2\beta^4\delta^{-4} = 0,$$

which implies $\beta^3 = -\delta^3$, contradicting the fourth equation.

Suppose now, for example, that $\alpha = 0$. We deduce from the second equation that $\gamma\delta = 0$. Moreover, $\gamma^3 = -1$ by the first equation, hence $\delta = 0$ and $\beta^3 = -1$ by the fourth equation. We thus get in this way the 9 lines with the equations

$$x_0 = \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2,$$

where $\beta, \gamma \in k$ are such that $\beta^3 = -1 = \gamma^3$. After permuting the variables, we obtain 2 more sets of lines on X, hence in total we have 27 lines.

We next discuss hypersurfaces that contain linear spaces of small codimension.

PROPOSITION 7.4.10. *We consider hypersurfaces in $\mathbf{P}^n$ of degree $d \geq 2$.*

i) *If X is a smooth such hypersurface containing a linear subspace $\Lambda \subseteq \mathbf{P}^n$ of dimension r, then $r \leq \frac{n-1}{2}$.*
ii) *If $\Lambda \subseteq \mathbf{P}^n$ is a linear subspace of dimension $r \leq \frac{n-1}{2}$, then a general hypersurface containing Λ is smooth.*

PROOF. After a suitable choice of coordinates on $\mathbf{P}^n$, we may assume that Λ is the linear subspace defined by

$$x_{r+1} = \ldots = x_n = 0.$$

Suppose that X is the hypersurface defined by a homogeneous polynomial F, of degree d. If X contains Λ, then we can write

$$F = \sum_{i=1}^{n-r} x_{r+i} f_i, \tag{7.4.4}$$

for some $f_i \in k[x_0, \ldots, x_n]$, homogeneous of degree $d-1$. For every i, with $1 \leq i \leq n-r$, consider the homogeneous polynomials of degree $d-1$

$$g_i(x_0, \ldots, x_r) = f_i(x_0, \ldots, x_r, 0, \ldots, 0).$$

If $n-r\leq r$, then a repeated application of Corollary 4.2.12 implies that there is a point $[u_0,\ldots,u_r]\in\mathbf{P}^r$ such that

$$g_i(u_0,\ldots,u_r)=0\quad\text{for}\quad 1\leq i\leq n-r.$$

In other words, there is a point $p\in\Lambda$ such that $f_i(p)=0$ for all $1\leq i\leq n-r$. In this case, it follows from (7.4.4) that $F(p)=0$ and $\frac{\partial F}{\partial x_j}(p)=0$ for $0\leq j\leq n$, hence p is a singular point of X. We thus deduce that if X is smooth, then $n-r\geq r+1$, giving i).

Suppose now that $r\leq\frac{n-1}{2}$ and consider the subset W of $\mathbf{P}^{N_d}$ consisting of those $[F]$ such that Λ is contained in the zero-locus $(F=0)$. This consists of those $[F]$ such that $F\in(x_{r+1},\ldots,x_n)$, which is a linear subspace in $\mathbf{P}^{N_d}$, of codimension $\binom{r+d}{d}$. Let U be the subset of W consisting of those $[F]$ such that there is no $p\in\mathbf{P}^n$, with

$$F(p)=0=\frac{\partial F}{\partial x_i}(p)\quad\text{for}\quad 0\leq i\leq n. \tag{7.4.5}$$

Note that such F generates a radical ideal (see Remark 6.3.12) and the corresponding degree d hypersurface contains Λ and is smooth. We need to show that U is open and non-empty.

As in Proposition 7.4.3, we consider the set $\mathcal{Y}_W$ of pairs $(p,[F])\in\mathbf{P}^n\times W$ such that (7.4.5) holds. This is a closed subset of $\mathbf{P}^n\times W$, hence it is a projective variety. Let $\alpha\colon\mathcal{Y}_W\to\mathbf{P}^n$ and $\beta\colon\mathcal{Y}_W\to W$ be the morphisms induced by the two projections. Since $U=W\smallsetminus\beta(\mathcal{Y}_W)$, it follows that U is open in W, and it is enough to show that $\beta(\mathcal{Y}_W)\neq W$.

We now describe the fiber $\alpha^{-1}(p)$ for $p\in\mathbf{P}^n$. Suppose first that $p\in\Lambda$. We may choose coordinates such that $p=[1,0,\ldots,0]$. The conditions in (7.4.5) are equivalent with the fact that the coefficients of $x_0^d,x_0^{d-1}x_1,\ldots,x_0^{d-1}x_n$ in F are 0. Since $F\in(x_{r+1},\ldots,x_n)$, we see that $\alpha^{-1}(p)\hookrightarrow W$ is a linear subspace of codimension $n-r$. Suppose now that $p\not\in\Lambda$, in which case we may choose coordinates such that $p=[0,\ldots,0,1]$, in which case the conditions in (7.4.5) are equivalent with the fact that the coefficients of $x_n^d,x_n^{d-1}x_{n-1},\ldots,x_n^{d-1}x_0$ are 0. We thus see that in this case $\alpha^{-1}(p)\hookrightarrow W$ is a linear subspace of codimension $n+1$. We deduce from Corollary 3.4.3 that

$$\dim\big(\alpha^{-1}(\Lambda)\big)=\dim(\Lambda)+\dim(W)-(n-r)=\dim(W)+(2r-n)$$

and

$$\dim\big(\alpha^{-1}(\mathbf{P}^n\smallsetminus\Lambda)\big)=\dim(\mathbf{P}^n\smallsetminus\Lambda)+\dim(W)-(n+1)=\dim(W)-1.$$

Since by assumption we have $2r-n\leq -1$, we deduce that $\dim(\mathcal{Y}_W)=\dim(W)-1$, hence $\dim\big(\beta(\mathcal{Y}_W)\big)\leq\dim(\mathcal{Y}_W)<\dim(W)$. This completes the proof of the proposition. $\square$

7.5. The variety of nilpotent matrices

Fix a positive integer n and let

$$\mathcal{N}_n=\{A\in M_n(k)\mid A\,\text{is nilpotent}\}.$$

The case $n=1$ is trivial ($\mathcal{N}_1$ consists of one point), hence from now on we will assume that $n\geq 2$.

Recall that a matrix $A\in M_n(k)$ is nilpotent if and only if $A^n=0$. Since the entries of A^n are homogeneous polynomials of degree n in the entries of A, it follows that $\mathcal{N}_n$ is a closed subset of $M_n(k)$, preserved by the standard k^*-action on $M_n(k)$. Note that there are nonzero nilpotent matrices (we use here the fact that $n \geq 2$). It follows that we have a non-empty projective variety $\mathcal{N}_n^{\mathrm{proj}}$ in the projective space $\mathbf{P} \simeq \mathbf{P}^{n^2-1}$ of lines in $M_n(k)$, such that $\mathcal{N}_n$ is the affine cone over $\mathcal{N}_n^{\mathrm{proj}}$.

In fact, we can define $\mathcal{N}_n$ by only n equations. Indeed, a matrix A is nilpotent if and only if its characteristic polynomial $\det(A - \lambda I)$ is equal to $(-\lambda)^n$. If we write

$$\det(A - \lambda I) = \sum_{i=0}^{n} (-1)^i p_i(A) \lambda^i,$$

then $p_n(A) = 1$ and for each i, with $0 \leq i \leq n-1$, $p_i(A)$ is a homogeneous polynomial of degree $n-i$ in the entries of A. We thus see that $\mathcal{N}_n$ is the zero-locus of the ideal $(p_0, \ldots, p_{n-1})$.

Our next goal is to show that $\mathcal{N}_n$ is irreducible and compute its dimension. For this, it is a bit more convenient to work with the corresponding projective variety $\mathcal{N}_n^{\mathrm{proj}}$.

The key observation is the following: a matrix $A \in M_n(k)$ is nilpotent if and only if there is a complete flag of subspaces

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n = V,$$

with $\dim_k(V_i) = i$ and $A(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq n$ (where we put $V_0 = 0$). Indeed, it is clear that if we have such a flag, then $A^n = 0$. Conversely, if $A^n = 0$, let $W_i = A^{n-i}(k^n)$. It follows from definition that

$$W_0 = 0 \subseteq W_1 \subseteq \ldots \subseteq W_n = k^n$$

and $A(W_i) \subseteq W_{i-1}$ for $1 \leq i \leq n$. If we refine this sequence of subspaces to a complete flag, this flag will satisfy the required conditions.

Motivated by this, we define

$$Z = \big\{ \big([A], (V_1, \ldots, V_n)\big) \in \mathbf{P} \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1} \,\mathrm{for}\, 1 \leq i \leq n \big\}$$

(where in the above formula we make the convention that $V_0 = \{0\}$). We leave it as an exercise for the reader to check that Z is a closed subset of $\mathbf{P} \times \mathrm{Fl}(k^n)$. In particular, we see that Z is a projective variety. The projections of $\mathbf{P} \times \mathrm{Fl}(k^n)$ onto the two components induce proper morphisms

$$\pi_1 \colon Z \to \mathbf{P} \quad \text{and} \quad \pi_2 \colon Z \to \mathrm{Fl}(k^n).$$

Let us consider the fiber of π_2 over a flag $V_\bullet = (V_1, \ldots, V_n)$. If we choose a basis $e_1, \ldots, e_n$ such that each V_i is generated by $e_1, \ldots, e_i$, it follows that $\pi_2^{-1}(V_\bullet)$ is isomorphic to the the subvariety of $\mathbf{P}$ consisting of classes of nonzero strictly upper-triangular matrices, hence it is isomorphic to $\mathbf{P}^{\frac{n(n-1)}{2}-1}$. Since $\mathrm{Fl}(k^n)$ is irreducible, of dimension $\frac{n(n-1)}{2}$, it follows from Proposition 5.5.1 that Z is an irreducible variety, of dimension $n^2 - n - 1$.

Consider now the morphism $\pi_1 \colon Z \to \mathbf{P}$, whose image is $\mathcal{N}_n^{\mathrm{proj}}$. This implies that $\mathcal{N}_n^{\mathrm{proj}}$ is irreducible. We next show that over a non-empty open subset of $\mathcal{N}_n^{\mathrm{proj}}$, each fiber of π_1 consists of just one point. Note that if $A \in M_n(k)$ is a nilpotent matrix, then its rank is $\leq n-1$. Let $\mathcal{U}_n^{\mathrm{proj}}$ be the open subset of $\mathcal{N}_n^{\mathrm{proj}}$ consisting of matrices of rank $n-1$. Note that this is a non-empty subset: for example, the nilpotent matrix $(a_{i,j})$ with $a_{\ell,\ell-1} = 1$ for $2 \leq \ell \leq n$ and all other $a_{i,j}$ equal to 0 has rank $n-1$. We note that if $[A] \in \mathcal{U}_n^{\mathrm{proj}}$, then $\pi^{-1}\big([A]\big)$ has only one element: if $(V_1, \ldots, V_n)$ is a flag in k^n such that $A(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq n$, then $V_i = A^{n-i}(V)$ for all i. Indeed, the condition on the flag implies that $A^{n-i}(k^n) \subseteq V_i$ and the condition on the rank of A implies easily, by descending induction on i, that $\dim_k A^{n-i}(k^n) = i$. Therefore $A^{n-i}(k^n) = V_i$ for $1 \leq i \leq n$.

Since π_1 has finite fibers over $\mathcal{U}_n$, we deduce from Theorem 3.4.2 that

$$\dim(\mathcal{N}_n^{\mathrm{proj}}) = \dim(Z) = n^2 - n - 1.$$

We thus conclude that $\mathcal{N}_n$ is an irreducible variety of dimension $n^2 - n$.

Remark 7.5.1. In fact, the above construction, but done for the affine cone $\mathcal{N}_n$, gives a resolution of singularities of $\mathcal{N}_n$. Indeed, let

$$W = \left\{ \big(A, (V_1, \ldots, V_{n-1})\big) \in M_n(k) \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1} \,\text{for}\, 1 \leq i \leq n \right\}.$$

One can check that the projection onto the second component induces a morphism $\pi_2 \colon W \to \mathrm{Fl}(k^n)$ that is locally trivial, with fiber $\mathbf{A}^{\frac{n(n-1)}{2}}$. In particular, it follows that W is smooth, irreducible, of dimension $n^2 - n$. The projection onto the first component induces a proper, surjective morphism $\pi_1 \colon W \to \mathcal{N}_n$. In order to see that this is birational, note that if

$$\mathcal{U}_n = \{A \in \mathcal{N}_n \mid \mathrm{rk}(A) = n - 1\},$$

then the induced morphism $\pi_1^{-1}(\mathcal{U}_n) \to \mathcal{U}_n$ is an isomorphism, whose inverse maps A to $\big(A, (A^{n-1}(k^n), \ldots, A(k^n), k^n)\big)$.

Remark 7.5.2. One can see that the ideal $(p_0, \ldots, p_{n-1}) \subseteq \mathcal{O}\big(M_n(k)\big)$ is a radical ideal, but we do not pursue this here, since the argument involves some deeper facts of commutative algebra than we have used so far.

CHAPTER 8

Coherent sheaves on algebraic varieties

In algebra, when one is interested in the study of rings, modules naturally appear: for example, as ideals and quotient rings. Because of this, it is more natural to study the whole category of modules over the given ring. This method becomes even more powerful with the introduction of cohomological techniques, since by working in the category of modules over a given ring, we can construct derived functors of familiar functors like Hom and the tensor product. Our goal in this chapter is to introduce objects that in the context of arbitrary varieties extend what (finitely generated) modules over a ring are in the case of an affine variety: these are the quasi-coherent (respectively, the coherent) sheaves. This will provide us with the language to treat in later chapters global objects, such as divisors, vector bundles, and projective morphisms. We begin with some general constructions for sheaves of R-modules, then discuss sheaves of $\mathcal{O}_X$-modules, and then introduce quasi-coherent and coherent sheaves. In particular, we use these to globalize the MaxSpec and MaxProj constructions. In the last section of this chapter we describe coherent sheaves on varieties of the form $\mathrm{MaxProj}(S)$.

8.1. General constructions with sheaves

In this section we discuss several general constructions involving sheaves. We fix a commutative ring R and consider presheaves and sheaves of R-modules. Important examples are the cases when $R = \mathbf{Z}$ or R is a field. Given a topological space X, we denote by $\mathcal{P}sh_X^R$ and $\mathcal{S}h_X^R$ the categories of presheaves, respectively sheaves, of R-modules on X. However, when R is understood, we simply write $\mathcal{P}sh_X$ and $\mathcal{S}h_X$.

8.1.1. The sheaf associated to a presheaf. Let R be a fixed commutative ring and consider a topological space X. We show that the inclusion functor $\mathcal{P}sh_X \hookrightarrow \mathcal{S}h_X$ has a left adjoint. Explicitly, this means that for every presheaf $\mathcal{F}$ on X, we have a sheaf $\mathcal{F}^+$, together with a morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{F}^+$ that satisfies the following universal property: given any morphism of presheaves $\psi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism of sheaves $\alpha\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\alpha \circ \phi = \psi$. In other words, ϕ induces a bijection

$$\mathrm{Hom}_{\mathcal{S}h_X}(\mathcal{F}^+, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{P}sh_X}(\mathcal{F}, \mathcal{G}).$$

Note that the universal property implies that given any morphism of presheaves $u\colon \mathcal{F} \to \mathcal{G}$, we obtain a unique morphism of sheaves $u^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "u"'] & \mathcal{F}^+ \arrow[d, "u^+"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd}$$

is commutative.

Given a presheaf $\mathcal{F}$, for every open subset $U \subseteq X$ we define $\mathcal{F}^+(U)$ to consist of all maps $t\colon U \to \bigsqcup_{x\in U} \mathcal{F}_x$ that satisfy the following properties:

i) We have $t(x) \in \mathcal{F}_x$ for all $x \in U$.
ii) For every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and $s \in \mathcal{F}(U_x)$, such that $t(y) = s_y$ for all $y \in U_x$.

Note that since each $\mathcal{F}_x$ is an R-module, addition and scalar multiplication of functions makes each $\mathcal{F}^+(U)$ an R-module. We also see that restriction of functions induces for every open subsets $U \subseteq V$ a map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$ that make $\mathcal{F}^+$ a presheaf of R-modules. In fact, it is straightforward to check that $\mathcal{F}^+$ is a sheaf: this is a consequence of the local characterization of the sections of $\mathcal{F}^+$. We have a morphism of presheaves of R-modules $\phi\colon \mathcal{F} \to \mathcal{F}^+$ that maps $s \in \mathcal{F}(U)$ to the map $U \to \bigsqcup_{x\in U} \mathcal{F}_x$ that takes x to s_x.

Let's check the universal property: consider a morphism of presheaves $\psi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf. Given $t \in \mathcal{F}^+(U)$, it follows from definition that we can cover U by open subsets U_i and we have $s_i \in \mathcal{F}(U_i)$ such that for every i and every $y \in U_i$, we have $t(y) = (s_i)_y \in \mathcal{F}_y$. This implies that the sections $t'_i := \psi(s_i) \in \mathcal{G}(U_i)$ have the property that $(t'_i)_y = (t'_j)_y$ for all $y \in U_i \cap U_j$. Using the fact that $\mathcal{G}$ is a sheaf, we first see that $t'_i|_{U_i\cap U_j} = t'_j|_{U_i\cap U_j}$ for all i and j, and then that there is a unique $t' \in \mathcal{G}(U)$ such that $t'|_{U_i} = t'_i$ for all i. We then define $\alpha(t) = t'$. It is straightforward to see that this gives a morphism of sheaves $\alpha\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\alpha \circ \phi = \psi$ and that in fact α is the unique morphism of sheaves with this property.

Remark 8.1.1. It is straightforward to check, using the definition, that if $\mathcal{F}$ is a sheaf, then the canonical morphism $\phi\colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

Remark 8.1.2. For every presheaf $\mathcal{F}$ and every $x \in X$, the morphism $\phi\colon \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism $\phi_x\colon \mathcal{F}_x \to \mathcal{F}^+_x$. The inverse map is defined as follows. Given an element $u \in \mathcal{F}^+_x$ represented by $\big(U, t \in \mathcal{F}^+(U)\big)$, by hypothesis we have an open neighborhood U_x of x and $s \in \mathcal{F}(U_x)$ such that $t(y) = s_y$ for all $y \in U_x$. We define $\tau(u) = s_x \in \mathcal{F}_x$ and leave it as an exercise for the reader to check that this is well-defined and that τ gives an inverse of ϕ_x.

Remark 8.1.3. Wherever we mention stalks in this section, the same results hold, with analogous proofs, for the stalks at irreducible closed subsets of the given topological space. For simplicity, we only give the statements at points of X, since this is sufficient for the study of sheaves on topological spaces; however, in the setting of algebraic varieties it is sometimes convenient to also consider more general stalks (corresponding to localizing a ring to a possibly non-maximal prime ideal).

Remark 8.1.4. It is clear from definition that if U is an open subset of X, then we have a canonical isomorphism

$$(\mathcal{F}|_U)^+ \simeq \mathcal{F}^+|_U.$$

EXAMPLE 8.1.5. If $\mathcal{F}$ is a sheaf and $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, then the inclusion morphism $i\colon \mathcal{G} \hookrightarrow \mathcal{F}$ induces a morphism of sheaves $j\colon \mathcal{G}^+ \to \mathcal{F}$. This gives an isomorphism of $\mathcal{G}^+$ with the subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that for an open subset U of X, $\mathcal{F}'(U)$ consists of those $s \in \mathcal{F}(U)$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x such that $s|_{U_x}$ lies in $\mathcal{G}(U_x)$. Indeed, it is easy to see that $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ and j induces a morphism of sheaves $\alpha\colon \mathcal{G}^+ \to \mathcal{F}'$ such that for all $x \in X$, the induced morphism $\mathcal{G}_x^+ \to \mathcal{F}'_x$ is an isomorphism; therefore α is an isomorphism (see Exercise 2.1.20).

EXAMPLE 8.1.6. If M is any R-module, then we have the *constant presheaf* on X that associates M to every open subset of X, the restriction maps being the identity maps. The associated sheaf is the *constant sheaf* M_X associated to M (though we sometimes drop the subscript, when it is clear that we refer to the sheaf and not to the module M). If X has the property that every open subset is a union of open connected subsets (for example, this is the case for an algebraic variety), then $\Gamma(U, M_X)$ can be identified with the set of maps $U \to M$ that are constant on every connected open subset of U.

8.1.2. Kernels and cokernels. Let R be a fixed commutative ring and X a fixed topological space. We first note that for every two sheaves $\mathcal{F}$ and $\mathcal{G}$, the set of morphisms $\mathrm{Hom}_{\mathcal{S}h_X}(\mathcal{F}, \mathcal{G})$ is an R-module. In particular, we have a zero morphism. We also note that composition of morphisms of sheaves is bilinear.

Given finitely many sheaves $\mathcal{F}_1, \ldots \mathcal{F}_n$ on X, we define $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ by

$$(\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n)(U) := \mathcal{F}_1(U) \oplus \ldots \oplus \mathcal{F}_n(U),$$

with the restriction maps being induced by those for each $\mathcal{F}_i$. It is straightforward to see that this is a sheaf. We have canonical sheaf morphisms $\mathcal{F}_i \to \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ that make $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ the coproduct of $\mathcal{F}_1, \ldots \mathcal{F}_n$ and we have sheaf morphisms $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n \to \mathcal{F}_i$ that make $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ the product of $\mathcal{F}_1, \ldots, \mathcal{F}_n$. Note that for every $x \in X$ we have a canonical isomorphism

$$(\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n)_x \simeq (\mathcal{F}_1)_x \oplus \ldots \oplus (\mathcal{F}_n)_x,$$

due to the fact that filtered direct limits commute with finite direct sums.

We now show that the category $\mathcal{S}h_X$ has kernels. Given a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, define for an open subset U of X

$$\ker(\phi)(U) := \ker\big(\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big).$$

The restriction maps of $\mathcal{F}$ induce restriction maps for $\ker(\phi)$ that make $\ker(\phi)$ a presheaf and it is straightforward to see that it is a sheaf (in fact, a subsheaf of $\mathcal{F}$). It is an easy exercise to see that the inclusion morphism $i\colon \ker(\phi) \hookrightarrow \mathcal{F}$ is a kernel of ϕ: this means that $\phi \circ i = 0$ and for every morphism of sheaves $u\colon \mathcal{F}' \to \mathcal{F}$ such that $\phi \circ u = 0$, there is a unique morphism of sheaves $v\colon \mathcal{F}' \to \ker(\phi)$ such that $u = i \circ v$. Note that since filtered inductive limits are exact functors, it follows that for every $x \in X$, we have

$$\ker(\phi)_x \simeq \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

We now define the cokernel of a morphism of sheaves of R-modules $\phi\colon \mathcal{F} \to \mathcal{G}$. For every open subset U of X, define

$$\widetilde{\mathrm{coker}}(\phi)(U) := \mathrm{coker}\big(\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big).$$

It is straightforward to see that the restriction maps of $\mathcal{G}$ induce restriction maps for $\widetilde{\mathrm{coker}}(\phi)$ which make it a presheaf. We define

$$\mathrm{coker}(\phi) := \widetilde{\mathrm{coker}}(\phi)^+.$$

Note that the composition map p

$$\mathcal{G} \to \widetilde{\mathrm{coker}}(\phi) \to \mathrm{coker}(\phi)$$

is a cokernel of ϕ; this means that $p \circ \phi = 0$ and for every morphism of sheaves $u\colon \mathcal{G} \to \mathcal{G}'$ such that $u \circ \phi = 0$, there is a unique morphism of sheaves $v\colon \mathrm{coker}(\phi) \to \mathcal{G}'$ such that $v \circ p = u$ (this follows using the corresponding property of cokernels of morphsms of R-modules and the universal property of the sheaf associated to a presheaf). Finally, we note that since filtering direct limits are exact and since passing to the associated sheaf preserves the stalks, for every $x \in X$ we have a canonical isomorphism

$$\mathrm{coker}(\phi)_x \simeq \mathrm{coker}(\mathcal{F}_x \to \mathcal{G}_x).$$

If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, we define $\mathcal{F}/\mathcal{F}'$ as the cokernel of the inclusion morphism $\mathcal{F}' \hookrightarrow \mathcal{F}$. It follows that for every $x \in X$, we have a short exact sequence

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to (\mathcal{F}/\mathcal{F}')_x \to 0.$$

The image $\mathrm{Im}(\phi)$ of a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is defined as the kernel of

$$\mathcal{G} \to \mathrm{coker}(\phi).$$

Using the universal property of the kernel and of the cokernel, we obtain a canonical morphism

$$\mathcal{F}/\ker(\phi) \to \mathrm{Im}(\phi). \tag{8.1.1}$$

This is an isomorphism: this follows by considering the induced morphisms at the levels of stalks, using the fact that a morphism of sheaves $\alpha\colon \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\alpha_x\colon \mathcal{A}_x \to \mathcal{B}_x$ is an isomorphism for every $x \in X$ (see Exercise 2.1.20). The existence of kernels and cokernels, together with the fact that the canonical morphism (8.1.1) is an isomorphism mean that Sh_X^R is an *Abelian category*.

EXAMPLE 8.1.7. Given a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the image $\mathrm{Im}(\phi)$ is the subsheaf of $\mathcal{G}$ described as follows: for every open subset $U \subseteq X$, the subset $\mathrm{Im}(\phi)(U) \subseteq \mathcal{G}(U)$ consists of those $s \in \mathcal{G}(U)$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x, such that $s|_{U_x}$ lies in the image of $\mathcal{F}(U_x) \to \mathcal{G}(U_x)$. This follows from Example 8.1.5.

A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if $\ker(\phi) = 0$. Equivalently, for every open subset U of X, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective; moreover, this holds if and only if $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for every $x \in X$. In this case, ϕ gives an isomorphism of $\mathcal{F}$ with a subsheaf of $\mathcal{G}$.

The morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if $\mathrm{coker}(\phi) = 0$, or equivalently, $\mathrm{Im}(\phi) = \mathcal{G}$ (in this case we say that $\mathcal{G}$ is a *quotient* of $\mathcal{F}$). Equivalently, for every $x \in X$, the morphism $\mathcal{F}_x \to \mathcal{G}_x$ is surjective. However, this does *not* imply that for an open subset U of X, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. What we can say in this case is that for every $s \in \mathcal{G}(U)$ and every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x such that $s|_{U_x}$ lies in the image of $\mathcal{F}(U_x) \to \mathcal{G}(U_x)$.

As in any Abelian category, we can consider exact sequences: given morphisms

$$\mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'',$$

this is exact if $\mathrm{Im}(u) = \ker(v)$; equivalently, for every $x \in X$, the sequence of R-modules

$$\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$$

is exact.

In particular, the sequence

$$0 \longrightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \longrightarrow 0$$

is exact if and only if v is surjective and u gives an isomorphism $\mathcal{F}' \simeq \ker(v)$; equivalently, u is injective and v induces an isomorphism $\mathrm{coker}(u) \simeq \mathcal{F}''$. Moreover, this is equivalent with the fact that for every $x \in X$, the sequence of R-modules

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

is exact. Note that in this case, for every open subset U of X, the induced sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

is exact. In other words, the functor $\Gamma(U, -)$ is left exact. However, in general this is not an exact functor.

Remark 8.1.8. Given morphisms of sheaves

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'', \tag{8.1.2}$$

if $\mathcal{B}$ is a basis of open subsets of X such that for every $U \in \mathcal{B}$, the sequence

$$\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \tag{8.1.3}$$

is exact, then (8.1.2) is exact. Indeed, for every $x \in X$, if we take the direct limit of the sequences (8.1.3) over those $U \in \mathcal{B}$, with $x \in U$, we conclude that the sequence

$$\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$$

is exact.

8.1.3. The sheaf $\mathcal{H}om$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of R-modules on X, then for every open subset U of X, we may consider the R-module $\mathrm{Hom}_{\mathcal{S}h_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. If $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a morphism of sheaves and $V \subseteq U$ is an open subset, then we clearly get an induced morphism $\phi|_V \colon \mathcal{F}|_V \to \mathcal{G}|_V$. We thus get a presheaf of R-modules denoted $\mathcal{H}om_R(\mathcal{F}, \mathcal{G})$. In fact, this is a sheaf: this follows from the fact that morphisms of sheaves can be uniquely patched together (see Exercise 2.1.22).

8.1.4. The functor f^{-1}. Recall that if $f\colon X \to Y$ is a continuous map, then we have the functor $f_*\colon \mathcal{S}h_X^R \to \mathcal{S}h_Y^R$ such that

$$\Gamma(V, f_*\mathcal{F}) = \Gamma\big(f^{-1}(V), \mathcal{F}\big) \quad \text{for every open subset} \quad V \subseteq Y.$$

A special case is that when Y is a point, in which case this functor gets identified with $\Gamma(X, -)$.

Like the special case of the functor $\Gamma(X, -)$, the functor f_* is left-exact. Indeed, given an exact sequence of sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and an open subset V in Y, the corresponding sequence

$$0 \to \mathcal{F}'\big(f^{-1}(V)\big) \to \mathcal{F}\big(f^{-1}(V)\big) \to \mathcal{F}''\big(f^{-1}(V)\big)$$

is exact.

We now construct a left adjoint of this functor. Given a sheaf of R-modules $\mathcal{G}$ on Y, consider the presheaf $\widetilde{G}$ on X given by

$$\widetilde{G}(U) := \varinjlim_{f(U)\subseteq V} \mathcal{G}(V),$$

where the direct limit is over the open subsets V of Y containing $f(U)$, ordered by reverse inclusion. Note that if $U_1 \subseteq U_2$, then for every open subset V in Y such that $f(U_2) \subseteq V$, we also have $f(U_1) \subseteq V$, which induces a restriction map

$$\widetilde{G}(U_2) \to \widetilde{G}(U_1)$$

and it is easy to see that these maps make $\widetilde{G}$ a presheaf. We define $f^{-1}(\mathcal{G}) := \widetilde{G}^+$.

If $\phi\colon \mathcal{G} \to \mathcal{G}'$ is a morphism of sheaves on Y, then for every open subset U of X, we have a morphism of R-modules

$$\varinjlim_{f(U)\subseteq V} \phi_V\colon \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) \to \varinjlim_{f(U)\subseteq V} \mathcal{G}'(V)$$

and these give a morphism of presheaves $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}'}$. This in turn induces a morphism of sheaves $f^{-1}(\mathcal{G}) \to f^{-1}(\mathcal{G}')$. This is compatible with composition of morphisms, hence we get a functor

$$f^{-1}\colon \mathcal{S}h_Y^R \to \mathcal{S}h_X^R.$$

Note that for every sheaf $\mathcal{G}$ on Y and every $x \in X$, we have canonical isomorphisms

$$f^{-1}(\mathcal{G})_x \simeq \widetilde{G}_x \simeq \varinjlim_{x\in U} \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) \simeq \varinjlim_{f(x)\in V} \mathcal{G}(V) \simeq \mathcal{G}_{f(x)}.$$

This immediately implies that f^{-1} is an exact functor.

Remark 8.1.9. With the above notation, note that for every open subset V of Y, we have a canonical morphism

$$\Gamma(V,\mathcal{G}) \to \Gamma\big(f^{-1}(V),\widetilde{G}\big) \to \Gamma\big(f^{-1}(V), f^{-1}(\mathcal{G})\big),$$

where the first map comes from the direct limit definition of the R-module in the middle and the fact that $f\big(f^{-1}(V)\big) \subseteq V$.

Example 8.1.10. Note that if U is an open subset of X and $i\colon U \hookrightarrow X$ is the inclusion, then we have a canonical isomorphism $i^{-1}(\mathcal{F}) \simeq \mathcal{F}|_U$.

An important property is that the pair (f^{-1}, f_*) is an adjoint pair of functors. This means that for every sheaves of R-modules $\mathcal{F}$ on X and $\mathcal{G}$ on Y, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{S}h_X}\big(f^{-1}(\mathcal{G}),\mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{S}h_Y}\big(\mathcal{G}, f_*(\mathcal{F})\big).$$

Indeed, giving a morphism of sheaves $f^{-1}(\mathcal{G}) \to \mathcal{F}$ is equivalent to giving a morphism of presheaves $\widetilde{\mathcal{G}} \to \mathcal{F}$, which is equivalent to giving for every open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$ morphisms of R-modules

$$\mathcal{G}(V) \to \mathcal{F}(U)$$

that are compatible with the maps induced by restriction. Because of this compatibility, it is enough to give such maps when $U = f^{-1}(V)$, and such a family of maps compatible with the restriction maps is precisely a morphism of sheaves $\mathcal{G} \to f_*(\mathcal{F})$.

8.2. Sheaves of $\mathcal{O}_X$-modules

Suppose now that $(X, \mathcal{O}_X)$ is a *ringed space*, that is, X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X. Our main example will be that when X is an algebraic variety and $\mathcal{O}_X$ is the sheaf of regular functions on X, but it is more natural to develop the notions that we need here in the general framework.

Definition 8.2.1. A *sheaf of $\mathcal{O}_X$-modules* (or, simply, *$\mathcal{O}_X$-module*) is a sheaf of Abelian groups $\mathcal{F}$ such that for every open subset U of X we have an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$, and these structures are compatible with restriction maps, in the sense that for every open sets $V \subseteq U$, we have

$$(a \cdot s)|_V = a|_V \cdot s|_V \quad \text{for all} \quad a \in \mathcal{O}_X(U) \quad \text{and} \quad s \in \mathcal{F}(U).$$

If $\mathcal{F}$ is a presheaf, instead of a sheaf, we call it a *presheaf of $\mathcal{O}_X$-modules*.

A morphism of sheaves (or presheaves) of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (respectively, presheaves) of Abelian groups such that for every open subset U of X, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. We write $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the set of such morphisms. It is clear that the $\mathcal{O}_X$-modules form a category that we will denote $\mathcal{O}_X$-mod.

Example 8.2.2. The sheaf $\mathcal{O}_X$ has an obvious structure of $\mathcal{O}_X$-module.

Example 8.2.3. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and U is an open subset of X, with $\mathcal{O}_U = \mathcal{O}_X|_U$, then $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module.

Remark 8.2.4. It is easy to see that a sheaf (presheaf) of $\mathcal{O}_X$-modules is the same as a sheaf (respectively, presheaf) of Abelian groups $\mathcal{F}$, together with a morphism of sheaves (respectively, presheaves)

$$\mathcal{O}_X \to \mathcal{H}om_{\mathbf{Z}}(\mathcal{F}, \mathcal{F}).$$

This easily implies that if $\mathcal{O}_X = \underline{R}$, for a ring R, then giving a sheaf of $\mathcal{O}_X$-modules is equivalent to giving a sheaf of R-modules.

Remark 8.2.5. Note that every $\mathcal{O}_X$-module $\mathcal{F}$ is in particular an $\mathcal{O}_X(X)$-module. Indeed, for every open subset U of X, the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ induces an $\mathcal{O}_X(X)$-module structure on $\mathcal{F}(U)$. We get in this way a functor from $\mathcal{O}_X$-mod to $\mathcal{S}h_X^{\mathcal{O}_X(X)}$.

Remark 8.2.6. It follows easily from definition that if $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then for every $x \in X$, the stalk $\mathcal{F}_x$ has a canonical structure of $\mathcal{O}_{X,x}$-module. More generally, if V is an irreducible, closed subset of X, then $\mathcal{F}_V$ has a canonical structure of $\mathcal{O}_{X,V}$-module.

Remark 8.2.7. Note that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \subseteq \mathrm{Hom}_{\mathbf{Z}}(\mathcal{F}, \mathcal{G})$$

is a subgroup. In fact, it follows from Remark 8.2.5 that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has a natural $\mathcal{O}_X(X)$-module structure.

Moreover, we have a subsheaf

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{H}om_{\mathbf{Z}}(\mathcal{F}, \mathcal{G}),$$

whose sections over an open subset $U \subseteq X$ consist of the morphisms of $\mathcal{O}_U$-modules $\mathcal{F}|_U \to \mathcal{G}|_U$. Since each $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an $\mathcal{O}_X(U)$-module, we see that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ becomes naturally an $\mathcal{O}_X$-module.

Note that for every $\mathcal{O}_X$-module $\mathcal{G}$, we have a canonical isomorphism of $\mathcal{O}_X(X)$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \simeq \mathcal{F}(X), \quad \phi \to \phi_X(1)$$

and therefore an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \simeq \mathcal{F}.$$

REMARK 8.2.8. For every $\mathcal{O}_X$-module $\mathcal{F}$, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ gives a left exact functor from the category of $\mathcal{O}_X$-modules to the category of R-modules (the left exactness follows immediately from the definition of the kernel). Similarly, the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ gives a left exact functor from the category of $\mathcal{O}_X$-modules to itself (this follows by using the previous assertion for every open subset U of X).

REMARK 8.2.9. It is clear that if $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are sheaves of $\mathcal{O}_X$-modules, then $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ has a natural structure of $\mathcal{O}_X$-module such that with respect to the obvious maps, it is both the coproduct and the product of the $\mathcal{F}_i$.

REMARK 8.2.10. It follows immediately from Remark 8.2.6 that if $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}^+$ has an induced structure of sheaf of $\mathcal{O}_X$-modules such that the canonical map $\mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves of $\mathcal{O}_X$-modules. Moreover, this satisfies an obvious universal property with respect to morphisms to sheaves of $\mathcal{O}_X$-modules.

REMARK 8.2.11. It follows from definitions and the previous remark that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then $\ker(\phi)$, $\mathrm{coker}(\phi)$, and $\mathrm{Im}(\phi)$ carry natural $\mathcal{O}_X$-module structures. In particular, $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are the kernel, respectively the cokernel, of ϕ in the category of $\mathcal{O}_X$-modules. Moreover, the isomorphism of sheaf of Abelian groups

$$\mathcal{F}/\ker(\phi) \to \mathrm{Im}(\phi)$$

is now an isomorphism in the category of $\mathcal{O}_X$-modules. Therefore $\mathcal{O}_X$-mod is an Abelian category.

The notions of *injective* and *surjective* morphisms of $\mathcal{O}_X$-modules are defined as in the case of sheaves of R-modules. We also have a notion of $\mathcal{O}_X$*-submodule*, which is an $\mathcal{O}_X$-module that is also a subsheaf. In particular, a *sheaf of ideals* is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

EXAMPLE 8.2.12. The following notion will play an important role later: an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* (which, for us, always means *of finite rank*) if for every $x \in X$, there is an open neighborhood U of x such that we have an isomorphism

$$\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus n}.$$

If the integer n does not depend on x, then we say that $\mathcal{F}$ *has rank* n.

EXERCISE 8.2.13. Show that if $(\mathcal{M}_i)_{i\in I}$ in an inverse system of $\mathcal{O}_X$-modules, then the inverse limit $\varprojlim_{i\in I} \mathcal{M}_i$ can be constructed as follows. For every open subset U of X, consider the $\mathcal{O}_X(U)$-module

$$\mathcal{M}(U) := \varprojlim_{i\in I} \mathcal{M}_i(U).$$

If $V \subseteq U$, then the inverse limit of the restriction maps induce a restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ and these maps make $\mathcal{M}$ an $\mathcal{O}_X$-module. Moreover, for every $j \in I$, the projection

$$\varprojlim_{i \in I} \mathcal{M}_i(U) \to \mathcal{M}_j(U)$$

defines a morphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{M}_j$ and $\mathcal{M}$, together with these morphisms, is the inverse limit of $(\mathcal{M}_i)_{i \in I}$.

EXERCISE 8.2.14. Show that if $(\mathcal{M}_i)_{i \in I}$ is a direct system of $\mathcal{O}_X$-modules, then the direct limit $\varinjlim_{i \in I} \mathcal{M}_i$ can be constructed as follows. For every open subset $U \subseteq X$, consider the $\mathcal{O}_X(U)$-module

$$\mathcal{M}(U) := \varinjlim_{i \in I} \mathcal{M}_i(U).$$

If V is an open subset of U, then the direct limit of the restriction maps induces a restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ and these maps make $\mathcal{M}$ a presheaf of $\mathcal{O}_X$-modules. Moreover, for every $j \in I$, the canonical morphisms $\mathcal{M}_j(U) \to \varinjlim_{i \in I} \mathcal{M}_i(U)$ give a morphism of presheaves $\mathcal{M}_j \to \mathcal{M}$.

i) Show that the compositions $\mathcal{M}_j \to \mathcal{M} \to \mathcal{M}^+$ make $\mathcal{M}^+$ the direct limit of the direct system $(\mathcal{M}_i)_{i \in I}$.
ii) Deduce that for every $x \in X$, we have a canonical isomorphism

$$(\varinjlim_{i \in I} \mathcal{M}_i)_x \simeq \varinjlim_{i \in I} \mathcal{M}_{i,x}.$$

8.2.1. Multilinear algebra for $\mathcal{O}_X$-modules. Operations like tensor product, exterior, and symmetric products have analogues for $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, the we can consider the presheaf that associates to an open subset U of X, the $\mathcal{O}_X(U)$-module

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

If V is an open subset of U, the restriction map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V)$$

is the tensor product of the restriction maps of $\mathcal{F}$ and $\mathcal{G}$. The associated sheaf is the *tensor product* of $\mathcal{F}$ and $\mathcal{G}$, and it is denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. It is easy to see that we have a bilinear map of sheaves

$$\mathcal{F} \oplus \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$$

that satisfies the same universal property in $\mathcal{O}_X$-mod as the usual tensor product in the category of R-modules.

While the sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ over some U are not vert explicit, the stalks of this sheaf are easier to understand. In fact, using the fact that a presheaf and its associated sheaf have the same stalks, and the fact that tensor product commutes with direct limits, we obtain for every $x \in X$ a canonical isomorphsim

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \varinjlim_{U \ni x} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x. \tag{8.2.1}$$

Similarly, given an $\mathcal{O}_X$-module $\mathcal{F}$ and a non-negative integer m, we define $\mathcal{O}_X$-modules $\wedge^m \mathcal{F}$ and $\mathrm{Sym}^m(\mathcal{F})$ by taking the sheaf associated to the presheaf that maps an open subset U to $\wedge^m_{\mathcal{O}_X(U)}\mathcal{F}(U)$, respectively to $\mathrm{Sym}^m_{\mathcal{O}_X(U)}\mathcal{F}(U)$. Again, for every $x \in X$, we have canonical isomorphisms

$$(\wedge^m\mathcal{F})_x \simeq \wedge^m_{\mathcal{O}_{X,x}}\mathcal{F}_x \quad \text{and} \quad \big(\mathrm{Sym}^m(\mathcal{F})\big)_x \simeq \mathrm{Sym}^m_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$

Similar isomorphisms hold for the stalks at irreducible closed subsets of X.

EXERCISE 8.2.15. Show that for every $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{M}$, and $\mathcal{N}$, we have a functorial isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{N})\big).$$

In particular, by taking global sections, we see that the functor $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is the left adjoint of the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$.

8.2.2. Push-forward and pull-back for $\mathcal{O}_X$-modules. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a pair $(f, f^\#)$, where $f\colon X \to Y$ is a continuous map and $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. By a slight abuse, $f^\#$ is sometimes dropped from the notation and the morphism is simply denoted by f. The main example for us is that given by a morphism of algebraic varieties. A special feature in this case is that $f^\#$ is determined by the continuous map f.

Note that morphisms of ringed spaces can be composed: if $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces, with associated morphisms of sheaves of rings

$$f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X \quad \text{and} \quad g^\#\colon \mathcal{O}_Z \to g_*\mathcal{O}_Y,$$

then the composition $(X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is given by the continuous map $g \circ f$ and the morphism of sheaves of rings

$$\mathcal{O}_Z \xrightarrow{g^\#} g_*\mathcal{O}_Y \xrightarrow{g_*(f^\#)} g_*(f_*\mathcal{O}_X).$$

It is easy to see that in this way the ringed spaces form a category.

Let $f\colon X \to Y$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we see that for every open subset V of Y, the Abelian group

$$\Gamma\big(V, f_*(\mathcal{F})\big) = \Gamma\big(f^{-1}(V), \mathcal{F}\big)$$

is a module over $\Gamma\big(f^{-1}(V), \mathcal{O}_X\big)$, hence via the given homomorphism $\Gamma(V, \mathcal{O}_Y) \to \Gamma\big(f^{-1}(V), \mathcal{O}_X\big)$ it becomes a module over $\Gamma(V, \mathcal{O}_V)$. This makes $f_*(\mathcal{F})$ an $\mathcal{O}_Y$-module. We thus obtain a left exact functor, the *push-forward* functor

$$f_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}.$$

We now construct a left adjoint of this functor, the *pull-back*. Recall that we have a left adjoint f^{-1} for the corresponding functor between the categories of sheaves of Abelian groups. Note also that by the adjointness of (f^{-1}, f_*) the structure morphism $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponds to a morphism of sheaves of rings $\psi\colon f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. It is straightforward to see that if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then $f^{-1}(\mathcal{G})$ has a natural structure of $f^{-1}(\mathcal{O}_Y)$-module. We put

$$f^*(\mathcal{G}) := f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

and this has a natural structure of $\mathcal{O}$-module. Again, it is not easy to describe the sections of $f^*(\mathcal{G})$ over an open subset of X, but for every $x \in X$, we have a homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induced by $f^\#$ and a canonical isomorphism

$$f^*(\mathcal{G})_x \simeq \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}. \tag{8.2.2}$$

Since the functor $-\otimes_{\mathcal{O}_{Y,f(x)}}\mathcal{O}_{X,x}$ is right-exact, it follows that the functor f^* is right exact. More generally, if V is an irreducible, closed subset of X and $W = \overline{f(V)}$, then for every $\mathcal{O}_Y$-module $\mathcal{G}$, we have a canonical isomorphism

$$f^*(\mathcal{G})_V \simeq \mathcal{G}_W \otimes_{\mathcal{O}_{Y,W}} \mathcal{O}_{X,V}.$$

EXAMPLE 8.2.16. It follows from definition that for every morphism of ringed spaces $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$, we have $f^*(\mathcal{O}_Y) = \mathcal{O}_X$. It is also straightforward to see that if $\mathcal{M}_1,\ldots,\mathcal{M}_n$ are $\mathcal{O}_Y$-modules, we have a canonical isomorphism

$$f^*(\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_n) \simeq f^*(\mathcal{M}_1) \oplus \ldots \oplus f^*(\mathcal{M}_n).$$

This easily implies that if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module (of rank r), then $f^*(\mathcal{E})$ is locally free (of rank r).

EXAMPLE 8.2.17. If U is an open subset of X and $\mathcal{O}_U = \mathcal{O}_X|_U$, then we have a morphism of ringed spaces $i\colon (U,\mathcal{O}_U) \to (X,\mathcal{O}_X)$, where $i\colon U \to X$ is the inclusion and the morphism of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_U$ maps $\phi \in \mathcal{O}_X(V)$ to $\phi|_{U\cap V}$. The corresponding morphism $i^{-1}\mathcal{O}_X = \mathcal{O}_U \to \mathcal{O}_U$ is the identity, so that we have a canonical isomorphism $i^*(\mathcal{F}) \simeq \mathcal{F}|_U$ for every $\mathcal{O}_X$-module $\mathcal{F}$. In particular, in this case the functor i^* is exact.

EXAMPLE 8.2.18. If $f\colon X \to Y$ is a flat morphism of algebraic varieties, then the functor f^* is exact. This follows from the fact that for every $\mathcal{O}_Y$-module $\mathcal{G}$ and every $x \in X$ we have the isomorphism (8.2.2) and $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

PROPOSITION 8.2.19. *The pair of functors (f^*, f_*) is an adjoint pair, that is, for every $\mathcal{O}_X$-module $\mathcal{F}$ and every $\mathcal{O}_Y$-module $\mathcal{G}$, we have a natural isomorphism of Abelian groups*

$$\mathrm{Hom}_{\mathcal{O}_X}\big(f^*(\mathcal{G}),\mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{G}, f_*(\mathcal{F})\big).$$

PROOF. The assertion follows easily from the fact that (f^{-1}, f_*) is an adjoint pair of functors between the corresponding categories of sheaves of Abelian groups, together with the universal property of the tensor product. $\square$

REMARK 8.2.20. The push-forward and pull-back functors are compatible with compositions of morphisms of ringed spaces: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of ringed spaces, then for every $\mathcal{O}_X$-modules $\mathcal{F}$ and every $\mathcal{O}_Z$-module $\mathcal{G}$, we have

$$(g\circ f)_*(\mathcal{F}) = g_*\big(f_*(\mathcal{F})\big)$$

and a natural isomorphism

$$(g\circ f)^*(\mathcal{G}) \simeq f^*\big(g^*(\mathcal{G})\big).$$

Indeed, the first assertion follows directly from definition, and the second one follows from the fact that both functors $(g\circ f)^*$ and $f^*\circ g^*$ are left adjoints of $(g\circ f)_*$.

REMARK 8.2.21. Given a morphism of ringed spaces $f\colon X \to Y$, for every $\mathcal{O}_Y$-module $\mathcal{F}$ and every open subset V of Y, we have an induced morphism of $\mathcal{O}_Y(V)$-modules

$$\Gamma(V,\mathcal{F}) \to \Gamma\big(f^{-1}(V), f^{-1}(\mathcal{F})\big) \to \Gamma\big(f^{-1}(V), f^*(\mathcal{F})\big),$$

where the first map is the one given in Remark 8.1.9 and the second map is induced by the canonical morphism $f^{-1}(\mathcal{F}) \to f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. We will refer to this as the *pull-back of sections* of $\mathcal{F}$. This operation is compatible, in an obvious sense, with composition of morphisms.

NOTATION 8.2.22. If Y is a closed subvariety of X with $i\colon Y \hookrightarrow X$ the inclusion, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, we often write $\mathcal{F}|_Y$ for $i^*(\mathcal{F})$.

We now show that the pull-back is compatible with multilinear operations. For example, we have the following:

PROPOSITION 8.2.23. *If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_Y$-modules, then we have a natural isomorphism*

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \simeq f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} f^*(\mathcal{G}).$$

PROOF. Note first that if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_X$-modules, we have a canonical morphism of $\mathcal{O}_Y$-modules

$$f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{N}) \to f_*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \tag{8.2.3}$$

defined as follows. Let $\mathcal{S}$ be the presheaf of $\mathcal{O}_Y$-modules such that for an open subset V of Y, we have

$$\mathcal{S}(V) = f_*(\mathcal{M})(V) \otimes_{\mathcal{O}_Y(V)} f_*(\mathcal{N})(V) = \mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}\big(f^{-1}(V)\big)$$

and $\mathcal{T}$ the presheaf of $\mathcal{O}_X$-modules such that for an open subset U of X, we have

$$\mathcal{T}(U) = \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U).$$

It thus follows from definition that

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \mathcal{T}^+ \quad \text{and} \quad f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{N}) = \mathcal{S}^+.$$

It is clear that we have a morphism of $\mathcal{O}_Y$-modules

$$\mathcal{S} \to f_*(\mathcal{T})$$

which for an open subset $V \subseteq Y$ is given by the canonical morphism

$$\mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}\big(f^{-1}(V)\big) \to \mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_X(f^{-1}(V))} \mathcal{N}\big(f^{-1}(V)\big).$$

mapping $u \otimes_{\mathcal{O}_Y(V)} v \to u \otimes_{\mathcal{O}_X(f^{-1}(V))} v$. By composing this with the morphism $f_*(\mathcal{T}) \to f_*(\mathcal{T}^+)$, we obtain a morphism $\mathcal{S} \to f_*(\mathcal{T}^+)$ and since the target is a sheaf, this corresponds to a unique morphism of $\mathcal{O}_Y$-modules

$$\mathcal{S}^+ \to f_*(\mathcal{T}^+)$$

which is the morphism in (8.2.3).

Note now that the adjoint property of (f^*, f_*) gives canonical morphisms $\alpha\colon \mathcal{F} \to f_*\big(f^*(\mathcal{F})\big)$ and $\beta\colon \mathcal{G} \to f_*\big(f^*(\mathcal{G})\big)$. We thus obtain the following composition

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to f_*\big(f^*(\mathcal{F})\big) \otimes_{\mathcal{O}_X} f_*\big(f^*(\mathcal{G})\big) \to f_*\big(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{G})\big),$$

where the first morphism is $\alpha \otimes \beta$ and the second morphism is given by (8.2.3). Using the fact that (f^*, f_*) is an adjoint pair, this corresponds to a morphism of $\mathcal{O}_X$-modules

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \longrightarrow f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} f^*(\mathcal{G}). \tag{8.2.4}$$

In order to complete the proof, it is enough to show that this is an isomorphism and this follows if we show that it induces an isomorphism at the level of stalks (see Exercise 2.1.20). This is a consequence of the formulas in (8.2.1) and (8.2.2). $\square$

REMARK 8.2.24. A similar argument shows that if $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then for every non-negative integer m, we have canonical isomorphisms

$$f^*\big(\mathrm{Sym}^m(\mathcal{F})\big) \simeq \mathrm{Sym}^m\big(f^*(\mathcal{F})\big) \quad \text{and} \quad f^*(\wedge^m \mathcal{F}) \simeq \wedge^m f^*(\mathcal{F}).$$

8.2.3. $\mathcal{O}_X$-algebras. A *commutative $\mathcal{O}_X$-algebra* is a sheaf of commutative rings $\mathcal{A}$ on X, together with a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{A}$. Note that in this case, every $\mathcal{A}$-module has an induced $\mathcal{O}_X$-module structure. A morphism of (commutative) $\mathcal{O}_X$-algebras $\mathcal{A} \to \mathcal{B}$ is a morphism of sheaves such that for every open subset U, the map $\mathcal{A}(U) \to \mathcal{B}(U)$ is a morphism of $\mathcal{O}_X(U)$-algebras.

One example that will come up is the *symmetric algebra* of an $\mathcal{O}_X$-module $\mathcal{F}$. This is defined as

$$\mathrm{Sym}^\bullet(\mathcal{F}) := \bigoplus_{m\geq 0} \mathrm{Sym}^m(\mathcal{F}).$$

For every open subset U of X we have a multiplication map

$$\mathrm{Sym}^p\big(\mathcal{F}(U)\big) \otimes_{\mathcal{O}_X(U)} \mathrm{Sym}^q\big(\mathcal{F}(U)\big) \to \mathrm{Sym}^{p+q}\mathcal{F}(U).$$

This gives a morphism of presheaves and by passing to the associated sheaves, we obtain a morphism of sheaves

$$\mathrm{Sym}^p(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathrm{Sym}^q(\mathcal{F}) \to \mathrm{Sym}^{p+q}(\mathcal{F}).$$

By putting these together, we obtain an $\mathcal{O}_X$-algebra structure on $\mathrm{Sym}^\bullet(\mathcal{F})$. As in the case of modules over a ring, this satisfies the following universal property: for every commutative $\mathcal{O}_X$-algebra $\mathcal{A}$, the inclusion $\mathcal{F} \hookrightarrow \mathrm{Sym}^\bullet(\mathcal{F})$ induces a bijection

$$\mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}\big(\mathrm{Sym}^\bullet(\mathcal{F}), \mathcal{A}\big) \simeq \mathrm{Hom}_{\mathcal{O}_X-\mathrm{mod}}(\mathcal{F}, \mathcal{A}).$$

8.3. Quasi-coherent sheaves on affine varieties

We now introduce quasi-coherent sheaves in the setting of affine varieties. We will see that these correspond to modules over the coordinate ring of the affine variety.

We begin with a general proposition about constructing sheaves in the presence of a suitable basis of open subsets. We will use it for the principal affine open subsets of an affine variety and later, for the principal affine open subsets of varieties of the form $\mathrm{MaxProj}(S)$. We state it for $\mathcal{O}_X$-modules, but the reader will see that a similar statement holds in other settings (for example, for sheaves of R-algebras).

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{P}$ a family of open subsets of X that satisfies the following two properties:

i) Every open subset of X is a union of subsets in $\mathcal{P}$ (that is, $\mathcal{P}$ gives a basis of open subsets), and
ii) For every $U, V \in \mathcal{P}$, we have $U \cap V \in \mathcal{P}$.

We define a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules on X to be a map α that associates to every $U \in \mathcal{P}$ an $\mathcal{O}_X(U)$-module $\alpha(U)$ and to every inclusion $U \subseteq V$ a map $\alpha(V) \to \alpha(U)$, $s \to s|_U$, such that

$$(a \cdot s)|_U = a|_U \cdot s|_U \quad \text{for every} \quad a \in \mathcal{O}_X(V), s \in \alpha(V).$$

These restriction maps are supposed to satisfy the usual compatibility conditions. Furthermore, the map α should satisfy the following gluing condition: for every cover $U = \bigcup_{i\in I} U_i$, with U and U_i in $\mathcal{P}$, and for every family $(s_i)_{i\in I}$, with $s_i \in \alpha(U_i)$ for all i, such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i and j, there is a unique $s \in \alpha(U)$ such that $s|_{U_i} = s_i$ for all i. If α and β are $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules, a *morphism* $g\colon \alpha \to \beta$ associates to every $U \in \mathcal{P}$ a morphism of $\mathcal{O}_X(U)$-modules $g_U\colon \alpha(U) \to \beta(U)$ and these are compatible with the restriction maps in the obvious sense. It is clear that $\mathcal{P}$-sheaves form a category.

PROPOSITION 8.3.1. *The functor from the category of sheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules, given by only recording the information for the open subsets in $\mathcal{P}$, is an equivalence of categories.*

PROOF. Given a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules α, we define a corresponding sheaf $\mathcal{F}_\alpha$, such that for an open subset $W \subseteq X$, we let $\mathcal{F}_\alpha(W)$ be the kernel of the map

$$\prod_{U\in\mathcal{P};U\subseteq W} \alpha(U) \to \prod_{U,V\in\mathcal{P};U,V\subseteq W} \alpha(U\cap V), \quad (s_U)_U \to (s_U|_{U\cap V} - s_V|_{U\cap V})_{U,V}.$$

Component-wise addition makes this an Abelian group and we get a structure of $\mathcal{O}_X(U)$-module by putting

$$a\cdot(s_U)_U = (a|_U \cdot s_U)_U \quad \text{for all} \quad a\in\mathcal{O}_X(W), (s_U)_U \in \mathcal{F}_\alpha(U).$$

Note that if $W' \subseteq W$, then we have a restriction map given by

$$(s_U)_U \to (s_V)_V,$$

where the second tuple is indexed by those V that lie inside W'. It is clear that this is compatible with scalar multiplication and makes $\mathcal{F}_\alpha$ a presheaf of $\mathcal{O}_X$-modules. Moreover, it is a straightforward (though somewhat tedious) to check that the glueing condition on α implies that $\mathcal{F}_\alpha$ is a sheaf.

Suppose now that $g\colon \alpha\to\beta$ is a morphism of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules. Given any open subset W of X, we have a commutative diagram

$$\begin{tikzcd}
\prod_{U\in\mathcal{P};U\subseteq W} \alpha(U) \arrow[r] \arrow[d, "\prod_U g_U"'] & \prod_{U,V\in\mathcal{P};U,V\subseteq W} \alpha(U\cap V) \arrow[d, "\prod_{U,V} g_{U\cap V}"] \\
\prod_{U\in\mathcal{P};U\subseteq W} \beta(U) \arrow[r] & \prod_{U,V\in\mathcal{P};U,V\subseteq W} \beta(U\cap V),
\end{tikzcd}$$

which induces a morphism of $\mathcal{O}_X(U)$-modules $\mathcal{F}_\alpha(U)\to\mathcal{F}_\beta(U)$. It is straightforward to check that these maps are compatible with the restriction maps and that in this way we get a functor from the category of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules to the category of sheaves of $\mathcal{O}_X$-modules. Checking that this is an inverse of the functor in the statement of the proposition is an easy exercise that we leave for the enthusiastic reader. $\square$

Suppose now that $(X,\mathcal{O}_X)$ is an affine variety and $A=\mathcal{O}_X(X)$. We consider the set $\mathcal{P}$ consisting of the principal affine open subsets of X. Recall that $D_X(f)\cap D_X(g) = D_X(fg)$. Let M be an A-module. Given any $U\in\mathcal{P}$, say $U=D_X(f)$, we put

$$\alpha(U) := M_f.$$

Note that if $D_X(f)\supseteq D_X(g)$, then $V(f)\subseteq V(g)$, hence $\sqrt{(f)}\supseteq\sqrt{(g)}$. We thus have a localization morphism $A_f\to A_g$ and a corresponding canonical morphism of A_f-modules $M_f\to M_g$. In particular, we see that $\alpha(U)$ only depends on U (up to a canonical isomorphism) and that we have restriction maps that satisfy the usual compatibility relations. The next lemma allows us to apply Proposition 8.3.1 to conclude that we have a sheaf of $\mathcal{O}_X$-modules on X, that we denote $\widetilde{M}$, such that for every $f\in A$, we have a canonical isomorphism

$$\Gamma\big(D_X(f),\widetilde{M}\big)\simeq M_f.$$

LEMMA 8.3.2. *If X is an affine variety, with $A = \mathcal{O}(X)$, and M is an A-module, then for every cover*

$$D_X(f) = \bigcup_{i \in I} D_X(g_i),$$

the sequence

$$0 \longrightarrow M_f \longrightarrow \prod_{i \in I} M_{g_i} \longrightarrow M_{g_i g_j}$$

is exact.

PROOF. The proof is similar to the proof of Proposition 1.4.7. After replacing X by $D_X(f)$ and M by M_f, we may assume that $f = 1$. The condition $X = \bigcup_{i \in I} D_X(g_i)$ is equivalent to the fact that the ideal $(g_i \mid i \in I)$ is the unit ideal. The injectivity of the map $M \to \prod_{i \in I} M_{g_i}$ is clear: if $u \in M$ is such that $\frac{u}{1} = 0$ in M_{g_i} for all i, then there is m_i such that $g_i^{m_i} \in \mathrm{Ann}_A(u)$. Since the elements $g_i^{m_i}$ generate the unit ideal, it follows that $\mathrm{Ann}_A(u) = A$, hence $u = 0$.

Suppose now that we have $u_i \in M_{g_i}$ for all $i \in I$, such that for all $i, j \in I$, the images of u_i and u_j in $M_{g_i g_j}$ coincide. Note first that we may assume that I is finite. Indeed, we may choose a finite subset $J \subseteq I$ such that $(g_i \mid i \in J) = A$. If we can find $u \in M$ such that $\frac{u}{1} = u_i \in M_{g_i}$ for all $i \in J$, then it follows that $\frac{u}{1} = u_i \in M_{g_i}$ also for all $i \in I$. Indeed, $D_X(g_i) = \bigcup_{j \in J} D_X(g_i g_j)$, and we deduce using the first part of the proof that it is enough to show that $\frac{u}{1}$ and u_i have the same image in $M_{g_i g_j}$ for all $j \in J$. This is a consequence of the fact that $\frac{u}{1} = u_j \in M_{g_j}$ and the fact that by hypothesis, u_i and u_j have the same image in $M_{g_i g_j}$.

Suppose now that I is finite and let us write

$$u_i = \frac{v_i}{g_i^{n_i}} \quad \text{for all} \quad i \in I.$$

After replacing each g_i by a suitable power, we may assume that $n_i = 1$ for all i. The condition that u_i and u_j have the same image in $M_{g_i g_j}$ implies that

$$(g_i g_j)^{q_{i,j}} (g_j v_i - g_i v_j) = 0 \quad \text{for some} \quad q_{i,j}.$$

After replacing one more time each g_i by a suitable power, we may assume that $g_i v_j = g_j v_i$ for all i and j. In this case, if we write $1 = \sum_{i \in I} a_i g_i$ and take $u = \sum_{i \in I} a_i v_i \in M$, we have $\frac{u}{1} = u_i \in M_{g_i}$ for all i. Indeed, we have

$$g_i u = \sum_{j \in I} a_j g_i u_j = \sum_{j \in I} a_j g_j u_i = u_i.$$

This completes the proof of the lemma. $\square$

EXAMPLE 8.3.3. With the above notation, the sheaf $\widetilde{A}$ is the structure sheaf $\mathcal{O}_X$. This follows from the fact that for every $f \in A$, the canonical morphism $\mathcal{O}_X(X)_f \to \mathcal{O}_X\big(D_X(f)\big)$ is an isomorphism.

REMARK 8.3.4. If $\mathcal{F} = \widetilde{M}$, then for every irreducible, closed subset $V \subseteq X$, we have a canonical isomorphism

$$\mathcal{F}_V \simeq M_{\mathfrak{p}},$$

where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to V. Indeed, it follows from definition that

$$\mathcal{F}_V = \varinjlim_{V \cap D_X(f) \neq \emptyset} \mathcal{F}\big(D_X(f)\big) = \varinjlim_{f \not\in \mathfrak{p}} M_f \simeq M_{\mathfrak{p}}.$$

Given a morphism of A-modules $\phi\colon M \to N$, for every $f \in A$, we have an induced morphism of A_f-modules $M_f \to N_f$ and these satisfy the obvious compatibility conditions with respect to inclusions of principal affine open subsets. By Proposition 8.3.1, we thus get a morphism of sheaves $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$ such that over every $D_X(f)$, this is given by $M_f \to N_f$. It is clear that in this way we get a functor from the category of A-modules to the category of $\mathcal{O}_X$-modules.

Definition 8.3.5. Let X be an affine variety and $A = \mathcal{O}_X(X)$. A *quasi-coherent* sheaf on X is an $\mathcal{O}_X$-module isomorphic to $\widetilde{M}$, for some A-module M. The sheaf is *coherent* if, in addition, M is a finitely generated A-module. The category of quasi-coherent (or coherent) sheaves on X is a full subcategory of the category of $\mathcal{O}_X$-modules on X.

Remark 8.3.6. It is clear from definition that if $\mathcal{M}$ is a quasi-coherent (coherent) sheaf on the affine variety X, then for every $f \in \mathcal{O}(X)$, the sheaf $\mathcal{M}|_{D_X(f)}$ is again quasi-coherent (respectively, coherent). In fact, if $\mathcal{M} \simeq \widetilde{M}$, then $\mathcal{M}|_{D_X(f)} \simeq \widetilde{M_f}$.

Remark 8.3.7. Note that if X is an affine variety, with $A = \mathcal{O}_X(X)$, and $f \in A$, then for every $\mathcal{O}_X$-module $\mathcal{M}$, the restriction map $\mathcal{M}(X) \to \mathcal{M}\big(D_X(f)\big)$ induces a morphism of A_f-modules

$$\mathcal{M}(X)_f \to \mathcal{M}\big(D_X(f)\big). \tag{8.3.1}$$

These morphisms are compatible with the restriction maps, hence by Proposition 8.3.1, we obtain a canonical morphism of $\mathcal{O}_X$-modules

$$\widetilde{\mathcal{M}(X)} \to \mathcal{M}.$$

This is an isomorphism if and only if $\mathcal{M}$ is quasi-coherent if and only if for every $f \in A$, the canonical morphism

$$\mathcal{M}(X)_f \to \mathcal{M}\big(D_X(f)\big)$$

is an isomorphism.

Proposition 8.3.8. *Let X be an affine variety, with $A = \mathcal{O}(X)$.*

i) *If $(M_i)_{i\in I}$ are A-modules, then*

$$\widetilde{\bigoplus_{i\in I} M_i} \simeq \bigoplus_{i\in I} \widetilde{M_i}.$$

ii) *The functor mapping M to $\widetilde{M}$, from the category of A-modules to the category of quasi-coherent $\mathcal{O}_X$-modules is exact.*

iii) *For every morphism of quasi-coherent (coherent) sheaves $\phi\colon \mathcal{M} \to \mathcal{N}$, the sheaves $\ker(\phi)$, $\operatorname{coker}(\phi)$, and $\operatorname{Im}(\phi)$ are quasi-coherent (respectively, coherent). In particular, the categories of quasi-coherent (coherent) sheaves are Abelian subcategories of the category of $\mathcal{O}_X$-modules.*

iv) *Every quasi-coherent subsheaf or quotient sheaf of a coherent sheaf is coherent.*

v) *The functor in* i) *gives an equivalence of categories with inverse $\Gamma(X, -)$; the latter functor is thus exact on quasi-coherent $\mathcal{O}_X$-modules.*

PROOF. Under the assumption in i), each inclusion

$$M_j \hookrightarrow \bigoplus_{i\in I} M_i$$

induces a morphism of $\mathcal{O}_X$-modules

$$\widetilde{M_j} \to \widetilde{\bigoplus_{i\in I} M_i},$$

and by the universal property of direct sum, a morphism

$$\bigoplus_{i\in I} \widetilde{M_i} \to \widetilde{\bigoplus_{i\in I} M_i}. \tag{8.3.2}$$

We show that it is an isomorphism by showing that it induces an isomorphism at the level of stalks. If $x \in X$ corresponds to the maximal ideal $\mathfrak{m}$, then it follows from Exercise 8.2.14 and Remark 8.3.4 that we have a canonical isomorphism

$$\left(\bigoplus_{i\in I} \widetilde{M_i}\right)_x \simeq \bigoplus_{i\in I} (M_i)_{\mathfrak{m}}$$

and similarly, a canonical isomorphism

$$\left(\widetilde{\bigoplus_{i\in I} M_i}\right)_x \simeq \left(\bigoplus_{i\in I} M_i\right)_{\mathfrak{m}} \simeq \bigoplus_{i\in I} (M_i)_{\mathfrak{m}},$$

where the second isomorphism follows from the fact that localization commutes with direct sums. Via these isomorphisms, the morphism induced by (8.3.2) for the stalks at x is the identity, which completes the proof of i).

We now consider ii). Given an exact sequence of A-modules

$$M' \to M \to M'',$$

for every $x \in X$, if $\mathfrak{m}$ is the maximal ideal of A corresponding to x, we have an exact sequence of $A_{\mathfrak{m}}$-modules

$$M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}}.$$

By Remark 8.3.4, this shows that the sequence

$$\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$$

is exact, proving the assertion in ii).

We next show that if $\phi\colon \widetilde{M} \to \widetilde{N}$ is a morphism of $\mathcal{O}_X$-modules and $u = \phi_X \colon M \to N$ is the induced morphism on the spaces of global sections, then $\phi = \widetilde{u}$. Since ϕ is a morphism of sheaves, it follows that for every $f \in A$, the diagram

$$\begin{tikzcd} M \arrow[r, "\phi_X"] \arrow[d] & N \arrow[d] \\ M_f \arrow[r, "\phi_{D_X(f)}"'] & N_f \end{tikzcd}$$

is commutative. Since $\phi_{D_X(f)}$ is a morphism of A_f-modules, it follows that $\phi_{D_X(f)}$ is the morphism induced by u via localization. We deduce that $\phi = \widetilde{u}$, since they induce the same morphisms on the principal affine open subsets.

Since the functor mapping M to $\widetilde{M}$ is exact, we can now conclude that

$$\ker(\phi) \simeq \widetilde{\ker(u)}, \quad \mathrm{coker}(\phi) \simeq \widetilde{\mathrm{coker}(u)}, \quad \text{and} \quad \mathrm{Im}(\phi) \simeq \widetilde{\mathrm{Im}(u)}.$$

This gives the assertion in iii) for quasi-coherent sheaves. The assertion for coherent sheaves, as well as the one in iv), now follows from the fact that a submodule or a quotient module of a finitely generated A-module is finitely generated (recall that A is a Noetherian ring).

By construction, for every A-module M, we have a canonical isomorphism $\Gamma(X, \widetilde{M}) \simeq M$. This implies that given any two A-modules M and N, the canonical map

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

is injective. Since we have already seen that this map is surjective, it follows that the functor mapping M to $\widetilde{M}$ is fully faithful, hence it gives an equivalence between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Since it is clear that the functor $\Gamma(X, -)$ gives an inverse, this completes the proof of the proposition. ☐

The following result compares certain operations on modules to the corresponding ones for sheaves.

PROPOSITION 8.3.9. *Let X be an affine variety and $A = \mathcal{O}_X(X)$.*

i) *If M and N are A-modules, then*

$$\widetilde{M \otimes_A N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

ii) *If M is an A-module, then for every non-negative integer p, we have*

$$\widetilde{\wedge^p M} \simeq \wedge^p \widetilde{M} \quad \text{and} \quad \widetilde{\mathrm{Sym}^p(M)} \simeq \mathrm{Sym}^p(\widetilde{M}).$$

iii) *If M and N are A-modules, with M finitely generated, then*

$$\widetilde{\mathrm{Hom}_A(M, N)} \simeq \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

PROOF. In order to prove i), note first that by the definition of the tensor product of two $\mathcal{O}_X$-modules, we have a canonical morphism of A-modules:

$$u\colon M \otimes_A N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) =: T.$$

Consider now the composition ϕ

$$\widetilde{M \otimes_A N} \to \widetilde{T} \to \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N},$$

where the first morphism is $\widetilde{u}$ and the second one is that given in Remark 8.3.7. In order to prove the assertion in i), it is enough to show that this induces an isomorphism at the level of stalks. By Remark 8.3.4, the stalk of the left-hand side at a point x corresponding to the maximal ideal $\mathfrak{m}$ is

$$(M \otimes_A N) \otimes_A A_{\mathfrak{m}} \simeq M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}},$$

while the stalk at x of the right-hand side is isomorphic by (8.2.1) to

$$\widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x \simeq M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Via these isomorphisms, the morphism ϕ_x is the identity, which completes the proof of i). The argument for ii) is entirely similar.

Let us prove iii). Note first that by Proposition 8.3.8, the canonical morphism

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}) = \Gamma\big(X, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})\big)$$

is an isomorphism. Therefore we are done if we show that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ is quasi-coherent. However, for every $f \in A$, we have a corresponding isomorphism

$$\mathrm{Hom}_{A_f}(M_f, N_f) \to \mathrm{Hom}_{\mathcal{O}_{D_X(f)}}(\widetilde{M}|_{D_X(f)}, \widetilde{N}|_{D_X(f)}) = \Gamma\big(D_X(f), \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})\big),$$

hence it is enough to show that for every multiplicative system S in A, the canonical morphism

$$\mathrm{Hom}_A(M,N) \otimes_A S^{-1}A \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \tag{8.3.3}$$

is an isomorphism. It is clear that this is the case if M is a free module of finite rank. The general case follows by taking an exact sequence

$$F_1 \to F_0 \to M \to 0,$$

with F_0 and F_1 free A-modules of finite rank (we use here the fact that M is a finitely generated A-module and A is Noetherian) and by noting that both sides of (8.3.3) are left exact functors in M, hence we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M,N)\otimes_A S^{-1}A \arrow[r] \arrow[d] & \mathrm{Hom}_A(F_0,N)\otimes_A S^{-1}A \arrow[r] \arrow[d] & \mathrm{Hom}_A(F_1,N)\otimes_A S^{-1}A \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}F_0, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}F_1, S^{-1}N).
\end{tikzcd}$$

Since the second and the third vertical maps are isomorphisms, the first one is an isomorphism, too. □

PROPOSITION 8.3.10. *Let $f\colon X \to Y$ be a morphism of affine algebraic varieties and*

$$f^{\#}\colon A = \mathcal{O}(Y) \to \mathcal{O}(X) = B$$

the corresponding homomorphism of k-algebras.

i) *If M is an B-module, then $f_*(\widetilde{M}) \simeq \widetilde{M}$, where on the right-hand side M is considered as an A-module via $f^{\#}$.*
ii) *If N is an A-module, then $f^*(\widetilde{N}) \simeq \widetilde{N \otimes_A B}$.*

PROOF. Note that by definition we have

$$\Gamma\big(Y, f_*(\widetilde{M})\big) = \Gamma(X, \widetilde{M}) \simeq M, \tag{8.3.4}$$

with the A-module structure induced by $f^{\#}$. Therefore in order to prove i) it is enough to show that $f_*(\widetilde{M})$ is quasi-coherent. Furthermore, in order to show this, it is enough to see that for every $a \in A$, the canonical morphism

$$\Gamma\big(Y, f_*(\widetilde{M})\big)_a \to \Gamma\big(D_Y(a), f_*(\widetilde{M})\big) \tag{8.3.5}$$

is an isomorphism. However, since $f^{-1}\big(D_Y(a)\big) = D_X\big(f^{\#}(a)\big)$, we have by definition

$$\Gamma\big(D_Y(a), f_*(\widetilde{M})\big) \simeq \Gamma\big(D_X(f^{\#}(a)), \widetilde{M}\big) \simeq M_{f^{\#}(a)}. \tag{8.3.6}$$

By combining (8.3.4) and (8.3.6), we deduce that (8.3.6) is an isomorphism.

The proof of ii) is entirely analogous to that of assertion i) in Proposition 8.3.9; we leave it as an exercise for the reader. □

8.4. Quasi-coherent sheaves on arbitrary varieties

We now globalize the previous discussion, by considering quasi-coherent and coherent sheaves on arbitrary algebraic varieties.

PROPOSITION 8.4.1. *Given an algebraic variety X and an $\mathcal{O}_X$-module $\mathcal{M}$, the following are equivalent:*

i) *For every affine open subset $U \subseteq X$, the restriction $\mathcal{M}|_U$ is a quasi-coherent sheaf.*

ii) *For every affine open subset $U \subseteq X$ and for every $f \in \mathcal{O}_X(U)$, the canonical morphism*

$$\Gamma(U, \mathcal{M})_f \to \Gamma\big(D_U(f), \mathcal{M}\big) \tag{8.4.1}$$

is an isomorphism.

iii) *There is an affine open cover $X = U_1 \cup \ldots \cup U_n$ such that $\mathcal{M}|_{U_i}$ is a quasi-coherent sheaf for $1 \leq i \leq n$.*

Moreover, the equivalence between i) *and* iii) *holds if we replace "quasi-coherent" by "coherent"*

A *quasi-coherent sheaf* on X is an $\mathcal{O}_X$-module $\mathcal{M}$ that satisfies the equivalent conditions i)-iii) in the above proposition. The $\mathcal{O}_X$-module $\mathcal{M}$ is *coherent* if it satisfies the equivalent conditions i) and iii), with "quasi-coherent" replaced by "coherent".

PROOF. The equivalence between i) and ii) is a consequence of the definition of quasi-coherent sheaves on affine varieties. Since the implication i)⇒iii) is trivial, it is enough to prove iii)⇒ii). It follows from Remark 8.3.6 that for every principal affine open subset V of one of the U_i, the restriction $\mathcal{M}|_V$ is quasi-coherent. Given an affine open subset U of X, we can cover U by subsets that are principal affine open subsets with respect to both U and one of the U_i (see Lemma 5.3.3). We are thus reduced to proving the following assertion. Suppose that X is an affine variety, with $\mathcal{O}(X) = A$, and $f \in A$. If $f_1, \ldots, f_r \in A$ are such that $X = \bigcup_{i=1}^r D_X(f_i)$ and $\mathcal{M}|_{D_X(f_i)}$ is quasi-coherent for every i, then the canonical morphism

$$\Gamma(X, \mathcal{M})_f \to \Gamma\big(D_X(f), \mathcal{M}\big)$$

is an isomorphism. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathcal{M})_f \arrow[r] \arrow[d, "u"'] & \prod_{i=1}^r \Gamma\big(D_X(f_i),\mathcal{M}\big)_f \arrow[r] \arrow[d, "v"'] & \prod_{i,j=1}^r \Gamma\big(D_X(f_if_j),\mathcal{M}\big)_f \arrow[d, "w"'] \\
0 \arrow[r] & \Gamma\big(D_X(f),\mathcal{M}\big) \arrow[r] & \prod_{i=1}^r \Gamma\big(D_X(ff_i),\mathcal{M}\big) \arrow[r] & \prod_{i,j=1}^r \Gamma\big(D_X(ff_if_j),\mathcal{M}\big).
\end{tikzcd}$$

Since each $\mathcal{M}|_{D_X(f_i)}$ is quasi-coherent, it follows that both v and w are isomorphisms, hence u is an isomorphism.

The equivalence between i) and iii) for coherent sheaves follows from the fact that if X is an affine variety, $f_1, \ldots, f_r \in A = \mathcal{O}(X)$ are such that they generate the unit ideal, and M is an A-module such that each M_{f_i} is finitely generated as an A_{f_i}-module, then M is a finitely generated A-module (see Corollary C.3.5). $\square$

EXAMPLE 8.4.2. For every algebraic variety X, the structure sheaf $\mathcal{O}_X$ is coherent. This follows from Example 8.3.3.

REMARK 8.4.3. If $\mathcal{M}$ is a coherent sheaf, then it follows from Remark 8.3.4 that for every irreducible, closed subset V of X, the $\mathcal{O}_{X,V}$-module $\mathcal{M}_V$ is finitely generated.

We denote by $\mathcal{Q}coh(X)$ and $\mathcal{C}oh(X)$ the full subcategories of the category of $\mathcal{O}_X$-modules consisting of the quasi-coherent, and respectively coherent, sheaves on X.

PROPOSITION 8.4.4. *Let X be an algebraic variety.*

i) *If $(\mathcal{M}_i)_{i\in I}$ are quasi-coherent sheaves on X, then $\bigoplus_{i\in I}\mathcal{M}_i$ is quasi-coherent; moreover, this is coherent if all $\mathcal{M}_i$ are coherent and I is finite.*
ii) *If $\phi\colon \mathcal{M}\to\mathcal{N}$ is a morphism of quasi-coherent (coherent) sheaves, then $\ker(\alpha)$, $\operatorname{coker}(\alpha)$, and $\operatorname{Im}(\alpha)$ are quasi-coherent (respectively, coherent). In particular, $\mathcal{Q}coh(X)$ and $\mathcal{C}oh(X)$ are Abelian categories.*
iii) *If $\mathcal{M}$ is a quasi-coherent subsheaf or quotient of the coherent sheaf $\mathcal{N}$, then $\mathcal{M}$ is coherent.*
iv) *If $\mathcal{M}$ and $\mathcal{N}$ are quasi-coherent (coherent) sheaves on X, then $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$ is quasi-coherent (respectively, coherent).*
vi) *If $\mathcal{M}$ is a quasi-coherent (coherent) sheaf on X, then $\wedge^p\mathcal{M}$ and $\operatorname{Sym}^p(\mathcal{M})$ are quasi-coherent (respectively, coherent) for every non-negative integer p.*
vii) *If $\mathcal{M}$ is a coherent sheaf and $\mathcal{N}$ is quasi-coherent (coherent), then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ is quasi-coherent (respectively, coherent).*

PROOF. The assertions regarding quasi-coherence follow immediately from Propositions 8.3.8 and 8.3.9. The ones regarding coherence follow from the fact that all operations involved preserve finitely generated modules. The only assertion that is not entirely obvious is the one concerning vii): if M and N are finitely generated modules over a Noetherian ring A, then $\operatorname{Hom}_A(M,N)$ is finitely generated. In order to see this, we choose a surjective morphism $A^{\oplus n}\to M$, which induces an injective morphism

$$\operatorname{Hom}_A(M,N)\hookrightarrow\operatorname{Hom}_A(A^{\oplus n},N)\simeq N^{\oplus n}.$$

Therefore $\operatorname{Hom}_A(M,N)$ is finitely generated, as a submodule of the finitely generated A-module $N^{\oplus n}$. □

We now discuss the behavior of quasi-coherence and coherence via push-forward and pull-back.

PROPOSITION 8.4.5. *Let $f\colon X\to Y$ be a morphism of algebraic varieties.*

i) *If $\mathcal{M}$ is a quasi-coherent (coherent) sheaf on Y, then $f^*(\mathcal{M})$ is a quasi-coherent (respectively, coherent) sheaf on X.*
ii) *If $\mathcal{N}$ is a quasi-coherent sheaf on X, then $f_*(\mathcal{N})$ is a quasi-coherent sheaf on Y.*

PROOF. Let us prove i). For every $x\in X$, we may choose an affine open neighborhood V of $f(x)$ and an affine open neighborhood $U\subseteq f^{-1}(V)$ of x. Let $g\colon U\to V$ be the morphism induced by f. It follows from Proposition 8.3.10 that if $\mathcal{M}|_V\simeq\widetilde{M}$, then $f^*(\mathcal{M})|_U\simeq g^*(\mathcal{M}|_V)$ is isomorphic to the sheaf associated to the $\mathcal{O}_X(U)$-module $M\otimes_{\mathcal{O}_Y(V)}\mathcal{O}_X(U)$. Therefore $f^*(\mathcal{M})$ is quasi-coherent if $\mathcal{M}$ is quasi-coherent. Moreover, if $\mathcal{M}$ is coherent, then $f^*(\mathcal{M})$ is coherent, since M

finitely generated over $\mathcal{O}_Y(V)$ implies that $M \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)$ is finitely generated over $\mathcal{O}_X(U)$.

Suppose now that $\mathcal{N}$ is a quasi-coherent sheaf on X. Given an affine open subset $W \subseteq Y$, we need to show that $f_*(\mathcal{N})|_W$ is quasi-coherent. If $h\colon f^{-1}(W) \to W$ is the induced morphism, then it is clear that

$$f_*(\mathcal{N})|_W \simeq h_*(\mathcal{N}|_{f^{-1}(W)}).$$

It follows that we may replace f by h and thus assume that Y is affine.

Consider a cover $X = U_1 \cup \ldots \cup U_r$, with U_i affine open subsets in X. Since X is separated, each intersection $U_i \cap U_j$ is again an affine open subset. If we denote by $\alpha_i\colon U_i \hookrightarrow X$ and $\alpha_{i,j}\colon U_i \cap U_j \hookrightarrow X$ the inclusions, then $\mathcal{N}$ being a sheaf implies that we have an exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{N} \to \bigoplus_i (\alpha_i)_*(\mathcal{N}|_{U_i}) \to \bigoplus_{i,j} (\alpha_{i,j})_*(\mathcal{N}|_{U_i \cap U_j}).$$

Since f_* is an exact functor, we obtain an exact sequence of $\mathcal{O}_Y$-modules:

$$0 \to f_*(\mathcal{N}) \to \bigoplus_i (f \circ \alpha_i)_*(\mathcal{N}|_{U_i}) \to \bigoplus_{i,j} (f \circ \alpha_{i,j})_*(\mathcal{N}|_{U_i \cap U_j}).$$

The second and third term are quasi-coherent by Proposition 8.3.10, and we deduce that $f_*(\mathcal{N})$ is quasi-coherent using assertion ii) in Proposition 8.4.4. □

REMARK 8.4.6. In general, it is not true that if $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{N}$ is a coherent sheaf on X, then $f_*(\mathcal{N})$ is a coherent sheaf on Y. For example, if $X = \mathbf{A}^1$ and Y is a point, then $f_*(\mathcal{O}_X)$ corresponds to the k-vector space $k[x]$, which is not finitely generated. An important finiteness result that we will prove later says that if f is a *proper* morphism, then $f_*(\mathcal{N})$ is coherent for every coherent sheaf $\mathcal{N}$ on X.

EXAMPLE 8.4.7. One easy case in which the push-forward of any coherent sheaf is coherent is that of a finite morphism: in this case the assertion follows directly from Proposition 8.3.10. In particular, this applies to any closed immersion.

REMARK 8.4.8. If $f\colon X \to Y$ is an affine morphism of algebraic varieties, then the functor f_* is exact on the category of quasi-coherent $\mathcal{O}_X$-modules. Indeed, given an exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

it is enough to show that for every affine open subset V of Y, the induced sequence

$$0 \to \Gamma\big(V, f_*(\mathcal{F}')\big) \to \Gamma\big(V, f_*(\mathcal{F})\big) \to \Gamma\big(V, f_*(\mathcal{F}'')\big) \to 0$$

is exact. This follows from the fact that $f^{-1}(V)$ is affine and thus the functor $\Gamma\big(f^{-1}(V), -\big)$ is exact on the category of quasi-coherent sheaves.

The following exercise shows that if $f\colon X \to Y$ is a finite morphism, then the functor f_* between the corresponding categories of coherent sheaves has a right adjoint.

EXERCISE 8.4.9. Let $f\colon X \to Y$ be a finite morphism.

i) Show that for every coherent (quasi-coherent) sheaf $\mathcal{G}$ on Y, there is a coherent (respectively, quasi-coherent) sheaf $f^!(\mathcal{G})$ on X such that we have an isomorphism
$$f_*\big(f^!(\mathcal{G})\big) \simeq \mathrm{Hom}_{\mathcal{O}_Y}\big(f_*(\mathcal{O}_X), \mathcal{G}\big).$$
Moreover, show that in this way we get a functor $f^!$ from $\mathcal{C}oh(Y)$ to $\mathcal{C}oh(X)$ (and similarly, from $\mathcal{Q}coh(Y)$ to $\mathcal{Q}coh(X)$).

ii) Show that for every quasi-coherent sheaf $\mathcal{F}$ on X and every quasi-coherent sheaf $\mathcal{G}$ on Y, we have a functorial isomorphism
$$f_*\mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{F}, f^!(\mathcal{G})\big) \simeq \mathcal{H}om_{\mathcal{O}_Y}\big(f_*(\mathcal{F}), \mathcal{G}\big).$$

iii) Deduce that the functor $f^! \colon \mathcal{C}oh(Y) \to \mathcal{C}oh(X)$ is the right adjoint of the functor $f_* \colon \mathcal{C}oh(X) \to \mathcal{C}oh(Y)$ (and similarly for the functors between the categories of quasi-coherent sheaves).

We end this section by extending some familiar constructions of ideals to the setting of coherent sheaves. We fix an algebraic variety X.

Definition 8.4.10. For every sheaf of Abelian groups $\mathcal{F}$ on X, the *support* of $\mathcal{F}$ is the subset
$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$
If $Y = \mathrm{Supp}(\mathcal{F})$, we also say that $\mathcal{F}$ is *supported on* Y.

Proposition 8.4.11. *If $\mathcal{M}$ is a coherent sheaf on X, then $\mathrm{Supp}(\mathcal{M})$ is a closed subset of X; in fact, if U is an affine open subset of X, then $\mathrm{Supp}(\mathcal{M}) \cap U$ is the zero-locus of $\mathrm{Ann}_{\mathcal{O}_X(U)}\mathcal{M}(U)$.*

Proof. Of course, it is enough to prove the last assertion. Since $\mathcal{M}|_U \simeq \widetilde{M}$, for some finitely generated $\mathcal{O}_X(U)$-module M, it is enough to check that for a maximal (or prime) ideal $\mathfrak{p}$ in $\mathcal{O}_X(U)$, we have $\mathrm{Ann}_{\mathcal{O}_X(U)}(M) \subseteq \mathfrak{p}$ if and only if $M_{\mathfrak{p}} \neq 0$. If $u_1, \ldots, u_r \in M$ form a system of generators, then $\mathrm{Ann}_R(M) = \bigcap_{i=1}^r \mathrm{Ann}_R(u_i)$, hence
$$\mathrm{Ann}_R(M) \subseteq \mathfrak{p} \quad \text{if and only if} \quad \mathrm{Ann}_R(u_i) \subseteq \mathfrak{p} \quad \text{for some} \quad i,$$
which is the case if and only if some $\frac{u_i}{1} \in M_{\mathfrak{p}}$ is non-zero. This completes the proof. $\square$

Definition 8.4.12. Given a coherent ideal $\mathcal{I} \hookrightarrow \mathcal{O}_X$, the *co-support* or *zero-locus* $V(\mathcal{I})$ of $\mathcal{I}$ is the support of the coherent sheaf $\mathcal{O}_X/\mathcal{I}$. It follows from the above proposition that if U is an affine open subset of X, then $V(\mathcal{I}) \cap U$ is the zero locus of $\mathcal{I}(U) \subseteq \mathcal{O}_X(U)$.

Definition 8.4.13. Given an $\mathcal{O}_X$-module $\mathcal{M}$ on X, its *annihilator* $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is the subsheaf of $\mathcal{O}_X$ given by
$$\Gamma\big(U, \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})\big) := \mathrm{Ann}_{\mathcal{O}_X(U)}\mathcal{M}(U)$$
for every open subset U of X. It is straightforward to check that this is, indeed, an ideal sheaf of $\mathcal{O}_X$.

Proposition 8.4.14. *If $\mathcal{M}$ is a coherent sheaf on X, then $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is a coherent ideal and for every irreducible closed subset $V \subseteq X$, we have*
$$\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})_V \simeq \mathrm{Ann}_{\mathcal{O}_{X,V}}(\mathcal{M}_V).$$

Proof. Both assertions follow from the fact that taking the annihilator of a finitely generated module commutes with localization: if M is a finitely generated A-module and $S \subseteq A$ is a multiplicative system, then the canonical morphism

$$S^{-1}\mathrm{Ann}_A(M) \to \mathrm{Ann}_{S^{-1}A} S^{-1}M$$

is an isomorphism. This is straightforward to check. □

Remark 8.4.15. Note that for every coherent sheaf $\mathcal{M}$, the zero-locus of $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is equal to $\mathrm{Supp}(\mathcal{M})$.

Definition 8.4.16. If $\mathcal{I}$ is a coherent sheaf of ideals in $\mathcal{O}_X$, then the *radical* $\mathrm{rad}(\mathcal{I})$ of $\mathcal{I}$ is the sheaf given by

$$\Gamma\big(U, \mathrm{rad}(\mathcal{I})\big) = \mathrm{rad}\big(\Gamma(U, \mathcal{I})\big) \subseteq \Gamma(U, \mathcal{O}_X).$$

It is straightforward to see that $\mathrm{rad}(\mathcal{I})$ is an ideal sheaf of $\mathcal{O}_X$.

Remark 8.4.17. It is easy to see that if I is an ideal in a ring A and S is a multiplicative system in A, then

$$\mathrm{rad}(S^{-1}I) = S^{-1}\big(\mathrm{rad}(I)\big).$$

This implies that if $\mathcal{I}$ is a coherent ideal sheaf, the sheaf $\mathrm{rad}(\mathcal{I})$ is coherent, and for every irreducible, closed subset $V \subseteq X$, we have

$$\mathrm{rad}(\mathcal{I})_V = \mathrm{rad}(\mathcal{I}_V) \subseteq \mathcal{O}_{X,V}.$$

Remark 8.4.18. If $i\colon Y \hookrightarrow X$ is a closed immersion, the canonical morphism of sheaves $\phi\colon \mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is surjective; indeed, for every affine open subset $U \subseteq X$, the corresponding homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_Y\big(i^{-1}(U)\big)$ is surjective. We get an ideal sheaf $\mathcal{I}$, which is the kernel of ϕ (equivalently, this can be described as $\mathrm{Ann}_{\mathcal{O}_X} i_*(\mathcal{O}_Y)$). Since $i_*(\mathcal{O}_Y)$ is coherent (see Example 8.4.7), it follows from Proposition 8.4.4 that $\mathcal{I}$ is coherent. Note that for every affine open subset U of X, we have, by definition

$$\mathcal{I}(U) = I_U\big(i(Y) \cap U\big).$$

This applies, in particular, if Y is a closed subvariety of X and i is the inclusion; in this case we write $\mathcal{I}_{Y/X}$ for $\mathcal{I}$.

Example 8.4.19. Given a point $x \in X$, we consider the ideal $\mathcal{I}_x$ defining x, and put $k(x) := \mathcal{O}_X/\mathcal{I}_x$. Note that if U is an affine open neighborhood of x, with $A = \mathcal{O}_X(U)$ and $\mathfrak{m} \subseteq A$ the maximal ideal corresponding to x, then $k(x) = \widetilde{A/\mathfrak{m}}$ (it is thus not surprising that we use the same notation for this sheaf as we do for the residue field of $\mathcal{O}_{X,x}$). Of course, we have

$$\Gamma\big(U, k(x)\big) = k \quad \text{if} \quad x \in U,$$

and $\Gamma\big(U, k(x)\big) = 0$, otherwise. In the former situation, the $\mathcal{O}_X(U)$-module structure is given by

$$f \cdot \lambda = f(x)\lambda \quad \text{for} \quad f \in \mathcal{O}_X(U), \lambda \in k.$$

Note that if $i_x\colon \{x\} \hookrightarrow X$ is the inclusion, we can also describe $k(x)$ as $(i_x)_*k$.

Remark 8.4.20. If Y is a closed subvariety of X and $\mathcal{I} = \mathcal{I}_{Y/X}$, then it follows from definition that $Y = V(\mathcal{I})$. Conversely, if $\mathcal{I}$ is any coherent ideal sheaf on X and $Y = V(\mathcal{I})$, then $\mathcal{I}_{Y/X} = \mathrm{rad}(\mathcal{I})$. This is clear, by considering the sections over affine open subsets of X.

REMARK 8.4.21. If $\mathcal{F}$ is a sheaf of Abelian groups on a topological space X and $i\colon Y \hookrightarrow X$ is the inclusion map of a closed subset such that $\mathcal{F}_x = 0$ for all $x \in X \smallsetminus Y$, then the canonical morphism $\mathcal{F} \to i_*\big(i^{-1}(\mathcal{F})\big)$ is an isomorphism (it is enough to check that we have isomorphisms at the level of stalks).

Suppose now that X is an algebraic variety and $i\colon Y \hookrightarrow X$ is the inclusion map of a closed subvariety. If $\mathcal{F}$ is a coherent sheaf on X such that $\mathrm{Supp}(\mathcal{F}) \subseteq Y$, then the canonical morphism $\mathcal{F} \to i_*\big(i^{-1}(\mathcal{F})\big)$ is an isomorphism. In general, this is not very useful, since $i^{-1}(\mathcal{F})$ is not an $\mathcal{O}_Y$-module.

In practice, a more useful fact is that $\mathcal{F}$ has a filtration by coherent sheaves such that the successive quotients are the push-forward of coherent sheaves on Y. Indeed, if $\mathcal{I}$ is the radical ideal in $\mathcal{O}_X$ corresponding to Y, then $\mathcal{I}^r \cdot \mathcal{F} = 0$ for some $r \geq 1$. If we consider the filtration

$$0 = \mathcal{F}_r \subseteq \mathcal{F}_{r-1} \subseteq \ldots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = \mathcal{F},$$

where $\mathcal{F}_j = \mathcal{I}^j \cdot \mathcal{F}$, then each $\mathcal{F}_j/\mathcal{F}_{j+1}$ is annihilated by $\mathcal{I}$, hence it is equal to $i_*(\mathcal{G}_j)$, for a coherent $\mathcal{O}_Y$-module $\mathcal{G}_j$.

Suppose now that $\mathcal{M}$ is a quasi-coherent sheaf and $\mathcal{M}_1, \ldots, \mathcal{M}_r$ are quasi-coherent subsheaves of $\mathcal{M}$.

DEFINITION 8.4.22. The *sum* $\sum_{i=1}^r \mathcal{M}_i$ is the sheaf associated to the presheaf that maps an open subset $U \subseteq X$ to

$$\sum_{i=1}^r \mathcal{M}_i(U) \subseteq \mathcal{M}(U).$$

REMARK 8.4.23. Equivalently, $\sum_{i=1}^r \mathcal{M}_i$ is the image of the morphism of sheaves

$$\bigoplus_{i=1}^r \mathcal{M}_i \to \mathcal{M}.$$

In particular, it is quasi-coherent (coherent) if $\mathcal{M}$ and all $\mathcal{M}_i$ are quasi-coherent (respectively, coherent) by Proposition 8.4.4.

DEFINITION 8.4.24. The *intersection* $\bigcap_{i=1}^r \mathcal{M}_i$ is the sheaf that associates to an open subset $U \subseteq X$ the $\mathcal{O}_X(U)$-module

$$\bigcap_{i=1}^r \mathcal{M}_i(U) \subseteq \mathcal{M}(U)$$

(it is straightforward to see that this is a subsheaf of $\mathcal{M}$).

REMARK 8.4.25. If $\mathcal{M}$ and all $\mathcal{M}_i$ are quasi-coherent, then $\bigcap_{i=1}^r \mathcal{M}_i$ is quasi-coherent. Indeed, arguing by induction, it is enough to show this when $r = 2$. In this case, it follows from Proposition 8.4.4 using the fact that $\mathcal{M}_1 \cap \mathcal{M}_2$ is the kernel of the canonical morphism

$$\mathcal{M}_1 \to (\mathcal{M}_1 + \mathcal{M}_2)/\mathcal{M}_2.$$

If, in addition, some $\mathcal{M}_j$ is coherent, then $\bigcap_{i=1}^n \mathcal{M}_i$ is coherent, as a subsheaf of $\mathcal{M}_j$.

DEFINITION 8.4.26. If X is an algebraic variety and $\mathcal{I}_1$ and $\mathcal{I}_2$ are coherent ideals of $\mathcal{O}_X$, then the *product* $\mathcal{I}_1 \cdot \mathcal{I}_2$ is the ideal of $\mathcal{O}_X$ which is the sheaf corresponding to the presheaf that associates to an open subset U the ideal $\mathcal{I}_1(U) \cdot \mathcal{I}_2(U) \subseteq$

$\mathcal{O}_X(U)$. Note that multiplication gives a canonical morphism $\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{I}_2 \to \mathcal{O}_X$, whose image is, by definition, $\mathcal{I}_1 \cdot \mathcal{I}_2$. We deduce that $\mathcal{I}_1 \cdot \mathcal{I}_2$ is a coherent sheaf of ideals and for every affine open subset U of X, we have

$$\Gamma(U, \mathcal{I}_1 \cdot \mathcal{I}_2) = \Gamma(U, \mathcal{I}_1) \cdot \Gamma(U, \mathcal{I}_2).$$

Definition 8.4.27. Suppose that $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{I}$ is a coherent sheaf of ideals on Y. The image of the induced morphism

$$f^*(\mathcal{I}) \to f^*(\mathcal{O}_Y) = \mathcal{O}_X$$

is a coherent sheaf of ideals on X, that we denote $\mathcal{I} \cdot \mathcal{O}_X$.

Remark 8.4.28. With the notation in the above definition, note that we have

$$V(\mathcal{I} \cdot \mathcal{O}_X) = f^{-1}\big(V(\mathcal{I})\big). \tag{8.4.2}$$

Indeed, it is enough to check this in a neighborhood of each $x \in X$. Let us choose an affine open neighborhood $V \subseteq Y$ of $f(x)$ and then an affine open neighborhood $U \subseteq f^{-1}(V)$ of x. In this case, it follows from the description of $f^*(\mathcal{I})$ in Proposition 8.3.10 that

$$\Gamma(U, \mathcal{I} \cdot \mathcal{O}_X) = \Gamma(V, \mathcal{I}) \cdot \mathcal{O}_X(U),$$

and we deduce the equality in (8.4.2) from Proposition 1.4.23.

One of the draw-backs of requiring the structure sheaves on our varieties to be sheaves of functions is that in performing certain geometric operations (for example, in intersecting closed subvarieties or, more generally, taking the inverse image of a closed subset via a morphism) we lose some information. In order to remedy this, we will keep track of what happens to the ideal sheaves when performing these operations. For example, if $f\colon X \to Y$ is a morphism and W is a closed subvariety of Y, then the ideal sheaf $\mathcal{I}_{W/Y} \cdot \mathcal{O}_X$ contains more information than its zero-locus, the geometric inverse image $f^{-1}(W)$.

Exercise 8.4.29. Show that if Y is a closed subvariety of X and $i\colon Y \hookrightarrow X$ is the inclusion, then the map $\mathcal{F} \to i_*(\mathcal{F})$ gives an equivalence of categories between $\mathcal{Q}coh(Y)$ (or $\mathcal{C}oh(Y)$) and the full subcategory of $\mathcal{Q}coh(X)$ (respectively, $\mathcal{C}oh(X)$) consisting of those $\mathcal{F}$ such that $\mathcal{I}_{Y/X} \subseteq \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$.

Exercise 8.4.30. Show that if $\mathcal{M}$ is a quasi-coherent sheaf on an algebraic variety X, we have $f \in \mathcal{O}_X(X)$, and $D_X(f) = \{x \in X \mid f(x) \neq 0\}$, then the restriction map $\Gamma(X, \mathcal{M}) \to \Gamma\big(D_X(f), \mathcal{M}\big)$ induces an isomorphism

$$\Gamma(X, \mathcal{M})_f \to \Gamma\big(D_X(f), \mathcal{M}\big).$$

Hint: you can argue as in the proof of Proposition 2.3.15.

8.5. Locally free sheaves

Let X be an algebraic variety. Recall that an $\mathcal{O}_X$-module $\mathcal{M}$ is locally free if there is an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{M}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r_i}$ for all i. If $r_i = r$ for all i, then we say that $\mathcal{F}$ *has rank* r. Note that a locally free $\mathcal{O}_X$-module $\mathcal{M}$ is coherent: this is an immediate consequence of assertion i) in Proposition 8.4.4 and Example 8.4.2. We note that if X is connected and $\mathcal{M}$ is locally free, then it has a well-defined rank. Indeed, for every r, the set of points $x \in X$ such that $\mathcal{M}|_U \simeq \mathcal{O}_U^{\oplus r}$ in some neighborhood U of x, is open in X. Since these sets are disjoint and X is connected, it follows that all but one of them is empty.

PROPOSITION 8.5.1. *If X is an affine variety with $A = \mathcal{O}(X)$, and M is a finitely generated A-module, then the following assertions are equivalent:*

i) *The coherent sheaf $\widetilde{M}$ is locally free.*
ii) *For every prime ideal $\mathfrak{p}$ in A, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free.*
iii) *For every maximal ideal $\mathfrak{p}$ in A, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free.*
iv) *The A-module M is projective.*

PROOF. Note first that M is projective if and only if $M_{\mathfrak{p}}$ is projective for every prime (maximal) ideal $\mathfrak{p}$ in A. Indeed, if we consider a surjective morphism $\phi\colon F \to M$, with F free, ϕ splits if and only if M is projective. On the other hand, we claim that ϕ splits if and only if it splits after localizing at every prime (maximal) ideal: the splitting condition is equivalent with the fact that the induced morphism

$$\mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(M, M)$$

is surjective, and since M is finitely generated over a Noetherian ring, for every prime ideal $\mathfrak{p}$, we have

$$\mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, F_{\mathfrak{p}}) \simeq \mathrm{Hom}_A(M, F) \otimes_A A_{\mathfrak{p}} \quad \text{and}$$

$$\mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, M_{\mathfrak{p}}) \simeq \mathrm{Hom}_A(M, M) \otimes_A A_{\mathfrak{p}}.$$

The claim then follows since a morphism is surjective if and only if it induces a surjective morphism after localizing at each prime (maximal) ideal (see Corollary C.3.4). Note also that a finitely generated module over a local ring is projective if and only if it is free (see Proposition C.2.1). This proves the equivalence of ii), iii), and iv) above. Since the implication i)⇒ii) is clear, it is enough to prove ii)⇒i).

Suppose that $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for some prime ideal $\mathfrak{p}$. We can choose a basis of $M_{\mathfrak{p}}$ of the form $\frac{u_1}{1}, \ldots, \frac{u_r}{1} \in M_{\mathfrak{p}}$. If we consider the morphism

$$\phi\colon A^{\oplus r} \to M, \quad \phi(e_i) = u_i,$$

then $\phi_{\mathfrak{p}}$ is an isomorphism. Since both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are finitely generated, this implies that there is $f \not\in \mathfrak{p}$ such that $\ker(\phi)_f = 0 = \mathrm{coker}(\phi)_f$, hence ϕ induces an isomorphism after tensoring with A_f. We thus have

$$\widetilde{M}|_{D_X(f)} \simeq \mathcal{O}_{D_X(f)}^{\oplus r}.$$

□

REMARK 8.5.2. The proof of the implication ii)⇒i) in the above proposition gives the following stronger statement: if $\mathcal{F}$ is a coherent sheaf on the algebraic variety X and $V \subseteq X$ is an irreducible closed subset such that $\mathcal{F}_V \simeq \mathcal{O}_{X,V}^{\oplus r}$, then there is an open subset U of X such that $U \cap V \neq \emptyset$ and $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$. In particular, the *locally free locus* of $\mathcal{F}$

$$\{x \in X \mid \mathcal{F}_x \text{ is free over } \mathcal{O}_{X,x}\}$$

is an open subset of X and for an irreducible, closed subset V of X, the $\mathcal{O}_{X,V}$-module $\mathcal{F}_V$ is free if and only if V intersects the locally free locus of $\mathcal{F}$.

EXAMPLE 8.5.3. Given any coherent sheaf $\mathcal{F}$ on the variety X, if $X_1, \ldots, X_r$ are the irreducible components of X, then each local ring $\mathcal{O}_{X,X_i} = k(X_i)$ is a field, hence $\mathcal{F}_{X_i}$ is a free $\mathcal{O}_{X,X_i}$-module. We deduce from the previous remark that there is an open dense subset U of X such that $\mathcal{F}|_U$ is locally free.

DEFINITION 8.5.4. For every coherent sheaf $\mathcal{F}$ on X and for every $x \in X$, the *fiber* of $\mathcal{F}$ at x is

$$\mathcal{F}_{(x)} := \mathcal{F}_x/\mathfrak{m}\mathcal{F}_x,$$

where $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is the maximal ideal in $\mathcal{O}_{X,x}$. Note that if $i_x \colon \{x\} \hookrightarrow X$ is the inclusion, we have

$$\mathcal{F}_{(x)} \simeq i_x^*(\mathcal{F})$$

(where we identify, as usual, a sheaf over a point with its global sections). It is clear that $\mathcal{F}_{(x)}$ is a finite-dimensional k-vector space, whose dimension is equal to the minimal number of generators of $\mathcal{F}_x$ by Nakayama's lemma (see Remark C.1.3).

More generally, if V is an irreducible closed subset of X, we put

$$\mathcal{F}_{(V)} := \mathcal{F}_V/\mathfrak{m}\mathcal{F}_V,$$

where $\mathfrak{m} \subseteq \mathcal{O}_{X,V}$ is the maximal ideal. This is a finite-dimensional vector space over $k(V)$.

The definition is functorial: given a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$, for every irreducible closed subset V of X, we have an induces $k(V)$-linear map

$$\mathcal{F}_{(V)} \to \mathcal{G}_{(V)}.$$

EXERCISE 8.5.5. Show that if $f \colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{F}$ is a coherent sheaf on Y, then for every point $x \in X$, we have a canonical isomorphism

$$f^*(\mathcal{F})_{(x)} \simeq \mathcal{F}_{(f(x))}.$$

The following criterion for a coherent sheaf to be locally free is often useful.

PROPOSITION 8.5.6. *Given a coherent sheaf $\mathcal{F}$ on X, the following are equivalent:*

i) *The sheaf $\mathcal{F}$ is locally free, of rank r.*
ii) *For every irreducible, closed subset V of X, we have $\dim_{k(V)} \mathcal{F}_{(V)} = r$.*
iii) *For every $x \in X$, we have $\dim_k \mathcal{F}_{(x)} = r$.*

PROOF. If $\mathcal{F}$ is locally free, of rank r, then for every irreducible closed subset V, we can find an open subset U, with $U \cap V \neq \emptyset$, such that $\mathcal{F}|_V \simeq \mathcal{O}_V^{\oplus r}$, hence $\mathcal{F}_V \simeq \mathcal{O}_{X,V}^{\oplus r}$, and thus $\dim_{k(V)} \mathcal{F}_{(V)} = r$. Therefore it is enough to show that if $\dim_k \mathcal{F}_{(x)} = r$ for every $x \in X$, then $\mathcal{F}$ is locally free, of rank r. Given $x \in X$, we need to find an open neighborhood U of x such that $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$. After replacing X by an affine neighborhood of x, we may assume that X is affine, with $A = \mathcal{O}_X(X)$, and $\mathcal{F} = \widetilde{M}$, for some finitely generated A-module M. If $\mathfrak{m}$ is the maximal ideal corresponding to x, it follows by assumption that we have a surjective morphism

$$A_{\mathfrak{m}}^{\oplus r} \to M_{\mathfrak{m}}.$$

After replacing X by $D_X(f)$, for some $f \not\in \mathfrak{m}$, we may assume that we have a surjective morphism

$$\phi \colon A^{\oplus r} \to M,$$

and let N be its kernel. For every maximal ideal $\mathfrak{n}$ of A, the minimal number of generators of $M_{\mathfrak{n}}$ is r, hence after tensoring the short exact sequence

$$0 \to N_{\mathfrak{n}} \to A_{\mathfrak{n}}^{\oplus r} \to M_{\mathfrak{n}} \to 0$$

with $A_\mathfrak{n}/\mathfrak{n}A_\mathfrak{n}$, we see that $N_\mathfrak{n} \subseteq \mathfrak{n} \cdot A_\mathfrak{n}^{\oplus r}$. This implies that

$$N \subseteq \bigcap_{\mathfrak{n}} \mathfrak{n} \cdot A^{\oplus r} = 0,$$

where we use the fact that the intersection of all maximal ideals in A is 0. $\square$

PROPOSITION 8.5.7. *Given an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of $\mathcal{O}_X$-modules, with $\mathcal{F}''$ locally free, for every $\mathcal{O}_X$-module $\mathcal{G}$, the sequence

$$0 \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}'' \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. In particular, for every $x \in X$, we have an exact sequence of k-vector spaces:

$$0 \to \mathcal{F}'_{(x)} \to \mathcal{F}_{(x)} \to \mathcal{F}''_{(x)} \to 0.$$

PROOF. It is enough to show that for every $x \in X$ the induced sequence of stalks

$$0 \to \mathcal{F}'_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \to \mathcal{F}''_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \to 0$$

is exact. However, by assumption we know that we have an exact sequence

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0;$$

moreover, this is split, since $\mathcal{F}''_x$ is a free $\mathcal{O}_{X,x}$-module. This implies that by tensoring with $\mathcal{G}_x$, the sequence is again split exact. The last assertion about fibers follows by taking $\mathcal{G}$ to be the sheaf $k(x)$. $\square$

COROLLARY 8.5.8. *Given an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on X, with $\mathcal{F}''$ locally free, then $\mathcal{F}'$ is locally free if and only if $\mathcal{F}''$ is locally free. Moreover, if two of $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ have a well-defined rank, then the third one does, and

$$\operatorname{rank}(\mathcal{F}') + \operatorname{rank}(\mathcal{F}'') = \operatorname{rank}(\mathcal{F}).$$

PROOF. After replacing X by each of its connected components, we may assume that X is connected. By the proposition, for every $x \in X$ we have a short exact sequence

$$0 \to \mathcal{F}'_{(x)} \to \mathcal{F}_{(x)} \to \mathcal{F}''_{(x)} \to 0,$$

hence

$$\dim_k \mathcal{F}_{(x)} = \dim_k \mathcal{F}'_{(x)} + \dim_k \mathcal{F}''_{(x)}.$$

The assertion then follows from Proposition 8.5.6. $\square$

We will use the term *vector bundle* on X as a synonym for *locally free sheaf.* A *line bundle* is a locally free sheaf of rank 1. A vector bundle of rank r is *trivial* if it is isomorphic to $\mathcal{O}_X^{\oplus r}$.

DEFINITION 8.5.9. Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$, a morphism of sheaves $\mathcal{E} \to \mathcal{F}$ is a *morphism of vector bundles* if the map

$$x \to \operatorname{rank}\big(E_{(x)} \to F_{(x)}\big)$$

is constant on each connected component of X.

PROPOSITION 8.5.10. *If* $\phi\colon \mathcal{E} \to \mathcal{F}$ *is a morphism of vector bundles, then* $\operatorname{coker}(\phi)$*,* $\operatorname{Im}(\phi)$*, and* $\ker(\phi)$ *are vector bundles.*

PROOF. By right exactness of the tensor product, for every $x \in X$, we have an exact sequence

$$\mathcal{E}_{(x)} \to \mathcal{F}_{(x)} \to \operatorname{coker}(\phi)_x \to 0$$

and using the fact that ϕ is a morphism of vector bundles, we conclude that the map $x \to \dim_k \operatorname{coker}(\phi)_x$ is constant on the connected components of X. Therefore $\operatorname{coker}(\phi)$ is locally free by Proposition 8.5.7. Using the exact sequences

$$0 \to \operatorname{Im}(\phi) \to \mathcal{F} \to \operatorname{coker}(\phi) \to 0$$

and

$$0 \to \ker(\phi) \to \mathcal{E} \to \operatorname{Im}(\phi) \to 0,$$

we deduce using Corollary 8.5.7 first that $\operatorname{Im}(\phi)$ is locally free and then that $\ker(\phi)$ is locally free. □

EXERCISE 8.5.11. Given a vector bundle $\mathcal{E}$, a *subbundle* of $\mathcal{E}$ is a subsheaf $\mathcal{F}$ of $\mathcal{E}$, which is a vector bundle, and such that the inclusion map $\mathcal{F} \hookrightarrow \mathcal{E}$ is a morphism of vector bundles. Show that a subsheaf $\mathcal{F}$ of $\mathcal{E}$ is a subbundle if and only if it is a vector bundle and for all $x \in X$, the induced map $\mathcal{F}_{(x)} \to \mathcal{E}_{(x)}$ is injective.

EXAMPLE 8.5.12. Note that the composition of two morphisms of vector bundles might not be a morphism of vector bundles. Suppose for example that $X = \mathbf{A}^1$, with corresponding ring $k[x]$. The morphisms

$$k[x] \xrightarrow{\phi} k[x]^{\oplus 2} \xrightarrow{\psi} k[x]$$

given by

$$\phi(f) = \big(xf, (1-x)f\big) \quad \text{and} \quad \psi(f,g) = f$$

give morphisms of vector bundles, but the composition does not.

REMARK 8.5.13. Several of the operations that we defined preserve locally free sheaves:

i) If $\mathcal{M}_1, \ldots, \mathcal{M}_r$ are locally free sheaves on X (with $\operatorname{rank}(\mathcal{M}_i) = m_i$), then $\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_r$ is locally free (of rank $m_1 + \ldots + m_r$).
ii) If $\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves on X (of ranks e and f, respectively), then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free (of rank ef).
iii) If $\mathcal{E}$ is a locally free sheaf on X (of rank m), then $\operatorname{Sym}^p(\mathcal{E})$ and $\wedge^p \mathcal{E}$ are locally free, for every $p \geq 0$ (with $\operatorname{rank}\big(\operatorname{Sym}^p(\mathcal{E})\big) = \binom{m+p-1}{m-1}$ and $\operatorname{rank}\big(\wedge^p \mathcal{E}\big) = \binom{m}{p}$, with the convention that this is 0 for $p > m$).

All assertions follow, for example, by considering the stalks, using Proposition 8.5.1, and the fact that all these operations take free modules to free modules of corresponding ranks.

DEFINITION 8.5.14. For every coherent sheaf $\mathcal{E}$ on X, we define its *dual*

$$\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

REMARK 8.5.15. It follows from Proposition 8.4.4 that if $\mathcal{E}$ is coherent, then $\mathcal{E}^\vee$ is coherent. The map taking $\mathcal{E}$ to $\mathcal{E}^\vee$ thus gives a contravariant functor from the category of coherent sheaves on X to itself. Moreover, if $\mathcal{E}$ is locally free (of rank r), then $\mathcal{E}^\vee$ has the same property: if $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$, then

$$\mathcal{E}^\vee|_U \simeq \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U) \simeq \mathcal{O}_U^r.$$

REMARK 8.5.16. For every coherent sheaves $\mathcal{E}$ and $\mathcal{F}$, given $(\phi, s) \in \mathcal{E}^\vee(U) \times \mathcal{F}(U)$, we obtain a morphism $\mathcal{E}|_U \to \mathcal{F}|_U$ that maps $a \in \mathcal{E}(V)$ to $\phi_V(a) \cdot s|_V$. This induces a natural morphism

$$\mathcal{E}^\vee \otimes \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}). \tag{8.5.1}$$

It is straightforward to see that this is an isomorphism if either $\mathcal{E}$ or $\mathcal{F}$ is locally free; in particular, if both $\mathcal{E}$ and $\mathcal{F}$ are locally free, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free.

REMARK 8.5.17. For every coherent sheaf $\mathcal{E}$, we have a canonical morphism of sheaves $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ corresponding to the identity morphism $1_E \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. If $\mathcal{E}$ is a line bundle, this is an isomorphism (consider the induced morphism at the level of stalks). By combining this with the previous remark, we thus see that if $\mathcal{E}$ is a line bundle, then we have a canonical isomorphism

$$\mathcal{O}_X \simeq \mathcal{E} \otimes \mathcal{E}^\vee.$$

REMARK 8.5.18. For every coherent sheaf $\mathcal{E}$, we have a canonical morphism of $\mathcal{O}_X$-modules

$$\mathcal{E} \to (\mathcal{E}^\vee)^\vee \tag{8.5.2}$$

that maps $s \in \mathcal{E}(U)$ to the morphism $\mathrm{ev}_s \colon \mathcal{H}om_{\mathcal{O}_U}(\mathcal{E}|_U, \mathcal{O}_U) \to \mathcal{O}_U$, that for an open subset $V \subseteq U$, maps $\phi \in \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{E}|_V, \mathcal{O}_V)$ to $\phi_V(s|_V)$. Of course, if U is an affine open subset such that $\mathcal{E}|_U \simeq \widetilde{M}$, where M is a module over $A = \mathcal{O}_X(U)$, then the above morphism is induced by the canonical morphism of A-modules

$$M \to \mathrm{Hom}_A\big(\mathrm{Hom}(M, A), A\big).$$

This is an isomorphism if M is a finitely generated, free A-module, which implies that (8.5.2) is an isomorphism if $\mathcal{E}$ is locally free.

REMARK 8.5.19. Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

with $\mathcal{F}''$ locally free, and a coherent sheaf $\mathcal{G}$, the induced sequences

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}') \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}'') \to 0$$

and

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to 0$$

are exact. Indeed, on any affine open subset, the original exact sequence corresponds to an exact sequence of modules; since the third module is projective, the sequence is split. It follows that after applying either of the functors $\mathrm{Hom}(G, -)$ or $\mathrm{Hom}(-, G)$, the resulting sequence is still exact.

An important way to describe a vector bundle of rank r is via transition functions. Suppose that $\mathcal{E}$ is locally free, of rank r, and let us choose a finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of X and *trivializations* (that is, isomorphisms) $\phi_i \colon \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$. In this case, for every $i, j \in I$, we have isomorphisms

$$\phi_{i,j} = \phi_i \circ \phi_j^{-1} \colon \mathcal{O}_{U_i \cap U_j}^{\oplus r} \to \mathcal{O}_{U_i \cap U_j}^{\oplus r}$$

that satisfy the following compatibility relations (known as *cocycle condition*):

i) $\phi_{i,i} = \mathrm{Id}$ for all $i \in U$.
ii) $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ on $U_i \cap U_j \cap U_k$, for all $i, j, k \in I$.

Note that a morphism of sheaves $\mathcal{O}_{U_i\cap U_j}^{\oplus r} \to \mathcal{O}_{U_i\cap U_j}^{\oplus r}$ is given by a matrix $a \in M_r\big(\mathcal{O}_X(U_i \cap U_j)\big)$. This is an isomorphism if and only if the matrix is invertible, that is, $\det(a) \in \mathcal{O}_X(U_i \cap U_j)$ is invertible. The isomorphisms $(\phi_{i,j})_{i,j\in I}$ (or the corresponding matrices $(a_{i,j})_{i,j\in I}$) are the *transition functions* associated to this cover and choice of trivializations. In fact, these transition functions can be used in order to identify the isomorphism class of a vector bundle, as follows.

Given a finite open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, a *refinement* of $\mathcal{U}$ is given by another such cover $\mathcal{V} = (V_j)_{j\in J}$, together with a map $\alpha\colon J \to I$, such that $V_j \subseteq U_{\alpha(j)}$ for all $j \in J$. Given a family of invertible matrices a_{i_1,i_2} that satisfy the cocycle condition and a refinement $\mathcal{V} = (V_j)_{j\in J}$ with $\alpha\colon J \to I$, as above, we obtain a new family of matrices $a_{j_1,j_2} = a_{\alpha(j_1),\alpha(j_2)}|_{V_{j_1}\cap V_{j_2}}$, which again satisfies the cocycle condition. We will refer to this operation as *passing to a refinement*. Given two families of matrices that satisfy the cocycle condition, corresponding to two covers $\mathcal{U}_1$ and $\mathcal{U}_2$, in order to compare them we may always pass to a common refinement $\mathcal{V}$.

Exercise 8.5.20. Let X be an algebraic variety.

i) Show that given a finite open cover $\mathcal{U} = (U_i)_{i\in I}$ and a family of invertible matrices $a_{i,j} \in M_r\big(\mathcal{O}_X(U_i \cap U_j)\big)$, for $i, j \in I$, that satisfy the cocycle condition, there is a vector bundle $\mathcal{E}$ of rank r, unique up to a canonical isomorphism, with associated transition functions with respect to the cover $\mathcal{U}$ given by $(a_{i,j})_{i,j\in I}$.
ii) Given two such families of matrices a and a', corresponding to possibly two different covers, the corresponding vector bundles are isomorphic if and only if they are *cohomologous*, in the following sense: after passing to a common refinement, both families of matrices are taken with respect to the same cover $\mathcal{U} = (U_i)_{i\in I}$, and for every $i \in I$, there is an invertible matrix $b_i \in M_r\big(\mathcal{O}_X(U_i)\big)$ such that

$$a_{i,j} = a'_{i,j} \cdot b_i \cdot b_j^{-1} \quad \text{on} \quad U_i \cap U_j, \quad \text{for all} \quad i, j \in I.$$

Example 8.5.21. Let us denote by $\mathcal{O}_X^*$ the subsheaf of $\mathcal{O}_X$ such that $\mathcal{O}_X^*(U)$ consists of the invertible elements in $\mathcal{O}_X(U)$ (hence $\mathcal{O}_X^*$ is a sheaf of Abelian groups with respect to multiplication). A line bundle $\mathcal{L}$ is described by a finite open cover $\mathcal{U} = (U_i)_{i\in I}$ of X and a family of functions $a_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j)$, which satisfy the cocycle condition. Note that for a positive integer m, the line bundle $\mathcal{L}^{\otimes m}$ is described by the family $(a_{i,j}^m)_{i,j\in I}$, while the dual $\mathcal{L}^\vee$ is described by the family $(a_{i,j}^{-1})_{i,j\in I}$. Given such a family of transition functions, the space of global sections $\Gamma(X, \mathcal{L})$ is isomorphic to

$$\left\{(f_i)_i \in \prod_{i\in I} \mathcal{O}_X(U_i) \mid a_{i,j} f_j = f_i \text{ on } U_i \cap U_j \text{ for all } i, j \in I\right\}.$$

We end this section by introducing an important invariant of an algebraic variety, its *Picard group*.

Definition 8.5.22. Let X be an algebraic variety. The *Picard group* of X, denoted $\mathrm{Pic}(X)$, is the set of isomorphism classes of line bundles on X, with the multiplication given by tensor product. Since the tensor product is associative and commutative, the operation on $\mathrm{Pic}(X)$ satisfies these two properties. Moreover, we have an identity element given by (the isomorphism class of) $\mathcal{O}_X$, and every $\mathcal{L}$ has an inverse given by $\mathcal{L}^{-1} := \mathcal{L}^\vee$ (see Remark 8.5.17). We thus see that $\mathrm{Pic}(X)$ is an Abelian group.

REMARK 8.5.23. For every morphism of algebraic varieties $f\colon X \to Y$ and every locally free sheaf $\mathcal{E}$ on Y, the pull-back $f^*(\mathcal{E})$ is locally free on X (see Example 8.2.16). Moreover, since for every $x \in X$, we have a canonical isomorphism

$$f^*(\mathcal{E})_{(x)} \simeq \mathcal{E}_{(f(x))},$$

it follows that the pull-back of a morphism of vector bundles is again a morphism of vector bundles.

In particular, we obtain a group homomorphism

$$f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X), \quad \mathcal{L} \to f^*(\mathcal{L}).$$

Note that this is compatible with composition of morphisms: if $g\colon Y \to Z$ is another morphism of algebraic varieties, then $(g \circ f)^* = f^* \circ g^*$ (see Remark 8.2.20).

EXAMPLE 8.5.24. If $\mathcal{E}$ is a vector bundle on X, then the *determinant* $\det(\mathcal{E})$ of $\mathcal{E}$ is obtained by taking on each connected component of X, the top exterior power of $\mathcal{E}$. This is a line bundle on X.

REMARK 8.5.25. If $\mathcal{E}$ is a vector bundle of rank r on X, then we have a morphism of $\mathcal{O}_X$-modules

$$\wedge^p \mathcal{E} \otimes_{\mathcal{O}_X} \wedge^{r-p}\mathcal{E} \to \wedge^r \mathcal{E} = \det(\mathcal{E}), \quad u \otimes v \to u \wedge v.$$

By considering local trivializations, we see that this induces an isomorphism

$$\wedge^p(\mathcal{E}^\vee) \simeq (\wedge^p \mathcal{E})^\vee \simeq \wedge^{r-p}\mathcal{E} \otimes_{\mathcal{O}_X} \det(\mathcal{E})^{-1}.$$

At this point we don't have the tools to compute the Picard group in any nontrivial examples. We will return to this topic in the next chapter, after discussing divisors.

EXERCISE 8.5.26. Let $\mathcal{E}$ be a locally free sheaf on the algebraic variety Y.

i) Show that if $f\colon X \to Y$ is a dominant morphism, then the map

$$\Gamma(Y, \mathcal{E}) \to \Gamma\big(X, f^*(\mathcal{E})\big)$$

given by pull-back of sections is injective.

ii) In particular, if Y is irreducible, then for every non-empty open subset $V \subseteq Y$, the restriction map

$$\Gamma(Y, \mathcal{E}) \to \Gamma(V, \mathcal{E})$$

is injective.

EXERCISE 8.5.27. Show that if $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is *invertible* (that is, there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{O}_X$) if and only if $\mathcal{F}$ is a line bundle.

EXERCISE 8.5.28. Show that if $f\colon X \to Y$ is a morphism of algebraic varieties, then for every locally free sheaf $\mathcal{E}$ on Y and for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a canonical isomorphism

$$f_*\big(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}\big) \simeq \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$$

(this is known as the *projection formula*).

EXERCISE 8.5.29. Consider an exact sequence of vector bundles on the algebraic variety X:

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

i) For every $p \geq 0$ and for every $0 \leq i \leq p$, let $\mathcal{F}_i$ be the image of the composition

$$S^i(\mathcal{E}') \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}) \to S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}) \to S^p(\mathcal{E}).$$

Show that for every $0 \leq i \leq p$, we have a sequence of subbundles of $S^p(\mathcal{E})$

$$0 = \mathcal{F}_{p+1} \hookrightarrow \mathcal{F}_p \hookrightarrow \mathcal{F}_{p-1} \hookrightarrow \ldots \hookrightarrow \mathcal{F}_0 = S^p(\mathcal{E})$$

and we have canonical isomorphisms

$$\mathcal{F}_i/\mathcal{F}_{i+1} \simeq S^i(\mathcal{E}') \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}'') \quad \text{for} \quad 0 \leq i \leq p.$$

ii) Similarly, show that for every $p \geq 0$ and $0 \leq i \leq p$, we have a sequence of subbundles

$$0 = \mathcal{G}_{p+1} \hookrightarrow \mathcal{G}_p \hookrightarrow \ldots \hookrightarrow \mathcal{G}_0 = \wedge^p \mathcal{E}$$

such that we have canonical isomorphisms

$$\mathcal{G}_i/\mathcal{G}_{i+1} \simeq \wedge^i \mathcal{E}' \otimes_{\mathcal{O}_X} \wedge^{p-i} \mathcal{E}'' \quad \text{for} \quad 0 \leq i \leq p.$$

In particular, we have a canonical isomorphism

$$\det(\mathcal{E}) \simeq \det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'').$$

8.6. The $\mathcal{M}axSpec$ and $\mathcal{M}axProj$ constructions

In this section we use quasi-coherent sheaves in order to globalize the MaxSpec and MaxProj constructions. This will allow us to describe varieties that are affine or projective over another variety. We also discuss one important class of affine morphisms, the geometric vector bundles.

8.6.1. Affine morphisms and quasi-coherent sheaves of algebras. Let X be an algebraic variety. All $\mathcal{O}_X$-algebras considered in this subsection will be commutative.

DEFINITION 8.6.1. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is *quasi-coherent* or *coherent* if it is so with respect to the induced $\mathcal{O}_X$-module structure. A *finitely generated* quasi-coherent $\mathcal{O}_X$-algebra is a quasi-coherent $\mathcal{O}_X$-algebra such that for every affine open subset U of X, the $\mathcal{O}_X(U)$-algebra $\mathcal{A}(U)$ is finitely generated. Similarly, a quasi-coherent $\mathcal{O}_X$-algebra is *reduced* if $\mathcal{A}(U)$ is a reduced ring for every affine open subset U.

REMARK 8.6.2. Arguing as in the proof of Proposition 2.3.16, we see that if $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, in order to check that it is finitely generated, it is enough to find an affine open cover $X = U_1 \cup \ldots \cup U_n$ such that $\mathcal{A}(U_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-algebra for all i. Note that since $\mathcal{O}_X(U_i)$ is a finitely generated k-algebra, then $\mathcal{A}(U_i)$ is finitely generated as an $\mathcal{O}_X(U_i)$-algebra if and only if it is finitely generated as a k-algebra.

EXERCISE 8.6.3. Show that if $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-algebra, then the following are equivalent:

i) For every open subset $U \subseteq X$, the ring $\mathcal{A}(U)$ is reduced.
ii) The $\mathcal{O}_X$-algebra $\mathcal{A}$ is reduced.

iii) There is an affine open cover $X = U_1 \cup \ldots \cup U_r$ such that $\mathcal{A}(U_i)$ is reduced for all i.

Consider a morphism of algebraic varieties $f\colon Y \to X$. The canonical morphism $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ makes $f_*(\mathcal{O}_Y)$ an $\mathcal{O}_X$-algebra, which is quasi-coherent by Proposition 8.4.5. Moreover, it is clearly reduced. If f is an affine morphism, then $f_*(\mathcal{O}_Y)$ is a finitely generated $\mathcal{O}_X$-algebra: for every affine open subset $U \subseteq X$, the inverse image $f^{-1}(U)$ is an affine variety, hence

$$\Gamma\big(U, f_*(\mathcal{O}_Y)\big) = \Gamma\big(f^{-1}(U), \mathcal{O}_Y\big)$$

is a finitely generated k-algebra, and therefore a finitely generated $\mathcal{O}_X(U)$-algebra.

Note that if $g\colon Z \to X$ is another variety over X, then for every morphism $\phi\colon Z \to Y$ of varieties over X (recall that this means that we have $f \circ \phi = g$), we have a canonical morphism $\mathcal{O}_Y \to \phi_*(\mathcal{O}_Z)$, and by pushing this forward via f, we get a canonical morphism of $\mathcal{O}_X$-algebras

$$f_*(\mathcal{O}_Y) \to f_*\big(\phi_*(\mathcal{O}_Z)\big) = g_*(\mathcal{O}_Z).$$

We get in this way a contravariant functor Φ from the category of varieties over X that are affine over X (that is, the structure morphism to X is affine) to the category of finitely generated, reduced, quasi-coherent $\mathcal{O}_X$-algebras.

Our next goal is to construct an inverse functor. Given a finitely generated, reduced, quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$, we will construct an algebraic variety $\mathcal{M}axSpec(\mathcal{A})$, together with an affine morphism

$$\pi_X\colon \mathcal{M}axSpec(\mathcal{A}) \to X.$$

If X is an affine variety, then $\mathcal{A}(X)$ is a finitely generated $\mathcal{O}_X(X)$-algebra, and therefore a finitely generated k-algebra; the variety we consider is $\mathrm{MaxSpec}\big(\mathcal{A}(X)\big)$ and the morphism π_X is the one corresponding to the canonical homomorphism $\mathcal{O}_X(X) \to \mathcal{A}(X)$. We note that if U is an affine open subset of X, then the following commutative diagram

$$\begin{tikzcd}
\mathrm{MaxSpec}\big(\mathcal{A}(U)\big) \arrow[r, "i"] \arrow[d, "\pi_U"'] & \mathrm{MaxSpec}\big(\mathcal{A}(X)\big) \arrow[d, "\pi_X"] \\
U \arrow[r, "j"] & X
\end{tikzcd}$$

is Cartesian, where j is the inclusion and i corresponds to the ring homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$. In other words, the canonical morphism of varieties over U

$$\phi\colon \mathrm{MaxSpec}\big(\mathcal{A}(U)\big) \to \pi_X^{-1}(U)$$

is an isomorphism. This is clear if $U = D_X(f)$ is a principal affine open subset of X, since in this case the canonical homomorphism $\mathcal{A}(X)_f \to \mathcal{A}(U)$ is an isomorphism. In the general case, we write $U = U_1 \cup \ldots \cup U_r$, with $U_i = D_X(f_i)$ for $1 \leq i \leq r$. Note that since $U_i = D_U(f_i|_U)$, it follows that we have an isomorphism

$$\phi^{-1}\big(\pi_X^{-1}(U_i)\big) = \pi_U^{-1}(U_i) \simeq \mathrm{MaxSpec}\big(\mathcal{A}(U_i)\big)$$

such that the restriction of ϕ to $\phi^{-1}\big(\pi_X^{-1}(U_i)\big)$ gets identified to the canonical morphism

$$\mathrm{MaxSpec}\big(\mathcal{A}(U_i)\big) \to \pi_X^{-1}(U_i)$$

of varieties over U_i, which we have seen that it is an isomorphism. Therefore ϕ is an isomorphism.

Suppose now that X is an arbitrary variety. For every affine open subset $U \subseteq X$, we consider the morphism

$$\pi_U : \mathrm{MaxSpec}\big(\mathcal{A}(U)\big) \to U.$$

If V is another affine open subset, then $U \cap V$ is again an affine open subset (see Proposition 2.5.5) and the above discussion shows that we have a canonical isomorphism

$$\phi_{U,V} : \pi_U^{-1}(U \cap V) \to \pi_V^{-1}(U \cap V).$$

It is straightforward to check that these isomorphisms satisfy the required compatibilities such that by Exercise 2.3.12 we obtain a prevariety $Y = \mathcal{M}ax\mathcal{S}pec(\mathcal{A})$. Moreover, we can glue the morphisms π_U to a morphism $\pi_X : Y \to X$ such that for every affine open subset $U \subseteq X$, we have an isomorphism

$$\pi_X^{-1}(U) \simeq \mathrm{MaxSpec}\big(\mathcal{A}(U)\big).$$

Note that $\mathcal{M}ax\mathcal{S}pec(\mathcal{A})$ is a variety: since X is a variety, it follows from Proposition 2.5.14 that it is enough to show that π_X is a separated morphism. This is a consequence of the fact that for every affine open subset U of X, the inverse image $\pi_X^{-1}(U)$ is affine, hence separated (see Example 2.5.13).

It is clear that π_X is an affine morphism. Moreover, it follows from the construction that

$$(\pi_X)_*(\mathcal{O}_Y) \simeq \mathcal{A}.$$

EXERCISE 8.6.4. Show that if U is an open subset of X, then we have an isomorphism of varieties over U

$$\mathcal{M}ax\mathcal{S}pec(\mathcal{A}|_U) \simeq (\pi_X)^{-1}(U).$$

If $\phi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of finitely generated, reduced $\mathcal{O}_X$-algebras, then for every affine open subset $U \subseteq X$, we have a morphism of finitely generated $\mathcal{O}_X(U)$-algebras $\mathcal{A}(U) \to \mathcal{B}(U)$, inducing a morphism

$$\mathrm{MaxSpec}\big(\mathcal{B}(U)\big) \to \mathrm{MaxSpec}\big(\mathcal{A}(U)\big)$$

over U. These glue together to give a morphism

$$\mathcal{M}ax\mathcal{S}pec(\mathcal{B}) \to \mathcal{M}ax\mathcal{S}pec(\mathcal{A})$$

of varieties over X. We thus obtain a contravariant functor $\mathcal{M}ax\mathcal{S}pec$ from the category of finitely generated, reduced, quasi-coherent $\mathcal{O}_X$-algebras to the category of varieties over X that are affine over X.

EXERCISE 8.6.5. Show that that the two functors Φ and $\mathcal{M}ax\mathcal{S}pec$ give inverse anti-equivalences of categories.

REMARK 8.6.6. If $\mathcal{A}$ is a reduced, finitely generated, quasi-coherent $\mathcal{O}_X$-algebra, then the morphism $\mathcal{M}ax\mathcal{S}pec(\mathcal{A}) \to X$ is finite if and only if $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module.

Consider an arbitrary variety over X given by $g\colon Z \to X$. Given a morphism $f\colon Z \to Y = \mathcal{M}ax\mathcal{S}pec(\mathcal{A})$ of varieties over X, we have a canonical morphism $\mathcal{O}_Y \to f_*(\mathcal{O}_Z)$, and by pushing forward via π_X, a morphism of $\mathcal{O}_X$-algebras

$$\mathcal{A} \simeq (\pi_X)_*(\mathcal{O}_Y) \to (\pi_X)_*\big(f_*(\mathcal{O}_Z)\big) = g_*(\mathcal{O}_Z).$$

REMARK 8.6.7. For every variety over X given by $g\colon Z \to X$, the above map

$$\mathrm{Hom}_{\mathrm{Var}/X}\big(Z, \mathcal{M}axSpec(\mathcal{A})\big) \to \mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}\big(\mathcal{A}, g_*(\mathcal{O}_Z)\big) \tag{8.6.1}$$

is a bijection. Note that if X is a point, then this is just Proposition 2.3.14. The proof in the general case follows by covering X by affine open subsets and using Proposition 2.3.14. We leave the details as an exercise for the reader. We note that by the adjoint property of g^* and g_*, the set on the right-hand side of (8.6.1) is in natural bijection with $\mathrm{Hom}_{\mathcal{O}_Z-\mathrm{alg}}\big(g^*(\mathcal{A}), \mathcal{O}_Z\big)$.

EXERCISE 8.6.8. Let X be an algebraic variety. If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-algebra, then a *quasi-coherent $\mathcal{A}$-module* is an $\mathcal{A}$-module, which is quasi-coherent when considered with the induced $\mathcal{O}_X$-module structure. Show that if $f\colon Y \to X$ is an affine morphism, then by mapping a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$ to $f_*(\mathcal{M})$, we obtain an equivalence of categories between the quasi-coherent sheaves on Y and the quasi-coherent $f_*(\mathcal{O}_Y)$-modules on X.

8.6.2. Locally free sheaves vs. geometric vector bundles. We now introduce the geometric counterpart for the notion of locally free sheaf. Let X be an algebraic variety.

DEFINITION 8.6.9. A geometric vector bundle on X is a variety over X

$$\pi\colon E \to X$$

such that each fiber $E(x) := \pi^{-1}(x)$ has a structure of vector space over k and E is *locally trivial* over X in the following sense: there is an open cover $X = \bigcup_{i\in I} U_i$ and for every i an isomorphism of varieties over U_i:

$$\phi_i\colon \pi^{-1}(U_i) \simeq U_i \times k^{r_i}$$

with the property that for every $x \in U_i$, the induced isomorphism $E(x) \to k^{r_i}$ is k-linear. If $\dim_x E(x) = r$ for all $x \in X$, then E *has rank* r.

REMARK 8.6.10. Note that for every geometric vector bundle E and every r, the set of points $x \in X$ such that $\dim_k E(x) = r$ is open in X. Since for different values of r, we get disjoint sets, it follows that the map $x \to \dim_k E(x)$ is constant on the connected components of X.

DEFINITION 8.6.11. We consider the category $\mathcal{V}ect(X)$, whose objects are geometric vector bundles on X; the morphisms[1] in this category are the morphisms of varieties over X that are k-linear on fibers over the points $x \in X$. In particular, we can talk about isomorphisms of geometric vector bundles. A vector bundle of rank r is *trivial* if it is isomorphic to $X \times k^r$, with the obvious k-vector space structure on the fibers.

Our main goal is to show that the category of geometric vector bundles is equivalent to the category of locally free sheaves. We now construct two functors between the two categories. Suppose first that E is a geometric vector bundle on X, given by $\pi\colon E \to X$. A *section* of E over an open subset $U \subseteq X$ is a morphism $s\colon U \to E$ such that $\pi\big(s(x)\big) = x$ for every $x \in U$. For every such U, we define $\mathcal{E}(U)$ to consist of all sections of E over U. It is clear that if V is an open subset of U, then we have a map $\mathcal{E}(U) \to \mathcal{E}(V)$ given by restriction and this makes $\mathcal{E}$ a

[1] We do not call these *morphisms of geometric vector bundles*, as that notion has a different meaning, see Remark 8.6.13.

sheaf (of sets, for now). In fact, we can put an $\mathcal{O}_X(U)$-module structure on $\mathcal{E}(U)$ such that $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module: for this, we use the addition and scalar multiplication in each fiber of E to define

$$(s_1 + s_2)(x) = s_1(x) + s_2(x) \quad \text{and} \quad (f \cdot s)(x) = f(x) \cdot s(x)$$

for every $s, s_1, s_2 \in \mathcal{E}(U)$ and $f \in \mathcal{O}_X(U)$. In order to see that as defined s_1+s_2 and fs are morphisms of algebraic varieties, we consider an open cover $X = \bigcup_{i\in I} U_i$ such that we have isomorphisms $\pi^{-1}(U_i) \simeq U_i \times k^r$. These isomorphisms clearly induce bijections

$$\mathcal{E}(U \cap U_i) \simeq \mathcal{O}_X(U \cap U_i)^{\oplus r}$$

that, with the above operations, become isomorphisms of $\mathcal{O}_X(U \cap U_i)$-modules. This easily implies that the operations on $\mathcal{E}(U)$ are well-defined and $\mathcal{E}$ becomes in this way an $\mathcal{O}_X$-module. The fact that it is locally free follows from the fact that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$.

If $f\colon E \to F$ is a morphism in $\mathcal{V}ect(X)$, and $\mathcal{E}$ and $\mathcal{F}$ are the sheaves of sections of E and F, respectively, then we have a morphism of $\mathcal{O}_X$-modules $\mathcal{E} \to \mathcal{F}$ that maps s to $f \circ s$. In this way we have a functor from $\mathcal{V}ect(X)$ to the full subcategory of $\mathcal{C}oh(X)$ consisting of locally free sheaves.

We now construct a functor going in the reverse direction. Given a locally free sheaf $\mathcal{E}$ on X, consider the corresponding symmetric algebra

$$\mathrm{Sym}^\bullet(\mathcal{E}) = \bigoplus_{m\geq 0} \mathrm{Sym}^m(\mathcal{E}).$$

Note that if U is an open subset of X such that $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$, then $\mathrm{Sym}^\bullet\mathcal{E}|_U \simeq \mathcal{O}_U[x_1, \ldots, x_r]$. In particular, we see that $\mathrm{Sym}^\bullet\mathcal{E}$ is a reduced, finitely generated $\mathcal{O}_X$-algebra. The geometric vector bundle associated to $\mathcal{E}$ is

$$\mathbf{V}(\mathcal{E}) := \mathcal{M}axSpec\big(\mathrm{Sym}^\bullet(\mathcal{E})\big) \xrightarrow{\pi} X.$$

If U is an open subset such that

$$\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}, \tag{8.6.2}$$

then we have an isomorphism

$$\pi^{-1}(U) \simeq \mathcal{M}axSpec\big(\mathcal{O}_U[x_1, \ldots, x_r]\big) \simeq U \times \mathbf{A}^r$$

of varieties over U. Using this isomorphism, we can put a k-vector space structure on each fiber of π. It is straightforward to see that this is independent of the choice of isomorphism (8.6.2). We thus see that $\pi\colon \mathbf{V}(\mathcal{E}) \to X$ is a geometric vector bundle on X.

This construction gives a contravariant functor: if $\mathcal{E} \to \mathcal{F}$ is a morphism of locally free sheaves, we get a morphism of $\mathcal{O}_X$-algebras

$$\mathrm{Sym}^\bullet(\mathcal{E}) \to \mathrm{Sym}^\bullet(\mathcal{F})$$

and thus a morphism in $\mathcal{V}ect(X)$

$$\mathbf{V}(\mathcal{F}) \to \mathbf{V}(\mathcal{E}).$$

We claim that the sheaf of sections of $\mathbf{V}(\mathcal{E})$ is canonically isomorphic to $\mathcal{E}^\vee$. Indeed, a section $s\colon U \to E$ is the same as a morphism $U \to E$ of varieties over X, where U is a variety over X via the inclusion $j\colon U \hookrightarrow X$. Using Remark 8.6.7, we can identify this set with the set of morphisms of $\mathcal{O}_U$-algebras

$$\mathrm{Sym}^\bullet(\mathcal{E}|_U) \simeq j^*\big(\mathrm{Sym}^\bullet\mathcal{E}\big) \to \mathcal{O}_U,$$

which is identified via the universal property of the symmetric algebra with the set of morphisms of $\mathcal{O}_U$-modules, $\mathcal{E}|_U \to \mathcal{O}_U$, hence to $\mathcal{E}^\vee|_U$. This proves our claim.

EXERCISE 8.6.12. Show that the functor from $\mathcal{V}ect(X)$ to the full subcategory of $\mathcal{C}oh(X)$ consisting of locally free sheaves, which maps a geometric vector bundle to the corresponding sheaf of sections, is an equivalence of categories; its inverse is the functor mapping $\mathcal{E}$ to $\mathbf{V}(\mathcal{E}^\vee)$.

REMARK 8.6.13. Let $\pi\colon E \to X$ be a geometric vector bundle and $\mathcal{E}$ the corresponding sheaf of sections. Given any $x \in X$ and any open neighborhood U of x, we have a map

$$\mathcal{E}(U) \to E(x), \quad s \to s(x).$$

This induces a map $\mathcal{E}_x \to E(x)$ and finally a k-linear map $\mathcal{E}_{(x)} \to E(x)$. This is an isomorphism, as can be seen by restricting to an open neighborhood U of x such that $\pi^{-1}(U) \simeq U \times k^r$.

This implies that if $\phi\colon \mathcal{E} \to \mathcal{F}$ is a morphism of locally free sheaves, then this is a morphism of vector bundles if and only if the corresponding morphism

$$f\colon \mathbf{V}(\mathcal{E}^\vee) \to \mathbf{V}(\mathcal{F}^\vee)$$

has the property that the map

$$x \to \operatorname{rank}\bigl(\mathbf{V}(\mathcal{E}^\vee)_{(x)} \to \mathbf{V}(\mathcal{F}^\vee)_{(x)}\bigr)$$

is constant on the connected components of X.

EXAMPLE 8.6.14. Given $n \geq 1$, let us consider again the blow-up of $\mathbf{A}^{n+1}$ at the origin (see Example 5.1.13):

$$\mathrm{Bl}_0(\mathbf{A}^{n+1}) := \bigl\{\bigl(P,\ell\bigr) \in \mathbf{A}^{n+1} \times \mathbf{P}^n \mid P \in \ell\bigr\}.$$

The first projection is the blow-up map of $\mathbf{A}^{n+1}$. Let us consider now the morphism $q\colon \mathrm{Bl}_0(\mathbf{A}^{n+1}) \to \mathbf{P}^n$ induced by the second projection $f\colon \mathbf{A}^{n+1} \times \mathbf{P}^n \to \mathbf{P}^n$. We claim that q gives a geometric vector bundle of rank 1, in fact a subbundle of the trivial rank $(n+1)$ bundle given by f. Indeed, each fiber $q^{-1}\bigl([\ell]\bigr)$ is a one-dimensional linear subspace of $f^{-1}\bigl([\ell]\bigr) \simeq k^{n+1}$. Moreover, if $x_0, \ldots, x_n$ are the coordinates on $\mathbf{A}^{n+1}$ and $y_0, \ldots, y_n$ are the homogeneous coordinates on $\mathbf{P}^n$, and $U_i = (y_i \neq 0)$, then we have an isomorphism

$$q^{-1}(U_i) = \left\{\bigl((x_0, \ldots, x_n), [y_0, \ldots, y_n]\bigr) \mid y_i \neq 0, x_j = x_i \cdot \frac{y_j}{y_i} \text{ for all } j\right\} \simeq k \times U_i$$

mapping $\bigl((x_0, \ldots, x_n), [y_0, \ldots, y_n]\bigr)$ to $\bigl(x_i, [y_0, \ldots, y_n]\bigr)$. This proves our claim.

The geometric vector bundle given by q is the *tautological subbundle* on $\mathbf{P}^n$. The sheaf of sections of this bundle is denoted $\mathcal{O}_{\mathbf{P}^n}(-1)$ and its dual by $\mathcal{O}_{\mathbf{P}^n}(1)$, while the corresponding m^{th} tensor powers (for $m > 0$) are denoted by $\mathcal{O}_{\mathbf{P}^n}(-m)$ and $\mathcal{O}_{\mathbf{P}^n}(m)$, respectively. Using the above trivializations, we see that the transition functions of $\mathcal{O}_{\mathbf{P}^n}(-1)$ are given by $(y_i/y_j)_{i,j}$.

Let us compute, using this, $\Gamma\bigl(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\bigr)$. It follows from Example 8.5.21 that for every $m \in \mathbf{Z}$, the transition functions of $\mathcal{O}_{\mathbf{P}^n}(m)$ are $(y_j^m/y_i^m)_{i,j}$ and thus

$$\Gamma\bigl(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\bigr) \simeq \left\{(s_0, \ldots, s_n) \in \prod_{i=0}^n \Gamma(U_i, \mathcal{O}_{\mathbf{P}^n}) \mid s_i = \frac{y_j^m}{y_i^m} s_j \text{ for all } i, j\right\}.$$

Using the isomorphism

$$\Gamma(U_i, \mathcal{O}_{\mathbf{P}^n}) \simeq k[y_0, \ldots, y_n]_{(y_i)}$$

for $0 \leq i \leq n$, we see that giving a tuple $(s_0, \ldots, s_n)$ as above is equivalent to giving $s \in k[y_0, \ldots, y_n]_{y_0 \cdots y_n}$ homogeneous of degree m, such that for all i, we have

$$s_i = \frac{s}{y_i^m} \in k[y_0, \ldots, y_n]_{y_i}.$$

By writing $s = \frac{P}{Q}$, with $P, Q \in k[y_0, \ldots, y_n]$ relatively prime, we see that the condition is that Q divides a suitable power of y_i for all i, hence $s \in k[y_0, \ldots, y_n]_m$. We thus conclude that

$$\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) \simeq k[y_0, \ldots, y_n]_m,$$

where the right-hand side is 0 for $m < 0$. We thus have

$$\dim_k \Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) = \binom{m+n}{n} \quad \text{for} \quad m \geq 0$$

(the formula for the number of monomials of fixed degree in $n+1$ variables follows easily by induction on n). The line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ and its restriction to the subvarieties of $\mathbf{P}^n$ will play an important role in later chapters.

8.6.3. Projective morphisms. Suppose now that $\mathcal{S}$ is an $\mathbf{N}$-*graded* $\mathcal{O}_X$-*algebra*, that is, $\mathcal{S}$ is an $\mathcal{O}_X$-algebra that has a decomposition

$$\mathcal{S} = \bigoplus_{m \in \mathbf{N}} \mathcal{S}_m,$$

where $\mathcal{S}_i \cdot \mathcal{S}_j \subseteq \mathcal{S}_{i+j}$ for all $i, j \in \mathbf{N}$. Suppose also that $\mathcal{S}$ is reduced and quasi-coherent. We assume, in addition, that $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent, and that the $\mathcal{S}_0$-algebra $\mathcal{S}$ is generated by $\mathcal{S}_1$; this means that for every affine open subset U of X, the $\mathcal{S}_0(U)$-algebra $\mathcal{S}(U)$ is generated by $\mathcal{S}_1(U)$, In particular, we see that $\mathcal{S}$ is a finitely generated $\mathcal{O}_X$-algebra.

EXERCISE 8.6.15. Show that if $\mathcal{S}$ is an $\mathbf{N}$-graded, quasi-coherent $\mathcal{O}_X$-algebra, then $\mathcal{S}$ is generated as an $\mathcal{S}_0$-algebra by $\mathcal{S}_1$ if and only if the canonical morphism

$$\mathrm{Sym}^{\bullet}_{\mathcal{S}_0}(\mathcal{S}_1) \to \mathcal{S}$$

is surjective. Deduce that if $X = U_1 \cup \ldots \cup U_n$ is an affine open cover, this condition holds if and only if each $\mathcal{S}_0(U_i)$-algebra $\mathcal{S}(U_i)$ is generated by $\mathcal{S}_1(U_i)$.

REMARK 8.6.16. It is easy to see that since $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent and $\mathcal{S}$ is generated over $\mathcal{S}_0$ by $\mathcal{S}_1$, all $\mathcal{S}_m$ are coherent $\mathcal{O}_X$-modules.

Under the above assumptions, we construct a variety over X

$$\pi \colon \mathcal{M}axProj(\mathcal{S}) \to X,$$

as follows. For every affine open subset U of X, we consider the composition

$$\pi_U \colon \mathrm{MaxProj}\big(\mathcal{S}(U)\big) \to \mathrm{MaxSpec}\big(\mathcal{S}_0(U)\big) \to U,$$

where the second morphism is the finite morphism induced by the homomorphism $\mathcal{O}_X(U) \to \mathcal{S}_0(U)$. Arguing as in § 8.6.1, we see that if $V \subseteq U$ are affine open subsets, then we have a Cartesian diagram

$$\begin{tikzcd}
\mathrm{MaxProj}\big(\mathcal{S}(V)\big) \arrow[r, "i"] \arrow[d, "\pi_V"'] & \mathrm{MaxProj}\big(\mathcal{A}(U)\big) \arrow[d, "\pi_U"] \\
V \arrow[r, "j"] & U.
\end{tikzcd}$$

In particular, given any affine open subsets $U_1, U_2 \subseteq X$, we have a canonical isomorphism

$$\pi_{U_1}^{-1}(U_1 \cap U_2) \simeq \pi_{U_2}^{-1}(U_1 \cap U_2)$$

of varieties over $U_1 \cap U_2$. We can thus glue the projective varieties $\mathrm{MaxProj}\big(\mathcal{S}(U)\big)$, where U varies over the affine open subsets of X, to obtain an algebraic prevariety $\mathcal{M}axProj(\mathcal{S})$, and the morphisms π_U glue to give a morphism $\pi\colon \mathcal{M}axProj(\mathcal{S}) \to X$.

In fact, $\mathcal{M}axProj(\mathcal{S})$ is an algebraic variety: by Proposition 2.5.14, it is enough to show that π is a separated morphism. This is a consequence of the fact that for every affine open subset U of X, the inverse image $\pi_X^{-1}(U)$ is isomorphic to $\mathrm{MaxProj}\big(\mathcal{S}(U)\big)$, hence it is separated (see Example 2.5.13).

DEFINITION 8.6.17. A morphism of algebraic varieties $f\colon Y \to X$ is *projective* if there is an $\mathcal{O}_X$-algebra $\mathcal{S}$ as above, such that Y is isomorphic to $\mathcal{M}axProj(\mathcal{S})$, as varieties over X.

REMARK 8.6.18. Note that every projective morphism $\pi\colon Y \to X$ is proper: this follows from the fact that we can cover X by finitely many affine open subsets $U_1, \ldots, U_n$ and $\pi^{-1}(U_i) \to U_i$ is proper (see assertion v) in Proposition 5.1.4 and Corollary 5.1.11).

EXAMPLE 8.6.19. Note that every finite morphism $f\colon X \to Y$ is projective. Indeed, it is straightforward to check that X is isomorphic to $\mathcal{M}axProj\big(f_*(\mathcal{O}_X)[x]\big)$ as varieties over Y.

EXAMPLE 8.6.20. An important example is given by the *blow-up* of X with respect to a coherent ideal $\mathcal{I}$. In this case, the $\mathcal{O}_X$-algebra we consider is

$$\mathcal{S} := \bigoplus_{m \geq 0} \mathcal{I}^m t^m \subseteq \mathcal{O}_X[t].$$

Note that if $\pi\colon \mathcal{M}axProj(\mathcal{S}) \to X$ is the corresponding morphism and $U \subseteq X$ is an affine open subset, then the induced morphism $\pi^{-1}(U) \to U$ is the one that we discussed in § 6.1.

EXAMPLE 8.6.21. Another example is provided by *projective bundles*. If $\mathcal{E}$ is a locally free sheaf on the algebraic variety X, then the corresponding projective bundle is

$$\mathbf{P}(\mathcal{E}) = \mathcal{M}axProj\big(\mathrm{Sym}^\bullet(\mathcal{E})\big).$$

In particular, if $\mathcal{E} = \mathcal{O}_Y^{\oplus(n+1)}$, then we obtain $\mathbf{P}(\mathcal{E}) = \mathbf{P}^n \times Y =: \mathbf{P}^n_Y$.

We will discuss projective morphisms in more detail in Chapter 11.

8.7. The cotangent sheaf

In this section we construct the cotangent sheaf of an algebraic variety. As we will see, this is a vector bundle of the appropriate rank if and only if X is smooth. More generally, we will construct a relative version of the cotangent sheaf, associated to a morphism $X \to Y$. We will use this later in order to characterize the smoothness of an arbitrary morphism.

We first treat the affine case. In fact, we do this under very general assumptions on the rings involved.

DEFINITION 8.7.1. Let R be a commutative ring, A a commutative R-algebra, and M an A-module. An R-*derivation* from A to M is a map $D\colon A \to M$ such that

i) D is a morphism of R-modules, and
ii) For every a, $b \in A$, we have the *Leibniz rule*
$$D(ab) = aD(b) + bD(a).$$

The set of such derivations will be denoted by $\mathrm{Der}_R(A, M)$. It is easy to see that this is an A-submodule of $\mathrm{Hom}_R(A, M)$, where the A-module structure on $\mathrm{Hom}_R(A, M)$ is induced by the one on M.

REMARK 8.7.2. In the presence of condition ii), condition i) above is equivalent to the fact that D is a morphism of Abelian groups such that
$$D(\lambda \cdot 1) = 0 \quad \text{for every} \quad \lambda \in R.$$

REMARK 8.7.3. If $f\colon M \to N$ is a morphism of A-modules, we obtain an induced morphism of A-modules
$$\mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A, N), \quad D \to f \circ D.$$
This shows that $\mathrm{Der}_R(A, -)$ gives a covariant functor from the category of A-modules to itself.

PROPOSITION 8.7.4. *The functor* $\mathrm{Der}_R(A, -)$ *is representable, that is, there is an A-module $\Omega_{A/R}$, together with an R-derivation $d = d_{A/R}\colon A \to \Omega_{A/R}$ such that for every R-derivation $D\colon A \to M$, there is a unique morphism of A-modules $\phi\colon \Omega_{A/R} \to M$ such that $\phi \circ d = D$; in other words, $d_{A/R}$ induces an isomorphism of A-modules*
$$\mathrm{Hom}_A(\Omega_{A/R}, M) \simeq \mathrm{Der}_R(A, M).$$

Of course, like every object representing a functor, the A-module $\Omega_{A/R}$ is unique, up to a canonical isomorphism that commutes with the derivation $d_{A/R}$; this is the *module of Kähler differentials* of A over R.

PROOF OF PROPOSITION 8.7.4. Let $\Omega_{A/R}$ be the quotient of the free A-module generated by the symbols $d(a)$, for $a \in A$, by the A-submodule generated by the following elements:

i) $d(a) + d(a') - d(a + a')$ for $a, a' \in A$,
ii) $\lambda \cdot d(a) - d(\lambda a)$ for $\lambda \in R$, $a \in A$, and
iii) $d(ab) - a \cdot d(b) - b \cdot d(a)$ for $a, b \in A$.

We define $d_{A/R}\colon A \to \Omega_{A/R}$ by mapping each $a \in A$ to $d(a)$. It is clear from the definition that $d_{A/R}$ is an R-derivation and it is straightforward to check that it satisfies the required universal property. $\square$

REMARK 8.7.5. It is clear from definition that $\Omega_{A/R}$ is generated as an A-module by $\{d_{A/R}(a) \mid a \in A\}$. In fact if $(a_i)_{i\in I}$ is a family of elements of A that generate A as an R-algebra, then $\{d_{R/A}(a_i) \mid i \in I\}$ generate $\Omega_{A/R}$: indeed, the Leibniz rule implies that every element of the form $d_{A/R}(a_{i_1} \cdot \ldots \cdot a_{i_r})$ lies in the linear span of $d_{A/R}(a_{i_1}), \ldots, d_{A/R}(a_{i_r})$. In particular, we see that if A is an R-algebra of finite type, then $\Omega_{A/R}$ is a finitely generated A-module.

EXAMPLE 8.7.6. If $A = R[x_1, \ldots, x_n]$, then $\Omega_{R/A}$ is a free A-module, with generators $dx_1, \ldots, dx_n$. Indeed, we have already seen that $dx_1, \ldots, dx_n$ generate $\Omega_{R/A}$. In order to see that they form a basis, consider for every i the derivation

$$\partial_{x_i} \colon A \to A, \quad f \to \frac{\partial f}{\partial x_i}.$$

It is clear that the induced morphism of A-modules $\phi \colon \Omega_{A/R} \to A$ maps dx_i to 1 and dx_j to 0 for all $j \neq i$. This easily implies that $dx_1, \ldots, dx_n$ are linearly independent. Note that $d_{A/R}$ is thus given by

$$d_{A/R}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for all $f \in A$.

The now prove some general properties of modules of Kähler differentials.

PROPOSITION 8.7.7. *Let A be an R-algebra as above. If S is a multiplicative system in A, then we have a canonical isomorphism of A-modules*

$$S^{-1}\Omega_{A/R} \simeq \Omega_{S^{-1}A/R}.$$

PROOF. Consider an arbitrary $S^{-1}A$-module M and let $i \colon A \to S^{-1}A$ be the canonical homomorphism. Note that for every R-derivation $D \colon A \to M$, there is a unique R-derivation $D' \colon S^{-1}A \to M$ such that $D' \circ i = D$: this is given by the "quotient rule":

$$D'\left(\frac{a}{s}\right) = \frac{1}{s}D(a) - \frac{a}{s^2}D(s).$$

This implies that the canonical morphism

$$\mathrm{Hom}_{S^{-1}A}(\Omega_{S^{-1}A/R}, M) \simeq \mathrm{Der}_R(S^{-1}A, M) \longrightarrow$$

$$\mathrm{Der}_R(A, M) \simeq \mathrm{Hom}_A(\Omega_{A/R}, M) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}\Omega_{A/R}, M)$$

that maps D to $D \circ i$, is an isomorphism. Since this holds for all M, we obtain the isomorphism in the proposition. $\square$

PROPOSITION 8.7.8. *If A and B are R-algebras, then we have a canonical isomorphism of $A \otimes_R B$ modules*

$$\Omega_{A/R} \otimes_R B \simeq \Omega_{A\otimes_R B/B}.$$

PROOF. Let $j \colon A \to A \otimes_R B$ be given by $j(a) = a \otimes 1$. Note that for every $A \otimes_R B$-module M, the canonical morphism

$$\mathrm{Der}_B(A \otimes_R B, M) \to \mathrm{Der}_R(A, M),$$

that maps D to $D \circ j$, is an isomorphism. This immediately implies that the morphism of $A \otimes_R B$-modules

$$\Omega_{A/R} \otimes_R B \to \Omega_{A\otimes_R B/B}, \quad d_{A/R}(a) \otimes b \to d_{A\otimes_R B/B}(a \otimes b)$$

is an isomorphism. □

The following two propositions contain two exact sequences that are very useful for computing modules of Kähler differentials.

PROPOSITION 8.7.9. *Given a morphism $\phi\colon A \to B$ of R-algebras, there is an exact sequence*

$$\Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0,$$

where $\alpha\big(d_{A/R}(a) \otimes b\big) = b \cdot d_{B/R}\big(\phi(a)\big)$ and $\beta\big(d_{B/R}(b)\big) = d_{B/A}(b)$ for all $a \in A$ and $b \in B$.

PROOF. Given any B-module M, it is straightforward to see that the sequence

$$0 \longrightarrow \mathrm{Der}_A(B, M) \xrightarrow{u_M} \mathrm{Der}_R(B, M) \xrightarrow{v_M} \mathrm{Der}_R(A, M),$$

in which u_M is the inclusion and $v_M(D) = D \circ \phi$, is exact. Since u_M and v_M are natural in M, it follows easily from the universal property of the module of Kähler differentials that the maps α and β as in the proposition are well defined, such that $\mathrm{Hom}_B(\beta, M) = u_M$ and $\mathrm{Hom}_B(\alpha, M) = v_M$ for all M. Since applying $\mathrm{Hom}_R(-, M)$ to the sequence in the proposition is exact for all M, it is well-known (and easy to see) that the original sequence is exact. □

EXAMPLE 8.7.10. It is clear that for every ring A, we have $\Omega_{A/A} = 0$, hence Proposition 8.7.7 implies that if S is a multiplicative system in the ring A, then $\Omega_{S^{-1}A/A} = 0$. If B is an $S^{-1}A$-algebra, we deduce from Proposition 8.7.9 that the canonical morphism

$$\Omega_{B/A} \to \Omega_{B/S^{-1}A}$$

is an isomorphism.

PROPOSITION 8.7.11. *If $\phi\colon A \to B$ is a surjective morphism of R-algebras, with $\ker(\phi) = I$, then we have an exact sequence*

$$I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \longrightarrow 0,$$

where α is the morphism defined in the previous proposition and $\delta(\overline{a}) = d_{A/R}(a) \otimes 1$ for every $a \in I$.

PROOF. Note that since ϕ is surjective, every derivation $B \to M$ over A, where M is a B-module, is 0. Therefore $\Omega_{B/A} = 0$. Given such M, we define

$$w_M \colon \mathrm{Der}_R(A, M) \to \mathrm{Hom}_B(I/I^2, M)$$

such that $w_M(D)$ maps $\overline{a}$ to $D(a) \in M$ for every $a \in I$. Note that if $a, b \in I$, then

$$D(ab) = \phi(a) \cdot D(b) + \phi(b) \cdot D(a) = 0,$$

hence w_M is well-defined. Moreover, $w_M(D) = 0$ if and only if $D(I) = 0$; equivalently, $D = \overline{D} \circ \phi$ for a (unique) $\overline{D} \in \mathrm{Der}_R(B, M)$. This says that we have an exact sequence:

$$0 \longrightarrow \mathrm{Der}_R(B, M) \xrightarrow{v_M} \mathrm{Der}_R(A, M) \xrightarrow{w_M} \mathrm{Hom}_B(I/I^2, M).$$

Arguing as in the proof of the previous proposition, we deduce that there is a map δ given by the formula in our statement and which makes the sequence exact. □

EXAMPLE 8.7.12. Suppose that B is a finitely generated R-algebra. Let $b_1, \ldots, b_n$ be generators of B as an R-algebra and consider the surjective homomorphism $\phi\colon A = R[x_1, \ldots, x_n] \to B$ given by $\phi(x_i) = b_i$ for all i. By Example 8.7.6, $\Omega_{A/R}$ is a free A-module with basis $dx_1, \ldots, dx_n$, and we deduce using Proposition 8.7.11 that $\Omega_{B/R}$ is the B-module generated by $db_1, \ldots, db_n$, with relations

$$\sum_{i=1}^{n} \phi\left(\frac{\partial f}{\partial x_i}\right) db_i.$$

EXERCISE 8.7.13. Show that if L/K is a finite field extension, then L/K is separable if and only if $\Omega_{L/K} = 0$.

We now globalize the above results in our geometric setting. In order to show that the modules of Kähler differentials define a quasi-coherent sheaf on X, we make use of the following general result.

LEMMA 8.7.14. *Let X be an algebraic variety. Suppose that for every affine open subset $U \subseteq X$, we have an $\mathcal{O}_X(U)$-module $\alpha(U)$ and for every inclusion $V \subseteq U$ of such subsets we have a restriction map $\alpha(U) \to \alpha(V)$, which is a morphism of $\mathcal{O}_X(U)$-modules, and these maps satisfy the usual compatibility conditions. If for every affine open subset $U \subseteq X$ and for every $f \in \mathcal{O}_X(U)$, the induced morphism $\alpha(U)_f \to \alpha\big(D_U(f)\big)$ is an isomorphism, then there is a quasi-coherent sheaf $\mathcal{F}$ on X, unique up to a canonical isomorphism, such that for every affine open subset U of X, we have an isomorphism $\mathcal{F}(U) \simeq \alpha(U)$, and these isomorphisms are compatible with the restriction maps.*

PROOF. For every affine open subset $U \subseteq X$, we consider the quasi-coherent sheaf $\mathcal{F}_U = \widetilde{\alpha(U)}$ on U. Note that if $V \subseteq U$ are two such subsets, then the morphism $\alpha(U) \to \alpha(V)$ induces a morphism

$$\alpha(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \alpha(V)$$

that corresponds to the morphism of $\mathcal{O}_V$-modules

$$\tau_{V,U}\colon \mathcal{F}_U|_V \to \mathcal{F}_V.$$

The compatibility of the restriction maps for α implies that if W is an affine open subset of V, then

$$\tau_{W,V} \circ \tau_{V,U}|_W = \tau_{W,U}. \tag{8.7.1}$$

The hypothesis implies that $\tau_{V,U}$ is an isomorphism if V is a principal affine open subset of U. By covering any V by such open subsets and using (8.7.1), we see that $\tau_{V,U}$ is always an isomorphism.

If U_1 and U_2 are any affine open subsets of X, we obtain isomorphisms

$$\mathcal{F}_{U_1}|_{U_1 \cap U_2} \simeq \mathcal{F}_{U_1 \cap U_2} \simeq \mathcal{F}_{U_2}|_{U_1 \cap U_2}$$

and these are compatible in the obvious sense. We can thus apply Exercise 2.1.23 to construct a sheaf $\mathcal{F}$, together with isomorphisms $\mathcal{F}|_U \simeq \mathcal{F}_U$ for every affine open subset U of X. Checking the fact that this satisfies the required condition, as well as the uniqueness of $\mathcal{F}$, is straightforward. □

REMARK 8.7.15. Given a morphism of algebraic varieties $f\colon X \to Y$, instead of considering on X all affine open subsets, we may consider only those affine open subsets $U \subseteq X$ such that there is an affine open subset W of Y such that $f(U) \subseteq W$. The statement of the above lemma holds also in this case, with the same proof.

REMARK 8.7.16. Suppose that $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules on the algebraic variety X with the property that for every affine open subset U of X and every $f \in \mathcal{O}_X(U)$, the induced morphism

$$\mathcal{F}(U)_f \to \mathcal{F}\big(D_U(f)\big)$$

is an isomorphism. In this case, the associated sheaf $\mathcal{F}^+$ is quasi-coherent and for every affine open subset U of X, the induced morphism $\mathcal{F}(U) \to \mathcal{F}^+(U)$ is an isomorphism. Indeed, we apply Lemma 8.7.14 to construct a quasi-coherent sheaf $\mathcal{G}$ such that for affine open subsets U of X, we have isomorphisms compatible with the restriction maps $\mathcal{F}(U) \simeq \mathcal{G}(U)$. It is straightforward to see that we have a unique morphism of presheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ that on affine open subsets is given by the above isomorphisms. This implies that for every $x \in X$, the induced morphism $\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, and thus the induced morphism of sheaves $\mathcal{F}^+ \to \mathcal{G}$ is an isomorphism.

LEMMA 8.7.17. *If $j \colon U \to V$ is an open immersion of affine varieties and if $\phi \colon A \to B$ is the corresponding k-algebra homomorphism, then $\Omega_{B/A} = 0$.*

PROOF. It is enough to show that for every maximal ideal $\mathfrak{n}$ in B, we have $\Omega_{B/A} \otimes_B B_{\mathfrak{n}} = 0$. Since j is an open immersion, if $\mathfrak{m} = \phi^{-1}(\mathfrak{n})$, then the induced morphism $A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ is an isomorphism. Using Proposition 8.7.7 and Example 8.7.10, we conclude that

$$\Omega_{B/A} \otimes_B B_{\mathfrak{n}} \simeq \Omega_{B_{\mathfrak{n}}/A} \simeq \Omega_{B_{\mathfrak{n}}/A_{\mathfrak{m}}} = 0.$$

$\square$

Given a morphism of algebraic varieties $f \colon X \to Y$ and affine open subsets $U \subseteq X$ and $W \subseteq Y$ such that $f(U) \subseteq W$, we consider the $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(W)}$. Note that this does not depend on W: if we have an affine open subset $W' \supseteq W$, then by applying Proposition 8.7.9 for the morphisms $\mathcal{O}(W') \to \mathcal{O}(W) \to \mathcal{O}(X)$, together with Lemma 8.7.17, we see that the canonical morphism

$$\Omega_{\mathcal{O}(X)/\mathcal{O}(W')} \to \Omega_{\mathcal{O}(X)/\mathcal{O}(W)}$$

is an isomorphism.

If V is an affine open subset of U, then Proposition 8.7.9 gives a canonical morphism

$$\Omega_{\mathcal{O}(U)/\mathcal{O}(W)} \to \Omega_{\mathcal{O}(V)/\mathcal{O}(W)}.$$

If $V = D_U(f)$, then the induced morphism

$$(\Omega_{\mathcal{O}(U)/\mathcal{O}(W)})_f \to \Omega_{\mathcal{O}(V)/\mathcal{O}(W)}$$

is an isomorphism by Proposition 8.7.7. We can thus apply Lemma 8.7.14 (in the formulation given in Remark 8.7.15), to conclude that we have a quasi-coherent sheaf $\Omega_{X/Y}$ on X such that for U and W as above, we have an isomorphism

$$\Omega_{X/Y}|_U \simeq \Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(W)}.$$

Since for such U and W, the $\mathcal{O}_Y(W)$-algebra $\mathcal{O}_X(U)$ is finitely generated, it follows from Remark 8.7.5 that $\Omega_{X/Y}$ is a coherent $\mathcal{O}_X$-module. We note that Proposition 8.7.7 also implies that given any $x \in X$, we have an isomorphism

$$(\Omega_{X/Y})_x \simeq \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

(and a similar isomorphism for arbitrary irreducible, closed subsets of X).

Definition 8.7.18. The sheaf $\Omega_{X/Y}$ is the *relative cotangent sheaf* of X over Y. If Y is a point, then we obtain the *cotangent sheaf* of X, that we simply denote by Ω_X. The *tangent sheaf* of X is the dual $T_X := \Omega_X^\vee$ (we note that in general, it is not the case that Ω_X is the dual of T_X). For $p \geq 0$, the *sheaf of p-differentials* of X is the sheaf $\wedge^p \Omega_X$.

Exercise 8.7.19. Show that if X and Y are algebraic varieties and $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the two projections, then there is an isomorphism

$$\Omega_{X\times Y} \simeq p^*(\Omega_X) \oplus q^*(\Omega_Y).$$

Propositions 8.7.9 and 8.7.11 immediately give the following global statements.

Proposition 8.7.20. *Given two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$, the following hold:*

i) *We have an exact sequence of $\mathcal{O}_X$-modules*

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

ii) *If f is a closed immersion, with corresponding coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, then we have an exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \to f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to 0.$$

Definition 8.7.21. If Y is a closed subvariety of X, with corresponding ideal $\mathcal{I} = \mathcal{I}_{Y/X}$, the *conormal sheaf* of Y in X is $\mathcal{I}/\mathcal{I}^2$. This is a sheaf on X whose annihilator contains $\mathcal{I}$, hence we consider it as a sheaf on Y. Its dual $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is the *normal sheaf* $N_{Y/X}$.

Remark 8.7.22. If $f\colon X \to Y$ is a morphism of algebraic varieties, then the canonical morphism $f^*(\Omega_Y) \to \Omega_X$ induces for every $p \geq 0$ a morphism $f^*(\Omega_Y^p) \to \Omega_X^p$. We thus obtain a pull-back morphism for p-forms

$$\Gamma(Y, \Omega_Y^p) \to \Gamma(X, \Omega_X^p), \quad \eta \to f^*(\eta)$$

given by the composition

$$\Gamma(Y, \Omega_Y^p) \to \Gamma\big(X, f^*(\Omega_Y^p)\big) \to \Gamma(X, \Omega_X^p),$$

where the first map is the canonical pull-back of sections and the second map is induced by the morphism described above. Of course, if $p = 0$, then this map is just the pull-back of regular functions. For every p, if f is an open immersion, this composition is given simply by restriction of sections. It is straightforward to check that if $g\colon Y \to Z$ is another morphism and $\eta \in \Gamma(Z, \Omega_Z^p)$, then $f^*\big(g^*(\eta)\big) = (g \circ f)^*(\eta)$.

The following proposition explains the name of Ω_X.

Proposition 8.7.23. *For every algebraic variety X and every $x \in X$, we have a canonical isomorphism*

$$(\Omega_X)_{(x)} \simeq (T_x X)^\vee.$$

Proof. Let $R = \mathcal{O}_{X,x}$, with maximal ideal $\mathfrak{m}$ and residue field k. We have

$$(\Omega_X)_{(x)}^\vee \simeq \mathrm{Hom}_R(\Omega_{R/k}, k) \simeq \mathrm{Der}_k(R, k).$$

Note that since $\mathfrak{m}$ annihilates k, it follows from the Leibniz rule that we have an isomorphism

$$\mathrm{Der}_k(R, k) = \mathrm{Der}_k(R/\mathfrak{m}^2, k).$$

Since $R/\mathfrak{m}^2 = k + \mathfrak{m}/\mathfrak{m}^2$, it is straightforward to check that restriction to $\mathfrak{m}/\mathfrak{m}^2$ induces an isomorphism

$$\mathrm{Der}_k(R/\mathfrak{m}^2, k) \simeq \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = T_x X.$$

This completes the proof of the proposition. □

We deduce the following characterization of smooth varieties.

PROPOSITION 8.7.24. *An algebraic variety X is smooth if and only if Ω_X is locally free (in which case, it has rank n on an irreducible component of X of dimension n).*

PROOF. Of course, we may assume that X is connected. If X is smooth and $\dim(X) = n$, then we know that X is irreducible, and by definition of smoothness and Proposition 8.7.23, we conclude that

$$\dim_k(\Omega_X)_{(x)} = n \quad \text{for every} \quad x \in X.$$

We then conclude that Ω_X is locally free, of rank n, by Proposition 8.5.6.

Let us assume that, conversely, Ω_X is locally free of rank n. By Theorem 6.3.7, the smooth locus X_{sm} is dense in X. If we consider the restriction of Ω_X to X_{sm}, we deduce from what we have already proved that every irreducible component of X has dimension n, hence $\dim(\mathcal{O}_{X,x}) = n$ for every $x \in X$. In this case, the hypothesis on Ω_X, together with Proposition 8.7.23 imply that every $x \in X$ is a smooth point. □

If X is a smooth variety, then Ω_X is locally free, hence so is its dual T_X. Moreover, we have $\Omega_X \simeq T_X^{\vee}$ by Remark 8.5.18. The *geometric tangent bundle* is the geometric vector bundle on X whose sheaf of sections is T_X, namely $\mathbf{V}(\Omega_X)$; similarly, the *geometric cotangent bundle* is $\mathbf{V}(T_X)$.

REMARK 8.7.25. Note that sections of the geometric tangent bundle are derivations of $\mathcal{O}_X$. Indeed, it follows from definition that a section over an affine open subset $U \subseteq X$ corresponds to an element of

$$\Gamma(U, \Omega_X^{\vee}) = \mathrm{Hom}_{\mathcal{O}(U)}\big(\Omega_{\mathcal{O}(U)/k}, \mathcal{O}(U)\big) \simeq \mathrm{Der}_k\big(\mathcal{O}(U), \mathcal{O}(U)\big).$$

Note also that if X is smooth, then each sheaf Ω_X^p is locally free by Remark 8.5.13. If, in addition, X is irreducible of dimension n, the line bundle $\omega_X := \Omega_X^n$ is the *canonical line bundle* of X. This line bundle governs governs much of the geometry of X.

PROPOSITION 8.7.26. *If Y is a closed subvariety of X, with both X and Y smooth, then the conormal sheaf of Y in X is a locally free sheaf on Y, and we have an exact short exact sequence of vector bundles on Y*

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0.$$

PROOF. It is clear that we may assume that both X and Y are irreducible. Let $\mathcal{I} = \mathcal{I}_{Y/X}$. If $r = \mathrm{codim}_X(Y)$, then $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf on Y, of rank r, by Proposition 6.3.21. On the other hand, it follows from Proposition 8.7.20 that we have an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\phi} \Omega_X|_Y \xrightarrow{\psi} \Omega_Y \longrightarrow 0.$$

Since all terms are locally free, with

$$\mathrm{rank}(\mathcal{I}/\mathcal{I}^2) = \mathrm{rank}(\Omega_X|_Y) - \mathrm{rank}(\Omega_Y),$$

it follows that ϕ induces a surjective morphism $\mathcal{I}/\mathcal{I}^2 \to \ker(\psi)$ between locally free sheaves of the same rank, which is thus an isomorphism. Therefore ϕ is injective. Applying $\mathcal{H}om_{\mathcal{O}_X}(-\mathcal{O}_X)$ to the above exact sequence, we obtain the exactness of the sequence in the proposition. $\square$

COROLLARY 8.7.27. *If Y is a closed subvariety of X, with both X and Y smooth, then*

$$\omega_Y \simeq \omega_X|_Y \otimes_{\mathcal{O}_Y} \det(N_{Y/X}).$$

PROOF. It follows from the proposition that we have an exact sequence

$$0 \to N_{Y/X}^{\vee} \to \Omega_X|_Y \to \Omega_Y \to 0$$

and the assertion in the proposition follows by taking determinants (see Exercise 8.5.29). $\square$

EXAMPLE 8.7.28. On the projective space $\mathbf{P}^n$, we have a short exact sequence

$$0 \to \Omega_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n} \to 0$$

(this is known as the *Euler exact sequence*).

Recall first that we have seen in Example 8.6.14 that we have a morphism of vector bundles

$$\mathcal{O}_{\mathbf{P}^n}(-1) \hookrightarrow \mathcal{O}_{\mathbf{P}^n}^{\oplus(n+1)}.$$

Since the cokernel is locally free, by dualizing this, we obtain a surjective morphism of vector bundles

$$\phi\colon \mathcal{O}_{\mathbf{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n}(1).$$

It is easy to see that via the isomorphism

$$H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) \simeq k[x_0, \ldots, x_n]_1$$

in Example 8.6.14, the morphism ϕ is given by $(x_0, \ldots, x_n)$. By tensoring with $\mathcal{O}_{\mathbf{P}^n}(-1)$, we obtain a surjective morphism

$$\psi\colon \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n}.$$

In order to complete the proof, we need to give an isomorphism

$$\Omega_{\mathbf{P}^n} \simeq \ker(\psi).$$

We define this on each of the charts $U_i = (x_i \neq 0)$, where $x_0, \ldots, x_n$ are the homogeneous coordinates on $\mathbf{P}^n$. We have $U_i \simeq \mathbf{A}^n$, with coordinates $y_i = x_j/x_i$ for $j \neq i$ and $\Omega_{\mathbf{P}^n}(U_i)$ is a free module over $\mathcal{O}(U_i)$, with basis $\{dy_j \mid j \neq i\}$. Recall that we have an isomorphism $\rho_i\colon \mathcal{O}_{\mathbf{P}^n}(-1)|_{U_i} \simeq \mathcal{O}_{U_i}$ such that for $\ell \neq i$, the composition

$$\rho_\ell|_{U_i \cap U_\ell} \circ (\rho_i|_{U_i \cap U_\ell})^{-1}$$

is given by multiplication by x_ℓ/x_i. If we write $e_0, \ldots, e_n$ for the standard basis of $k^{\oplus(n+1)}$, then we have

$$\Gamma\big(U_i, \ker(\psi)\big) = \left\{ \sum_{j=0}^{n} \rho_i^{-1}(f_j)e_j \mid f_0, \ldots, f_n \in \Gamma(U_i, \mathcal{O}_{\mathbf{P}^n}), \sum_{j=0}^{n} x_j f_j = 0 \right\}.$$

This is a free $\mathcal{O}(U_i)$-module, with basis

$$\{\rho_i^{-1}(1)e_j - \rho_i^{-1}(y_j)e_i \mid 0 \leq j \leq n, j \neq i\}.$$

We thus have an isomorphism

$$\tau_i \colon \Omega_{\mathbf{P}^n}|_{U_i} \to \ker(\psi)|_{U_i}, dy_j \to \rho_i^{-1}(1)e_j - \rho_i^{-1}(y_j)e_i.$$

In order to complete the proof, we need to show that these morphisms are compatible. Choose $\ell \neq i$ and let $z_j = x_j/x_\ell$ for $j \neq \ell$. Note that we have

$$dy_j = d(x_j/x_i) = d(z_j/z_i) = \frac{1}{z_i}dz_j - \frac{z_j}{z_i^2}dz_i \quad \text{for} j \neq i, \ell$$

and

$$dy_\ell = d(x_\ell/x_i) = d(1/z_i) = -\frac{1}{z_i^2}dz_i.$$

It follows that if $j \neq i, \ell$, then dy_j is mapped by τ_ℓ to

$$\frac{1}{z_i}\cdot\big(\rho_\ell^{-1}(1)e_j - \rho_\ell^{-1}(z_j)e_\ell\big) - \frac{z_j}{z_i^2}\cdot\big(\rho_\ell^{-1}(1)e_i - \rho_\ell^{-1}(z_i)e_\ell\big) = \frac{1}{z_i}\cdot\left(\rho_\ell^{-1}(1)e_j - \frac{z_j}{z_i}\rho_\ell^{-1}(1)e_i\right)$$
$$= \rho_i^{-1}(1)e_j - y_je_i.$$

We also see that dy_ℓ is mapped by τ_ℓ to

$$-\frac{1}{z_i^2}\cdot\big(\rho_\ell^{-1}(1)e_i - \rho_\ell^{-1}(z_i)e_\ell\big) = \frac{1}{z_i}\cdot\big(\rho_\ell^{-1}(1)e_\ell - \rho_\ell^{-1}(y_\ell)e_i\big) = \rho_i^{-1}(1)e_\ell - \rho_i^{-1}(y_\ell)e_i.$$

This completes the proof.

EXAMPLE 8.7.29. It follows from the above example and Exercise 8.5.29 that

$$\omega_{\mathbf{P}^n} \simeq \mathcal{O}_{\mathbf{P}^n}(-n-1).$$

CHAPTER 9

Normal varieties and divisors

In the first section, we discuss normal varieties and prove some important properties. In the second section we show that a smooth variety is locally factorial; in particular, it is normal. In the remaining sections we introduce Weil divisors and Cartier divisors and discuss their connection with the Picard group.

9.1. Normal varieties

Recall that an irreducible affine variety X is normal if the corresponding ring $\mathcal{O}(X)$ is integrally closed. We want to extend the definition to arbitrary algebraic varieties, not necessarily irreducible. We begin with the following lemma concerning the behavior of integral closure under localization:

LEMMA 9.1.1. *Let R be an integral domain, with fraction field K, and R' the integral closure of R in K.*

i) *If S is a multiplicative system in R, then the integral closure of $S^{-1}R$ in K is $S^{-1}R'$.*
ii) *In particular, R is integrally closed in K if and only if $R_{\mathfrak{p}}$ is integrally closed in K for every prime (maximal) ideal $\mathfrak{p}$ in R.*

PROOF. We first prove i). if $u \in R'$, then we can find a positive integer n and $a_1, \ldots, a_n \in R$ such that

$$u^n + \sum_{i=1}^{n} a_i u^{n-i} = 0.$$

In this case, for every $s \in S$, we have

$$\left(\frac{u}{s}\right)^n + \sum_{i=1}^{n} \frac{a_i}{s^i} \cdot \left(\frac{u}{s}\right)^{n-i} = 0,$$

hence $\frac{u}{s}$ lies in the integral closure of $S^{-1}R$.

Conversely, suppose that $v \in K$ lies in the integral closure of $S^{-1}R$. We can thus find a positive integer n and $b_i \in S^{-1}R$ such that

$$v^n + \sum_{i=1}^{n} b_i v^{n-i} = 0.$$

We can find $s \in S$ such that $sv \in R$ and $sb_i \in R$ for all i, in which case we see that

$$(sv)^n + \sum_{i=1}^{n} (s^i b_i)(sv)^{n-i} = 0,$$

hence $sv \in R'$ and thus $v \in S^{-1}R'$. This completes the proof of i).

The assertion in ii) follows immediately from the fact that $R = R'$ if and only if $R_{\mathfrak{p}} = R'_{\mathfrak{p}}$ for all prime (maximal) ideals $\mathfrak{p}$ in R (see Corollary C.3.3). $\square$

PROPOSITION 9.1.2. *Given an algebraic variety X, the following assertions are equivalent:*

i) *For every affine open subset $U \subseteq X$ and every connected component V of U, V is irreducible and $\mathcal{O}_X(V)$ is integrally closed.*
ii) *There is an affine open cover $X = U_1 \cup \ldots \cup U_n$ such that each U_i is irreducible and $\mathcal{O}_X(U_i)$ is integrally closed.*
iii) *For every irreducible closed subset V of X, the local ring $\mathcal{O}_{X,V}$ is an integrally closed domain.*
iv) *For every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.*

The variety X is *normal* if it satisfies the above equivalent conditions. Note that for an irreducible affine variety, we recover our old definition.

PROOF OF PROPOSITION 9.1.2. The implications i)⇒ii) and iii)⇒iv) are trivial, while the implication ii)⇒iii) follows from the lemma. Suppose now that iv) holds. If U is an affine open subset of X, we first deduce from iv) that every point of U lies on a unique irreducible component of U; in other words, the connected components of U are irreducible. If V is such a connected component, then it follows from iv) that $\mathcal{O}_X(V)_{\mathfrak{p}}$ is integrally closed for every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_X(V)$, hence $\mathcal{O}_X(V)$ is integrally closed by the lemma. This completes the proof. □

At this point, we don't have many examples of normal varieties. An important result is that smooth varieties are normal. The usual proof makes use of the fact that regular rings are Cohen-Macaulay, a fact that we postpone until Chapter 12. We give, instead, a different proof in the next section, which shows a stronger assertion: the local rings of a smooth variety are UFDs.

We now discuss some important geometric properties of normal varieties. These rely on the characterization of normal domains in Proposition E.5.1. We say that an algebraic variety is *smooth in codimension 1* if every irreducible component of X_{sing} has codimension ≥ 2 in X.

REMARK 9.1.3. Recall that by Corollaries 6.3.19 and 6.3.23, given an irreducible closed subset V of X, we have $V \subseteq X_{\mathrm{sing}}$ if and only if the local ring $\mathcal{O}_{X,V}$ is not regular; if V has codimension 1, then this is the case if and only if $\mathcal{O}_{X,V}$ is not a DVR. It follows that X is smooth in codimension 1 if and only if for every irreducible, closed subset V of X, of codimension 1, the local ring $\mathcal{O}_{X,V}$ is a DVR. We thus deduce from Proposition E.5.1 that every normal variety is smooth in codimension 1 (note that the reduction to the case when X is affine and irreducible is straightforward).

PROPOSITION 9.1.4. *If X is a normal variety and U is an open subset of X such that* $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, *then for every locally free sheaf $\mathcal{E}$ on X, the restriction map*

$$\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E})$$

is an isomorphism.

PROOF. We may assume that X is an irreducible affine variety and $\mathcal{E} = \mathcal{O}_X$. Indeed, assuming this case, we cover X by finitely many irreducible affine open subsets U_i such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r_i}$ for all i. It follows from the sheaf condition that the horizontal sequences in the following commutative diagram are exact:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathcal{E}) \arrow[r] \arrow[d, "\alpha"'] & \prod_i \Gamma(U_i,\mathcal{E}) \arrow[r] \arrow[d, "\beta"'] & \prod_{i,j} \Gamma(U_i\cap U_j,\mathcal{E}) \arrow[d, "\gamma"'] \\
0 \arrow[r] & \Gamma(U,\mathcal{E}) \arrow[r] & \prod_i \Gamma(U\cap U_i,\mathcal{E}) \arrow[r] & \prod_{i,j} \Gamma(U\cap U_i\cap U_j,\mathcal{E}).
\end{tikzcd}$$

Since we know that β and γ are isomorphisms, it follows that α is an isomorphism as well.

We assume from now on that X is affine, irreducible, with $A = \mathcal{O}(X)$, and $\mathcal{E} = \mathcal{O}_X$. The restriction map in the proposition is injective since U is a dense open subset of X (otherwise we would have $\operatorname{codim}_X(X \smallsetminus U) = 0$). If $\phi \in \mathcal{O}_X(U)$, then ϕ is a rational function on X. For every prime ideal $\mathfrak{p}$ in A with $\operatorname{codim}(\mathfrak{p}) = 1$, if V is the corresponding irreducible closed subset of X, then the hypothesis on U implies that $V \cap U \neq \emptyset$; therefore $\phi \in A_{\mathfrak{p}}$. Since $A = \bigcap_{\operatorname{codim}(\mathfrak{p})=1} A_{\mathfrak{p}}$ by Proposition E.5.1, we conclude that $\phi \in A$, completing the proof. $\square$

Corollary 9.1.5. *If X is an irreducible, normal variety and $\phi \in k(X)$ is a rational function, with domain U, then every irreducible component of $X \smallsetminus U$ has codimension 1.*

Proof. If V is an irreducible component of $X \smallsetminus U$ with $\operatorname{codim}_X(V) \geq 2$, let V' be the union of the irreducible components of $X \smallsetminus U$ different from V. In this case, $X \smallsetminus V'$ is an open subset of X (in particular, it is a normal variety), it contains U, and we deduce from the proposition that the restriction map

$$\mathcal{O}_X(X \smallsetminus V') \to \mathcal{O}_X(U)$$

is surjective. Therefore ϕ extends to $X \smallsetminus V'$, contradicting the fact that U is the domain of ϕ. $\square$

The next proposition shows that if instead of considering rational maps to $\mathbf{A}^1$, we consider rational maps to a complete variety, the opposite is true: the complement of the domain has codimension ≥ 2.

Proposition 9.1.6. *Let X be an irreducible normal variety (or, more generally, a variety that is smooth in codimension 1).*

i) *For every rational map $f\colon X \dashrightarrow Y$, where Y is a complete variety, if U is the domain of f, then $\operatorname{codim}_X(X \smallsetminus U) \geq 2$.*
ii) *More generally, given a rational map $f\colon X \dashrightarrow Y$ and a proper morphism $g\colon Y \to Z$ such that the composition $g \circ f$ is a morphism, if U is the domain of f, then $\operatorname{codim}_X(X \smallsetminus U) \geq 2$.*

Before giving the proof, we introduce one notion that is needed in the proof, and which will play an important role in this chapter. Given an irreducible variety X that is smooth in codimension 1 and an irreducible, closed subvariety V of X, with $\operatorname{codim}_X(V) = 1$, the local ring $\mathcal{O}_{X,V}$ is a DVR. We denote by ord_V the corresponding discrete valuation of $k(X)$ (see § C.5). Note that for $\phi \in k(X)$, we have $\operatorname{ord}_V(\phi) \geq 0$ if and only if $\phi \in \mathcal{O}_{X,V}$, that is, if and only if ϕ is defined on an open subset that intersects V non-trivially.

We say that $\phi \in k(V)$ has a *pole* along V if $\operatorname{ord}_V(\phi) < 0$. This is the case if and only if ϕ^{-1} is defined in an open subset U with $U \cap V \neq \emptyset$ and $\phi^{-1}|_{U\cap V} = 0$.

If $\mathrm{ord}_V(\phi) = -m$, for a positive integer m, we say that ϕ has a *pole of order* m along V; similarly, if $\mathrm{ord}_V(\phi) = m > 0$, we say that ϕ has a *zero of order* m along V. Note that by Corollary 9.1.5, if X is normal, then for every $\phi \in k(X)$, the complement of the domain of ϕ is a union of codimension 1 irreducible subvarieties: these are precisely those V with $\mathrm{ord}_V(\phi) < 0$.

Proof of Proposition 9.1.6. Of course, it is enough to prove the second assertion. Note first that we may assume that f is dominant: indeed, we have a closed subvariety Y' of Y such that f induces a dominant rational map $X \dashrightarrow Y'$ and we may replace g by the composition $Y' \hookrightarrow Y \longrightarrow Z$. By the relative version of Chow's lemma (see Theorem 5.2.2), we can find a proper, birational morphism $h\colon \widetilde{Y} \to Y$ such that the composition $g \circ h$ factors as

$$\widetilde{Y} \overset{i}{\hookrightarrow} Z \times \mathbf{P}^n \xrightarrow{p} Z$$

for some n, with p being the projection onto the first component and i a closed immersion. Note that we may replace f by $\widetilde{f} = h^{-1} \circ f$: if this is defined on the complement of a closed subset of codimension ≥ 2, the same will hold for the composition $h \circ \widetilde{f} = f$. Moreover, it is enough to show that $i \circ \widetilde{f}$ satisfies the same property. We can write $i \circ f = (g \circ f, f_1)$, for a rational map $f_1\colon X \dashrightarrow \mathbf{P}^n$. By assumption, $g \circ f$ is a morphism, hence it is enough to show that f_1 can be defined on the complement of a closed subset of codimension ≥ 2. In other words, it is enough to prove the assertion in i), with $Y = \mathbf{P}^n$.

Note that we can find an open subset U of X and $\phi_0, \ldots, \phi_n \in \mathcal{O}_X(U)$ not all 0, such that f is defined on U and

$$f(x) = [\phi_0(x), \ldots, \phi_n(x)] \quad \text{for all} \quad x \in U.$$

Indeed, given a point p in the domain of f, we can choose i such that $f(p)$ lies in the affine open subset U_i of $\mathbf{P}^n$ given by $x_i \neq 0$. In this case, by taking U to be the inverse image of U_i, we can find $\phi_0, \ldots, \phi_n$ as above, with $\phi_i = 1$.

We need to show that given any irreducible closed subset V of X, with $\mathrm{codim}_X(V) = 1$, V intersects the domain of f. Given such V, let j be such that

$$\mathrm{ord}_V(\phi_j) = \min\{\mathrm{ord}_V(\phi_0), \ldots, \mathrm{ord}_V(\phi_n)\}.$$

In particular, $\phi_j \neq 0$. In this case, each rational function ϕ_i/ϕ_j is defined on an open subset that intersects V. We can thus find such an open subset intersecting V on which all ϕ_i/ϕ_j are defined and f is also defined on this open subset, given by

$$x \to \left(\frac{\phi_0}{\phi_j}(x), \ldots, \frac{\phi_n}{\phi_j}(x)\right).$$

This completes the proof of the proposition. $\square$

Remark 9.1.7. A special case of the above proposition says that if X is a smooth curve and Y is a complete variety, then every rational map $X \dashrightarrow Y$ is a morphism.

A nice feature of normality is that it can be arranged by a canonical operation. The main ingredient is the following result about algebras of finite type over a field.

Theorem 9.1.8. *Let A be an algebra of finite type over a field k, with A an integral domain. If K is the fraction field of A and L is a finite field extension of K, then the integral closure B of A in L is finite over A.*

PROOF. We give the proof following [**Eis95**]. Note that since A is Noetherian, it is enough to show that B is a submodule of a finitely generated A-module. In particular, we may replace at any point L by a finite extension L': if the integral closure of A in L' is finite over A, then so is B.

The first step in the proof is to show that we may assume that A is normal and the field extension L/K is a separable extension. We apply Noether's Normalization lemma to find a subring R of A that is isomorphic to a polynomial ring $k[x_1, \ldots, x_n]$ and such that A is finite over R. In this case, B is also the integral closure of R in L, hence after replacing A by R, we may assume that $A = k[x_1, \ldots, x_n]$. In particular, A is normal, and $K = k(x_1, \ldots, x_n)$.

After possibly replacing L by a suitable finite extension, we may assume that the extension L/K is normal. Let us show that we may assume that the extension is also separable. If this is not separable, then let $p = \mathrm{char}(k) > 0$, $G = G(L/K)$, and K' the subfield of L fixed by G. In this case the extension L/K' is separable and K'/K is purely inseparable. If we show that the integral closure A' of A in K' is finite over A, then we only need to show that the integral closure of A' in L is finite over A'; since A' is normal, this would complete the proof of the reduction step. Since K'/K is purely inseparable, we can find $e > 0$ such that for every $f \in K'$, we have $f^{p^e} \in K = k(x_1, \ldots, x_n)$. We can thus find a finite extension k' of k such that $K' \subseteq K'' = k'(x_1^{1/p^e}, \ldots, x_n^{1/p^e})$. Note that the integral closure of A in K'' is $k'[x_1^{1/p^e}, \ldots, x_n^{1/p^e}]$ (indeed, this is a finite extension of $k[x_1, \ldots, x_n]$ and it is a normal ring); this is clearly finite over A and since it contains A', it follows that A' is finite over A.

We conclude that in order to complete the proof it is enough to treat the case when A is normal and the extension L/K is separable. After possibly enlarging L, we may assume that L/K is a Galois extension, with group G. Let $\sigma_1, \ldots, \sigma_r$ be the elements of G and let $u_1, \ldots, u_r \in L$ be a basis of L over K. After multiplying each u_i by a suitable element of A, we may assume that $u_i \in B$ for every i. In this case we have $\sigma_i(u_j) \in B$ for all i, j and we consider the matrix $M = \big(\sigma_i(u_j)\big) \in M_r(B)$, with $D = \det(M)$.

Note first that $D \neq 0$. Indeed, if $D = 0$, then there are $\lambda_1, \ldots, \lambda_r \in L$, not all 0, such that $\left(\sum_{i=1}^r \lambda_i \sigma_i\right)(u_j) = 0$ for all j. Hence $\sum_{i=1}^r \lambda_i \sigma_i = 0$. This can't happen since distinct field automorphisms of L are linearly independent over L. We recall the argument: after relabeling the σ_i, we may assume that $\sum_{i=1}^s \lambda_i \sigma_i = 0$, with all $\lambda_i \neq 0$, and that s is minimal with the property that we have such a relation. Note that $s \geq 2$. For every $a, b \in L$, we have

$$0 = \sum_{i=1}^s \lambda_i \sigma_i(ab) = \left(\sum_{i=1}^s \lambda_i \sigma_i(a) \sigma_i\right)(b),$$

hence

$$\sum_{i=1}^r \lambda_i \sigma_i(a) \sigma_i = 0.$$

Choose a such that $\sigma_1(a) \neq \sigma_2(a)$ and note that we have

$$\sum_{i=2}^s \big(\sigma_1(a) - \sigma_i(a)\big) \lambda_i \sigma_i = 0.$$

Since the coefficient of σ_2 is non-zero, this contradicts the minimality of s.

We thus have $D \neq 0$. Note that for every i, $\sigma_i(D)$ is the determinant of a matrix obtained by permuting the rows of M, hence $\sigma_i(D) = \pm D$. This implies that $\sigma_i(D^2) = D^2$ for all i, hence $D^2 \in K$.

We will show that $B \subseteq \frac{1}{D^2} \cdot \sum_{i=1}^{r} A \cdot u_i$, which is a finitely generated A-module. This would imply that B is finite over A, completing the proof. Given any $u \in B$, we can write $u = \sum_{j=1}^{r} \alpha_j u_j$, with $\alpha_j \in K$. In order to obtain our assertion, we need to show that $D^2\alpha_j \in A$ for all j. Note that since $u \in B$, we have $\sigma_i(u) \in B$ for all i, hence

$$\sigma_i(u) = \sum_{j=1}^{r} \sigma_i(u_j)\alpha_j \in B.$$

Since the matrix $M \cdot (\alpha_1, \ldots, \alpha_r)^{\mathsf{T}}$ has entries in B, after multiplying with the classical adjoint of M, we deduce that $D \cdot \alpha_j \in B$ for all j. Since we have $D \in B$, it follows that

$$D^2\alpha_j \in B \cap K = A \quad \text{for all} \quad j,$$

where we use the fact that A is integrally closed in K. This completes the proof of the theorem. $\square$

Suppose now that X is an irreducible variety. For every affine open subset $U \subseteq X$, consider the $\mathcal{O}_X(U)$-algebra $\mathcal{A}(U)$, given by the integral closure of $\mathcal{O}_X(U)$ in its field of fractions $k(X)$. Given two such affine open subsets $V \subseteq U$, we have an injective homomorphism $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V)$ (this is injective since V is dense in U), which induces an injective homomorphism $\mathcal{A}(U) \hookrightarrow \mathcal{A}(V)$. Moreover, it follows from Lemma 9.1.1 that if U is as above and $f \in \mathcal{O}_X(U)$, then the induced homomorphism $\mathcal{A}(U)_f \to \mathcal{A}\big(D_U(f)\big)$ is an isomorphism. It thus follows from Lemma 8.7.14 that $\mathcal{A}$ can be extended to a quasi-coherent sheaf (in fact, to an $\mathcal{O}_X$-algebra). In fact, it follows from Theorem 9.1.8 that $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module. Since $\mathcal{A}(U)$ is reduced (in fact, a domain) for every affine open subset U, we can thus define the *normalization* of X to be given by $X^{\mathrm{norm}} = \mathcal{M}axSpec(\mathcal{A})$. Note that this comes with a canonical morphism $\pi\colon X^{\mathrm{norm}} \to X$ which is a finite morphism.

Note that X^{norm} is an irreducible variety. Indeed, if X is covered by the affine open subsets $U_1, \ldots, U_n$, then each $\pi^{-1}(U_i)$ is an irreducible variety and for every i and j, we have

$$\pi^{-1}(U_i) \cap \pi^{-1}(U_j) = \pi^{-1}(U_i \cap U_j) \neq \emptyset.$$

Moreover, it follows by construction that X^{norm} is a normal variety and that π is a birational morphism.

We have the following universal property of normalization:

PROPOSITION 9.1.9. *Let X be an irreducible variety and $\pi\colon X^{\mathrm{norm}} \to X$ the normalization morphism. For every dominant morphism $f\colon Z \to X$, with Z irreducible and normal, there is a unique morphism $g\colon Z \to X^{\mathrm{norm}}$ such that $\pi \circ g = f$.*

PROOF. Suppose first that X is affine, with $\mathcal{O}(X) = A$ and let B denote the integral closure of A in its field of fractions, so that $\mathcal{O}(X^{\mathrm{norm}}) = B$. By Proposition 2.3.14, f corresponds to a k-algebra homomorphism $f^{\#}\colon A \to \mathcal{O}_Z(Z)$ and giving a morphism g as in the statement is equivalent to giving a k-algebra homomorphism $g^{\#}\colon B \to \mathcal{O}_Z(Z)$, whose restriction to A is equal to $f^{\#}$. Since f is a dominant morphism, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f^{\#}"] \arrow[d] & \mathcal{O}_Z(Z) \arrow[d] \\ k(X) \arrow[r, "j"] & k(Z) \end{tikzcd}$$

in which the vertical maps are the natural inclusions. It is clear that if $g^{\#}$ exists, then it is given by the restriction of j to B, and existence is equivalent to the condition $j(B) \subseteq \mathcal{O}_Z(Z)$.

Consider an affine open cover $Z = V_1 \cup \ldots \cup V_r$. For every i, we have $j(A) \subseteq \mathcal{O}(V_i)$, and since B is integral over A and $\mathcal{O}(V_i)$ is normal, it follows that $j(B) \subseteq \mathcal{O}(V_i)$. Since this holds for every i, we conclude that $j(B) \subseteq \mathcal{O}_Z(Z) = \bigcap_{i=1}^{r} \mathcal{O}_Z(V_i)$. This proves the existence of $g^{\#}$.

The case when X is not necessarily affine, follows by taking an affine open cover: the corresponding morphisms patch due to the uniqueness we have already proved, giving existence, while uniqueness again follows since g has to induce the field homomorphism $j\colon k(X) \hookrightarrow k(Z)$. $\square$

Remark 9.1.10. If X is a reducible variety, with irreducible components $X_1, \ldots, X_n$ and if $X_i^{\mathrm{norm}} \to X_i$ are the normalization morphisms, then we get a morphism

$$X^{\mathrm{norm}} := \bigsqcup_{i=1}^{n} X_i^{\mathrm{norm}} \to X,$$

which is, by definition, the normalization of the reducible variety X.

Remark 9.1.11. The construction of the normalization can be generalized as follows. Given an irreducible variety X and a finite field extension $k(X) \hookrightarrow K$, one constructs a finite morphism $f\colon Y \to X$ such that for every affine open subset U of X, the homomorphism $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_Y\big(f^{-1}(U)\big)$ is the inclusion in the integral closure of $\mathcal{O}_X(U)$ in K. This follows verbatim as in the construction of the normalization. Note that the variety Y is normal, by construction.

Exercise 9.1.12. Show that if X is an irreducible normal variety and U is an affine open subset of X, then every irreducible component of $X \smallsetminus U$ has codimension 1 in X.

Exercise 9.1.13. Compute the normalization morphism for each of the following varieties:

i) X is the curve in $\mathbf{A}^2$ given by $x^2 - y^3 = 0$.
ii) Y is the surface in $\mathbf{A}^3$ given by $x^2 - y^2 z = 0$.

Exercise 9.1.14. Let $f \in k[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be an irreducible polynomial of the form

$$f = x_1^{a_1} \cdots x_r^{a_r} - y_1^{b_1} \cdots y_s^{b_s}$$

for non-negative integers $a_1, \ldots, a_r, b_1 \ldots, b_s$. Let $Z \subseteq \mathbf{A}^{r+s}$ be the hypersurface defined by f.

i) Show that if Z is smooth in codimension 1, then either $a_i \in \{0, 1\}$ for $1 \leq i \leq r$ or $b_j \in \{0, 1\}$ for $1 \leq j \leq s$.

ii) Show that conversely, if say $a_i \in \{0,1\}$ for all i, then f is an irreducible polynomial that defines a normal hypersurface Z in $\mathbf{A}^{r+s}$. Hint: you can show that $\mathcal{O}(Z) \simeq k[S]$ for an integral, finitely generated, saturated semigroup S, and then use Proposition 1.7.30.

9.2. Smooth varieties are locally factorial

Definition 9.2.1. A variety X is *locally factorial* if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a UFD.

Remark 9.2.2. Since the localization of a UFD at a prime ideal is again a UFD, it follows that if a variety X is locally factorial, then for every irreducible closed subset V of X, the local ring $\mathcal{O}_{X,V}$ is a UFD. However, we emphasize that if X is a locally factorial, irreducible affine variety, it *does not* imply that $\mathcal{O}(X)$ is a UFD.

The following is the main result of this section:

Theorem 9.2.3. *Every smooth variety is locally factorial.*

We note that it is a general theorem of Auslander and Buchsbaum that every regular local ring is a UFD (see [**Eis95**, Theorem 19.19]). The proof in the general case makes use of homological algebra techniques. We give a proof in our geometric setting using completions, following [**Mum88**, Chapter III.7].

For the definitions and basic facts regarding completion, we refer to Appendix G. Given a variety X and a point $x \in X$, we will be interested in the completion $\widehat{\mathcal{O}_{X,x}}$ of the local ring $\mathcal{O}_{X,x}$, with respect to the maximal ideal. This is a local ring and we have a local, injective, flat homomorphism $\psi\colon \mathcal{O}_{X,x} \to \widehat{\mathcal{O}_{X,x}}$.

While the local ring $\mathcal{O}_{X,x}$ remembers properties of X in small, Zariski open neighborhoods of x, the completion of this ring encodes properties of a "more local" nature at $x \in X$. If we work over the complex numbers, we will see that we can consider small neighborhoods of the point x in the classical topology and in this way, we can identify points that are different from the algebraic point of view. A typical example is that a smooth variety becomes a complex manifold and thus every point has a neighborhood in the classical topology that is homeomorphic to an open ball in the affine space of the same dimension. Over an arbitrary ground field, this can be modeled algebraically by considering the completion $\widehat{\mathcal{O}_{X,x}}$. More generally, if X is an affine variety with $\mathcal{O}(X) = R$ and Y is a closed subvariety defined by the ideal I, then the completion $\widehat{R}$ of R with respect to I can be considered as an algebraic analogue of a tubular neighborhood of Y in X.

A manifestation of the above point of view is the following

Proposition 9.2.4. *Given a variety X over k and a smooth point $x \in X$, with $\dim(\mathcal{O}_{X,x}) = d$, we have an isomorphism of k-algebras $\widehat{\mathcal{O}_{X,x}} \simeq k[\![t_1, \ldots, t_d]\!]$.*

Proof. Recall that if $R = \mathcal{O}_{X,x}$, with maximal ideal $\mathfrak{m}$, and if $u_1, \ldots, u_d \in \mathfrak{m}$ form a minimal system of generators, then the smoothness of X at x implies that the k-algebra homomorphism

$$\phi\colon k[t_1, \ldots, t_d] \to \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}, \quad \phi(t_j) = \overline{u_j} \in \mathfrak{m}/\mathfrak{m}^2$$

is an isomorphism (see Proposition 6.3.9). In order to simplify the notation, let $S = k[t_1, \ldots, t_d]$ and $\mathfrak{n} = (t_1, \ldots, t_d) \subseteq S$.

We also consider, for every $i \geq 1$, the k-algebra homomorphism

$$\phi_i \colon S/\mathfrak{n}^i \to R/\mathfrak{m}^i, \quad \phi_i(t_j) = u_j \bmod \mathfrak{m}^i.$$

Note that we have a commutative diagram of k-vector spaces, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}^i/\mathfrak{n}^{i+1} \arrow[r] \arrow[d] & S/\mathfrak{n}^{i+1} \arrow[r] \arrow[d, "\phi_{i+1}"'] & S/\mathfrak{n}^i \arrow[r] \arrow[d, "\phi_i"'] & 0 \\
0 \arrow[r] & \mathfrak{m}^i/\mathfrak{m}^{i+1} \arrow[r] & R/\mathfrak{m}^{i+1} \arrow[r] & R/\mathfrak{m}^i \arrow[r] & 0.
\end{tikzcd}$$

Since we know that the left-most vertical map is an isomorphism, arguing by induction on i, we see that all ϕ_i are isomorphisms. By taking the inverse limit of the ϕ_i, we obtain an isomorphism

$$k[\![t_1, \ldots, t_d]\!] = \varprojlim S/\mathfrak{n}^i \simeq \varprojlim R/\mathfrak{m}^i = \widehat{R}.$$

$\square$

Remark 9.2.5. The converse of the above proposition also holds: if $x \in X$ is a point with $\dim(\mathcal{O}_{X,x}) = n$ and such that $\widehat{\mathcal{O}_{X,x}} \simeq k[\![x_1, \ldots, x_n]\!]$, then x is a smooth point of X. Indeed, if $(A, \mathfrak{m}, k)$ is a Noetherian local ring and $\widehat{A}$ is its completion, then the maximal ideal of $\widehat{A}$ is $\mathfrak{m}\widehat{A}$ and we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}\widehat{A}/\mathfrak{m}^2\widehat{A}$. We thus see that our hypothesis implies that $\dim_k T_xX = n$, hence x is a smooth point of X.

We also need the following lemma that deduces the UFD property of a local Noetherian ring from that of its completion.

Lemma 9.2.6. *If $(A, \mathfrak{m})$ is a local Noetherian ring and the completion $\widehat{A}$ of A with respect to $\mathfrak{m}$ is a UFD, then A is a UFD.*

Proof. Note first that A is a domain, since $\widehat{A}$ is a domain, and the canonical homomorhism $\psi \colon A \to \widehat{A}$ is injective. Since A is a Noetherian domain, in order to check that it is a UFD it is enough to show that for every $f, g \in A$, the ideal

$$J = fA : gA = \{h \in A \mid hg \in fA\}$$

is principal (conversely, this condition holds in any UFD); see Proposition F.1.3.

We claim that since $\widehat{A}$ is flat over A, we have $J\widehat{A} = f\widehat{A} \colon g\widehat{A}$. Indeed, we have an exact sequence of A-modules

$$0 \to J \to A \xrightarrow{\nu} A/(f),$$

where $\nu(a) = ga$, and by tensoring with $\widehat{A}$ and using the flatness of $\widehat{A}$, we obtain an exact sequence

$$0 \to J\widehat{A} \to \widehat{A} \to \widehat{A}/f\widehat{A}.$$

This gives our claim.

Since $\widehat{A}$ is a UFD, it follows that $J\widehat{A}$ is a principal ideal. Therefore we have $\dim_k J\widehat{A}/J\mathfrak{m}\widehat{A} = 1$, where $k = A/\mathfrak{m} = \widehat{A}/\mathfrak{m}\widehat{A}$. Using again the flatness of $\widehat{A}$ over A, we see that we have an isomorphism

$$J\widehat{A}/J\mathfrak{m}\widehat{A} \simeq (J/\mathfrak{m}J) \otimes_A \widehat{A} \simeq \widehat{J/\mathfrak{m}J} \simeq J/\mathfrak{m}J,$$

where the second and the third isomorphisms follow from Corollary G.2.2 and, respectively, Remark G.1.3. Therefore $\dim_k(J/\mathfrak{m}J) = 1$, hence J is a principal ideal by Nakayama's lemma. $\square$

We can now prove the main result of this section (assuming the factoriality of formal power series rings over a field).

Proof of Theorem 9.2.3. Given $x \in X$, with $\dim(\mathcal{O}_{X,x}) = d$, it follows from Proposition 9.2.4 that $\widehat{\mathcal{O}_{X,x}} \simeq k[\![t_1, \ldots, t_d]\!]$. This ring is a UFD (for an elementary proof of this fact, see [ZS75, Chapter VII, §1, Theorem 6]). We then deduce that $\mathcal{O}_{X,x}$ is a UFD by Lemma 9.2.6. $\square$

9.3. Weil divisors and the class group

In this section we consider an irreducible variety X that is smooth in codimension 1 (for example, a normal variety).

9.3.1. Weil divisors and linear equivalence. A *prime divisor* on X is an irreducible, codimension 1, closed subvariety V of X. The group $\mathrm{Div}(X)$ of Weil divisors on X is the free Abelian group on the set of prime divisors of X; thus a *Weil divisor* (or simply, a *divisor*) is a formal linear combination

$$D = n_1 D_1 + \ldots + n_r D_r,$$

with each D_i a prime divisor and $n_i \in \mathbf{Z}$. Such a divisor is *effective* if it is a linear combination of prime divisors, with *non-negative* coefficients; in this case we write $D \geq 0$. Given divisors D and E, we write $D \geq E$ if $D - E \geq 0$.

Given a non-zero rational function $\phi \in k(X)$, we obtain a corresponding divisor

$$\mathrm{div}(\phi) := \sum_V \mathrm{ord}_V(\phi) \cdot V,$$

where the sum is over all prime divisors on X (when X is not understood from the context, we also write $\mathrm{div}_X(\phi)$). Note that this is well-defined (in other words, there are only finitely many such V with $\mathrm{ord}_V(\phi) \neq 0$). Indeed, let U be an open subset such that ϕ is defined on U and let $U' = \{x \in U \mid \phi(x) \neq 0\}$. If $V \cap U' \neq \emptyset$, then ϕ is an invertible element of $\mathcal{O}_{X,V}$, hence $\mathrm{ord}_V(\phi) = 0$. Since there are only finitely many codimension 1 irreducible closed subsets of $X \smallsetminus U'$, we see that $\mathrm{div}(\phi)$ is well-defined.

Note that the map

$$\mathrm{div}\colon k(X)^* \to \mathrm{Div}(X)$$

is a group homomorphism. This follows from the fact that for every prime divisor V on X and for every non-zero $\phi_1, \phi_2 \in k(X)$, we have

$$\mathrm{ord}_V(\phi_1\phi_2) = \mathrm{ord}_V(\phi_1) + \mathrm{ord}_V(\phi_2).$$

A divisor of the form $\mathrm{div}(\phi)$, for some $\phi \in k(X)^*$, is a *principal* divisor. For two divisors D and E, we say that D and E are *linearly equivalent* (and we write $D \sim E$) if $D - E$ is a principal divisor. The quotient of $\mathrm{Div}(X)$ by the subgroup $\mathrm{PDiv}(X)$ of principal divisors, is the *class group* $\mathrm{Cl}(X)$. We write $[D]$ for the class of the divisor D in $\mathrm{Cl}(X)$.

Remark 9.3.1. If X is a normal variety, then for every $\phi \in k(X)^*$, we have $\phi \in \mathcal{O}(X)$ if and only if $\mathrm{div}(\phi) \geq 0$. Indeed, the condition $\mathrm{div}(\phi) \geq 0$ is equivalent to the fact that if U is the domain of ϕ, then $\mathrm{codim}_X(X \smallsetminus U) \geq 2$. However, in this case it follows from Proposition 9.1.5 that $\phi \in \mathcal{O}(X)$.

Applying this remark to both ϕ and ϕ^{-1}, we see that $\mathrm{div}(\phi) = 0$ if and only if $\phi \in \mathcal{O}_X^*(X)$, that is, ϕ is an invertible function on X.

PROPOSITION 9.3.2. *If X is an irreducible normal affine variety, then* $\mathrm{Cl}(X) = 0$ *if and only if* $A = \mathcal{O}(X)$ *is a UFD.*

PROOF. Note that A is a UFD if and only if every codimension 1 prime ideal $\mathfrak{p}$ in A is principal (see Proposition F.1.4). On the other hand, $\mathrm{Cl}(X) = 0$ if and only if every prime divisor is principal. This means that for every codimension 1 prime ideal $\mathfrak{p}$ in $\mathcal{O}(X)$, with corresponding subvariety Y, there is a non-zero ϕ in the fraction field K of A, such that $\mathrm{div}(\phi) = Y$. Note that since $\mathrm{div}(\phi)$ is effective, it follows from Remark 9.3.1 that $\phi \in \mathcal{O}(X)$. The condition on ϕ is thus that $\phi \cdot A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, while $\phi \not\in \mathfrak{q}$ for every codimension 1 prime ideal $\mathfrak{q} \neq \mathfrak{p}$.

If A is a UFD, then for every $\mathfrak{p}$ as above, $\mathfrak{p}$ is principal, and it is clear that if $\mathfrak{p} = (\pi)$, then $\mathrm{div}(\pi)$ is the prime divisor corresponding to $\mathfrak{p}$. Conversely, given a codimension 1 prime ideal $\mathfrak{p}$, let us choose $\phi \in A$ such that $\phi \cdot A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ and $\phi \not\in \mathfrak{q}$, for every codimension 1 prime ideal $\mathfrak{q} \neq \mathfrak{p}$. We claim that in this case $\mathfrak{p} = (\phi)$. It is clear that we have $\phi \in \mathfrak{p}$. Suppose now that $u \in \mathfrak{p}$. The hypothesis on ϕ says that $\mathrm{div}(u/\phi) \geq 0$, hence $u \in (\phi)$ by Remark 9.3.1. Therefore $\mathfrak{p} = (\phi)$, hence it is principal, and we see that A is a UFD. $\square$

EXAMPLE 9.3.3. It follows from the above proposition that $\mathrm{Cl}(\mathbf{A}^n) = 0$.

EXAMPLE 9.3.4. Consider the case $X = \mathbf{P}^n$. Recall that if Y is an irreducible closed subset of $\mathbf{P}^n$ of codimension 1, then the ideal of Y in $\mathbf{P}^n$ is generated by a homogeneous polynomial f (see Exercise 9.3.1). We put $\deg(Y) := \deg(f)$. For a divisor $D = \sum_{i=1}^r n_i Y_i$ in $\mathbf{P}^n$, we put $\deg(D) := \sum_{i=1}^r n_i \cdot \deg(Y_i)$. It is clear that we have a group homomorphism $\deg\colon \mathrm{Div}(\mathbf{P}^n) \to \mathbf{Z}$. This is surjective: if H is a hyperplane in $\mathbf{P}^n$, then $\deg(H) = 1$.

We claim that if D is a principal divisor, then $\deg(D) = 0$. Indeed, if $D = \mathrm{div}(\phi)$, for a non-zero rational function ϕ on $\mathbf{P}^n$, then we know that we can write $\phi = \frac{F}{G}$, where $F, G \in S = k[x_0 \ldots, x_n]$ are non-zero homogeneous polynomials of the same degree. It is easy to check that if we consider the irreducible decompositions

$$F = c_F \cdot F_1^{m_1} \cdots F_r^{m_r} \quad \text{and} \quad G = c_G \cdot G_1^{n_1} \cdots G_s^{n_s},$$

with $c_F, c_G \in k^*$, then $\mathrm{div}(\phi) = \sum_{i=1}^r m_i V(F_i) - \sum_{j=1}^s n_j V(G_j)$, hence

$$\deg\bigl(\mathrm{div}(\phi)\bigr) = \sum_{i=1}^r m_i \cdot \deg(F_i) - \sum_{j=1}^s n_j \cdot \deg(G_j) = \deg(F) - \deg(G) = 0.$$

We thus obtain an induced surjective group homomorphism

$$\deg\colon \mathrm{Cl}(\mathbf{P}^n) \to \mathbf{Z}.$$

We claim that this is an isomorphism. Indeed, it follows from the above discussion that if $D = \sum_{i=1}^d n_i V(f_i)$, for irreducible homogeneous polynomials $f_1, \ldots, f_d$, is such that $\deg(D) = 0$, then if we put

$$\phi := \frac{\prod_{n_i > 0} f_i^{n_i}}{\prod_{n_i < 0} f_i^{-n_i}},$$

then ϕ gives a rational function on $\mathbf{P}^n$ and $\mathrm{div}(\phi) = D$.

EXAMPLE 9.3.5. Given a divisor D on X and an open subset U of X, we define $D|_U$ as follows: if $D = \sum_{i=1}^r n_i D_i$, then $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ (note that if $D_i \cap U \neq \emptyset$, then $D_i \cap U$ is an irreducible, closed subvariety of U, of codimension 1).

With this notation, if the irreducible components of $X \smallsetminus U$ that have codimension 1 in X are $Z_1, \ldots, Z_s$, then we have a short exact sequence

$$\mathbf{Z}^s \xrightarrow{\alpha} \mathrm{Cl}(X) \xrightarrow{\beta} \mathrm{Cl}(U) \longrightarrow 0,$$

where

$$\alpha(m_1, \ldots, m_s) = [m_1 Z_1 + \ldots + m_s Z_s] \quad \text{and} \quad \beta\big([D]\big) = [D|_U].$$

It is clear that $\beta \circ \alpha = 0$. Moreover, β is surjective: if $Y_1, \ldots, Y_r$ are irreducible closed subsets of U, of codimension 1, and $D = \sum_{i=1}^r n_i Y_i$, then $\overline{Y_1}, \ldots, \overline{Y_r}$ are irreducible, closed subsets of X of codimension 1, and if $E = \sum_{i=1}^r n_i \overline{Y_i}$, then $\beta\big([E]\big) = [D]$. Finally suppose that F is a divisor on X such that for some $\phi \in k(X)^*$, we have $D|_U \simeq \mathrm{div}_U(\phi)$. In this case, we have

$$F - \mathrm{div}_X(\phi) = -\sum_{i=1}^s \mathrm{ord}_{Z_i}(\phi),$$

hence $[F]$ lies in the image of α.

Example 9.3.6. Let Y be a hypersurface in $\mathbf{P}^n$. If the irreducible components of Y are $Y_1, \ldots, Y_r$ and $\deg(Y_i) = d_i$, then it follows from Examples 9.3.4 and 9.3.5 that $\mathrm{Cl}(\mathbf{P}^n \smallsetminus Y) \simeq \mathbf{Z}/d\mathbf{Z}$, where d is the greatest common divisor of $d_1, \ldots, d_r$.

Remark 9.3.7. Since we assume that X is smooth in codimension 1, it follows from Example 9.3.5 that by restricting divisor classes to the smooth locus X_{sm}, we get an isomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X_{\mathrm{sm}})$.

9.3.2. The sheaf associated to a Weil divisor. Suppose now that X is a normal variety. To a divisor D on X, we associate a subsheaf $\mathcal{O}_X(D)$ of the constant sheaf $k(X)$, as follows. For every open subset U of X, we put

$$\Gamma\big(U, \mathcal{O}_X(D)\big) = \{0\} \cup \{\phi \in k(X)^* \mid \mathrm{div}(\phi)|_U + D|_U \geq 0\}.$$

Explicitly, if $D = \sum_{i=1}^r a_i D_i$, then $\phi \in k(X)$ lies in $\Gamma\big(U, \mathcal{O}_X(D)\big)$ if and only if $\phi = 0$ or $\mathrm{ord}_V(\phi) \geq -a_i$ for all V such that $V \cap U \neq \emptyset$. It is clear that the identity on $k(X)$ induces restriction maps such that $\mathcal{O}_X(D)$ is a subsheaf of $k(X)$. In fact, it is an $\mathcal{O}_X$-submodule of $k(X)$, since it is clear that if $\phi \in \Gamma\big(U, \mathcal{O}_X(D)\big)$ and $f \in \mathcal{O}_X(U)$, then $f\phi \in \Gamma\big(U, \mathcal{O}_X(D)\big)$.

Example 9.3.8. If $D = 0$, then it follows from Remark 9.3.1 that $\mathcal{O}_X(D) = \mathcal{O}_X$.

Remark 9.3.9. Note that if $D \geq E$, then $\mathcal{O}_X(E) \subseteq \mathcal{O}_X(D)$. In particular, it follows from the previous remark that if $E \leq 0$, then $\mathcal{O}_X(E)$ is a subsheaf of $\mathcal{O}_X$.

Proposition 9.3.10. *Let X be a normal variety. For every divisor D on X, the sheaf $\mathcal{O}_X(D)$ is coherent and its stalk at X is isomorphic to $k(X)$.*

Proof. Consider an open subset U of X and $f \in \mathcal{O}_X(U)$. The morphism

$$\Gamma\big(U, \mathcal{O}_X(D)\big)_f \to \Gamma\big(V, \mathcal{O}_X(D)\big),$$

where $V = D_U(f)$, is clearly injective since $k(X)$ is a domain. In order to check surjectivity, consider a non-zero $\phi \in \Gamma\big(V, \mathcal{O}_X(D)\big)$. Since $\mathrm{div}(\phi)|_V + D|_V$ is effective, it follows that the only possible negative coefficients of $D' := \mathrm{div}(\phi)|_U + D|_U$ are for the irreducible components $W_1, \ldots, W_r$ of $V(f) \subseteq U$. Let m be a non-negative integer such that for every i, the coefficient of W_i in D' is $\geq -m$. Since $\mathrm{ord}_{W_i}(f) \geq 1$ for all i, we conclude that

$$\mathrm{div}(f^m\phi)|_U + D|_U \geq 0,$$

hence $f^m\phi \in \Gamma\big(U, \mathcal{O}_X(D)\big)$. This shows that $\mathcal{O}_X(D)$ is a quasi-coherent sheaf.

We now show that $\mathcal{O}_X(D)$ is coherent. Let U be an affine open subset of X. If $Z_1, \ldots, Z_r$ are the prime divisors in U that appear with positive coefficient in $D|_U$, let $g \in \prod_{i=1}^r I_U(Z_i)$. For $m \gg 0$, we have $D|_U \leq \mathrm{div}(g^m)$, hence

$$\Gamma\big(U, \mathcal{O}_X(D)\big) \subseteq \{0\} \cup \{\phi \mid \mathrm{div}(\phi g^m) \geq 0\} = \frac{1}{g^m} \cdot \Gamma(U, \mathcal{O}_X).$$

Therefore $\Gamma\big(U, \mathcal{O}_X(D)\big)$ is a finitely generated $\Gamma(U, \mathcal{O}_X)$-module.

If $D = \sum_{i=1}^r n_i V_i$ and $U = X \smallsetminus \bigcup_{i=1}^r V_i$, then it follows from definition that $\mathcal{O}_X(D)|_U \simeq \mathcal{O}_U$. In particular, the stalk of $\mathcal{O}_X(D)$ at X is equal to $k(X)$. $\square$

Proposition 9.3.11. *Let X be an irreducible, normal variety. For two divisors D and E, we have $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ if and only if D and E are linearly equivalent.*

Proof. If $D - E = \mathrm{div}(\phi)$, then we have an isomorphism $\mathcal{O}_X(D) \to \mathcal{O}_X(E)$, given over every open subset $U \subseteq X$ by multiplication by ϕ. Conversely, suppose that we have an isomorphism $\alpha\colon \mathcal{O}_X(D) \to \mathcal{O}_X(E)$. Given a non-empty affine open subset U, we claim that the isomorphism

$$\alpha_U \colon \Gamma\big(U, \mathcal{O}_X(D)\big) \to \Gamma\big(U, \mathcal{O}_X(E)\big)$$

is given by multiplication with some unique non-zero $\phi_U \in k(X)$. In order to show this, it is enough to show that for every two non-zero $u, v \in \Gamma\big(U, \mathcal{O}_X(D)\big)$, we have $\frac{\alpha_U(u)}{u} = \frac{\alpha_U(v)}{v}$. This follows from the fact that α_U is a morphism of $\mathcal{O}_X(U)$-modules:

$$v \cdot \alpha_U(u) = \alpha_U(uv) = u \cdot \alpha_U(v).$$

This implies the existence of ϕ_U. The fact that ϕ_U is non-zero and unique follows from the fact that $k(X)$ is a domain and $\Gamma\big(U, \mathcal{O}_X(D)\big)$ and $\Gamma\big(U, \mathcal{O}_X(E)\big)$ are non-zero (this, in turn, follows from the fact that the stalks of $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ at $k(X)$ are isomorphic to $k(X)$). Since any two non-empty open subsets U and U' of X intersect non-trivially, it follows that all ϕ_U are equal; we thus simply denote this by ϕ.

By definition of ϕ, we know that for every affine open subset U of X and every $\psi \in k(X)^*$, we have

$$\mathrm{div}(\psi)|_U + D|_U \geq 0 \quad \text{if and only if} \quad \mathrm{div}(\phi\psi)|_U + E|_U \geq 0.$$

We will show that in this case we have $D = E + \mathrm{div}(\phi)$. Given a prime divisor V, let a_V and b_V be the coefficients of V in D and E, respectively, In order to show that $a_V = b_V + \mathrm{ord}_V(\phi)$ we may restrict to a suitable affine open subset of X and thus assume that X is smooth, V is smooth and $I_X(V) = (h)$, while $D = a_V V$ and $E = b_V V$. Note that $\mathrm{ord}_V(h) = 1$, hence by taking $\psi = h^{-a_V}$, we conclude that $b_V + \mathrm{ord}_V(\phi) - a_V \geq 0$, and by taking $\psi = h^{-b_V - \mathrm{ord}_V(\phi)}$, we conclude that $a_V - b_V - \mathrm{ord}_V(\phi) \geq 0$. Therefore $a_V = b_V + \mathrm{ord}_V(\phi)$, completing the proof. $\square$

Remark 9.3.12. Note that for every two divisors D and E, multiplication in $k(X)$ induces a canonical morphism

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D + E).$$

9.3.3. Push-forward by finite maps. We now discuss the push-forward of Weil divisors by finite, surjective morphisms. Let $f\colon X \to Y$ be a morphism of irreducible varieties that are smooth in codimension 1. We assume that f is finite and surjective. If V is a closed, irreducible subvariety of X of codimension 1, then $f(V)$ is a closed, irreducible subvariety of Y, also of codimension 1. We put $\deg\big(V/f(V)\big)$ for the degree of the induced map $V \to f(V)$ (recall that this is the degree of the field extension $k\big(f(V)\big)/k(V)$. We define a group homomorphism

$$f_*\colon \mathrm{Div}(X) \to \mathrm{Div}(Y)$$

that maps each prime divisor V to $\deg\big(V/f(V)\big) \cdot V$.

PROPOSITION 9.3.13. *With the above notation, for every non-zero $\phi \in k(X)$, we have*

$$f_*\big(\mathrm{div}_X(\phi)\big) = \mathrm{div}_Y\big(N_{k(X)/k(Y)}(\phi)\big).$$

In particular, we get an induced morphism of Abelian groups $f_*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$.

PROOF. The second assertion follows immediately from the first one, hence we focus on this one. The assertion to prove is equivalent to the following: for every irreducible, closed subvariety W of Y, of codimension 1, we have

$$\sum_{f(V)=W} \mathrm{ord}_V(\phi) \cdot \deg(V/W) = \mathrm{ord}_W\left(N_{k(X)/k(Y)}(\phi)\right), \tag{9.3.1}$$

where the sum on the left-hand side is over the irreducible, closed subvarieties V of X, with $f(V) = W$. After replacing Y by an open subset U that intersects W and X by $f^{-1}(U)$, we may assume that X and Y are affine varieties, with $A = \mathcal{O}(Y)$ and $B = \mathcal{O}(X)$. Moreover, after writing $\phi = \frac{b_1}{b_2}$, with $b_1, b_2 \in B$, we easily see that we may assume that $\phi \in B$. If $\mathfrak{p}$ is the prime ideal in A corresponding to W, then f induces a finite, injective ring homomorphism $A \hookrightarrow B$, which in turn induces a finite, injective ring homomorphism $\phi\colon A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$. The irreducible closed subsets V in X, of codimension 1, with $f(V) = W$ are in bijection with the prime ideals $\mathfrak{q}$ in B that whose inverse image in A is $\mathfrak{p}$, and thus with the maximal ideals in $B_{\mathfrak{p}}$. By assumption, $A_{\mathfrak{p}}$ is a DVR and for every ideal $\mathfrak{q}$ as above, the ring $B_{\mathfrak{q}}$ is a DVRs, hence

$$\mathrm{ord}_{V_i}(\phi) = \ell\big(B_{\mathfrak{q}_i}/(\phi)\big),$$

where $V_i = V(\mathfrak{q}_i)$. The equality in (9.3.1) then follows from Proposition H.2.2. $\square$

REMARK 9.3.14. If $V_1 \to V_2 \to V_3$ are finite, surjective morphisms of irreducible varieties, it is clear from definition that

$$\deg(V_1/V_3) = \deg(V_1/V_2) \cdot \deg(V_2/V_3).$$

This implies that given two surjective, finite morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of irreducible varieties, all of them being smooth in codimension 1, we have $(g \circ f)_* = g_* \circ f_*$ as maps $\mathrm{Div}(X) \to \mathrm{Div}(Z)$ (hence also as maps $\mathrm{Cl}(X) \to \mathrm{Cl}(Z)$).

REMARK 9.3.15. One can define a push-forward for arbitrary proper, generically finite morphisms such that the assertions in Proposition 9.3.13 still hold, see [**Ful98**, Chapter 1.4].

9.4. Cartier divisors

9.4.1. Cartier divisors on normal varieties. If X is a normal variety, then we distinguish on X a nice class of divisors such that the corresponding sheaves are line bundles.

DEFINITION 9.4.1. Given an irreducible, normal variety X, a divisor D on X is a *Cartier divisor* if it is *locally principal*, that is, there is an open cover $X = \bigcup_{i \in I} U_i$ of X such that $D|_{U_i}$ is a principal divisor for all i. It is clear that the Cartier divisors on X form a subgroup of $\mathrm{Div}(X)$, that we denote $\mathrm{Cart}(X)$.

PROPOSITION 9.4.2. *A divisor D on the irreducible, normal variety X is a Cartier divisor if and only if $\mathcal{O}_X(D)$ is a line bundle.*

PROOF. If $D|_U = \mathrm{div}_U(\phi)$, then it follows from Proposition 9.3.11 that

$$\mathcal{O}_X(D)|_U \simeq \mathcal{O}_U(D|_U) \simeq \mathcal{O}_U.$$

This shows that if D is a Cartier divisor, then $\mathcal{O}_X(D)$ is a line bundle. Conversely, suppose that $\mathcal{O}_X(D)$ is a line bundle. In this case, around every point there is an open neighborhood U such that $\mathcal{O}_X(D)|_U \simeq \mathcal{O}_U$. In this case, it follows from Proposition 9.3.11 that $D|_U$ is a principal divisor. □

REMARK 9.4.3. We note that if D is a Cartier divisor and if on an open subset U of X we have $D|_U = \mathrm{div}_U(\phi)$, then $\mathcal{O}_X(D)|_U = \frac{1}{\phi} \cdot \mathcal{O}_X \subseteq k(X)$. Indeed, given an open subset V of U and $\psi \in k(X)$ non-zero, we have $\mathrm{div}(\psi)|_V + D|_V \geq 0$ if and only if $\phi\psi \in \mathcal{O}_X(V)$.

It follows from Proposition 9.4.2 that we have a map $\mathrm{Cart}(X) \to \mathrm{Pic}(X)$ that maps D to the isomorphism class of $\mathcal{O}_X(D)$. The next lemma implies that this is a group homomorphism.

LEMMA 9.4.4. *If D and E are Cartier divisors on the normal variety X, then the canonical morphism*

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D+E)$$

is an isomorphism.

PROOF. It is enough to show that the morphism is an isomorphism on every affine open subset U of X such that $D|_U$ and $E|_U$ are principal divisors. Let us write $D|_U = \mathrm{div}_U(\alpha)$ and $E|_U = \mathrm{div}_U(\beta)$, with $\alpha, \beta \in k(X)^*$, in which case $D + E = \mathrm{div}_U(\alpha \cdot \beta)$. The morphism

$$\Gamma\big(U, \mathcal{O}_X(D)\big) \otimes_{\mathcal{O}_X(U)} \Gamma\big(U, \mathcal{O}_X(E)\big) \to \Gamma\big(U, \mathcal{O}_X(D+E)\big)$$

maps $\phi \otimes \psi$ to $\phi \cdot \psi$. The assertion now follows from the fact that

$$\Gamma\big(U, \mathcal{O}_X(D)\big) = \frac{1}{\alpha} \cdot \mathcal{O}_X(U), \quad \Gamma\big(U, \mathcal{O}_X(E)\big) = \frac{1}{\beta} \cdot \mathcal{O}_X(U), \quad \text{and}$$

$$\Gamma\big(U, \mathcal{O}_X(D+E)\big) = \frac{1}{\alpha\beta} \cdot \mathcal{O}_X(U).$$

□

If D is a principal divisor, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$, hence we get an induced homomorphism

$$\mathrm{Cart}(X)/\mathrm{PDiv}(X) \to \mathrm{Pic}(X),$$

and Proposition 9.3.11 implies that this is injective. It is also surjective: we will prove this in a more general context in Proposition 9.4.11 below.

Remark 9.4.5. Arguing as in the proof of Proposition 9.3.2, we see that every Weil divisor on X is Cartier if and only if X is locally factorial. In particular, Weil and Cartier divisors coincide on smooth varieties by Theorem 9.2.3.

Exercise 9.4.6. Let X be the affine cone over a smooth quadric surface in $\mathbf{P}^3$, that is, $X \subseteq \mathbf{A}^4$ is defined by the equation $x_1x_2 - x_3x_4 = 0$.

i) Show that X is smooth in codimension 1 (in fact, it is normal, see Exercise 9.1.14).
ii) Show that if L_1 and L_2 are the planes in X given by $x_1 = x_3 = 0$ and, respectively $x_1 = x_4 = 0$, then L_1 and L_2 are prime divisors on X which are not Cartier, but $L_1 + L_2$ is Cartier.

Exercise 9.4.7. Let $X \subseteq \mathbf{A}^3$ be the cone over the smooth conic in $\mathbf{P}^2$ with equation $x^2 = yz = 0$.

i) Show that X is smooth in codimension 1 (in fact, it is normal, see Exercise 9.1.14).
ii) Show that if Y is the subset of X defined by $x = y = 0$, then Y is a prime divisor which is not Cartier, but $2Y$ is Cartier.

9.4.2. Cartier divisors on arbitrary varieties. The notion of Cartier divisor, that we have encountered so far on normal varieties, admits an extension to arbitrary varieties. For simplicity, though, we will only deal with irreducible varieties.

Let X be an irreducible variety and consider the constant sheaf of Abelian groups $k(X)$ and its subsheaf $\mathcal{O}_X^*$. We have an exact sequence of sheaves of Abelian groups

$$0 \to \mathcal{O}_X^* \to k(X)^* \to k(X)^*/\mathcal{O}_X^* \to 0.$$

Definition 9.4.8. The group of *Cartier divisors* on X is the Abelian group

$$\mathrm{Cart}(X) := \Gamma\big(X, k(X)^*/\mathcal{O}_X^*\big).$$

We will see below that in fact, when X is a normal variety, we recover the definition in the previous section.

By definition, a Cartier divisor is described by giving an open cover $(U_i)_{i\in I}$ of X (which we can always assume to be finite) and $\phi_i \in k(X)^*$ for every $i \in I$, such that ϕ_i/ϕ_j is an invertible regular function on $U_i \cap U_j$ for every i and j. Given another family $(\psi_i)_{i\in I}$ associated to the same cover, then the corresponding Cartier divisors are equal if and only if each $\frac{\phi_i}{\psi_i}$ is an invertible regular function on U_i. One can compare two such sets of data, corresponding to two different finite covers, by passing to a common refinement. We note that if D and D' are defined by $(\phi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$, corresponding to the same cover $(U_i)_{i\in I}$, then $D + D'$ is defined by $(\phi_i\psi_i)_{i\in I}$.

DEFINITION 9.4.9. The group of *principal Cartier divisors* is the image of the canonical morphism

$$k(X)^* = \Gamma\big(X, k(X)^*\big) \to \Gamma\big(X, k(X)^*/\mathcal{O}_X^*\big) = \mathrm{Cart}(X).$$

We denote this by $\mathrm{PCart}(X)$.

To a Cartier divisor D we associate a line bundle $\mathcal{O}_X(D)$, as follows. Suppose that D is defined by an open cover $(U_i)_{i\in I}$ and the family $(\phi_i)_{i\in I}$ of non-zero rational functions. For every i, we have a subsheaf $\frac{1}{\phi_i}\cdot\mathcal{O}_{U_i}\subseteq k(X)$, which is isomorphic to $\mathcal{O}_{U_i}$. Given any i and j, the rational function $\frac{\phi_i}{\phi_j}$ lies in $\mathcal{O}_X^*(U_i\cap U_j)$, hence

$$\frac{1}{\phi_i}\cdot\mathcal{O}_{U_i\cap U_j} = \frac{1}{\phi_j}\cdot\mathcal{O}_{U_i\cap U_j}.$$

We thus have a subsheaf $\mathcal{O}_X(D)$ of the constant sheaf $k(X)$ such that

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{\phi_i}\cdot\mathcal{O}_{U_i}.$$

Moreover, it follows from construction that $\mathcal{O}_X(D)$ is a line bundle. It is straightforward to see that the sheaf $\mathcal{O}_X(D)$ does not depend on the choice of cover $(U_i)_{i\in I}$ and family $(\phi_i)_{i\in I}$ that describes D.

EXERCISE 9.4.10. Let X be an irreducible variety.

i) Show that for every Cartier divisors D and E, we have an isomorphism
$$\mathcal{O}_X(D)\otimes_{\mathcal{O}_X}\mathcal{O}_X(E)\to\mathcal{O}_X(D+E).$$
ii) Show that if D is a principal Cartier divisor, then we have an isomorphism $\mathcal{O}_X(D)\simeq\mathcal{O}_X$.
iii) Deduce that we have a group morphism
$$\mathrm{Cart}(X)/\mathrm{PCart}(X)\to\mathrm{Pic}(X) \tag{9.4.1}$$
that maps the class of D to the (isomorphism class) of $\mathcal{O}_X(D)$. Show that this morphism is injective.

The next proposition shows that the morphism (9.4.1) is, in fact, an isomorphism.

PROPOSITION 9.4.11. *If X is an irreducible variety, then for every line bundle $\mathcal{L}$ on X, there is a Cartier divisor D such that $\mathcal{O}_X(D)\simeq\mathcal{L}$. We thus have an isomorphism*

$$\mathrm{Cart}(X)/\mathrm{PCart}(X)\simeq\mathrm{Pic}(X).$$

PROOF. Consider a finite affine open cover $X=U_1\cup\ldots\cup U_n$, such that we have for every i an isomorphism $\alpha_i\colon\mathcal{L}|_{U_i}\to\mathcal{O}_{U_i}$. For every i and j, the isomorphism

$$\alpha_i|_{U_i\cap U_j}\circ\alpha_j^{-1}|_{U_i\cap U_j}\colon\mathcal{O}_{U_i\cap U_j}\to\mathcal{O}_{U_i\cap U_j}$$

is given by multiplication with an element $\alpha_{i,j}\in\mathcal{O}_X^*(U_i\cap U_j)$. Note that these rational functions satisfy the cocycle condition

$$\alpha_{i,j}\cdot\alpha_{j,k}=\alpha_{i,k}\quad\text{in}\quad k(X)$$

for all i,j,k, while $\phi_{i,i}=1$ for all i.

For every i, with $1 \leq i \leq n$, let $\phi_i = \alpha_{i,1}$. Note that in this case, the cocycle condition says that for every i and j, we have

$$\frac{\phi_i}{\phi_j} = \frac{\alpha_{i,1}}{\alpha_{j,1}} = \alpha_{i,j}.$$

This implies that the family $(\phi_i)_{i\in I}$ defines a Cartier divisor D and it is straightforward to see that $\mathcal{O}_X(D) \simeq \mathcal{L}$. $\square$

We now show that for normal varieties, our current definition agrees with the previous one. More generally, if X is smooth in codimension one, then we define a map $\alpha\colon \mathrm{Cart}(X) \to \mathrm{Div}(X)$, as follows. If D is a Cartier divisor on X described with respect to an open cover $(U_i)_{i\in I}$ by a family $(\phi_i)_{i\in I}$, then $\mathrm{div}(\phi_i)|_{U_i\cap U_j} = \mathrm{div}(\phi_j)|_{U_i\cap U_j}$ (this follows from the fact that $\phi_i/\phi_j \in \mathcal{O}_X^*(U_i \cap U_j)$). Therefore we have a divisor $\alpha(D)$ on X such that $D|_{U_i} = \mathrm{div}(\phi_i)|_{U_i}$ for all $i \in I$. It is easy to see that the divisor $\alpha(D)$ is independent of the choice of open cover and family of rational functions that describes D. It follows immediately from definitions that we have the equality $\mathcal{O}_X\big(\alpha(D)\big) = \mathcal{O}_X(D)$ of subsheaves of $k(X)$. It is also easy to check that for every two Cartier divisors D and E, we have $\alpha(D+E) = \alpha(D)+\alpha(E)$, hence α is a group homomorphism.

If X is a normal variety, then α is injective. Indeed, if D is as above and $\alpha(D) = 0$, then $\phi_i \in \mathcal{O}_X^*(U_i)$ for all i, hence $D = 0$. Moreover, a divisor is in the image of α if and only if D is locally principal. Indeed, it is clear from definition that $\alpha(D)$ is locally principal for every $D \in \mathrm{Cart}(X)$. Conversely, if E is a locally principal divisor on X, then we can find an cover $X = \bigcup_{i\in I} U_i$ and for every i a non-zero rational function ϕ_i such that $E|_{U_i} = \mathrm{div}(\phi_i)|_{U_i}$. In particular, we see that $\mathrm{div}(\phi_i)|_{U_i\cap U_j} = \mathrm{div}(\phi_j)|_{U_i\cap U_j}$ for every i and j, hence $\phi_i/\phi_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We thus see that the family $(\phi_i)_{i\in I}$ defines a Cartier divisor D such that $\alpha(D) = E$. Therefore, via the isomorphism α, we may and will identify the Cartier divisors, as defined in this section with those defined in the previous one.

Example 9.4.12. Since $\mathbf{P}^n$ is a smooth variety, every divisor on $\mathbf{P}^n$ is Cartier. By Example 9.3.4, we thus conclude that

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) \simeq \mathbf{Z}.$$

Moreover, if H is a hyperplane in $\mathbf{P}^n$, then $\mathcal{O}_{\mathbf{P}^n}(H)$ generates $\mathrm{Pic}(X)$. Note that the line bundle $\mathcal{O}_{\mathbf{P}^n}(H)$ is isomorphic to the line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ introduced in Example 8.6.14 (and therefore, if Y is a divisor in $\mathbf{P}^n$ of degree d, then $\mathcal{O}_{\mathbf{P}^n}(Y) \simeq \mathcal{O}_{\mathbf{P}^n}(d)$). In order to see this, note that if we consider the homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbf{P}^n$ and $h \in k[x_0, \ldots, x_n]_1$ is an equation of H, then on the open subset $U_i = (x_i \neq 0)$, we have $H|_{U_i} = \mathrm{div}(h/x_i)|_{U_i}$, and we thus have an isomorphism

$$\phi_i\colon \mathcal{O}_X(H)|_{U_i} \simeq \mathcal{O}_{U_i}$$

given by multiplication by h/x_i. It follows that the transition function $\phi_i|_{U_i\cap U_j} \circ \phi_j^{-1}|_{U_i\cap U_j}$ is given by multiplication by x_j/x_i, hence we have the same transition functions as for $\mathcal{O}_{\mathbf{P}^n}(1)$ (see Example 8.6.14). This proves our assertion.

Remark 9.4.13. If X is a smooth, irreducible variety, then we have seen that we have a canonical isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$. We deduce using Example 9.3.5 that if U is an open subset in X, then the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective, and it is an isomorphism if $\mathrm{codim}_X(X \smallsetminus U) \geq 2$.

9.4.3. Pull-back of Cartier divisors. We now describe a pull-back operation for Cartier divisors. Let $f\colon X \to Y$ be a dominant morphism between irreducible varieties, so that we have an induced field extension $\nu\colon k(Y) \hookrightarrow k(X)$. Using this, we obtain a morphism of sheaves of Abelian groups

$$f^{-1}(k(Y)^*_Y) = k(Y)^*_X \to k(X)^*_X,$$

which induces the canonical morphism of sheaves $f^{-1}(\mathcal{O}^*_Y) \to \mathcal{O}^*_X$. By taking the quotient, we obtain a morphism of sheaves of Abelian groups

$$f^{-1}\big(k(Y)^*/\mathcal{O}^*_Y\big) \to k(X)^*/\mathcal{O}^*_X.$$

Note that for every sheaf $\mathcal{F}$ on Y, we have a canonical map $\Gamma(Y,\mathcal{F}) \to \Gamma\big(X, f^{-1}(\mathcal{F})\big)$. We thus obtain a morphism of Abelian groups given by the composition

$$f^*\colon \mathrm{Cart}(Y) = \Gamma\big(Y, k(Y)^*/\mathcal{O}^*_Y\big) \to \Gamma\big(X, f^{-1}(k(Y)^*/\mathcal{O}^*_Y)\big)$$

$$\to \Gamma\big(X, k(X)^*/\mathcal{O}^*_X\big) = \mathrm{Cart}(X).$$

This can be described explicitly as follows. If the Cartier divisor D on Y is described by an open cover $Y = \bigcup_{i\in I} U_i$ and a family $(\phi_i)_{i\in I}$ with $\phi_i \in k(Y)^*$ such that $\phi_i/\phi_j \in \mathcal{O}^*_Y(U_i \cap U_j)$ for all i and j, then $f^*(D)$ is described with respect to the open cover $X = \bigcup_{i\in I} f^{-1}(U_i)$ by the family $\big(\nu(\phi_i)\big)_{i\in I}$.

It is clear from this explicit description that if $D = \mathrm{div}_Y(\phi)$, then $f^*(D) = \mathrm{div}_X\big(\nu(\phi)\big)$. We thus obtain an induced morphism of Abelian groups

$$f^*\colon \mathrm{Cart}(Y)/\mathrm{PCart}(Y) \to \mathrm{Cart}(X)/\mathrm{PCart}(X).$$

In fact, it is straightforward to check that via the isomorphisms with the corresponding Picard groups given by Proposition 9.4.11, this gets identified with the morphism

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(Y), \quad \mathcal{L} \to f^*(\mathcal{L}).$$

Remark 9.4.14. It follows from the explicit description in terms of a cover that if $f\colon X \to Y$ and $g\colon Y \to Z$ are dominant morphisms of irreducible varieties, then for every Cartier divisor D on Z, we have

$$f^*\big(g^*(D)\big) = (g\circ f)^*(D).$$

The following proposition gives a compatibility property between push-forward and pull-back of divisors, known as the *projection formula*.

Proposition 9.4.15. *If $f\colon X \to Y$ is a finite surjective morphism between irreducible varieties, both of them being smooth in codimension 1, then for every Cartier divisor D on Y, we have the following equality*[1] *in* $\mathrm{Div}(Y)$*:*

$$f_*\big(f^*(D)\big) = \deg(f)\cdot D.$$

Proof. It is enough to check the assertion locally on Y, so that we may assume that D is a principal divisor, corresponding to $\phi \in k(Y)^*$. If the degree of the field extension $k(X)/k(Y)$ is n, then $N_{k(X)/k(Y)}(\phi) = \phi^n$, hence the equality in the statement follows from Proposition 9.3.13. ☐

[1]On the left-hand side, we apply f_* to the Weil divisor corresponding to $f^*(D)$, while on the right-hand side, we consider the Weil divisor corresponding to D.

9.4.4. Effective Cartier divisors. We now consider those Cartier divisors that are defined by regular functions. Let X be an irreducible algebraic variety.

DEFINITION 9.4.16. A Cartier divisor on X is *effective* if, when described by a family $(\phi_i)_{i\in I}$ associated to an open cover $X=\bigcup_{i\in I}U_i$, we have $\phi_i\in\mathcal{O}_X(U_i)$ for all $i\in I$ (note that this condition does not depend on the choice of cover and family $(\phi_i)_{i\in I}$).

REMARK 9.4.17. If X is a variety that is smooth in codimension 1 and D is an effective Cartier divisor, then it is clear that the corresponding Weil divisor $\alpha(D)$ is an effective divisor. If X is normal, the converse is also true, since in this case, given an open subset U of X and $\phi\in k(X)$, we have $\phi\in\mathcal{O}_X(U)$ if and only if $\mathrm{ord}_V(\phi)\geq 0$ for every codimension 1 irreducible subvariety V of X, with $V\cap U\neq\emptyset$.

We will often identify an effective Cartier divisor with a coherent sheaf on X, as follows. A *locally principal* ideal of $\mathcal{O}_X$ is a coherent ideal such that there is an affine open cover $X=\bigcup_{i\in I}U_i$, with each $\mathcal{I}(U_i)\subseteq\mathcal{O}_X(U_i)$ a principal non-zero ideal.

PROPOSITION 9.4.18. *The map that associates to an effective Cartier divisor D on X the sheaf $\mathcal{O}_X(-D)$ gives a bijection between the effective Cartier divisors on X and the locally principal ideals on X.*

PROOF. If D is given by an open cover $X=\bigcup_{i\in I}U_i$, where we may assume that all U_i are affine, and by a family $(\phi_i)_{i\in I}$, with $\phi_i\in\mathcal{O}_X(U_i)$, then

$$\Gamma\big(U_i,\mathcal{O}_X(-D)\big)=\Gamma(U_i,\mathcal{O}_X)\cdot\phi_i,$$

which is a non-zero principal ideal in $\Gamma(U_i,\mathcal{O}_X)$. It follows that $\mathcal{O}_X(-D)$ is a locally principal ideal of $\mathcal{O}_X$. Conversely, if $\mathcal{I}\subseteq\mathcal{O}_X$ is a locally principal ideal and $X=\bigcup_{i\in I}U_i$ is an affine open cover such that $\Gamma(U_i,\mathcal{I})$ is the ideal of $\Gamma(U_i,\mathcal{O}_X)$ generated by f_i, then for every i and j, we have $f_i/f_j\in\Gamma(U_i\cap U_j,\mathcal{O}_X^*)$. Therefore $(f_i)_{i\in I}$ defines an effective Cartier divisor on X. It is straightforward to see that the two maps we defined are mutual inverses. $\square$

Given an effective Cartier divisor D on X, we will denote by $\mathcal{O}_D$ the quotient of $\mathcal{O}_X$ by $\mathcal{O}_X(-D)$, so that we have an exact sequence

$$0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to\mathcal{O}_D\to 0.$$

We define the *support* $\mathrm{Supp}(D)$ of D as the closed subvariety $V\big(\mathcal{O}_X(-D)\big)$ defined by $\mathcal{O}_X(-D)$.

REMARK 9.4.19. We can define the pull-back of effective Cartier divisors in a slightly more general setting than the pull-back of arbitrary Cartier divisors. Suppose that $f\colon X\to Y$ is a morphism of irreducible varieties and D is an effective Cartier divisor on Y. If $\mathcal{O}_Y(-D)\cdot\mathcal{O}_X$ is a locally principal ideal, then we have an effective Cartier divisor $f^*(D)$ defined as follows: if D is defined on an affine open subset $V\subseteq Y$ by $\phi\in\mathcal{O}_Y(V)$, then $f^*(D)$ is defined on any affine open subset $U\subseteq f^{-1}(V)$ by $\phi\circ f|_U$. It is clear that if f is dominant, then this definition agrees with the previous one.

REMARK 9.4.20. In order to define effective Cartier divisors it is not necessary to restrict to irreducible varieties. For example, from the point of view of ideal sheaves, an effective Cartier divisor on an arbitrary variety is a coherent ideal that is locally generated by one element which is a non-zero divisor. In fact, also general Cartier divisors can be defined on arbitrary varieties, but for the sake of simplicity, we preferred to stick to irreducible varieties, which is the only setting in which we will use Cartier divisors in these notes.

EXAMPLE 9.4.21. Let X be an irreducible algebraic variety and $\mathcal{I}$ a non-zero ideal on X. If $\pi\colon \widetilde{X} \to X$ is the blow-up along $\mathcal{I}$, then the inverse image $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ is a locally principal ideal. Indeed, by covering X with affine open subsets, we reduce to the case when X is an affine variety. In this case the assertion was shown in the proof of Proposition 6.1.3. We thus have an effective Cartier divisor E on $\widetilde{X}$ (the *exceptional divisor* of the blow-up) such that $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X(-E)$.

DEFINITION 9.4.22. Given any coherent sheaf $\mathcal{F}$ on the algebraic variety X and an irreducible component V of $\mathrm{Supp}(\mathcal{F})$, the *multiplicity* of $\mathcal{F}$ along V is defined as $\ell_{\mathcal{O}_{X,V}}(\mathcal{F}_V)$. By the assumption on V, the maximal ideal in $\mathcal{O}_{X,V}$ is minimal over $\mathrm{Ann}(\mathcal{F}_V)$, hence $\mathcal{F}_V$ is an $\mathcal{O}_{X,V}$-module of finite length (see Proposition H.1.5).

In particular, given a coherent ideal of $\mathcal{O}_X$, we may consider the multiplicity of $\mathcal{O}_X/\mathcal{I}$ along each of the irreducible components of $V(\mathcal{I})$. Note that if $\mathcal{I}$ is a radical ideal, then each such multiplicity is 1 (however, the converse does not hold in general, due to the possible presence of embedded associated primes).

REMARK 9.4.23. If D is an effective Cartier divisor on the variety X which is irreducible and smooth in codimension 1, then for every irreducible component V of $\mathrm{Supp}(D)$, the multiplicity of $\mathcal{O}_D$ along V is equal to the coefficient of V in the Weil divisor associated to V. Note that if X is normal, then $\mathcal{O}_X(-D)$ is a radical ideal if and only if all coefficients of D are equal to 1 (indeed, note that if the latter condition holds, then $\mathcal{O}_X(-D)$ is the subsheaf of $\mathcal{O}_X$ consisting of the regular functions that vanish on $\mathrm{Supp}(D)$); in this case, we say that D is *reduced*.

We now discuss a third point of view on effective Cartier divisors, as zero-loci of sections of line bundles. Suppose first that X is an arbitrary algebraic variety and $\mathcal{L}$ is a line bundle on X. Given a section $s \in \Gamma(X, \mathcal{L})$, we have a morphism $\mathcal{O}_X \to \mathcal{L}$ that maps 1 to s. By tensoring this with $\mathcal{L}^{-1}$, we obtain a morphism $\mathcal{L}^{-1} \to \mathcal{O}_X$, whose image is an ideal sheaf that we denote $\mathcal{I}(s)$.

In order to describe this ideal and the closed subset $V(s)$ of X it defines, consider an affine open subset U of X such that $\mathcal{L}|_U \simeq \mathcal{O}_U$. Via this isomorphism, $s|_U$ corresponds to some $f \in \mathcal{O}(U)$. Moreover, we have an induced isomorphism $\mathcal{L}^{-1}|_U \simeq \mathcal{O}_U$, so that the morphism $\mathcal{L}^{-1}|_U \to \mathcal{O}_U$ corresponding to s gets identified with the morphism $\mathcal{O}_U \to \mathcal{O}_U$ given by multiplication by f. Therefore the restriction of $\mathcal{I}_{Z(s)}$ to U is the ideal generated by f. In particular, we see that a point $x \in X$ lies in $V(s)$ if and only if $s(x) \in \mathcal{L}_{(x)}$ is 0. We will refer to $V(s)$ as the *zero-locus* of s.

From now on we assume that X is irreducible and s is non-zero. In this case, the above local description of the morphism $\mathcal{L}^{-1} \to \mathcal{O}_X$ associated to s shows that this is an injective morphism and its image $\mathcal{I}(s)$ is a locally principal ideal, corresponding to an effective Cartier divisor $Z(s)$, whose support is $V(s)$.

PROPOSITION 9.4.24. *If $\mathcal{L}$ is a line bundle on the irreducible variety X, then the following hold:*

i) *For every non-zero $s \in \Gamma(X, \mathcal{L})$, we have an isomorphism $\mathcal{O}_X\big(Z(s)\big) \simeq \mathcal{L}$.*

ii) *Given two non-zero $s, s' \in \Gamma(X, \mathcal{L})$, we have $Z(s') = Z(s)$ if and only if $s' = hs$ for some $h \in \mathcal{O}(X)^*$.*
iii) *If D is an effective Cartier divisor such that $\mathcal{O}_X(D) \simeq \mathcal{L}$, then there is a non-zero $s \in \Gamma(X, \mathcal{L})$ such that $D = Z(s)$.*
iv) *If $s_1 \in \Gamma(X, \mathcal{L}_1)$ and $s_2 \in \Gamma(X, \mathcal{L}_2)$ are non-zero sections, for line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, and if $s_1 \otimes s_2 \in \Gamma(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$ is the tensor product of s_1 and s_2, then $Z(s_1 \otimes s_2) = Z(s_1) + Z(s_2)$.*

PROOF. It follows from the definition that $\mathcal{L}^{-1}$ is isomorphic to the ideal $\mathcal{O}_X\big(-Z(s)\big)$ corresponding to $Z(s)$. By taking duals, we get an isomorphism $\mathcal{L} \simeq \mathcal{O}_X\big(Z(s)\big)$, giving i). The assertion in ii) follows immediately from the local description of $Z(s)$.

Suppose now that D is an effective Cartier divisor and we have an isomorphism $\alpha\colon \mathcal{O}_X(D) \simeq \mathcal{L}$. It follows from the definition of $\mathcal{O}_X(D)$ that $1 \in k(X)$ is a global section of this sheaf: if D is defined with respect to a cover $X = \bigcup_{i\in I} U_i$ by $(\phi_i)_{i\in I}$, by assumption $\phi_i \in \mathcal{O}_X(U_i)$, hence $1 \in \frac{1}{\phi_i}\mathcal{O}_X(U_i)$. If we take $s = \alpha(1)$, then it is straightforward to see that $Z(s) = D$. Finally, the assertion in iv) follows directly from definition. $\square$

REMARK 9.4.25. If X is a complete variety, then we will see in Chapter 11 that for every line bundle $\mathcal{L}$ on X, the k-vector space $\Gamma(X, \mathcal{L})$ is finite dimensional. The set of effective Cartier divisors D such that $\mathcal{O}_X(D) \simeq \mathcal{L}$ is the *linear system* $|\mathcal{L}|$. It follows from the above proposition that this can be identified with the projective space parametrizing the lines in $\Gamma(X, \mathcal{L})$. We will see in Chapter 11 that the linear systems on a variety are closely related to the rational maps from that variety to projective spaces.

EXERCISE 9.4.26. Show that if $\mathcal{L}$ is a line bundle on the irreducible, complete variety X, such that $\Gamma(X, \mathcal{L}) \neq 0$ and $\Gamma(X, \mathcal{L}^{-1}) \neq 0$, then $\mathcal{L} \simeq \mathcal{O}_X$.

We end this section with some examples concerning class groups of products.

EXAMPLE 9.4.27. If X and Y are irreducible varieties that are smooth in codimension 1, then the same holds for $X \times Y$. Indeed, it follows from Exercise 6.3.13 that $(X \times Y)_{\rm sm} = X_{\rm sm} \times Y_{\rm sm}$, hence its complement in $X \times Y$ is

$$\big((X \smallsetminus X_{\rm sm}) \times Y\big) \cup \big(X \times (Y \smallsetminus Y_{\rm sm})\big),$$

whose codimension in $X \times Y$ is ≥ 2.

Let us denote by $p\colon X \times Y \to X$ the first projection. Note that if V is a prime divisor in X, then $p^{-1}(V) = V \times Y$ is a prime divisor in $X \times Y$. We can thus define a group homomorphism $p^*\colon \mathrm{Div}(X) \to \mathrm{Div}(X \times Y)$, by mapping $\sum_{i=1}^r a_i D_i$ to $\sum_{i=1}^r a_i p^{-1}(D_i)$.

We claim that this induces a group homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times Y)$. For this, it is enough to show that if $\phi \in k(X)$ is a non-zero rational function, then $p^*\big(\mathrm{div}(\phi)\big) = \mathrm{div}(\phi \circ p)$. In order to see this, let $U \subseteq X$ be an open subset such that $\phi \in \mathcal{O}_X^*(U)$. In this case $\phi \circ p \in \mathcal{O}_{X\times Y}^*(U \times Y)$, hence the only prime divisors that appear with non-zero coefficient in $\mathrm{div}(\phi \circ p)$ are contained in $(X \smallsetminus U) \times Y$, hence they are of the form $V \times Y$, where V is an irreducible component of $X \smallsetminus U$ of codimension 1 in X. Our assertion thus follows if we show that for every such V, we have

$$\mathrm{ord}_V(\phi) = \mathrm{ord}_{V\times Y}(\phi \circ p) \quad \text{for every} \quad \phi \in k(X) \smallsetminus \{0\}. \tag{9.4.2}$$

For this, we may assume that both X and Y are affine, with coordinate rings A and B, respectively. We may also assume that the ideal defining V in X is generated by one element π. Note that the ideal in $A\otimes_k B$ defining $V\times Y$ is generated by $\pi\otimes 1$. In order to prove (9.4.2), we may clearly assume that $\phi\in A\smallsetminus\{0\}$. If $r=\mathrm{ord}_V(\phi)$, then we can write $\phi=\pi^r\cdot\psi$, where $\psi\in A\smallsetminus(\pi)$. We deduce that $\mathrm{ord}_{V\times Y}(\phi\otimes 1)=r$ from the fact that $(\pi)=\{a\in A\mid a\otimes 1\in(\pi\otimes 1)\}$, which is straightforward to check.

It is clear that the above definition is compatible with the definition of pull-back of Cartier divisors: if $\alpha\colon \mathrm{Cart}(X)\to\mathrm{Div}(X)$ is the canonical group homomorphism, then for every Cartier divisor D on X, we have $p^*\big(\alpha(D)\big)=\alpha\big(p^*(D)\big)$.

Of course, if $q\colon X\times Y\to Y$ is the second projection, then we have a similar group homomorphism $q^*\colon \mathrm{Cl}(Y)\to\mathrm{Cl}(X\times Y)$. We claim that the homomorphism

$$\mathrm{Cl}(X)\oplus\mathrm{Cl}(Y)\to\mathrm{Cl}(X\times Y),\quad (\alpha,\beta)\to p^*(\alpha)+q^*(\beta) \tag{9.4.3}$$

is injective. In order to check this, we may assume that both X and Y are smooth: we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Cl}(X)\oplus\mathrm{Cl}(Y) \arrow[r] \arrow[d] & \mathrm{Cl}(X\times Y) \arrow[d] \\
\mathrm{Cl}(X_{\mathrm{sm}})\oplus\mathrm{Cl}(Y_{\mathrm{sm}}) \arrow[r] & \mathrm{Cl}(X_{\mathrm{sm}}\times Y_{\mathrm{sm}}),
\end{tikzcd}$$

and the vertical maps induced by restriction are isomorphisms by Example 9.3.5. Note now that if X and Y are smooth, then the morphism (9.4.3) is identified with the morphism

$$\mathrm{Pic}(X)\oplus\mathrm{Pic}(Y)\to\mathrm{Pic}(X\times Y),\quad (\mathcal{L},\mathcal{M})\to p^*(\mathcal{L})\otimes_{\mathcal{O}_X}q^*(\mathcal{M}).$$

Suppose that $(\mathcal{L},\mathcal{M})$ lies in the kernel of this map, that is

$$p^*(\mathcal{L})\otimes_{\mathcal{O}_X}q^*(\mathcal{M})\simeq\mathcal{O}_{X\times Y}. \tag{9.4.4}$$

Let $x_0\in X$ and $y_0\in Y$ be any points, and let us consider the closed immersions $i\colon X\hookrightarrow X\times Y$ and $j\colon Y\hookrightarrow X\times Y$ given by $i(x)=(x,y_0)$ and $j(y)=(x_0,y)$. Note that $p\circ i=\mathrm{id}_X$ and $q\circ j=\mathrm{id}_Y$, while both $q\circ i$ and $p\circ j$ are constant maps. We thus see that by applying i^* and j^* to the isomorphism (9.4.4), we obtain $\mathcal{L}\simeq\mathcal{O}_X$ and $\mathcal{M}\simeq\mathcal{O}_Y$. This proves our claim.

Example 9.4.28. If X is an irreducible variety that is smooth in codimension 1 and $p\colon X\times\mathbf{A}^n\to X$ is the projection, then $p^*\colon\mathrm{Cl}(X)\to\mathrm{Cl}(X\times\mathbf{A}^n)$ is an isomorphism for every $n\geq 1$. Indeed, we have seen in the previous example that p^* is injective. In order to prove surjectivity, arguing by induction on n, it is clear that it is enough to consider the case $n=1$. Let V be a prime divisor in $X\times\mathbf{A}^1$. For every open subset U of X, if we show that $[V\cap(U\times\mathbf{A}^1)]$ lies in the image of $\mathrm{Cl}(U)\to\mathrm{Cl}(U\times\mathbf{A}^1)$, then it follows from Example 9.3.5 that if $Z_1,\ldots,Z_r$ are the irreducible components of $X\smallsetminus U$ of codimension 1 in X, then there are $m_1,\ldots,m_r\in\mathbf{Z}$ such that $[V]-\sum_{i=1}^r m_i[Z_i\times\mathbf{A}^1]$ lies in the image of p^*. In this case $[V]$ lies in the image of p^*.

In particular, we may assume that X is affine. We put $A=\mathcal{O}(X)$ and let $\mathfrak{q}$ be the ideal defining V in $A[x]$. We can also assume that $\overline{p(V)}=X$, since otherwise we can take $U=X\smallsetminus\overline{p(V)}$. Therefore $\mathfrak{q}\cap A=0$ and if $K=\mathrm{Frac}(A)=k(X)$, then $\mathfrak{q}\cdot K[x]=f\cdot K[x]$ for some nonzero $f\in\mathfrak{q}$. In particular, we have $\mathrm{ord}_V(f)=1$. If

$\mathfrak{q}' \neq \mathfrak{q}$ is a prime ideal in $A[x]$ of codimension 1 such that $f \in \mathfrak{q}'$, then $\mathfrak{q}' \cap A \neq 0$ (otherwise the inclusion $q \cdot K[x] \subseteq q' \cdot K[x]$ implies $\mathfrak{q} \subseteq \mathfrak{q}'$, a contradiction). We thus see that we can write $\mathrm{div}(\phi) = V + \sum_{i=1}^{r} m_i Z_i$, with each $[Z_i]$ in the image of p^*, hence $[V]$ also lies in the image of p^*. This completes the argument.

EXAMPLE 9.4.29. Suppose that X is an irreducible variety which is smooth in codimension 1. If $p\colon X \times \mathbf{P}^n \to X$ and $q\colon X \times \mathbf{P}^n \to \mathbf{P}^n$ are the two projections, then we claim that the induced morphism

$$\tau\colon \mathrm{Cl}(X) \oplus \mathbf{Z} = \mathrm{Cl}(X) \oplus \mathrm{Cl}(\mathbf{P}^n) \to \mathrm{Cl}(X \times \mathbf{P}^n), \quad (a,b) \to p^*(a) + q^*(b)$$

is an isomorphism.

The injectivity of τ follows directly from Example 9.4.27. In order to show that τ is surjective, consider a hyperplane $H \subseteq \mathbf{P}^n$, so that $U = \mathbf{P}^n \smallsetminus H \simeq \mathbf{A}^n$. Given any $\beta \in \mathrm{Cl}(X)$, it follows from Example 9.4.28 that we can write $\beta|_U = p_0^*(\alpha)$ for some $\alpha \in \mathrm{Cl}(X)$, where $p_0\colon X \times U \to X$ is the projection onto the first component. We thus deduce from Example 9.3.5 that there is $m \in \mathbf{Z}$ such that $\beta - p^*(\alpha) = m \cdot [X \times H] = q^*\big(m \cdot [H]\big)$. We thus conclude that τ is an isomorphism.

CHAPTER 10

Cohomology of coherent sheaves

10.1. Derived functors on the category of $\mathcal{O}_X$-modules

In this section we discuss the right derived functors of a left exact functor. For the sake of concreteness, we only consider categories of sheaves of modules on a ringed space, though it will be clear that everything we do generalizes in an obvious way to arbitrary Abelian categories that have enough injective objects in the sense of Definition 10.1.6 below.

10.1.1. Complexes of $\mathcal{O}_X$-modules. Let $(X, \mathcal{O}_X)$ be a fixed ringed space and let us consider the category $\mathcal{O}_X$-mod.

Definition 10.1.1. A *complex* of $\mathcal{O}_X$-modules is given by a sequence $(\mathcal{F}^i)_{i\in\mathbf{Z}}$ of $\mathcal{O}_X$-modules and maps of $\mathcal{O}_X$-modules

$$\ldots \to \mathcal{F}^{i-1} \overset{d^{i-1}}{\to} \mathcal{F}^i \overset{d^i}{\to} \mathcal{F}^{i+1} \to \ldots$$

such that $d^i \circ d^{i-1} = 0$ for all i. We typically denote a complex as above by $\mathcal{F}^\bullet$ and we denote all d^i simply by d.

Given two complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ as above, a morphism of complexes $u\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ consists of a sequence of morphisms of $\mathcal{O}_X$-modules $(u^i)_{i\in\mathbf{Z}}$, with $u^i\colon \mathcal{F}^i \to \mathcal{G}^i$, such that $d \circ u^i = u^{i+1} \circ d$ for all i. We have an obvious notion of composition of morphisms and in this way we obtain the category of complexes of $\mathcal{O}_X$-modules.

Remark 10.1.2. We will encounter sometimes families $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ with maps $d\colon \mathcal{F}_i \to \mathcal{F}_{i-1}$ such that $d \circ d = 0$. This can be viewed as a complex, in the above sense, if we use the following convention for lifting the indices: $\mathcal{F}^i = \mathcal{F}_{-i}$.

It is easy to see that the category of complexes is an Abelian category, with kernels and cokernels defined component-wise. In particular, we may consider exact sequences of complexes.

Definition 10.1.3. The *cohomology functor* $\mathcal{H}^i$ defined on the above category of complexes and taking values in $\mathcal{O}_X$-mod is given by

$$\mathcal{H}^i(\mathcal{F}^\bullet) := \ker(\mathcal{F}^i \to \mathcal{F}^{i+1})/\mathrm{Im}(\mathcal{F}^{i-1} \to \mathcal{F}^i).$$

Note that a morphism $u\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces a morphism $\mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{G}^\bullet)$, making $\mathcal{H}^i$ into a functor. Whenever we use lower indexing for complexes, it is convenient to put $\mathcal{H}_i = \mathcal{H}^{-i}$

A basic ingredient in setting up derived functors is the following lemma:

Lemma 10.1.4. *Given a short exact sequence of complexes*

$$0 \longrightarrow \mathcal{E}^\bullet \overset{u}{\longrightarrow} \mathcal{F}^\bullet \overset{v}{\longrightarrow} \mathcal{G}^\bullet \longrightarrow 0,$$

we have a long exact sequence of $\mathcal{O}_X$-modules

$$\ldots \longrightarrow \mathcal{H}^i(\mathcal{E}^\bullet) \xrightarrow{\mathcal{H}^i(u)} \mathcal{H}^i(\mathcal{F}^\bullet) \xrightarrow{\mathcal{H}^i(v)} \mathcal{H}^i(\mathcal{G}^\bullet) \xrightarrow{\delta} \mathcal{H}^{i+1}(\mathcal{E}^\bullet) \longrightarrow \ldots.$$

Moreover, the "connecting homomorphisms" δ are functorial, in an obvious sense, with respect to morphisms of short exact sequences of complexes.

Proof. We only sketch the proof. Given $\overline{s} \in \Gamma\big(U, \mathcal{H}^i(\mathcal{G}^\bullet)\big)$, we can choose around each point $x \in X$, a section $s(x)$ of $\ker(\mathcal{G}^i \to \mathcal{G}^{i+1})$ over an open neighborhood $U(x) \subseteq U$ of x, such that $s(x)$ lifts $\overline{s}|_{U(x)}$ Since v^i is surjective, after possibly shrinking $U(x)$, we can find $\widetilde{s}(x) \in \Gamma\big(U(x), \mathcal{F}^i\big)$ such that $v^i\big(\widetilde{s}(x)\big) = s(x)|_{U(x)}$. The hypothesis on $s(x)$ and the fact that $d \circ v^i = v^{i+1} \circ d$ implies that $d\big(\widetilde{s}(x)\big) = u^{i+1}\big(t(x)\big)$ for a unique $t(x) \in \Gamma\big(U(x), \mathcal{E}^{i+1}\big)$. It is clear that $d\big(t(x)\big) \in \Gamma\big(U(x), \mathcal{E}^{i+2}\big)$ is 0. While the sections $t(x)$ do not glue, in general, their images in $\Gamma\big(U(x), \mathcal{H}^{i+1}(\mathcal{E}^\bullet)\big)$ agree on overlaps and thus define a section $\delta(s) \in \Gamma\big(U, \mathcal{H}^{i+1}(\mathcal{E}^\bullet)\big)$. It is easy to see that the definition is independent of all choices made and it gives a morphism of $\mathcal{O}_X$-modules $\delta\colon \mathcal{H}^i(\mathcal{G}^\bullet) \to \mathcal{H}^{i+1}(\mathcal{E}^\bullet)$.

Checking the exactness of the long sequence is a matter of diagram chasing. Alternatively, it is enough to check exactness after passing to stalks, which reduces the assertion to the case of modules over a fixed ring. We leave the details as an exercise for the reader. $\square$

Definition 10.1.5. Two morphisms of complexes $u, v\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ are *homotopic*, written $u \approx v$ if there is a sequence $(\theta^i)_{i \in \mathbf{Z}}$ of morphisms $\theta_i\colon \mathcal{E}^i \to \mathcal{F}^{i-1}$ such that $u^i - v^i = d \circ \theta^i + \theta^{i+1} \circ d$ for all i. This is an equivalence relation (it is the congruence relation modulo those morphisms homotopic to 0). It is easy to see that if u and v are homotopic, then $\mathcal{H}^i(u) = \mathcal{H}^i(v)$ for all $i \in \mathbf{Z}$.

Definition 10.1.6. Recall that if $\mathcal{C}$ is an Abelian category, an object Q of $\mathcal{C}$ is *injective* if the functor $\mathrm{Hom}_{\mathcal{C}}(-, Q)$ from $\mathcal{C}$ to the category of Abelian groups is exact (as opposed to left exact, which is the case for general Q). The category $\mathcal{C}$ *has enough injectives* if for every object A of $\mathcal{C}$, there is an injective map (that is, a map with 0 kernel) $\iota\colon A \to I$, where I is injective.

Remark 10.1.7. It follows from the definition of injective objects and from the universal property of a direct product that a direct product of injective objects is again injective.

Remark 10.1.8. Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

with $\mathcal{F}'$ injective, by applying $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F}')$, we see that the sequence is split.

Proposition 10.1.9. *The category $\mathcal{O}_X$-mod has enough injectives.*

Proof. We will make use of the following construction. For every point $x \in X$ and every $\mathcal{O}_{X,x}$-module A, we define an $\mathcal{O}_X$-module $A(x)$ as follows. For an open subset U of X, we put $\Gamma\big(U, A(x)\big) = A$ if $x \in U$ and $\Gamma\big(U, A(x)\big) = 0$ otherwise. The restriction maps are either the identity maps or the 0 maps and the $\mathcal{O}_X(U)$-module structure on $\Gamma\big(U, A(x)\big)$, when $x \in U$, is induced by the canonical homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$. It is straightforward to see that for every $\mathcal{O}_X$-module $\mathcal{M}$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{M}, A(x)\big) \simeq \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, A).$$

In particular, this implies that if A is an injective $\mathcal{O}_{X,x}$-module, then $A(x)$ is an injective $\mathcal{O}_X$-module.

Given an arbitrary $\mathcal{O}_X$-module $\mathcal{M}$, for every $x \in X$, we consider the stalk $\mathcal{M}_x$. This is an $\mathcal{O}_{X,x}$-module and since the category of $\mathcal{O}_{X,x}$-modules has enough injectives (see Proposition I.1.1), we can find an injective morphism of $\mathcal{O}_{X,x}$-modules $j_x \colon \mathcal{M}_x \hookrightarrow Q^x$, where Q^x is an injective $\mathcal{O}_{X,x}$-module. Since each $\mathcal{O}_X$-module Q^x is injective, it follows that the direct product $\prod_{x\in X} Q^x(x)$ is injective, and we have an injective morphism of $\mathcal{O}_X$-modules

$$\mathcal{M} \to \prod_{x\in X} Q^x(x), \quad \mathcal{M}(U) \ni s \to \big(j_x(s_x)\big).$$

This completes the proof of the proposition. $\square$

Definition 10.1.10. A *resolution* of an $\mathcal{O}_X$-module $\mathcal{F}$ is given by a complex $\mathcal{I}^\bullet$ with $\mathcal{I}^q = 0$ for $q < 0$, together with a morphism of complexes $\mathcal{F} \to \mathcal{I}^\bullet$ (where we think of $\mathcal{F}$ as a complex concentrated in degree 0) that induces an isomorphism in cohomology; equivalently, the induced complex

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$$

is exact. This is an *injective* resolution if, in addition, all $\mathcal{O}_X$-modules $\mathcal{I}^j$ are injective.

The following result will allow us to construct derived functors.

Proposition 10.1.11. *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

i) *There is an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$.*
ii) *Given a morphism $\tau \colon \mathcal{F} \to \mathcal{G}$, a resolution $\mathcal{F} \to \mathcal{A}^\bullet$ and a morphism of complexes $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$, where $\mathcal{I}^i = 0$ for $i < 0$ and $\mathcal{I}^i$ is injective for all i, there is a morphism of complexes $u \colon \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\tau"'] & \mathcal{A}^\bullet \arrow[d, "u"] \\ \mathcal{G} \arrow[r] & \mathcal{I}^\bullet \end{tikzcd}$$

is commutative.
iii) *If u and v both satisfy the conclusion in ii), then they are homotopic.*

Proof. In order to prove i), we begin by using Proposition 10.1.9 to find an injective $\mathcal{O}_X$-module $\mathcal{I}^0$ and an injective morphism $\mathcal{F} \hookrightarrow \mathcal{I}^0$. If $\mathcal{C}$ is the cokernel of this map, we use the same proposition to find an injective $\mathcal{O}_X$-module $\mathcal{I}^1$ and an injective homomorphism $\mathcal{C} \hookrightarrow \mathcal{I}^1$. We thus have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1.$$

Continuing in this way we obtain the injective resolution $\mathcal{I}^\bullet$.

For ii), we construct the morphisms $u^i \colon \mathcal{A}^i \to \mathcal{I}^i$ recursively, as follows. Since the morphism $\mathcal{F} \to \mathcal{A}^0$ is injective and $\mathcal{I}^0$ is an injective $\mathcal{O}_X$-module, we can find

$u^0 \colon \mathcal{A}^0 \to \mathcal{I}^0$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d, "\tau"'] & \mathcal{A}^0 \arrow[d, "u^0"] \\ \mathcal{G} \arrow[r] & \mathcal{I}^0 \end{tikzcd}$$

is commutative. Since $\mathcal{H}^1(\mathcal{A}^\bullet) = 0$, we have an injective morphism $\mathrm{coker}(\phi) \hookrightarrow \mathcal{A}^1$. On the other hand, u^0 induces a morphism $\overline{u_0} \colon \mathrm{coker}(\phi) \to \mathcal{I}^1$, and since $\mathcal{I}^1$ is injective, there is a morphism $u^1 \colon \mathcal{A}^1 \to \mathcal{I}^1$ such that the diagram

$$\begin{tikzcd} \mathcal{A}^0 \arrow[r] \arrow[d, "u^0"'] & \mathcal{A}^1 \arrow[d, "u^1"] \\ \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \end{tikzcd}$$

is commutative. Iterating this argument, we obtain the assertion in ii).

Finally, suppose that u and v both satisfy the condition in ii). We construct recursively morphisms $\theta^i \colon \mathcal{A}^i \to \mathcal{I}^{i-1}$ for $i \geq 1$ such that $u^i - v^i = d \circ \theta^i + \theta^{i+1} \circ d$. The assumption implies that u^0 and v^0 agree on the image of $\mathcal{F} \to \mathcal{A}^0$, hence $u^0 - v^0$ induces a morphism $\mathrm{coker}(\phi) \to \mathcal{I}^0$. Using the injectivity of $\mathcal{I}^0$, we obtain a morphism $\theta^1 \colon \mathcal{A}^1 \to \mathcal{I}^0$ such that $u^0 - v^0 = \theta^1 \circ d$. Note now that $u^1 - v^1 - d \circ \theta^1$ vanishes on the image of $\mathcal{A}^1 \to \mathcal{A}^2$, and thus induces induces a morphism from the cokernel of this morphism to $\mathcal{I}^1$. Since $\mathcal{H}^1(\mathcal{A}^\bullet) = 0$, this cokernel embeds in $\mathcal{A}^2$, and the morphism has an extension as a morphism $\theta^2 \colon \mathcal{A}^2 \to \mathcal{I}^1$ by the injectivity of $\mathcal{I}^2$. We thus have $\theta^2 \circ d + d \circ \theta^1 = u^1 - v^1$. Iterating this argument, we obtain $u \approx v$. $\square$

We will also need the following lemma about injective resolutions for $\mathcal{O}_X$-modules in an exact sequence. This is what will allow us to apply Lemma 10.1.4 to obtain long exact sequences in cohomology.

LEMMA 10.1.12. *Given an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

and injective resolutions $\mathcal{E} \to \mathcal{I}^\bullet$ and $\mathcal{G} \to \mathcal{J}^\bullet$, we can find a commutative diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}^\bullet \arrow[r] & \mathcal{Q}^\bullet \arrow[r] & \mathcal{J}^\bullet \arrow[r] & 0, \end{tikzcd}$$

such that for every i, the sequence

$$0 \to \mathcal{I}^i \to \mathcal{Q}^i \to \mathcal{J}^i \to 0$$

is split exact. In particular, the middle vertical arrow in the above commutative diagram gives an injective resolution of $\mathcal{F}$.

PROOF. For every $i \geq 0$, we put $\mathcal{Q}^i = \mathcal{I}^i \oplus \mathcal{J}^i$ and take the maps $u^i \colon \mathcal{I}^i \to \mathcal{Q}^i$ and $v^i \colon \mathcal{Q}^i \to \mathcal{J}^i$ to be the canonical injection and surjection, respectively. We will show that we can find morphisms $\mathcal{F} \to \mathcal{Q}^0$ and $\mathcal{Q}^i \to \mathcal{Q}^{i+1}$ for $i \geq 0$ such that we have a commutative diagram of complexes as in the lemma. We define $(\alpha, \beta) \colon \mathcal{F} \to \mathcal{I}^0 \oplus \mathcal{J}^0$, where β is the composition $\mathcal{F} \to \mathcal{G} \to \mathcal{J}^0$ and $\alpha \colon \mathcal{F} \to \mathcal{I}^0$

is an extension of the map $\mathcal{E} \to \mathcal{I}^0$ (we use here the fact that $\mathcal{I}^0$ is injective). We thus obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{Q}^0 \arrow[r] & \mathcal{J}^0 \arrow[r] & 0, \end{tikzcd}$$

and an application of the 3×3 lemma[1] gives a short exact sequence

$$0 \to \operatorname{coker}(\mathcal{E} \to \mathcal{I}^0) \to \operatorname{coker}(\mathcal{F} \to \mathcal{Q}^0) \to \operatorname{coker}(\mathcal{G} \to \mathcal{J}^0) \to 0.$$

We can now repeat the construction to obtain the commutative diagram of complexes in the statement.

Finally, it is clear, by construction, that $\mathcal{Q}^i$ is an injective $\mathcal{O}_X$-module for every $i \geq 0$. The fact that $\mathcal{F} \to \mathcal{Q}^\bullet$ is a resolution follows from Lemma 10.1.4. $\square$

10.1.2. Right derived functors. Suppose now that $F \colon \mathcal{C} \to \mathcal{D}$ is a (covariant) left exact functor[2], where $\mathcal{C}$ and $\mathcal{D}$ are, respectively, the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules, where $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two ringed spaces. Here are the main examples for us.

EXAMPLE 10.1.13. Suppose that $\mathcal{O}_X$ is a sheaf of R-algebras, Y is a point and $\mathcal{O}_Y(Y) = R$. In this case, we have the global sections functor

$$\Gamma(X, -) \colon \mathcal{O}_X\text{-mod} \to R\text{-mod}.$$

This is left exact.

EXAMPLE 10.1.14. More generally, if $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, we have the left exact functor

$$f_* \colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}.$$

EXAMPLE 10.1.15. Suppose that $\mathcal{O}_X$ is a sheaf of R-algebras. For every $\mathcal{O}_X$-module $\mathcal{F}$, we have the left exact functor

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) \colon \mathcal{O}_X\text{-mod} \to R\text{-mod}.$$

EXAMPLE 10.1.16. For every ringed space $(X, \mathcal{O}_X)$, we have the left exact functor

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) \colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}.$$

In order to measure the failure of a left exact functor F as above to be exact, we extend F to a sequence of functors, as follows.

DEFINITION 10.1.17. A δ-*functor* is a sequence of functors $(F^i)_{i \geq 0}$ from $\mathcal{C}$ to $\mathcal{D}$, together with the following data: for every short exact sequence in $\mathcal{C}$

$$0 \longrightarrow \mathcal{A}' \xrightarrow{u} \mathcal{A} \xrightarrow{v} \mathcal{A}'' \longrightarrow 0,$$

[1] Note that this also holds for $\mathcal{O}_X$-modules, as can be seen by considering the assertions for the stalks at the points of X.

[2] All functors we consider are *additive*: a functor F between additive categories is *additive* if for any two objects A and B, the corresponding map $\operatorname{Hom}(A, B) \to \operatorname{Hom}\big(F(A), F(B)\big)$ is a group homomorphism.

we have "connecting morphisms" $\delta\colon F^i(\mathcal{A}'') \to F^{i+1}(\mathcal{A}')$ for $i \geq 0$, such that the complex

$$\ldots \longrightarrow F^i(\mathcal{A}') \xrightarrow{F^i(u)} F^i(\mathcal{A}) \xrightarrow{F^i(v)} F^i(\mathcal{A}'') \xrightarrow{\delta} F^{i+1}(\mathcal{A}') \xrightarrow{F^{i+1}(u)} \ldots$$

is exact. Moreover, the connecting morphisms are required to be functorial: given a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}' \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{B}'' \arrow[r] & 0, \end{tikzcd}$$

for every $i \geq 0$ we have a commutative diagram

$$\begin{tikzcd} F^i(\mathcal{A}'') \arrow[r, "\delta"] \arrow[d] & F^{i+1}(\mathcal{A}') \arrow[d] \\ F^i(\mathcal{B}'') \arrow[r, "\delta"] & \mathcal{F}^{i+1}(\mathcal{B}'). \end{tikzcd}$$

Definition 10.1.18. Given two δ-functors $(F_i)_{i\geq 0}$ and $(G_i)_{i\geq 0}$ from $\mathcal{C}$ to $\mathcal{D}$, a *morphism of δ-functors* is given by natural transformations $(F_i \to G_i)_{i\geq 0}$ such that for every short exact sequence in $\mathcal{C}$

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0,$$

we have a commutative diagram

$$\begin{tikzcd} F^i(\mathcal{A}'') \arrow[r, "\delta"] \arrow[d] & F^{i+1}(\mathcal{A}') \arrow[d] \\ G^i(\mathcal{A}'') \arrow[r, "\delta"] & G^{i+1}(\mathcal{A}''). \end{tikzcd}$$

Note that in this case, by the functoriality of the transformations $F^i \to G^i$, we have a morphism of long exact sequences.

The following is the fundamental result in the construction of derived functors. We keep the above convention about the categories $\mathcal{C}$ and $\mathcal{D}$ (though the statement generalizes immediately to the case when $\mathcal{C}$ and $\mathcal{D}$ are Abelian categories, with $\mathcal{C}$ having enough injective objects).

Theorem 10.1.19. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a left exact functor, then there is a δ-functor $(R^iF)_{i\geq 0}$ such that the following two conditions are satisfied:*

i) *We have a natural isomorphism $R^0F \simeq F$, and*
ii) *$R^iF(\mathcal{I}) = 0$ for every injective object $\mathcal{I} \in \mathcal{C}$ and every $i \geq 1$.*

Such a δ-functor is unique up to a unique isomorphism that corresponds to the identity on $R^0F \simeq F$. Moreover, if $(G^i)_{i\geq 0}$ is any δ-functor and we have a natural transformation $F \to G^0$, then there is a unique extension of this to a morphism of δ-functors $(R^iF)_{i\geq 0} \to (G^i)_{i\geq 0}$.

Proof. For every object $\mathcal{A}$ in $\mathcal{C}$, we choose an injective resolution $\mathcal{A} \to \mathcal{I}^\bullet$ and put

$$R^iF(\mathcal{A}) := \mathcal{H}^i\big(F(\mathcal{I}^\bullet)\big).$$

Given a morphism $\tau\colon \mathcal{A} \to \mathcal{B}$, if $\mathcal{A} \to \mathcal{I}^\bullet$ and $\mathcal{B} \to \mathcal{J}^\bullet$ are the chosen injective resolutions, then it follows from Proposition 10.1.11 that there is a morphism $u\colon \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ such that we have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\tau"'] & \mathcal{I}^\bullet \arrow[d, "u"] \\ \mathcal{B} \arrow[r] & \mathcal{J}^\bullet. \end{tikzcd}$$

We put $R^iF(\tau) = \mathcal{H}^i\big(F(u)\big)$. Note that u is not unique, but if v is another such morphism, then it follows from Proposition 10.1.11 that $u \approx v$, hence $F(u) \approx F(v)$, and thus $F(u)$ and $F(v)$ induce the same morphism in cohomology. Using this, it is straightforward to see that as defined R^iF is a functor. This also shows that if $\mathcal{A} \to \mathcal{I}'^\bullet$ is another injective resolution, then we have a canonical isomorphism $R^iF(A) \simeq \mathcal{H}^i\big(F(\mathcal{I}'^\bullet)\big)$.

We now show that we can put on the sequence $(R^iF)_{i\geq 0}$ the structure of a δ-functor. Suppose that we have an exact sequence

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

and that the chosen injective resolutions are $\mathcal{A}' \to \mathcal{I}^\bullet$, $\mathcal{A} \to \mathcal{I}^\bullet$, and $\mathcal{A}'' \to \mathcal{I}''^\bullet$. It follows from Lemma 10.1.12 that there is an injective resolution $\mathcal{A} \to \mathcal{J}^\bullet$ such that we have a commutative diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}' \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}'^\bullet \arrow[r] & \mathcal{J}^\bullet \arrow[r] & \mathcal{I}''^\bullet \arrow[r] & 0, \end{tikzcd}$$

such that for every i, the sequence

$$0 \to \mathcal{I}'^m \to \mathcal{J}^m \to \mathcal{I}''^m \to 0$$

is split. Since applying F preserves split exact sequences, we obtain a short exact sequence of complexes

$$0 \to F(\mathcal{I}'^\bullet) \to F(\mathcal{J}^\bullet) \to F(\mathcal{I}''^\bullet) \to 0,$$

and Lemma 10.1.4 gives a long exact sequence

$$\ldots \longrightarrow R^iF(\mathcal{A}') \longrightarrow \mathcal{H}^i\big(F(\mathcal{J}^\bullet)\big) \longrightarrow R^iF(\mathcal{A}'') \xrightarrow{\delta} R^{i+1}F(\mathcal{A}') \longrightarrow \ldots.$$

Since we have a canonical isomorphism $R^iF(\mathcal{A}) \simeq \mathcal{H}^i\big(F(\mathcal{J}^\bullet)\big)$ and since the connecting homomorphisms that we constructed are functorial with respect to morphisms of short exact sequences, we see that $(R^iF)_{i\geq 0}$ form a δ-functor.

The fact that we have a functorial isomorphism $R^0F \simeq F$ follows from definition and the fact that F is a left exact functor. In order to see that if $\mathcal{Q}$ is an injective object in $\mathcal{C}$, then $R^iF(\mathcal{Q}) = 0$ for $i \geq 1$, we consider the injective resolution $\mathcal{I}^\bullet$ of $\mathcal{Q}$ such that $\mathcal{Q} \to \mathcal{I}^0$ is the identity and $\mathcal{I}^i = 0$ for $i \geq 1$. In this case, the assertion is clear.

Note now that the uniqueness of the sequence $(R^iF)_{i\geq 0}$ follows if we show that properties i) and ii) imply the last assertion in the theorem. Given an object $\mathcal{A} \in \mathcal{C}$, choose an exact sequence

$$0 \to \mathcal{A} \to \mathcal{I} \to \mathcal{B} \to 0,$$

where $\mathcal{I}$ is injective. We thus obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F(\mathcal{A}) \arrow[r] \arrow[d] & F(\mathcal{I}) \arrow[r] \arrow[d] & F(\mathcal{B}) \arrow[r] \arrow[d] & R^1F(\mathcal{A}) \arrow[r] & 0 \\ 0 \arrow[r] & G^0(\mathcal{A}) \arrow[r] & G^0(\mathcal{I}) \arrow[r] & G^0(\mathcal{B}) \arrow[r] & G^1(\mathcal{A}) & \end{tikzcd}$$

and since the top row is exact (we use here that $R^1F(\mathcal{I}) = 0$), we obtain an induced morphism $R^1F(A) \to G^1(A)$ that makes the square commutative. It is easy to see, arguing as before, that this is independent of the choice of $\mathcal{I}$ and gives a natural transformation of functors. We then argue by induction on $i \geq 1$ to construct the natural transformation $R^iF \to G^i$. Note that for $i \geq 1$, we also obtain from the above exact sequence the horizontal maps below

$$\begin{tikzcd} R^iF(\mathcal{B}) \arrow[r] \arrow[d] & R^{i+1}F(\mathcal{A}) \\ G^i(\mathcal{B}) \arrow[r] & G^{i+1}(\mathcal{A}), \end{tikzcd}$$

while the vertical map is given by the inductive assumption. Since the top horizontal map is an isomorphism, it follows that we have a unique map $R^{i+1}F(\mathcal{A}) \to G^{i+1}(\mathcal{A})$ that makes the square commutative. It is then not hard to see that the transformations $(R^iF \to G^i)_{i\geq 0}$ constructed in this way give a morphism of δ-functors and that this is the unique such morphism that extends $F \to G^0$. $\square$

REMARK 10.1.20. By inspecting the above proof, one notices that for the construction of the morphism of δ-functors $(R^iF)_{i\geq 0} \to (G^i)_{i\geq 0}$ we only needed the fact that the sequence of functors $(G^i)_{i\geq 0}$ associates to every short exact sequence a complex (which does not have to be exact).

EXAMPLE 10.1.21. If $\eta\colon F \to G$ is a natural transformation between left exact functors, then it follows from Proposition 10.1.19 that we have unique natural transformations $\eta^i\colon R^iF \to R^iG$ that give a morphism of δ-functors and such that η^0 gets identified with η. These transformations can be computed as follows: if $\mathcal{A}$ is an object in $\mathcal{C}$ and $\mathcal{A} \to \mathcal{I}^\bullet$ is an injective resolution, then $\eta^i(\mathcal{A})$ corresponds to the morphism obtained by applying $\mathcal{H}^i(-)$ to the morphism of complexes $F(\mathcal{I}^\bullet) \to G(\mathcal{I}^\bullet)$ induced by η. This can be checked either by comparing with the argument in the proof of the proposition, or by showing that as described, we have a morphism of δ-functors that coincides with η for $i = 0$.

DEFINITION 10.1.22. The functor R^iF in the above theorem is the ith *right derived functor* of F.

In practice, one never uses injective resolutions to compute the derived functors. A situation that occurs more often is to identify a special class of objects that can be used to compute derived functors, as follows.

DEFINITION 10.1.23. Given a left exact functor $F\colon \mathcal{C} \to \mathcal{D}$ as above, an object $\mathcal{M}$ of $\mathcal{C}$ is *F-acyclic* if $R^iF(\mathcal{M}) = 0$ for all $i \geq 1$.

PROPOSITION 10.1.24. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a left exact functor and for an object $\mathcal{A}$ in $\mathcal{C}$ we have a resolution $\mathcal{A} \to \mathcal{M}^\bullet$ such that all $\mathcal{M}^i$ are acyclic, then we have a* *canonical isomorphism*

$$R^iF(\mathcal{A}) \simeq \mathcal{H}^i\big(F(\mathcal{M}^\bullet)\big).$$

Proof. We argue by induction on i. The case $i=0$ is clear, since

$$R^0F(\mathcal{A}) \simeq F(\mathcal{A}) \simeq \ker\big(F(\mathcal{M}^0) \to F(\mathcal{M}^1)\big)$$

by the left exactness of F. For $i \geq 1$, let $\mathcal{A}' = \mathrm{coker}(\mathcal{A} \hookrightarrow \mathcal{M}^0)$. Using the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{M}^0 \to \mathcal{A}' \to 0$$

and the fact that $\mathcal{M}^0$ is F-acyclic, we obtain an exact sequence

$$0 \to F(\mathcal{A}) \to F(\mathcal{M}^0) \to F(\mathcal{A}') \to R^1F(\mathcal{A}) \to 0 \tag{10.1.1}$$

and isomorphisms

$$R^pF(\mathcal{A}') \simeq R^{p+1}F(\mathcal{A}) \quad \text{for} \quad p \geq 1. \tag{10.1.2}$$

It follows from (10.1.1) that we have a canonical isomorphism

$$R^1F(\mathcal{A}) \simeq F(\mathcal{A}')/\mathrm{Im}\big(F(\mathcal{M}^0) \to F(\mathcal{A}')\big) \simeq \mathcal{H}^1\big(F(\mathcal{M}^\bullet)\big),$$

where the second isomorphism follows from the left exactness of F and the exact sequence

$$0 \to \mathcal{A}' \to \mathcal{M}^1 \to \mathcal{M}^2.$$

On the other hand, we have a resolution

$$0 \to \mathcal{A}' \to \mathcal{M}^1 \to \mathcal{M}^2 \to \ldots,$$

hence using (10.1.2) and induction, we see that if the assertion in the proposition holds for $i \geq 1$, it also holds for $i+1$. $\square$

10.2. Cohomology of sheaves and higher direct images

The discussion in the previous section applies generally to left exact functors between Abelian categories that have enough injective objects. We now specialize to the functors on categories of sheaves that we are interested in.

10.2.1. Cohomology of sheaves. Let $(X, \mathcal{O}_X)$ be a ringed space, where $\mathcal{O}_X$ is a sheaf of R-algebras (for example, we can always choose $R = \mathbf{Z}$) and consider the left exact global sections functor:

$$\Gamma(X,-)\colon \mathcal{O}_X\text{-mod} \to R\text{-mod}.$$

Its right-derived functors are denoted by $H^i(X,-)$, for $i \geq 0$. The R-modules $H^i(X,\mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$ (in our setting, it might make more sense to call them *cohomology R-modules*, but we prefer to follows the standard terminology). Note that we have a functorial isomorphism $\Gamma(X,-) \simeq H^0(X,-)$. Moreover, for every short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we have a *long exact sequence in cohomology*

$$\ldots \to H^i(X,\mathcal{F}') \to H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F}'') \to H^{i+1}(X,\mathcal{F}') \to \ldots.$$

Remark 10.2.1. It is clear that the Abelian groups underlying the $H^i(X,\mathcal{F})$ are independent of which R we choose.We can always take $R = \Gamma(X,\mathcal{O}_X)$, which shows that for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a natural $\Gamma(X,\mathcal{O}_X)$-module structure on the cohomology groups $H^i(X,\mathcal{F})$.

DEFINITION 10.2.2. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flasque* if for every open subset U of X, the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective.

REMARK 10.2.3. It is clear from definition that if $\mathcal{F}$ is flasque, then its restriction $\mathcal{F}|_V$ to any open subset V, is again flasque.

As we will see, the flasque sheaves are $\Gamma(X, -)$-acyclic. We begin with a couple of lemmas.

LEMMA 10.2.4. *Given a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$ flasque, we have a short exact sequence

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0.$$

PROOF. We only need to show that for every $s \in \mathcal{F}''(X)$, there is $t \in \mathcal{F}(X)$ such that $\beta(t) = s$. Consider the set $\mathcal{P}$ consisting of pairs (U, t_U), where U is an open subset of X and $t_U \in \mathcal{F}(U)$ is such that $\beta(t_U) = s|_U$. We order the elements of $\mathcal{P}$ by putting $(U, \phi_U) \leq (V, \phi_V)$ if $U \subseteq V$ and $\phi_U = \phi_V|_U$. It is clear that any totally ordered subset of $\mathcal{P}$ has a supremum (given by taking the union of the corresponding open subsets and by gluing the corresponding sections of $\mathcal{F}$). By Zorn's lemma, we may thus choose a maximal element (W, ϕ_W). In order to complete the proof, it is enough to show that if $W \neq X$, then the pair (W, ϕ_W) is not maximal.

Suppose that $x \in X \smallsetminus W$. By the surjectivity of β, we can choose an open neighborhood V of x and $t_V \in \mathcal{F}(V)$ such that $\beta(t_V) = s|_V$. In particular, we have $\beta(t_W|_{V \cap W}) = \beta(t_V|_{V\cap W})$, and thus $t_W|_{V\cap W} - t_V|_{V\cap W} = \alpha(t')$, for some $t' \in \mathcal{F}'(V \cap W)$. Since $\mathcal{F}'$ is flasque, we can find $t'' \in \mathcal{F}'(X)$ such that $t''|_{V\cap W} = t'$. After replacing t_V by $t_V + \alpha(t''|_V)$, we see that we may assume that $\phi_U|_{U\cap V} = \phi_V|_{V\cap W}$. If we take $W' = W \cup V$, there is a unique $t_{W'} \in \mathcal{F}(W')$ such that $t_{W'}|_W = t_W$ and $t_{W'}|_V = t_V$. It is then clear that $\beta(t_{W'}) = s|_{W'}$, this contradicts the maximality of (W, t_W). $\square$

COROLLARY 10.2.5. *Given a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

with $\mathcal{F}'$ flasque, we have $\mathcal{F}$ flasque if and only if $\mathcal{F}''$ is flasque.

PROOF. Since $\mathcal{F}'$ is flasque, the restriction $\mathcal{F}'|_U$ is flasque too, and we deduce from Lemma 10.2.4 that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\beta"] & \mathcal{F}''(X) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{F}''(U) \arrow[r] & 0,
\end{tikzcd}$$

in which the vertical maps are given by restriction. Since α is surjective by assumption, it follows that β is surjective if and only if γ is surjective. $\square$

EXERCISE 10.2.6. Let U be an open subset of X and $i\colon U \hookrightarrow X$ the inclusion map. For an $\mathcal{O}_U$-module $\mathcal{G}$ on U, we consider the presheaf of $\mathcal{O}_X$-modules $\mathcal{G}_0$, defined as follows: $\Gamma(V, \mathcal{G}_0) = \mathcal{G}(V)$ if $V \subseteq U$ and $\Gamma(V, \mathcal{G}_0) = 0$, otherwise (with the non-zero restriction maps given by those for $\mathcal{G}$). The *extension by* 0 *of* $\mathcal{G}$ is the $\mathcal{O}_X$-module $i_!(\mathcal{G})$ associated to $\mathcal{G}_0$.

i) Show that for $x \in X$, we have $i_!(\mathcal{G})_x \simeq \mathcal{G}_x$ if $x \in U$ and $i_!(\mathcal{G})_x = 0$, otherwise. In particular, the map taking $\mathcal{G}$ to $i_!(\mathcal{G})$ is an exact functor from the category of $\mathcal{O}_U$-modules to that of $\mathcal{O}_X$-modules.
ii) Show that for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a functorial isomorphism
$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}|_U) \simeq \mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{G}), \mathcal{F}\big).$$
In other words, $(i_!, i^*)$ is an adjoint pair.
iii) In particular, for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical morphism $i_!(\mathcal{F}|_U) \to \mathcal{F}$. Show that this is injective.

LEMMA 10.2.7. *Every injective $\mathcal{O}_X$-module is flasque.*

PROOF. Let U be an open subset of X and let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. We need to show that the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective.

We use the definition in the above exercise. Note that for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism
$$\mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{O}_U), \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) \simeq \mathcal{F}(U).$$
Since $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, the inclusion $i_!(\mathcal{O}_U) \hookrightarrow \mathcal{O}_X$ induces a surjective map
$$\mathcal{I}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{O}_U), \mathcal{I}\big) \simeq \mathcal{I}(U),$$
which is the map given by restriction. This completes the proof. $\square$

PROPOSITION 10.2.8. *If $\mathcal{M}$ is a flasque $\mathcal{O}_X$-module, then $H^i(X, \mathcal{M}) = 0$ for all $i \geq 1$. In particular, for every $\mathcal{O}_X$-module $\mathcal{F}$, if $\mathcal{F} \to \mathcal{M}^\bullet$ is a flasque resolution (that is, all $\mathcal{M}^i$ are flasque $\mathcal{O}_X$-modules), then we have a canonical isomorphism*
$$H^i(X, \mathcal{F}) \simeq \mathcal{H}^i\big(\Gamma(X, \mathcal{M}^\bullet)\big).$$

PROOF. Consider an injective resolution $\mathcal{M} \to \mathcal{I}^\bullet$. If $\mathcal{M}' = \mathrm{coker}(\mathcal{M} \to \mathcal{I}^0)$, then we have a short exact sequence
$$0 \to \mathcal{M} \to \mathcal{I}^0 \to \mathcal{M}' \to 0.$$
Since $\mathcal{M}$ is flasque and $\mathcal{I}^0$ is injective (hence flasque, by Lemma 10.2.7), we conclude that $\mathcal{M}'$ is flasque by Corollary 10.2.5. On the other hand, the long exact sequence in cohomology associated to the above short exact sequence gives
$$H^i(X, \mathcal{M}') \simeq H^{i+1}(X, \mathcal{M}) \quad \text{for} \quad i \geq 1, \quad \text{and} \tag{10.2.1}$$
$$H^1(X, \mathcal{M}) = \mathrm{coker}\big(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{M}')\big) = 0, \tag{10.2.2}$$
where the last equality in (10.2.2) follows from the fact that $\mathcal{M}$ is flasque, by Lemma 10.2.4. We thus have the vanishing in the statement for $i = 1$ and the assertion for $i \geq 2$ follows by induction using (10.2.1). The last assertion in the proposition now follows from Proposition 10.1.24. $\square$

EXAMPLE 10.2.9. While it is basically impossible to write down an explicit injective resolution of an $\mathcal{O}_X$-module $\mathcal{F}$, it is very easy to write down a flasque resolution. In fact, we obtain a canonical such resolution, as follows. Given an $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{M}^0$ be given by

$$\Gamma(U, \mathcal{M}^0) := \prod_{x \in U} \mathcal{F}_x,$$

with the restriction maps being given by projections onto the corresponding factors. Note that $\Gamma(U, \mathcal{M}^0)$ is a module over $\mathcal{O}_X(U)$, where $f \in \mathcal{O}_X(U)$ acts via $(f_x)_{x \in U}$. This makes $\mathcal{M}^0$ an $\mathcal{O}_X$-module, which is clearly flasque. Finally, we have an injective morphism

$$\mathcal{F} \to \mathcal{M}^0, \quad s \to (s_x)_{x \in U}.$$

We can then iterate this construction for $\operatorname{coker}(\mathcal{F} \to \mathcal{M}^0)$ and we thus obtain a flasque resolution $\mathcal{F} \to \mathcal{M}^\bullet$. It is clear that this is a functorial construction.

REMARK 10.2.10. Note that the flasque resolution constructed in the above example is independent of the structure sheaf $\mathcal{O}_X$. Since we can compute cohomology using flasque resolutions, we see that for every $\mathcal{O}_X$-module $\mathcal{F}$, the R-modules $H^i(X, \mathcal{F})$ are independent of $\mathcal{O}_X$ (as long as $\mathcal{O}_X$ is a sheaf of R-algebras).

REMARK 10.2.11. For every $\mathcal{O}_X$-module $\mathcal{F}$ on X and every open subset U of X, consider $H^i(U, \mathcal{F}|_U)$, that with a slight abuse of notation we will denote by $H^i(U, \mathcal{F})$. Note that the functors $\big(H^i(U, -)\big)_{i \geq 0}$ are the derived functors of $\Gamma(U, -)$ on $\mathcal{O}_X$-mod. Indeed, it is clear that they form a δ-functor, and if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque by Lemma 10.2.7, hence $\mathcal{I}|_U$ is flasque, and thus $H^i(U, \mathcal{I}) = 0$ for $i \geq 1$ by Proposition 10.2.8. The natural transformation $\Gamma(X, -) \to \Gamma(U, -)$ given by restriction of sections thus extends uniquely to a morphism of δ-functors given by

$$H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \quad \text{for} \quad i \geq 0 \tag{10.2.3}$$

(see Example 10.1.21). Explicitly, the morphisms in (10.2.3) can be described as follows: given an injective (or flasque) resolution $\mathcal{F} \to \mathcal{I}^\bullet$, we have a flasque resolution $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$, and the canonical morphism of complexes

$$\Gamma(X, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{I}^\bullet|_U)$$

induces after applying $\mathcal{H}^i(-)$ the required morphisms.

We note that these morphisms (10.2.3) are functorial with respect to the inclusion maps between the open subsets of X. This follows, for example, from the above explicit description.

EXAMPLE 10.2.12. Let X be an irreducible algebraic variety and consider the short exact sequence of sheaves of Abelian groups on X:

$$0 \to \mathcal{O}_X^* \to k(X) \to k(X)/\mathcal{O}_X^* \to 0.$$

Since X is irreducible, every constant sheaf on X is flasque, and thus $H^1\big(X, k(X)\big) = 0$ by Proposition 10.2.8. The long exact sequence in cohomology for the abort short exact sequence therefore gives

$$\Gamma\big(X, k(X)\big) \to \Gamma\big(X, k(X)/\mathcal{O}_X^*\big) = \operatorname{Cart}(X) \to H^1(X, \mathcal{O}_X^*) \to 0.$$

We thus see that

$$H^1(X, \mathcal{O}_X^*) \simeq \operatorname{Cart}(X)/\operatorname{PCart}(X) \simeq \operatorname{Pic}(X).$$

We note that the isomorphism $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$ holds even if X is not irreducible; the argument in the general setting makes use of Čech cohomology (see for example [Har77, Exercise III.4.5])

10.2.2. Higher-direct images. Suppose now that $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. We have the left exact functor

$$f_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}.$$

Its derived functors are denoted by $R^i f_*$, for $i \geq 0$; the functor $R^i f_*$ is the i^{th} *higher direct image* functor. We have a canonical isomorphism $R^0 f_* \simeq f_*$ and for every short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we have a *long exact sequence for higher direct images*

$$\ldots \to R^i f_*(\mathcal{F}') \to R^i f_*(\mathcal{F}) \to R^i f_*(\mathcal{F}'') \to R^{i+1} f_*(\mathcal{F}') \to \ldots.$$

Remark 10.2.13. If Y is a point and $\mathcal{O}_Y$ is given by the commutative ring R, then a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ corresponds to an R-algebra structure for $\mathcal{O}_X$, and the functor f_* gets identified with $\Gamma(X, -)$. Therefore in this case the functors $R^i f_*$ get identified to $H^i(X, -)$.

Remark 10.2.14. It is clear from definition that if $\mathcal{F}$ is a flasque $\mathcal{O}_X$-module, then $f_*(\mathcal{F})$ is flasque, too.

Given a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as above and an $\mathcal{O}_X$-module $\mathcal{F}$, for every open subset U of Y, we have on $H^i\big(f^{-1}(U), \mathcal{F}\big)$ a structure of $\mathcal{O}_X\big(f^{-1}(U)\big)$-module, and thus a structure of $\mathcal{O}_Y(U)$-module. Moreover, if V is an open subset of U, then we have a canonical morphism of $\mathcal{O}_Y(U)$-modules

$$H^i\big(f^{-1}(U), \mathcal{F}\big) \to H^i\big(f^{-1}(V), \mathcal{F}\big)$$

(see Remark 10.2.11). This shows that we get a presheaf of $\mathcal{O}_Y$-modules that we denote $\widetilde{R}^i f_*(\mathcal{F})$. It is clear that each $\widetilde{R}^i f_*(-)$ is a functor.

Proposition 10.2.15. *With the above notation, we have a functorial isomorphism*

$$R^i f_*(\mathcal{F}) \simeq \widetilde{R}^i f_*(\mathcal{F})^+.$$

Proof. Note that the sequence $\big(\widetilde{R}^i f_*(-)^+\big)_{i \geq 0}$ has a natural structure of δ-functor. Indeed, given any exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

for every open subset U of X, we have a long exact sequence

$$\ldots \to H^i\big(f^{-1}(U), \mathcal{F}'\big) \to H^i\big(f^{-1}(U), \mathcal{F}\big) \to H^i\big(f^{-1}(U), \mathcal{F}''\big) \to \tag{10.2.4}$$
$$\to H^{i+1}\big(f^{-1}(U), \mathcal{F}'\big) \to \ldots.$$

By varying the open subset U, we see that we get a morphism of presheaves

$$\widetilde{R}^i f_*(\mathcal{F}'') \to \widetilde{R}^{i+1} f_*(\mathcal{F}'),$$

and an induced morphism between the associated sheaves. The fact that the corresponding complex of sheaves is exact follows from the exact sequences (10.2.4) by passing to stalks.

Note that for $i = 0$ we have

$$\widetilde{R}^0 f_*(\mathcal{F})^+ \simeq f_*(\mathcal{F}) \simeq R^0 f_*(\mathcal{F}) \quad \text{for all} \quad \mathcal{F}.$$

Moreover, if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then it is flasque by Lemma 10.2.7 and thus $\widetilde{R}^i f_*(\mathcal{I}) = 0$ for $i \geq 1$ by Proposition 10.2.8, hence the same vanishing holds for the associated sheaves. The uniqueness assertion in Theorem 10.1.19 then gives an isomorphism of δ-functors between $\big(R^i f_*(-)\big)_{i\geq 0}$ and $\big(\widetilde{R}^i f_*(-)^+\big)_{i\geq 0}$. This implies, in particular, the assertion in the proposition. $\square$

Corollary 10.2.16. *For every morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and every flasque $\mathcal{O}_X$-module $\mathcal{M}$, we have $R^i f_*(\mathcal{M}) = 0$ for all $i \geq 1$. In particular, given any $\mathcal{O}_X$-module $\mathcal{F}$ and any flasque resolution $\mathcal{F} \to \mathcal{M}^\bullet$, we have a canonical isomorphism*

$$R^i f_*(\mathcal{F}) \simeq \mathcal{H}^i\big(f_*(\mathcal{M}^\bullet)\big) \quad \textit{for all} \quad i \geq 0.$$

Proof. The first assertion follows from the proposition, since $\widetilde{R}^i f_*(\mathcal{M}) = 0$ for all $i \geq 1$ by Proposition 10.2.8. The second assertion is then a consequence of Proposition 10.1.24. $\square$

10.2.3. Higher direct images of quasi-coherent sheaves. Our next goal is to show that for morphisms of algebraic varieties, higher direct images preserve quasi-coherence. The key result is the following

Proposition 10.2.17. *For every algebraic variety X and every quasi-coherent sheaf $\mathcal{F}$, there is an injective morphism $\mathcal{F} \hookrightarrow \mathcal{G}$, where $\mathcal{G}$ is a flasque quasi-coherent sheaf.*

The proof is somewhat involved. The argument we give follows [Har77, Chapter III.3]. We begin with two lemmas.

Lemma 10.2.18. *If A is a Noetherian ring and Q is an injective A-module, then for every ideal $\mathfrak{a}$ in A, the submodule*

$$\Gamma_{\mathfrak{a}}(Q) := \{u \in Q \mid \mathfrak{a}^r \cdot u = 0 \textit{ for some } r \geq 1\}$$

is injective, too

Proof. By Baer's criterion (see Proposition I.1.2), it is enough to show that for every ideal $\mathfrak{b}$ in A, the induced morphism

$$\Gamma_{\mathfrak{a}}(Q) = \mathrm{Hom}_A\big(A, \Gamma_{\mathfrak{a}}(Q)\big) \to \mathrm{Hom}_A\big(\mathfrak{b}, \Gamma_{\mathfrak{a}}(Q)\big) \tag{10.2.5}$$

is surjective. Let $\phi\colon \mathfrak{b} \to \Gamma_{\mathfrak{a}}(Q)$. Since $\mathfrak{b}$ is finitely generated, it follows that there is $r \geq 1$ such that

$$\phi(\mathfrak{a}^r \cdot \mathfrak{b}) = \mathfrak{a}^r \cdot \phi(\mathfrak{b}) = 0.$$

On the other hand, it follows from the Artin-Rees lemma (see Lemma C.4.2) that there is n such that $\mathfrak{a}^n \cap \mathfrak{b} \subseteq \mathfrak{a}^r \cdot \mathfrak{b}$, hence ϕ induces a morphism

$$\overline{\phi}\colon \mathfrak{b}/(\mathfrak{a}^n \cap \mathfrak{b}) \to \Gamma_{\mathfrak{a}}(Q) \subseteq Q.$$

Since Q is an injective A-module, the injective morphism

$$\iota\colon \mathfrak{b}/(\mathfrak{a}^n \cap \mathfrak{b}) \hookrightarrow A/\mathfrak{a}^n$$

induces a surjective morphism

$$\mathrm{Hom}_A(A/\mathfrak{a}^n, Q) \to \mathrm{Hom}_A\big(\mathfrak{b}/(\mathfrak{a}^n \cap \mathfrak{b}), Q\big).$$

We can thus find a morphism $\overline{\psi}\colon A/\mathfrak{a}^n \to Q$ such that $\overline{\psi} \circ \iota = \phi$. Note that

$$\operatorname{Im}(\overline{\psi}) \subseteq \{u \in Q \mid \mathfrak{a}^n \cdot u = 0\} \subseteq \Gamma_{\mathfrak{a}}(Q),$$

hence by composing $\overline{\psi}$ with the projection $A \to A/\mathfrak{a}^n$, we obtain a morphism $\psi\colon A \to \Gamma_{\mathfrak{a}}(Q)$, whose restriction to $\mathfrak{b}$ is equal to ϕ. $\square$

LEMMA 10.2.19. *If X is an affine variety, with $A = \mathcal{O}(X)$, then for every injective A-module Q, the quasi-coherent sheaf $\widetilde{Q}$ is flasque.*

PROOF. We need to show that for every open subset $U \subseteq X$, the restriction map

$$Q = \Gamma(X, \widetilde{Q}) \to \Gamma(U, \widetilde{Q}) \tag{10.2.6}$$

is surjective. Let us consider first the case when $U = D_X(f)$, for some $f \in A$, in which case $\Gamma(U, \widetilde{Q}) = Q_f$. Consider the following non-decreasing chain of ideals in A:

$$\operatorname{Ann}_A(f) \subseteq \operatorname{Ann}_A(f^2) \subseteq \ldots \subseteq \operatorname{Ann}_A(f^n) \subseteq \ldots.$$

Since A is Noetherian, it follows that there is $r \geq 1$ such that $\operatorname{Ann}_A(f^r) = \operatorname{Ann}_A(f^n)$ for all $n \geq r$. Given an element $u \in Q_f$, we can write it as $\frac{v}{f^s}$, for some $v \in Q$ and $s \geq 0$. We define a morphism $\phi\colon (f^{r+s}) \to Q$ given by $\phi(af^{r+s}) = af^r v$. Note that this is well-defined: if $af^{r+s} = bf^{r+s}$, then our choice of r implies $af^r = bf^r$, hence $af^r v = bf^r v$. Since Q is an injective module, there is a morphism $\psi\colon A \to Q$ that extends ϕ. If $w = \psi(1)$, then

$$f^r v = \phi(f^{r+s}) = \psi(f^{r+s}) = f^{r+s} w,$$

hence $u = \frac{v}{f^s} = \frac{w}{1}$ lies in the image of Q. We are thus done if U is a principal affine open subset of X.

We now consider the general case. Consider $u \in \Gamma(U, \widetilde{Q})$. If $f \in A$ is such that $D_X(f) \subseteq U$, it follows from the case we have already proved that there is $s \in Q$ such that $s|_{D_X(f)} = u|_{D_X(f)}$. After replacing u by $u - s|_U$, we may thus assume that $u|_{D_X(f)} = 0$. Since $D_X(f) = D_U(f|_U)$, it follows from Exercise 8.4.30 that u is annihilated by some power of f. Moreover, we claim that if our original u was annihilated by some power of an element $g \in A$, we may choose s that is also annihilated by some power of g, and thus the same will be true for $u - s$.

In order to prove the claim, consider the subsheaf $\mathcal{G}$ of $\widetilde{Q}$ given by

$$\Gamma(V, \mathcal{G}) = \{v \in \Gamma(V, \widetilde{Q}) \mid g^r v = 0 \text{ for some } r \geq 0\}.$$

It is straightforward to see that $\mathcal{G}$ is a quasi-coherent sheaf and thus, with the notation in Lemma 10.2.18, it is equal to $\widetilde{\Gamma_{(g)}(Q)}$. Since $\Gamma_{(g)}(Q)$ is injective by the lemma and $u \in \Gamma(U, \mathcal{G})$, it follows from what we have already proved that we can find $s \in \Gamma_{(g)}(Q)$ such that $s|_{D_X(f)} = u|_{D_X(f)}$. This proves our claim.

Suppose now that $U = D_X(f_1) \cup \ldots \cup D_X(f_n)$, for some $f_1, \ldots, f_n \in A$. Using repeatedly the above claim, we see that we may assume that for every i, u is annihilated by some power of f_i; equivalently, we have $u|_{D_X(f_i)} = 0$ for all i. Therefore $u = 0$, in which case it is trivially in the image of $\Gamma(X, \widetilde{Q})$. This completes the proof of the lemma. $\square$

We can now show that every quasi-coherent sheaf can be embedded in a flasque quasi-coherent sheaf.

Proof of Proposition 10.2.17. Note first that the assertion is clear if X is affine by Lemma 10.2.19, since every $\mathcal{O}(X)$-module can be embedded in an injective module. For an arbitrary variety X, consider an affine cover $X = U_1 \cup \ldots \cup U_r$ and let $\alpha_i \colon U_i \hookrightarrow X$ be the inclusion map for $1 \leq i \leq r$. Since $\mathcal{F}$ is quasi-coherent, the restriction $\mathcal{F}|_{U_i}$ is quasi-coherent, and since U_i is affine, we can find an injective morphism $\mathcal{F}|_{U_i} \hookrightarrow \mathcal{G}_i$, with $\mathcal{G}_i$ quasi-coherent and flasque on U_i. Each sheaf $\alpha_{i*}(\mathcal{G})$ is quasi-coherent by Proposition 8.4.5 and it is clearly flasque. The assertion in the proposition now follows from the injective homomorphism:

$$\mathcal{F} \hookrightarrow \bigoplus_{i=1}^{r} \alpha_{i*}(\mathcal{F}|_{U_i}) \hookrightarrow \bigoplus_{i=1}^{r} \alpha_{i*}(\mathcal{G}_i).$$

$\square$

Proposition 10.2.20. *If $f \colon X \to Y$ is a morphism of algebraic varieties, then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the sheaves $R^i f_*(\mathcal{F})$ are quasi-coherent for all $i \geq 0$. Moreover, for every open subset U of Y, we have isomorphisms*

$$\Gamma\big(U, R^i f_*(\mathcal{F})\big) \simeq H^i\big(f^{-1}(U), \mathcal{F}\big) \quad \textit{for all} \quad i \geq 0. \tag{10.2.7}$$

Proof. A straightforward inductive argument based on Proposition 10.2.17 shows that we can find a resolution $\mathcal{F} \to \mathcal{M}^\bullet$, with each $\mathcal{M}^i$ quasi-coherent and flasque. In this case Corollary 10.2.16 implies that we have isomorphisms

$$R^i f_*(\mathcal{F}) \simeq \mathcal{H}^i\big(f_*(\mathcal{M}^\bullet)\big).$$

Since all sheaves $f_*(\mathcal{M}^p)$ are quasi-coherent and kernels and cokernels of morphisms of quasi-coherent sheaves are quasi-coherent, we conclude that $R^i f_*(\mathcal{F})$ is quasi-coherent. Moreover, since the functor $\Gamma(U, -)$ is exact on quasi-coherent sheaves, it follows that

$$\Gamma\big(U, R^i f_*(\mathcal{F})\big) \simeq \Gamma\big(U, \mathcal{H}^i\left(f_*(\mathcal{M}^\bullet)\right)\big) \simeq \mathcal{H}^i\big(\Gamma\left(U, f_*(\mathcal{M}^\bullet)\right)\big)$$
$$\simeq \mathcal{H}^i\left(\Gamma\left(f^{-1}(U), \mathcal{M}^\bullet\right)\right) \simeq H^i\big(f^{-1}(U), \mathcal{F}\big),$$

where the last isomorphism follows from the fact that the $\mathcal{M}^\bullet|_{f^{-1}(U)}$ is a flasque resolution of $\mathcal{F}|_U$. $\square$

Remark 10.2.21. In fact we can be more precise about the isomorphisms in (10.2.7). Note that it follows from Proposition 10.2.15 that for every $\mathcal{O}_X$-module $\mathcal{F}$ and every affine open subset $U \subseteq X$, we have functorial morphisms of $\mathcal{O}_X(U)$-modules

$$H^i\big(f^{-1}(U), \mathcal{F}\big) \to \Gamma\big(U, R^i f_*(\mathcal{F})\big) \quad \text{for} \quad i \geq 0.$$

We claim that these, in fact, are isomorphisms. Indeed, note that the isomorphisms (10.2.7) are compatible with the maps induced by restrictions to open subsets. We thus deduce that for every affine open subset $U \subseteq Y$ and every $a \in \mathcal{O}_Y(U)$, the induced morphism

$$H^i\big(f^{-1}(U), \mathcal{F}\big)_a \to H^i\big(f^{-1}(D_U(a)), \mathcal{F}\big)$$

is an isomorphism. Then the assertion follows from Proposition 10.2.15 and Remark 8.7.16

Exercise 10.2.22. Show that if $\mathcal{F}$ is a sheaf (say, of Abelian groups) on X and there is an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i}$ is flasque for every i, then $\mathcal{F}$ is flasque.

EXERCISE 10.2.23. Let $f\colon (X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Show that if $\mathcal{E}$ is a locally free sheaf on Y, then for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a canonical isomorphism

$$R^pf_*\big(f^*(\mathcal{E})\otimes_{\mathcal{O}_X}\mathcal{F}\big)\simeq\mathcal{E}\otimes_{\mathcal{O}_Y}R^pf_*(\mathcal{F})\quad\text{for all}\quad p\geq 0.$$

10.3. Singular vs. sheaf cohomology, and the de Rham theorem

In this section we show that on nice spaces, singular cohomology can be computed as sheaf cohomology, and use this approach to relate singular cohomology and De Rham cohomology on smooth manifolds. We follow the approach in [**God73**]. For the basic facts on singular cohomology we refer to [**Hat02**] and for those on smooth manifolds to [**War83**].

10.3.1. The étale space of a presheaf. Since we will be dealing with sections of sheaves over closed subsets of the ambient space, it will be convenient to interpret these in terms of the étale space of a sheaf (or presheaf) that we now introduce. Let X be a topological space and $\mathcal{F}$ a presheaf on X (say, of Abelian groups). We let $\mathrm{Et}(\mathcal{F})=\bigsqcup_{x\in X}\mathcal{F}_x$ and consider the map $\pi\colon\mathrm{Et}(\mathcal{F})\to X$ that maps the stalk $\mathcal{F}_x$ to $x\in X$. For every open subset U of X and every $s\in\mathcal{F}(U)$, consider the map $\widetilde{s}\colon U\to\mathrm{Et}(\mathcal{F})$ given by $\widetilde{s}(x)=s_x\in\mathcal{F}_x$ (therefore $\pi\big(\widetilde{s}(x)\big)=x$ for all $x\in U$).

We consider on $\mathrm{Et}(\mathcal{F})$ the strongest topology that makes all maps $\widetilde{s}$ continuous; explicitly, a subset $V\subseteq\mathrm{Et}(\mathcal{F})$ is open if and only if for every map $\widetilde{s}$ as above, the subset $\widetilde{s}^{-1}(V)$ of U is open. Note that π is continuous: for every U and s as above, and for every open subset W in X, we see that $\widetilde{s}^{-1}\big(\pi^{-1}(W)\big)=U\cap W$ is an open subset of U. In the next remarks, we record some properties of $\mathrm{Et}(\mathcal{F})$.

REMARK 10.3.1. For every open subset U of X and every $s\in\mathcal{F}(U)$, the subset $\widetilde{s}(U)$ of $\mathrm{Et}(\mathcal{F})$ is open. Indeed, given any other open subset V of X and $t\in\mathcal{F}(V)$, the subset

$$\widetilde{t}^{-1}\big(\widetilde{s}(U)\big)=\{x\in U\cap V\mid s_x=t_x\}$$

is clearly open in V.

REMARK 10.3.2. The map π is a local homeomorphism onto X. Indeed, every point $u\in\mathrm{Et}(\mathcal{F})$ lies in some subset of the form $\widetilde{s}(U)$, and this is mapped by π homeomorphically onto U, with inverse $\widetilde{s}$.

REMARK 10.3.3. The subsets $\widetilde{s}(U)$, with U open in X and $s\in\mathcal{F}(U)$, give a basis for the topology of $\mathrm{Et}(\mathcal{F})$. Indeed, if V is an open subset in $\mathrm{Et}(\mathcal{F})$ and $v\in V$, then $v=s_x$ for some $s\in\mathcal{F}(U)$, where U is an open subset of X. If $U'=\widetilde{s}^{-1}(V)$ and $s'=s|_{U'}$, then $v\in\widetilde{s'}(U')\subseteq V$.

REMARK 10.3.4. It follows from definition that we have a morphism of presheaves $\phi\colon\mathcal{F}\to\mathcal{G}$, where $\mathcal{G}$ is the sheaf of continuous sections of π (see Example 2.1.7); this maps $s\in\mathcal{F}(U)$ to $\widetilde{s}$. We claim that $\mathcal{G}$ is the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ and ϕ is the canonical morphism. Indeed, it follows from the previous remark that a continuous section of π over an open subset $U\subseteq X$ consists of a map $f\colon U\to\bigsqcup_{x\in U}\mathcal{F}_x$ such that $f(x)\in\mathcal{F}_x$ for every $x\in U$ and for every open subset V of X and section $s\in\mathcal{F}(V)$, the inverse image

$$f^{-1}\big(\widetilde{s}(V)\big)=\{x\in U\cap V\mid f(x)=s_x\}$$

is open in U. It is clear that this condition holds if and only if for every $x \in U$, there is an open neighborhood $V \subseteq U$ of x and $s \in \mathcal{F}(V)$ such that $f(x) = s_x$ for all $x \in V$. We thus see that $\mathcal{G}(U) = \mathcal{F}^+(U)$, as defined in §8.1.1, and ϕ is the canonical morphism.

REMARK 10.3.5. It is clear that the construction of the étale space of a presheaf is functorial: if $u\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then the induced maps at the level of stalks give a map $\mathrm{Et}(u)\colon \mathrm{Et}(\mathcal{F}) \to \mathrm{Et}(\mathcal{G})$. This is continuous: for this, it is enough to show that its restriction to each subset $\widetilde{s}(U)$ is continuous, where $s \in \mathcal{F}(U)$, but the restriction of $\mathrm{Et}(u)$ to this open subset is equal to $\widetilde{u(s)} \circ \pi$.

It is clear that if $\phi\colon \mathcal{F} \to \mathcal{F}^+$ is the canonical morphism, then $\mathrm{Et}(\phi)$ is a homeomorphism.

REMARK 10.3.6. Suppose now that $\mathcal{F}$ is a sheaf and Z is a subset of X, with $i\colon Z \hookrightarrow X$ being the inclusion. We put $\mathcal{F}|_Z = i^{-1}(\mathcal{F})$. We claim that $\mathrm{Et}(\mathcal{F}|_Z)$ is homeomorphic to $\pi^{-1}(Z)$, with the homeomorphism being compatible with the maps to Z. Indeed, recall first that for every $x \in Z$, we have a canonical isomorphism $i^{-1}(\mathcal{F})_x \simeq \mathcal{F}_x$, hence we may and will identify $\mathrm{Et}(\mathcal{F}|_Z)$ and $\pi^{-1}(Z)$. Moreover, by definition, $\mathcal{F}|_Z$ is the sheaf associated to the presheaf that maps $V \subseteq Z$ to $\varinjlim_{V \subseteq U} \mathcal{F}(U)$. It is thus easy to see that a basis for the topology on $\mathrm{Et}(\mathcal{F}|_Z)$ is given by the sets $\widetilde{s}(U \cap Z) = \pi^{-1}(Z) \cap \widetilde{s}(U)$, where $U \subseteq X$ is open and $s \in \mathcal{F}(U)$. Therefore the topology on $\mathrm{Et}(\mathcal{F}|_Z)$ is the induced topology.

If $\mathcal{F}$ is a sheaf on X and Z is an arbitrary subset of X, we put

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) := \Gamma(Z, \mathcal{F}|_Z)$$

(of course, if U is open, we don't get anything new). It follows from Remark 10.3.6 that we can identify $\mathcal{F}(Z)$ with the set continuous maps $f\colon Z \to \mathrm{Et}(\mathcal{F})$ such that $\pi\big(f(x)\big) = x$ for all $x \in Z$. Moreover, if W is a subset of Z, then the canonical map $\mathcal{F}(Z) \to \mathcal{F}(W)$ is given by the restriction of sections of π.

REMARK 10.3.7. It follows from the above description of the sections of $\mathcal{F}$ on arbitrary subsets of X that if $X = \bigcup_{i \in I} Z_i$ is a locally finite[3] cover of X by closed subsets, then the following sequence induced by the restriction maps:

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(Z_i) \to \prod_{i,j \in I} \mathcal{F}(Z_i \cap Z_j)$$

is exact. Indeed, given a family of sections $s_i \in \mathcal{F}(Z_i)$ such that

$$s_i|_{Z_i \cap Z_j} = s_j|_{Z_i \cap Z_j} \quad \text{for all} \quad i, j \in I,$$

it is clear that we get a unique map $f\colon X \to \mathrm{Et}(X)$ such that $f(x) = s_i(x)$ for all $i \in I$ and $x \in Z_i$. In order to check that f is continuous, it is enough to do this locally, and we thus reduce to the case where we have only finitely many Z_j. In this case, f is continuous since each $f|_{Z_j}$ is continuous (the inverse image of a closed subset is a finite union of closed subsets, and thus closed).

[3] A cover $X = \bigcup_{j \in J} V_j$ is *locally finite* if every $x \in X$ has an open neighborhood V that intersects only finitely many of the V_j.

10.3.2. Soft sheaves on paracompact spaces. We now assume that X is a *paracompact* topological space; recall that this means that X is Hausdorff and for every open cover $X = \bigcup_{i\in I} U_i$, there is a locally finite open cover $X = \bigcup_{j\in J} V_j$ that refines it. A useful property is that if $X = \bigcup_{j\in J} V_j$ is a locally finite open cover of a paracompact space X, then there is another open cover $X = \bigcup_{j\in J} W_j$ such that $\overline{W_j} \subseteq V_j$ for all $j \in J$. A special case of this says that if $F \subseteq U$ are subsets of X, with F closed and U open, then there is an open subset V of X such that $F \subseteq V \subseteq \overline{V} \subseteq U$ (this means that X is a *normal space*). We also note that every closed subset of a paracompact space is paracompact: this is easy to see using the definition.

EXAMPLE 10.3.8. Every topological manifold (assumed to be Hausdorff and with countable basis of open subsets) is paracompact. Every CW-complex is paracompact.

DEFINITION 10.3.9. A sheaf of Abelian groups $\mathcal{F}$ on X is *soft* if for every closed subset Z of X, the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(Z)$$

is surjective.

LEMMA 10.3.10. *Let X be a paracompact topological space and $\mathcal{F}$ a sheaf of Abelian groups on X.*

i) *For every closed subset Z of X and every $s \in \mathcal{F}(Z)$, there is an open subset U containing Z and $t \in \mathcal{F}(U)$ such that $t|_Z = s$.*
ii) *If $\mathcal{F}$ is flasque, then it is soft.*

PROOF. Let us prove i). By definition of $\mathcal{F}|_Z$, we see that we have a family of open subsets $(U_i)_{i\in I}$, with $Z \subseteq \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap Z} = s|_{U_i\cap Z}$ for all $i \in I$. We consider the cover of X by the U_i and by $X \smallsetminus Z$; after passing to a suitable refinement, we may assume that this a locally finite cover. We can then find open subsets V_i, with $\overline{V_i} \subseteq U_i$, and such that $Z \subseteq \bigcup_{i\in I} V_i$.

Given any $x \in Z$, we choose an open neighborhood $U(x)$ of x that intersects only finitely many of the U_j and such that $U(x)$ is contained in some V_i. We put $s^{(x)} = s_i|_{U(x)}$. In particular, $s^{(x)}$ and s take the same value at x. If $x \not\in \overline{V_j}$ for some j, we may replace $U(x)$ by $U(x) \smallsetminus \overline{V_j}$. Since $U(x)$ intersects only finitely many U_j, it follows that after repeating this operation finitely many times, we may assume that whenever $U(x) \cap \overline{V_j} \neq \emptyset$, we have $x \in \overline{V_j} \subseteq U_j$. After further shrinking $U(x)$, we may thus assume, in addition, that for such j we have $U(x) \subseteq U_j$. Since $s^{(x)}$ and s_j take the same value at x, after further shrinking $U(x)$, we may assume that for all such j, we have $s^{(x)} = s_j|_{U(x)}$.

We put $U = \bigcup_{x\in Z} U(x)$. It is clear that U is an open neighborhood of Z. We claim that

$$s^{(x)}|_{U(x)\cap U(y)} = s^{(y)}|_{U(x)\cap U(y)} \quad \text{for all} \quad x, y \in Z. \tag{10.3.1}$$

If $z \in U(x) \cap U(y)$, then $z \in V_\ell$, for some ℓ. Since $U(x) \cap \overline{V_\ell} \neq \emptyset$ and $U(y) \cap \overline{V_\ell} \neq \emptyset$, then by construction we have $U(x), U(y) \subseteq U_\ell$ and

$$s_\ell|_{U(x)} = s^{(x)} \quad \text{and} \quad s_\ell|_{U(y)} = s^{(y)},$$

which gives (10.3.1). We can thus find $t \in \mathcal{F}(U)$ such that $t|_{U(x)} = s^{(x)}$ for all $x \in Z$. In particular, we have $t_x = s_x$ for every $x \in Z$, and thus $t|_Z = s$.

□

LEMMA 10.3.11. *Let X be a paracompact topological space. Given a short exact sequence of sheaves of Abelian groups*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$ soft, the corresponding sequence of global sections

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact.

PROOF. The proof is similar to that of Lemma 10.2.4. We only need to prove that for every section $s'' \in \mathcal{F}''(X)$, there is $s \in \mathcal{F}(X)$ such that $\psi(s) = s''$. By definition, we can find an open cover $X = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\psi(s_i) = s|_{U_i}$ for all i. After passing to a refinement, we may assume that the cover is locally finite. We may now find another cover $X = \bigcup_{i \in I} V_i$ such that $\overline{V_i} \subseteq U_i$ for all $i \in I$.

For every $J \subseteq I$, put $Z_J = \bigcup_{i \in J} \overline{V_j}$. We consider the set $\mathcal{P}$ of pairs (J, t), where $J \subseteq I$ and $t \in \mathcal{F}(Z_J)$ is such that $\psi(t) = s|_{Z_J}$. We order it by $(J_1, t_1) \leq (J_2, t_2)$ if $J_1 \subseteq J_2$ and $t_2|_{Z_{J_1}} = t_1$. It is straightforward to see that we may apply Zorn's lemma to choose a maximal element (J, s) of $\mathcal{P}$. If $J = I$, then $Z_J = X$, and $\psi(s) = s''$.

Suppose now that $J \neq I$ and let $i \in I \smallsetminus J$. Since $\psi(s_i|_{\overline{V_i} \cap Z_J}) = \psi(s|_{\overline{V_i} \cap Z_J})$, it follows that

$$s_i|_{\overline{V_i} \cap Z_J} - s|_{\overline{V_i} \cap Z_J} = \phi(s'),$$

for some $s' \in \mathcal{F}'(\overline{V_i} \cap Z_J)$. Since $\mathcal{F}'$ is soft, we can find $v \in \mathcal{F}'(X)$ such that $v|_{\overline{V_i} \cap Z_J} = s'$. After replacing s_i by $s_i - \phi(v|_{U_i})$, we may thus assume that $s_i|_{\overline{V_i} \cap Z_J} = s|_{\overline{V_i} \cap Z_J} = \phi(s')$, hence by Remark 10.3.7 we can find a section $t' \in \mathcal{F}(Z_{J \cup \{i\}})$ such that $t'|_{Z_J} = t$ and $t'|_{\overline{V_i}} = s_i|_{\overline{V_i}}$. In this case $\psi(t') = s''|_{Z_{J \cup \{i\}}}$, contradicting the maximality of J. □

LEMMA 10.3.12. *If X is a paracompact topological space and we have a short exact sequence of sheaves of Abelian groups on X*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

with $\mathcal{F}'$ soft, then $\mathcal{F}$ is soft if and only if $\mathcal{F}''$ is soft.

PROOF. If Z is a closed subset of X, then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}''(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}'(Z) \arrow[r] & \mathcal{F}(Z) \arrow[r] & \mathcal{F}''(Z) \arrow[r] & 0. \end{tikzcd}$$

The rows are exact by Lemma 10.3.11 (note that if $\mathcal{F}'$ is soft, then clearly $\mathcal{F}'|_Z$ is soft, and Z is paracompact, being closed in X). Moreover, since $\mathcal{F}'$ is soft, the first vertical map is surjective, hence the second one is surjective if and only if the third one is. □

PROPOSITION 10.3.13. *If X is a paracompact topological space and $\mathcal{E}$ is a soft sheaf of Abelian groups on X, then*

$$H^i(X,\mathcal{E}) = 0 \quad \textit{for all} \quad i \geq 1.$$

In particular, if $\mathcal{O}_X$ is a sheaf of rings on X and an $\mathcal{O}_X$-module $\mathcal{F}$ has a resolution $\mathcal{F} \to \mathcal{E}^\bullet$, with all $\mathcal{E}^i$ soft $\mathcal{O}_X$-modules, then we have a canonical isomorphism

$$H^i(X,\mathcal{F}) \simeq \mathcal{H}^i\big(\Gamma(X,\mathcal{E}^\bullet)\big).$$

PROOF. We argue by induction on $i \geq 1$. Consider a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E} \to \mathcal{A} \to \mathcal{B} \to 0,$$

with $\mathcal{A}$ flasque. By Lemma 10.3.10, we see that $\mathcal{A}$ is soft, hence $\mathcal{B}$ is soft by Lemma 10.3.12. The long exact sequence in cohomology for the above short exact sequence gives

$$0 \to \Gamma(X,\mathcal{E}) \to \Gamma(X,\mathcal{A}) \xrightarrow{\alpha} \Gamma(X,\mathcal{B}) \to H^1(X,\mathcal{E}) \to H^1(X,\mathcal{A}).$$

Note that the map α is surjective by Lemma 10.3.11, and since $\mathcal{A}$ is flasque, we have $H^i(X,\mathcal{A}) = 0$ for $i \geq 1$ by Proposition 10.2.8. First, we conclude that $H^1(X,\mathcal{E}) = 0$, completing the proof of the case $i = 1$ in the induction.

Moreover, the long exact sequence in cohomology gives isomorphisms

$$H^i(X,\mathcal{E}) \simeq H^{i-1}(X,\mathcal{B}) \quad \text{for all} \quad i \geq 2.$$

Since $\mathcal{B}$ is soft, we have $H^{i-1}(X,\mathcal{B}) = 0$ by induction, and thus $H^i(X,\mathcal{E}) = 0$.

The last assertion in the proposition is now a direct consequence of Proposition 10.1.24. □

10.3.3. Singular cohomology as sheaf cohomology. Given a topological space X and an Abelian group A, we temporarily denote by $H^i_{\rm sing}(X,A)$ the $i^{\rm th}$ singular cohomology group of X with coefficients in A. If R is a commutative ring and A is an R-module, then $H^i_{\rm sing}(X,A)$ has a natural structure of R-module.

Our goal is to prove is to prove the following result relating sheaf cohomology and singular cohomology on “nice” topological spaces.

THEOREM 10.3.14. *If X is a paracompact, locally contractible*[4] *topological space, then for every commutative ring R and every R-module A, we have a canonical isomorphism of R-modules*

$$H^i(X,A) \simeq H^i_{\rm sing}(X,A).$$

REMARK 10.3.15. Note that one can't hope to have an isomorphism as in the above theorem for all X. For example, we have $H^0(X,\mathbf{Z}) \simeq \mathbf{Z}^{(I_X)}$, where I_X is the set of connected components of X, while $H^0_{\rm sing}(X,\mathbf{Z}) \simeq \mathbf{Z}^{(J_X)}$, where J_X is the set of path-wise connected components of X.

REMARK 10.3.16. In fact, it is possible to prove the above theorem without assuming that X is paracompact, see [**Sel16**].

REMARK 10.3.17. An obvious example of a locally contractible space is a topological manifold. Other examples are provided by CW-complexes (see [**Hat02**, Proposition A.4]).

[4]A topological space is *locally contractible* if every point has a basis of contractible open neighborhoods.

The key ingredient in the proof of the above theorem is the following general proposition about certain presheaves on paracompact spaces.

PROPOSITION 10.3.18. *Let X be a paracompact topological space and $\mathcal{F}$ a presheaf of Abelian groups on X that satisfies the following condition: for every open cover $X = \bigcup_{i \in I} U_i$ and for every $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, there is $s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$ for all i. If $\mathcal{F} \to \mathcal{F}^+$ is the canonical morphism to the associated sheaf, then the morphism $\mathcal{F}(X) \to \mathcal{F}^+(X)$ is surjective.*

PROOF. A section $s \in \mathcal{F}^+(X)$ is given by a map $s \colon X \to \sqcup_{x \in X} \mathcal{F}_x$ such that we have an open cover $X = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $s(x) = (s_i)_x$ for every $x \in U_i$. After passing to a refinement, we may assume that the cover is locally finite. We choose another open cover $X = \bigcup_{i \in I} U'_i$ with $\overline{U'_i} \subseteq U_i$ for all i. Note that if $x \in U_i \cap U_j$, then $(s_i)_x = (s_j)_x$, hence there is an open neighborhood $V_{i,j}(x) \subseteq U_i \cap U_j$ such that $s_i|_{V_{i,j}(x)} = s_j|_{V_{i,j}(x)}$.

Given any $x \in X$, we choose an open neighborhood $V(x)$ of x, such that the following conditions are satisfied:

1) If $x \in U_i \cap U_j$, then $V(x) \subseteq V_{i,j}(x)$.
2) If $x \in U_i$, then $V(x) \subseteq U_i$.
3) If $x \in U'_i$, then $V(x) \subseteq U'_i$.
4) If $V(x) \cap \overline{U'_i} \neq \emptyset$, then $x \in \overline{U'_i}$.

This is possible since the cover given by the U_i is locally finite, hence every x lies in only finitely many U_i. Note that in this case we also have: if $x, y \in X$ are such that $V(x) \cap V(y) \neq \emptyset$, then there is i such that $V(x), V(y) \subseteq U_i$. Indeed, if $x \in U'_i$, then by 3) we have $V(x) \subseteq U'_i$; therefore $V(y) \cap \overline{U'_i} \neq \emptyset$, and thus $y \in \overline{U'_i}$ by 4). We thus get $V(y) \subseteq U_i$ by 2).

For every $x \in X$, it follows from 2) that if $x \in U_i$, then $V(x) \subseteq U_i$, and we put $\alpha^{(x)} = s_i|_{V(x)}$; this does not depend on i by 1). Moreover, we have seen that if $V(x) \cap V(y) \neq \emptyset$, then there is i such that $V(x), V(y) \subseteq U_i$, in which case it is clear that

$$\alpha^{(x)}|_{V(x) \cap V(y)} = s_i|_{V(x) \cap V(y)} = \alpha^{(y)}|_{V(x) \cap V(y)}.$$

By hypothesis, we can find $t \in \mathcal{F}(X)$ such that $t|_{V(x)} = \alpha^{(x)}$ for all $x \in X$. In particular, we have $t_x = \alpha^{(x)}_x = s(x)$ for every $x \in X$, and thus $s = \phi(t)$. $\square$

We can now relate sheaf cohomology and singular cohomology.

PROOF OF THEOREM 10.3.14. Recall that for every $p \geq 0$, a p-simplex in X is a continuous map $\Delta^p \to X$ from the standard p-dimensional simplex to X. The group of p-chains in X, denoted $\mathcal{C}_p(X)$, is the free Abelian group on the set of p-simplices and the R-module of p-cochains with values in A, denoted $\mathcal{C}^p(X, A)$, is equal to $\mathrm{Hom}_{\mathbf{Z}}\big(\mathcal{C}_p(X), A\big)$. Therefore a p-cochain can be identified to a map from the set of p-simplices in X to A. For every $p \geq 0$ we have maps $\partial \colon \mathcal{C}^p(X, A) \to \mathcal{C}^{p+1}(X, A)$ induced by corresponding maps $C_{p+1}(X) \to C_p(X)$. Then $\mathcal{C}^{\bullet}(X, A)$ is a complex and we have

$$H^p(X, A) = \mathcal{H}^p\big(\mathcal{C}^{\bullet}(X, A)\big). \tag{10.3.2}$$

Note that if $f \colon Y \to X$ is a continuous map, then we have a morphism of complexes $\mathcal{C}^{\bullet}(X, A) \to \mathcal{C}^{\bullet}(Y, A)$.

Since A is fixed, we will denote by $\mathcal{C}_X^p$ the presheaf that associates to an open subset of X the Abelian group $\mathcal{C}^p(U,A)$, with the restriction map corresponding to $U \subseteq V$ given by the map $\mathcal{C}_X^p(V,A) \to \mathcal{C}_X^p(U,A)$ induced by the inclusion. It is clear that we have a complex $\mathcal{C}_X^\bullet$ of presheaves on X. For every p, let $\mathcal{S}_X^p := (\mathcal{C}_X^p)^+$, so that we also have a complex $\mathcal{S}_X^\bullet$ of sheaves of R-modules on X. Note that we have a morphism of sheaves $A \to \mathcal{C}_X^0$ that associates to $s \in \Gamma(X,A)$, viewed as a locally constant function $X \to A$, the cocycle which associates to every 0-simplex in A, viewed as a point $x \in X$, the element $s(x) \in A$.

We claim that $A \to \mathcal{S}_X^\bullet$ is a resolution. Note first that if U is a contractible open subset of X, then $H^p(U,A) = 0$ for all $p \geq 1$ and $H^0(U,A) = A$, hence $\Gamma(U,A) \to \Gamma(U,\mathcal{C}_X^\bullet)$ is a resolution. Since X is locally contractible, we conclude that for every $x \in X$, at the level of stalks we have a resolution $A \to (\mathcal{C}_X^\bullet)_x = (\mathcal{S}_X^\bullet)_x$. This implies our claim.

If we are in a situation in which every open subset of X is paracompact (for example, if X is a topological manifold), then it is easy to deduce from Proposition 10.3.18 that each sheaf $\mathcal{S}_X^p$ is flasque. In general, we will show only that each sheaf $\mathcal{S}_X^p$ is soft, and the argument is a bit more involved. Note first that if Y is any subspace of X, with $i\colon Y \hookrightarrow X$ being the inclusion map, then for every open subset U of X, we have a canonical morphism of R-modules $\mathcal{C}^p(U,A) \to \mathcal{C}^p(U \cap Y, A)$. We thus obtain a morphism of presheaves $\mathcal{C}_X^p \to i_*\mathcal{C}_Y^p$ and thus a morphism of sheaves of R-modules $\mathcal{S}_X^p \to i_*\mathcal{S}_Y^p$. By the adjoint property of (i^{-1}, i_*), this corresponds to a morphism of sheaves $\mathcal{S}_X^p|_Y \to \mathcal{S}_Y^p$. It is clear that if we restrict this to an open subset V of X that is contained in Y, then both sides are canonically isomorphic to $\mathcal{S}_V^p$ and the map is the identity.

We can now show that $\mathcal{S}_X^p$ is soft. Suppose that Z is a closed subset of X and $s \in \mathcal{S}_X^p(Z)$. By assertion i) in Lemma 10.3.10, there is an open subset U of X containing Z, and $s_U \in \mathcal{S}_X^p(U)$ such that $s_U|_Z = s$. Let us choose an open subset V of X, with $Z \subseteq V \subseteq \overline{V} \subseteq U$. Let $t \in \mathcal{S}_{\overline{V}}^p(\overline{V})$ be the image of $(s_U)|_{\overline{V}}$ via the morphism $\mathcal{S}_X^p|_{\overline{V}} \to \mathcal{S}_{\overline{V}}^p$. Since X is paracompact, $\overline{V}$ is paracompact, too. It is straightforward to see that $\mathcal{C}_{\overline{V}}^p$ satisfies the hypothesis of Proposition 10.3.18: given an open cover $\overline{V} = \bigcup_{i\in I} U_i$ and cochains $\alpha_i \in \mathcal{S}_{\overline{V}}^p(U_i)$ such that $\alpha_i|_{U_i\cap U_j} = \alpha_j|_{U_i\cap U_j}$ for all i and j, we define $\alpha \in \mathcal{S}_{\overline{V}}^p(\overline{V})$ such that for a p-simplex σ in $\overline{V}$, we have $\alpha(\sigma) = \alpha_i(\sigma)$ if the image of σ lies in some U_i, and 0 otherwise; it is clear that α is well-defined and $\alpha|_{U_i} = \alpha_i$ for all i. We conclude, using the proposition, that t is the image of some $t' \in \mathcal{C}_{\overline{V}}^p(\overline{V})$. Since the map $\mathcal{C}_X^p(X) \to \mathcal{C}_{\overline{V}}^p(\overline{V})$ is clearly surjective, there is $s' \in \mathcal{C}_X^p(X)$ that maps to t'. Since $t|_V = s_U|_V$, it is straightforward to see that the image of s' in $\mathcal{S}_X^p(X)$ restricts to $(s_U)|_V \in \mathcal{S}_X^p(V)$, and thus farther to $s \in \mathcal{S}_X^p(Z)$. This shows that $\mathcal{S}_X^p$ is soft.

We thus have a soft resolution $A \to \mathcal{S}_X^\bullet$ of sheaves of R-modules, hence Proposition 10.3.13 gives a canonical isomorphism

$$H^p(X,A) \simeq \mathcal{H}^p\big(\mathcal{S}_X^\bullet(X)\big). \tag{10.3.3}$$

Applying as above Proposition 10.3.18 for the sheaves $\mathcal{C}_X^p$, we see that for every p, we have a surjection

$$\mathcal{C}_X^p(X) \to \mathcal{S}_X^p(X).$$

Let V^p be the kernel. This consists of the p-cochains β with the property that there is some open cover $X = \bigcup_{i\in I} U_i$ such that β vanishes on each p-simplex whose image is contained in some of the U_i. By considering the long exact sequence associated to the exact sequence of complexes

$$0 \to V^\bullet \to \mathcal{C}_X^\bullet(X) \to \mathcal{S}_X^\bullet(X) \to 0,$$

we see that if we show that $\mathcal{H}^p(V^\bullet) = 0$ for all p, then we are done by the isomorphisms (10.3.2) and (10.3.3),

By definition, $V^\bullet$ is the filtering direct limit of the complexes $V^\bullet(\mathcal{U})$, where $\mathcal{U} = (U_i)_{i\in I}$ is an open cover of X and where $V^p(\mathcal{U})$ consists of the p-cochains that vanish on $\mathcal{U}$-*small* simplices, that is, p-simplexes in X whose image is contained in some of the U_i. Since filtering direct limits form an exact functor, it is enough to show that $\mathcal{H}^p\big(V^\bullet(\mathcal{U})\big) = 0$ for all $\mathcal{U}$ and all p.

If $\mathcal{C}_p^{\mathcal{U}}(X)$ is the subgroup of $\mathcal{C}_p(X)$ generated by simplices whose image is contained in some open subset in $\mathcal{U}$, then $\mathcal{C}_\bullet^{\mathcal{U}}(X)$ is a subcomplex of $\mathcal{C}_\bullet(X)$. A basic result, proved using barycentric subdivisions, says that the inclusion

$$\mathcal{C}_\bullet^{\mathcal{U}}(X) \hookrightarrow \mathcal{C}_\bullet(X)$$

is a homotopy equivalence[5] (see [**Hat02**, Proposition 2.21]). In this case, applying $\mathrm{Hom}_{\mathbf{Z}}(-,A)$ gives a homotopy equivalence

$$u\colon \mathcal{C}^\bullet(X,A) \to \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}_\bullet^{\mathcal{U}}, A),$$

which thus induces isomorphisms in cohomology. On the other hand, u is a surjective morphism of complexes, whose kernel is equal to $V^\bullet(\mathcal{U})$. We thus conclude using Lemma 10.1.4 that $\mathcal{H}^p\big(V^\bullet(\mathcal{U})\big) = 0$ for all p. This completes the proof of the theorem. $\square$

10.3.4. The De Rham theorem. In this section we consider a smooth manifold X. Recall that by assumption X is assumed to be Hausdorff and with a countable basis of open subsets, hence it is paracompact. Let $n = \dim(X)$. We denote by $\mathcal{C}_X^\infty$ the sheaf of smooth functions on X with values in $\mathbf{R}$. For every $p \geq 0$, the sheaf $\mathcal{E}_X^p$ of smooth p-forms on X is a $\mathcal{C}_X^\infty$-module. Note that $\mathcal{E}_X^0 = \mathcal{C}_X^\infty$ and $\mathcal{E}_X^p = 0$ if $p > n$. For every $p \geq 0$ we have a morphism of sheaves $d\colon \mathcal{E}_X^p \to \mathcal{E}_X^{p+1}$ given by exterior differentiation such that $d \circ d = 0$. Moreover, we also have an inclusion $\iota\colon \mathbf{R} \hookrightarrow \mathcal{E}_X^0$, where the sections of the constant sheaf $\mathbf{R}$ are viewed as locally constant functions on X.

Definition 10.3.19. With the above notation, the De Rham complex of X is the complex of $\mathbf{R}$-vector spaces

$$0 \to \mathcal{E}_X^0(X) \to \mathcal{E}_X^1(X) \to \ldots \to \mathcal{E}_X^n(X) \to 0.$$

The p^{th} cohomology of this complex is the *De Rham cohomology* $\mathbf{R}$-vector space $H^p_{\mathrm{DR}}(X)$.

The following is the main result of this section.

Theorem 10.3.20. *For every smooth manifold X, we have a canonical isomorphism*

$$H^p_{\mathrm{DR}}(X) \simeq H^p(X,\mathbf{R}) \quad \textit{for every} \quad p \geq 0.$$

By combining this with Theorem 10.3.14, we obtain the following corollary, known as De Rham's theorem.

[5]This means that there is a morphism of complexes in the opposite direction such that both compositions are homotopic to the respective identity maps.

COROLLARY 10.3.21. *For every smooth manifold X, we have a canonical isomorphism*

$$H^p_{\rm DR}(X) \simeq H^p_{\rm sing}(X, \mathbf{R}).$$

The proof of the above theorem will follow from the following two lemmas.

LEMMA 10.3.22. *Every $\mathcal{C}^\infty_X$-module is a soft sheaf.*

PROOF. Let $\mathcal{F}$ be a $\mathcal{C}^\infty_X$-module and let $s \in \mathcal{F}(Z)$, where Z is a closed subset of X. By assertion i) in Lemma 10.3.10, there is an open subset U of X, containing Z, and $t \in \mathcal{F}(U)$, such that $t|_Z = s$. Let us choose open subsets U_1 and U_2 such that

$$Z \subseteq U_1 \subseteq \overline{U_1} \subseteq U_2 \subseteq \overline{U_2} \subseteq U.$$

Since X is a smooth manifold, by the smooth version of Urysohn's lemma, we can find a global section $f \in \mathcal{C}^\infty_X(X)$ such that $f = 1$ on $\overline{U_1}$ and $f = 0$ on $X \smallsetminus \overline{U_2}$. Since $\mathcal{F}$ is a sheaf and $(ft)|_{U \smallsetminus \overline{U_2}} = 0$, we can find $v \in \mathcal{F}(X)$ such that $v|_U = ft$ and $v|_{X \smallsetminus \overline{U_2}} = 0$. It is clear that $v|_{U_1} = t|_{U_1}$, hence $v|_Z = s$. □

LEMMA 10.3.23. *(Poincaré) On $\mathbf{R}^n$, the complex*

$$0 \longrightarrow \mathbf{R} \xrightarrow{\iota} \mathcal{E}^0_{\mathbf{R}^n}(\mathbf{R}^n) \xrightarrow{d} \mathcal{E}^1_{\mathbf{R}^n}(\mathbf{R}^n) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^n_{\mathbf{R}^n}(\mathbf{R}^n) \longrightarrow 0$$

is exact.

PROOF. We argue by induction on $n \geq 0$, the case $n = 0$ being trivial. We denote by $\mathrm{DR}^\bullet_{\mathbf{R}^n}$ the complex in the statement. Consider the smooth maps $i \colon \mathbf{R}^{n-1} \to \mathbf{R}^n$ and $\pi \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ given by

$$i(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n) \quad \text{and} \quad \pi(x_1, \ldots, x_n) = (x_2, \ldots, x_n).$$

The pull-back of differential forms gives morphisms of complexes

$$\pi^* \colon \mathrm{DR}^\bullet_{\mathbf{R}^{n-1}} \to \mathrm{DR}^\bullet_{\mathbf{R}^n} \quad \text{and} \quad i^* \colon \mathrm{DR}^\bullet_{\mathbf{R}^n} \to \mathrm{DR}^\bullet_{\mathbf{R}^{n-1}}$$

(by convention, both i^* and π^* act as the identity on $\mathbf{R}$). Since $\pi \circ i = \mathrm{id}_{\mathbf{R}^{n-1}}$, the composition $i^* \circ \pi^*$ is the identity on $\mathrm{DR}^\bullet_{\mathbf{R}^{n-1}}$. If we show that $\pi^* \circ i^*$ is homotopic to the identity on $\mathrm{DR}^\bullet_{\mathbf{R}^n}$, then we are done by induction.

Given a p-differential form ω on $\mathbf{R}^n$, we write it as $\omega = \sum_{|J|=p} f_J dx_J$, where J varies over the subsets of $\{1, \ldots, n\}$ with p elements, and if we order the elements of J as $j_1 < \ldots < j_p$, then $dx_J = dx_{j_1} \wedge \ldots dx_{j_p}$. We define an $\mathbf{R}$-linear map

$$\theta^p \colon \mathrm{DR}^p_{\mathbf{R}^n}(\mathbf{R}^n) \to \mathrm{DR}^{p-1}_{\mathbf{R}^n}(\mathbf{R}^n)$$

such that if $1 \in J$ and $J' = J \smallsetminus \{1\}$, then

$$\theta^p(f dx_J) = \left(\int_0^{x_1} f(t, x_2, \ldots, x_n) dt\right) dx_{J'},$$

and if $1 \not\in J$, then $\theta^p(f dx_J) = 0$. We make the convention that $\theta^0 = 0$. We claim that the θ^p give a homotopy between the identity and $\pi^* \circ i^*$.

Indeed, if $p \geq 1$, $1 \in J$ and $J' = J \smallsetminus \{1\}$, then it follows from the fundamental theorem of calculus and the fact that we can commute integration and differentiation, that

$$\theta^{p+1}\big(d(f dx_J)\big) + d\big(\theta^p(f dx_J)\big) =$$

$$\theta^{p+1}\left(\sum_{i \not\in J} \frac{\partial f}{\partial x_i} dx_i \wedge dx_1 \wedge dx_{J'}\right) + d\left(\left(\int_0^{x_1} f(t, x_2, \ldots, x_n) dt\right) dx_{J'}\right)$$

$$= -\sum_{i\not\in J}\left(\int_0^{x_1}\frac{\partial f}{\partial x_i}(t,x_2,\ldots,x_n)dt\right)dx_i\wedge dx_{J'}+\left(\frac{\partial}{\partial x_1}\int_0^{x_1}f(t,x_2,\ldots,x_n)dt\right)dx_1\wedge dx_{J'}+$$
$$\sum_{i\not\in J}\left(\frac{\partial}{\partial x_i}\int_0^{x_1}f(t,x_2,\ldots,x_n)dt\right)dx_i\wedge dx_{J'}=f(x_1,\ldots,x_n)dx_J.$$

Note also that in this case $\pi^*\big(i^*(fdx_J)\big)=0$.

On the other hand, if $p\geq 1$ and $1\not\in J$, then

$$\theta^{p+1}\big(d(fdx_J)\big)+d\big(\theta^p(fdx_J)\big)=\theta^{p+1}\left(\frac{\partial f}{\partial x_1}dx_1\wedge dx_J\right)$$
$$=\left(\int_0^{x_1}\frac{\partial f}{\partial x_1}(t,x_2,\ldots,x_n)dt\right)\cdot dx_J=\big(f(x_1,\ldots,x_n)-f(0,x_2\ldots,x_n)\big)dx_J$$
$$=fdx_J-\pi^*\big(i^*(fdx_J)\big).$$

Similarly, if $p=0$, then

$$\theta^1(df)+d\big(\theta^0(f)\big)=\theta^1\left(\frac{\partial f}{\partial x_1}dx_1\right)=\int_0^{x_1}\frac{\partial f}{\partial x_1}(t,x_2,\ldots,x_n)dt$$
$$=f(x_1,\ldots,x_n)-f(0,x_2,\ldots,x_n)=f-f\circ i\circ\pi.$$

Since it is also clear that $\theta^0\circ\iota(a)=0=a-\pi^*\big(i^*(a)\big)$, this completes the proof of the lemma. ☐

We can now relate De Rham cohomology and sheaf cohomology with coefficients in $\mathbf{R}$.

Proof of Theorem 10.3.20. Since every point of X has a basis of open neighborhoods diffeomorphic to an open ball in $\mathbf{R}^n$, and thus to $\mathbf{R}^n$, it follows from Lemma 10.3.23 that $\mathbf{R}\to\mathcal{E}_X^\bullet$ is a resolution. Since all sheaves $\mathcal{E}_X^p$ are soft by Lemma 10.3.22, we obtain the assertion in the theorem from Proposition 10.3.13. ☐

10.4. Cohomology of quasi-coherent sheaves on affine varieties

We prove the following theorem, due to Serre, characterizing affine varieties in terms of the vanishing of the higher cohomology of quasi-coherent sheaves.

Theorem 10.4.1. *Given an algebraic variety X, the following are equivalent:*
- i) *X is affine.*
- ii) *For every quasi-coherent sheaf $\mathcal{F}$ on X, we have $H^i(X,\mathcal{F})=0$ for all $i\geq 1$.*
- iii) *For every coherent ideal sheaf $\mathcal{I}\subseteq\mathcal{O}_X$, we have $H^1(X,\mathcal{I})=0$.*

Proof. We first prove the implication i)⇒ii). If X is affine, with $\mathcal{O}(X)=A$, and $\mathcal{F}=\widetilde{M}$, for an A-module M, consider an injective resolution $Q^\bullet$ of M. By Lemma 10.2.19, the induced complex $\widetilde{Q^\bullet}$ is a flasque resolution of $\widetilde{M}$. It follows from Proposition 10.2.8 that

$$H^i(X,\mathcal{F})\simeq\mathcal{H}^i\big(\Gamma(X,\widetilde{Q^\bullet})\big)=\mathcal{H}^i(Q^\bullet),$$

hence the left-hand side vanishes for $i>0$.

Since the implication ii)⇒iii) is trivial, in order to complete the proof, it is enough to show that if condition iii) holds, then X is affine. Given a point $x\in X$, let U be an affine open neighborhood of x, and consider the closed subset $Z=$

$\{x\} \cup (X \smallsetminus U)$ of X. If $\mathcal{I}_Z$ is the radical ideal sheaf corresponding to Z, then we have an exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

The long exact sequence in cohomology gives

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z) = \Gamma(\{x\}, \mathcal{O}_{\{x\}}) \oplus \Gamma(X \smallsetminus U, \mathcal{O}_{X \smallsetminus U}) \to H^1(X, \mathcal{I}_Z) = 0,$$

where the vanishing of the right-most term follows by iii). We thus conclude that there is $f_x \in \Gamma(X, \mathcal{O}_X)$ such that $f_x(x) \neq 0$, but $f_x|_{X \smallsetminus U} = 0$. This implies that the open subset $U_x := D_X(f_x)$ of X is contained in U; since U is affine, it follows that U_x is a principal affine open subset of U, hence in particular it is affine. By quasi-compactness, we conclude that we can find $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ such that

$$X = D_X(f_1) \cup \ldots \cup D_X(f_r) \tag{10.4.1}$$

and each $D_X(f_i)$ is an affine subset. If we show that $f_1, \ldots, f_r$ span the unit ideal in $\Gamma(X, \mathcal{O}_X)$, then we conclude that X is affine by Proposition 2.3.16.

Consider the morphism $p\colon \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X$, that maps the i^{th} element of the standard basis to f_i. It follows from (10.4.2) that p is surjective. If we put $\mathcal{F} = \ker(p)$, then it is enough to show that $H^1(X, \mathcal{F}) = 0$. Indeed, the long exact sequence in cohomology for

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X \to 0$$

then implies that $\mathcal{O}_X(X) = (f_1, \ldots, f_r)$. In order to prove the vanishing of $H^1(X, \mathcal{F})$, consider the filtration of $\mathcal{O}_X^{\oplus r}$ by

$$\mathcal{M}_0 = 0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}_r = \mathcal{O}_X^{\oplus r},$$

where $\mathcal{M}_i$ is generated by the first i elements of the standard basis of $\mathcal{O}_X^{\oplus r}$. If we put $\mathcal{F}_i = \mathcal{F} \cap \mathcal{M}_i$, then for every i, with $1 \leq i \leq r$, we have a short exact sequence

$$0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0. \tag{10.4.2}$$

Since we have an injection $\mathcal{F}_i/\mathcal{F}_{i-1} \hookrightarrow \mathcal{M}_i/\mathcal{M}_{i-1} \simeq \mathcal{O}_X$, condition iii) gives $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$. Using the long exact sequence in cohomology corresponding to (10.4.2), we deduce arguing by induction on $i \geq 0$ that $H^1(X, \mathcal{F}_i) = 0$ for $0 \leq i \leq r$. By taking $i = r$, we conclude that $H^1(X, \mathcal{F}) = 0$, completing the proof of the theorem. $\square$

Corollary 10.4.2. *Given a short exact sequence of $\mathcal{O}_X$-modules on the algebraic variety X:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

if $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent (coherent), then $\mathcal{F}$ is quasi-coherent (respectively, coherent).

Proof. For every affine open subset U of X and every $f \in \mathcal{O}_X(U)$, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}') \arrow[r] \arrow[d, "\alpha"'] & H^0(U, \mathcal{F}) \arrow[r] \arrow[d, "\beta"'] & H^0(U, \mathcal{F}'') \arrow[r] \arrow[d, "\gamma"'] & H^1(U, \mathcal{F}') \arrow[d] \\
0 \arrow[r] & H^0\big(D_U(f), \mathcal{F}'\big) \arrow[r] & H^0\big(D_U(f), \mathcal{F}\big) \arrow[r] & H^0\big(D_U(f), \mathcal{F}''\big) \arrow[r] & H^1\big(D_U(f), \mathcal{F}'\big).
\end{tikzcd}$$

Since $\mathcal{F}'$ is quasi-coherent and U and $D_U(f)$ are affine, we have

$$H^1(U, \mathcal{F}') = 0 = H^1\big(D_U(f), \mathcal{F}'\big).$$

Since $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent, it follows that the maps α and γ in the above diagram are isomorphisms, hence β is an isomorphism, too. Therefore $\mathcal{F}$ is quasi-coherent.

If $\mathcal{F}'$ and $\mathcal{F}''$ are coherent, then $\mathcal{F}$ is coherent, too: once we know that it is quasi-coherent, the assertion follows from the fact that given a short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0,$$

if M' and M'' are finitely generated, then M is finitely generated, too. $\square$

10.5. Introduction to spectral sequences

In this section we give an introduction to spectral sequences. We begin by discussing the spectral sequence associated to a filtered complex and then specialize to the case of the spectral sequence of a double complex. We apply this framework to construct Grothendieck's spectral sequence computing the right derived functors of a composition of two left exact functors.

10.5.1. The formalism of spectral sequences. In order to fix the ideas, we let $\mathcal{C}$ be the category of $\mathcal{O}_X$-modules, where $(X, \mathcal{O}_X)$ is a fixed ringed space. Suppose that we have a complex $(K^\bullet, d)$ of objects in $\mathcal{C}$, together with a filtration $F_\bullet K^\bullet = (F_p K^\bullet)_{p\in\mathbf{Z}}$. This means that we have a non-increasing sequence of subcomplexes of $K^\bullet$:

$$K^\bullet \supseteq \ldots \supseteq F_p K^\bullet \supseteq F_{p+1} K^\bullet \supseteq \ldots.$$

The filtration on $K^\bullet$ induces a filtration on the cohomology of $K^\bullet$ given by

$$F_p \mathcal{H}^n(K^\bullet) = \mathrm{Im}\big(\mathcal{H}^n(F_p K^\bullet) \to \mathcal{H}^n(K^\bullet)\big).$$

Instead of describing the cohomology sheaves $\mathcal{H}^n(K^\bullet)$, we will only describe the graded pieces with respect to the above filtration, that is,

$$\mathrm{gr}_p \mathcal{H}^n(K^\bullet) = F_p \mathcal{H}^n(K^\bullet)/F_{p+1}\mathcal{H}^n(K^\bullet).$$

The main idea is to describe these using a sequence of approximations built out of the complexes $F_p K^\bullet$.

This is achieved by the *spectral sequence* associated to the filtration on $K^\bullet$. For every $r \geq 0$ and every $p, q \in \mathbf{Z}$, we will define an object $E_r^{p,q}$ in $\mathcal{C}$. For each r, the $(E_r^{p,q})_{p,q\in\mathbf{Z}}$ form the r^{th} *page* of the spectral sequence. We note that in this notation, the index p is related to the level of the filtration, while the sum $p+q$ keeps track of the cohomological degree in the complex.

For every $r \in \mathbf{Z}$ and $p, q \in \mathbf{Z}$, we put

$$Z_r^{p,q} = \{u \in F_p K^{p+q} \mid d(u) \in F_{p+r} K^{p+q+1}\}.$$

Note that for $r \leq 0$, we have $Z_0^{p,q} = F_p K^{p+q}$. It is clear that we have

$$Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2}) \subseteq Z_r^{p,q}$$

and for $r \geq 0$, we put

$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})}.$$

It is instructive to look at the first two pages. For $r = 0$, we have

$$E_0^{p,q} = \frac{F_p K^{p+q}}{F_{p+1} K^{p+q} + d(F_{p+1} K^{p+q-1})} = \frac{F_p K^{p+q}}{F_{p+1} K^{p+q}}.$$

For $r = 1$, we have

$$E_1^{p,q} = \frac{\{u \in F_pK^{p+q} \mid d(u) \in F_{p+1}K^{p+q+1}\}}{F_{p+1}K^{p+q} + d(F_pK^{p+q-1})}.$$

Note that d induces morphisms $d_0 \colon E_0^{p,q} \to E_0^{p,q+1}$ such that $d_0 \circ d_0 = 0$ and

$$E_1^{p,q} = \frac{\ker(E_0^{p,q} \to E_0^{p,q+1})}{\operatorname{Im}(E_0^{p,q-1} \to E_0^{p,q})}.$$

We now show that a similar picture holds also for higher r. For every $r \geq 0$, we claim that d induces a map d_r of bidgree $(r, 1-r)$, that is

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

for every p and q. Indeed, it is clear that

$$d(Z_r^{p,q}) \subseteq Z_r^{p+r,q-r+1} \quad \text{and} \quad d\big(Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})\big) = d(Z_{r-1}^{p+1,q-1}),$$

which implies our claim. Since $d \circ d = 0$, it is clear that we have $d_r \circ d_r = 0$. The sequence $\big((E_r^{p,q})_{p,q}, d_r\big)_{r\geq 0}$ is the *spectral sequence* associated to the given filtered complex.

We now show that the $(r+1)^{\text{st}}$ page of the spectral sequence is obtain by taking the cohomology with respect to d^r.

Proposition 10.5.1. *With the above notation, for every* $r \geq 0$, *we have* $E_{r+1} = \mathcal{H}(E_r, d_r)$, *in the sense that for every* $p, q \in \mathbf{Z}$, *we have a canonical isomorphism*

$$E_{r+1}^{p,q} \simeq \frac{\ker(d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{Im}(d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

Proof. The assertion follows, with a bit of care, directly from definitions. Note first that if $u \in Z_r^{p,q}$, then $d_r(\overline{u}) = 0$ in $E_r^{p+r,q-r+1}$ if and only if

$$d(u) \in Z_{r-1}^{p+r+1,q-r} + d(Z_{r-1}^{p+1,q-1}).$$

This is the case if and only if there is $v \in Z_{r-1}^{p+1,q-1}$ such that $d(u-v) \in Z_{r-1}^{p+r+1,q-r}$. This last condition is equivalent to $d(u-v) \in F_{p+r+1}K^{p+q+1}$, and since we have in any case $u - v \in F_pK^{p+q}$, the condition is equivalent to $u - v \in Z_{r+1}^{p,q}$. We thus see that

$$\ker(d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}) \simeq \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})},$$

and thus

$$\begin{aligned}
\frac{\ker(d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{Im}(d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q})} &\simeq \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})} \\
&= \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1}},
\end{aligned}$$

where the equality follows from the inclusion $Z_{r-1}^{p-r+1,q+r-2} \subseteq Z_r^{p-r,q+r-1}$. We thus obtain

$$\frac{\ker(d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{Im}(d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q})} \simeq \frac{Z_{r+1}^{p,q}}{\big(d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1}\big) \cap Z_{r+1}^{p,q}}.$$

Note now that $\big(d(Z_r^{p-r,q+r-1})+Z_{r-1}^{p+1,q-1}\big)\cap Z_{r+1}^{p,q}$ consists of those $d(a)+b$ such that the following hold:

$$a\in F_{p-r}K^{p+q-1},\quad d(a)\in F_pK^{p+q},\quad \text{and}$$
$$b\in F_{p+1}K^{p+q},\quad d(b)\in F_{p+r}K^{p+q+1},\quad d(b)\in F_{p+r+1}K^{p+q+1}.$$

We thus see that the conditions on a and b are precisely that $b\in Z_r^{p+1,q-1}$ and $a\in Z_r^{p-r,q+r-1}$, and we obtain

$$\frac{\ker(d_r\colon E_r^{p,q}\to E_r^{p+r,q-r+1})}{\operatorname{Im}(d_r\colon E_r^{p-r,q+r-1}\to E_r^{p,q})}\simeq\frac{Z_{r+1}^{p,q}}{Z_r^{p+1,q-1}+d(Z_r^{p-r,q+r-1})}=E_{r+1}^{p,q}.$$

□

We now show that under suitable conditions on the filtration, we can recover the filtration on the cohomology of $K^\bullet$ from the spectral sequence.

Definition 10.5.2. The filtration on $K^\bullet$ is *pointwise finite* if for every $n\in\mathbf{Z}$, we have $F_pK^n=K^n$ for $p\ll 0$ and $F_pK^n=0$ for $p\gg 0$. Note that in this case a similar property holds for the filtration on the cohomology: for every n, we have $F_p\mathcal{H}^n(K^\bullet)=0$ for $p\gg 0$ and $F_p\mathcal{H}^n(K^\bullet)=\mathcal{H}^n(K^\bullet)$ for $p\ll 0$.

Proposition 10.5.3. *Given a pointwise finite filtration $(F_pK^\bullet)_{p\in\mathbf{Z}}$ on the complex $K^\bullet$, for every p and q, there is $r(p,q)$ such that $E_r^{p,q}=E_{r+1}^{p,q}$ for all $r\geq r(p,q)$. If we denote this stable value by $E_\infty^{p,q}$, then for every p and q we have a canonical isomorphism*

$$E_\infty^{p,q}\simeq \operatorname{gr}_p\mathcal{H}^{p+q}(K^\bullet). \tag{10.5.1}$$

We will refer to the conclusion of the above proposition by saying that the *spectral sequence converges to* (or *abuts to*) $\mathcal{H}^{p+q}(K^\bullet)$, and this is written as

$$E_r^{p,q}\Rightarrow_p \mathcal{H}^{p+q}(K^\bullet).$$

With a typical abuse of notation, we often write this by only recording one page of the spectral sequence, usually the one for E_1 or E_2.

Proof of Proposition 10.5.3. Let p and q be fixed. For $r\gg 0$, we have $F_{p+r}K^{p+q}=0$, hence $Z_r^{p+r,q-r+1}=0$ and thus $E_r^{p+r,q-r+1}=0$. In particular, the map $d_r\colon E_r^{p,q}\to E_r^{p+r,q-r+1}$ is 0. Similarly, for $r\gg 0$, we have

$$F_{p-r}K^{p+q-1}=F_{p-r+1}K^{p+q-1}=K^{p+q-1},$$

hence $Z_r^{p-r,q+r-1}=Z_{r-1}^{p-r+1,q+r-2}$, and thus $E_r^{p-r,q+r-1}=0$. In particular, the map $d_r\colon E_r^{p-r,q+r-1}\to E_r^{p,q}$ is 0. We thus conclude via Proposition 10.5.1 that for $r\gg 0$, we have a canonical isomorphism $E_r^{p,q}\simeq E_{r+1}^{p,q}$. This gives the first assertion in the proposition.

In fact, we can describe $E_\infty^{p,q}$ very explicitly. If $r\gg 0$, then $F_{p+r}K^{p+q}=0$, hence

$$Z_r^{p,q}=\{u\in F_pK^{p+q}\mid du=0\}\quad\text{and}\quad Z_{r-1}^{p+1,q-1}=\{u\in F_{p+1}K^{p+q}\mid du=0\}.$$

Moreover, for $r\gg 0$, we have $F^{p-r+1}K^{p+q-1}=K^{p+q-1}$, hence

$$d(Z_{r-1}^{p-r+1,q+r-2})=d(K^{p+q-1})\cap F_pK^{p+q}.$$

We thus conclude that for $r\gg 0$, we have

$$E_\infty^{p,q}\simeq\frac{\{u\in F_pK^{p+q}\mid d(u)=0\}}{\{u\in F_{p+1}K^{p+q}\mid d(u)=0\}+\big(F_pK^{p+q}\cap d(K^{p+q-1})\big)}.$$

On the other hand, it follows from definition that

$$\frac{F_p\mathcal{H}^{p+q}(K^\bullet)}{F_{p+1}\mathcal{H}^{p+q}(K^\bullet)} \simeq \frac{\{u \in F_pK^{p+q} \mid d(u)=0\} + d(K^{p+q-1})}{\{u \in F_{p+1}K^{p+q} \mid d(u)=0\} + d(K^{p+q-1})}$$
$$\simeq \frac{\{u \in F_pK^{p+q} \mid d(u)=0\}}{\{u \in F_{p+1}K^{p+q} \mid d(u)=0\} + \big(F_pK^{p+q} \cap d(K^{p+q-1})\big)}.$$

This completes the proof of the proposition. $\square$

DEFINITION 10.5.4. Suppose that the filtration on $K^\bullet$ is pointwise finite. We say that the spectral sequence $(E_r^{p,q})_{r\geq 0}$ *degenerates* at level $r_0 \geq 0$ if $d_r = 0$ for all $r \geq r_0$. In this case, it follows from Proposition 10.5.3 that

$$E_{r_0}^{p,q} \simeq \mathrm{gr}_p\mathcal{H}^{p+q}(K^\bullet) \quad \text{for all} \quad p, q \in \mathbf{Z}.$$

REMARK 10.5.5. There are two important cases in which the sequence degenerates at level r_0. Suppose, for example, that $r_0 \geq 1$ and there is $a \in \mathbf{Z}$ such that $E_{r_0}^{p,q} = 0$ unless $p = a$. This clearly implies, using Proposition 10.5.1 that $E_r^{p,q} = 0$ for all $r \geq r_0$ if $p \neq a$. Since every d_r has bidegree $(r, 1-r)$, we see that $d_r = 0$ for all $r \geq r_0$. Moreover, we see that in this case we have $\mathrm{gr}_p\mathcal{H}^{p+q}(K^\bullet) = 0$, unless $p = a$. We thus conclude that

$$\mathcal{H}^n(K^\bullet) \simeq E_{r_0}^{a,n-a} \quad \text{for all} \quad n \in \mathbf{Z}.$$

Another special case is that when $r_0 \geq 2$ and there is $b \in \mathbf{Z}$ such that $E_{r_0}^{p,q} = 0$, unless $q = b$. Again, we see that $d_r = 0$ for all $r \geq r_0$, and we have

$$\mathcal{H}^n(K^\bullet) \simeq E_{r_0}^{n-b,b} \quad \text{for all} \quad n \in \mathbf{Z}.$$

We now discuss two situations, that occur rather often, when we get some canonical maps out of the spectral sequence. We assume that all filtrations are pointwise finite. Suppose first that $E_1^{p,q} = 0$ if $p < 0$. For example, this condition holds if $F_0K^\bullet = K^\bullet$ (in this case, if $p < 0$, then $Z_r^{p,q} = Z_{r-1}^{p+1,q-1}$, hence $E_r^{p,q} = 0$ for all r). Under our assumption, it follows from Proposition 10.5.1 that $E_r^{p,q} = 0$ for all $r \geq 1$ if $p < 0$. Moreover, since $E_r^{-r,q+r-1} = 0$ for every $r \geq 1$, we deduce that we have canonical injective homomorphisms

$$E_\infty^{0,q} = E_r^{0,q} \hookrightarrow E_{r-1}^{0,q} \hookrightarrow \ldots \hookrightarrow E_2^{0,q} \hookrightarrow E_1^{0,q},$$

where $r \gg 0$. On the other hand, since

$$F_p\mathcal{H}^{p+q}(K^\bullet)/F_{p+1}\mathcal{H}^{p+q}(K^\bullet) \simeq E_\infty^{p,q} = 0 \quad \text{for} \quad p < 0,$$

it follows that $F_p\mathcal{H}^n(K^\bullet) = \mathcal{H}^n(K^\bullet)$ for every $p \leq 0$ and we have a canonical surjective homomorphism

$$\mathcal{H}^q(K^\bullet) \to E_\infty^{0,q}.$$

By composing the above morphisms, we see that we have canonical morphisms $\mathcal{H}^q(K^\bullet) \to E_2^{0,q} \hookrightarrow E_1^{0,q}$ for all q.

Suppose now that $E_2^{p,q} = 0$ if $q < 0$. This is the case, for example, if the filtration satisfies the condition that $F_pK^n = 0$ for $p > n$; indeed, this condition implies that if $q < 0$, then $Z_r^{p,q} = 0$, and thus $E_r^{p,q} = 0$ for all r. Under our assumption, it follows from Proposition 10.5.1 that $E_r^{p,q} = 0$ if $q < 0$ and $r \geq 2$. In particular, $E_r^{p+r,1-r} = 0$ for every $r \geq 2$, and we deduce using the same proposition that we have canonical surjective homomorphisms

$$E_2^{p,0} \to E_3^{p,0} \to \ldots \to E_r^{p,0} \to E_\infty^{p,0}.$$

On the other hand, we have

$$F_p\mathcal{H}^{p+q}(K^\bullet)/F_{p+1}\mathcal{H}^{p+q}(K^\bullet) \simeq E_\infty^{p,q} = 0 \quad \text{for} \quad q < 0,$$

and thus $F_pH^n(K^\bullet) = 0$ for $p > n$. This implies that we have canonical injective homomorphisms

$$E_\infty^{p,0} \hookrightarrow \mathcal{H}^p(K^\bullet)$$

for all p. By combining the above morphisms, we obtain for all p a canonical morphism

$$E_2^{p,0} \to \mathcal{H}^p(K^\bullet).$$

REMARK 10.5.6. All constructions in this section are functorial. More precisely, suppose that we have a morphism of filtered complexes $u\colon K^\bullet \to \overline{K}^\bullet$ (this is a morphism of complexes such that $u(F_pK^\bullet) \subseteq F_p\overline{K}^\bullet$ for all p). If $E_r^{p,q}$ and $\overline{E}_r^{p,q}$ are the terms of the spectral sequences associated to $(K^\bullet, F_\bullet K^\bullet)$ and $(\overline{K}^\bullet, F_\bullet K^\bullet)$, respectively, then u induces morphisms $E_r^{p,q} \to \overline{E}_r^{p,q}$ that commute with the morphisms d_r. Moreover, for every n, the induced morphism $\mathcal{H}^n(u)\colon \mathcal{H}^n(K^\bullet) \to \mathcal{H}^n(\overline{K}^\bullet)$ maps $F_p\mathcal{H}^n(K^\bullet)$ to $F_p\mathcal{H}^n(\overline{K}^\bullet)$. If both filtrations are pointwise finite, we have a commutative diagram

$$\begin{array}{ccc}
E_\infty^{p,q} & \longrightarrow & \mathrm{gr}_p\mathcal{H}^{p+q}(K^\bullet) \\
\downarrow & & \downarrow \\
\overline{E}_\infty^{p,q} & \longrightarrow & \mathrm{gr}_p\mathcal{H}^{p+q}(\overline{K}^\bullet),
\end{array}$$

in which the horizontal maps are isomorphisms.

10.5.2. The spectral sequences associated to a double complex. We now discuss an important example of spectral sequence that arises from a double complex.

DEFINITION 10.5.7. A *double complex* $A^{\bullet,\bullet}$ in $\mathcal{C}$ is given by a family of objects $(A^{p,q})_{p,q\in\mathbf{Z}}$ of objects in $\mathcal{C}$, together with morphisms $d_1^{p,q}\colon A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q}\colon A^{p,q} \to A^{p,q+1}$, such that the following conditions hold:

$$d_1 \circ d_1 = 0, \quad d_2 \circ d_2 = 0, \quad \text{and} \quad d_1 \circ d_2 = d_2 \circ d_1.$$

Note that in this case, for every p, we have a complex $(A^{p,\bullet}, d_2)$ and similarly, for every q, we have a complex $(A^{\bullet,q}, d_1)$.

The *total complex* of a double complex $A^{\bullet,\bullet}$ is the complex $K^\bullet = \mathrm{Tot}(A^{\bullet,\bullet})$, with $K^n = \bigoplus_{i+j=n} K^{i,j}$, and with map $d\colon K^n \to K^{n+1}$ defined on $A^{i,j}$ by $d_1^{i,j} + (-1)^i d_2^{i,j}$ (it is straightforward to check that $d \circ d = 0$). The complex $K^\bullet$ admits two natural filtrations, as follows. The first filtration is given by

$$F'_pK^n = \bigoplus_{i\geq p} A^{i,n-i};$$

note that indeed, we have $d(F'_pK^n) \subseteq F'_pK^{n+1}$. The second filtration is given by

$$F''_pK^n = \bigoplus_{j\geq p} A^{n-j,j};$$

note that again we have $d(F''_pK^n) \subseteq F''_pK^{n+1}$.

Let us compute the first terms of the spectral sequences associated to the two filtrations. We do the computation for the first filtration, the other one following by symmetry. We denote the terms of this spectral sequence by ${}'E_r^{p,q}$. By the general computation of the 0^{th} page, we see that

$$'E_0^{p,q} = F_pK^{p+q}/F_{p+1}K^{p+q} = A^{p,q} \quad \text{for all} \quad p,q \in \mathbf{Z}.$$

Moreover, the map induced by d on $F_pK^{p+q}/F_{p+1}K^{p+q}$ is equal to the map induced by $(-1)^p d_2$, hence

$$'E_1^{p,q} = \mathcal{H}^q(A^{p,\bullet}).$$

Note that for every p, the map d_1 induces a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$, and thus for every q, it induces a morphism $\mathcal{H}^q(A^{p,\bullet}) \to \mathcal{H}^q(A^{p+1,\bullet})$. An easy computation shows that this map gets identified with the morphism ${}'E_1^{p,q} \to {}'E_1^{p+1,q}$ associated to the spectral sequence We thus deduce from Proposition 10.5.1 that for every p and q we have

$$'E_2^{p,q} = \mathcal{H}^p_{d_1}\big(\mathcal{H}^q_{d_2}(A^{\bullet,\bullet})\big),$$

where the right-hand side stands for the cohomology of the complex

$$\mathcal{H}^q(A^{p-1,\bullet}) \to \mathcal{H}^q(A^{p,\bullet}) \to \mathcal{H}^q(A^{p+1,\bullet}),$$

with the maps induced by d_1.

Similarly, for the other filtration, we obtain the first terms of the exact sequence given by

$$''E_0^{p,q} = A^{q,p}, \quad ''E_1^{p,q} = \mathcal{H}^q(A^{\bullet,p}), \quad \text{and}$$
$$''E_2^{p,q} = \mathcal{H}^p_{d_2}\big(\mathcal{H}^q_{d_1}(A^{\bullet,\bullet})\big),$$

where the right-hand side stands for the cohomology of the complex

$$\mathcal{H}^q(A^{\bullet,p-1}) \to \mathcal{H}^q(A^{\bullet,p}) \to \mathcal{H}^p(A^{\bullet,p+1}),$$

with the maps induced by d_2.

REMARK 10.5.8. We will be interested in the case when the double complex satisfies the following extra condition: for every n, there are only finitely many pairs (p,q) with $i+j=n$ and with $A^{p,q} \neq 0$. In this case, both filtrations on $\mathrm{Tot}(A^{\bullet,\bullet})$ are pointwise finite, hence both spectral sequences satisfy Proposition 10.5.3. This condition is satisfied, in particular, for the first-quadrant or third-quadrant double complexes introduced below.

DEFINITION 10.5.9. A *first-quadrant* double complex $A^{\bullet,\bullet}$ is a double complex such that $A^{p,q} = 0$ unless $p,q \geq 0$. A *third-quadrant* double complex $A^{\bullet,\bullet}$ is a double complex such that $A^{p,q} = 0$ unless $p,q \leq 0$.

If $A^{\bullet,\bullet}$ is a first-quadrant double complex, then both filtrations on $\mathrm{Tot}(A^{\bullet,\bullet})$ are non-negative and lower than the grading. Therefore we have canonical morphisms

$$'E_2^{q,0} \to \mathcal{H}^q\big(\mathrm{Tot}(A^{\bullet})\big) \to {}'E_2^{0,q} \to {}'E_1^{0,q} \quad \text{for all} \quad q \geq 0,$$

and similar morphisms for the spectral sequence corresponding to the second filtration.

EXERCISE 10.5.10. Suppose that we have a spectral sequence

$$E_2^{p,q} \Rightarrow_p \mathcal{H}^{p+q}.$$

Show that if this is a first-quadrant spectral sequence (that is, $E_2^{p,q}=0$ unless $p\geq 0$ and $q\geq 0$), then we have an associated *five-term exact sequence*:

$$0\to E_2^{1,0}\to \mathcal{H}^1\to E_2^{0,1}\to E_2^{2,0}\to \mathcal{H}^2.$$

10.5.3. The spectral sequence of a composition of two functors. We will apply the formalism of spectral sequences for double complexes to prove the following result of Grothendieck about the right derived functors of a composition of two left exact functors. We consider two left exact functors

$$G\colon \mathcal{C}_1\to \mathcal{C}_2\quad\text{and}\quad F\colon \mathcal{C}_2\to\mathcal{C}_3$$

(where, as usual, the categories $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are categories of sheaves of modules on suitable ringed spaces).

THEOREM 10.5.11. *With the above notation, suppose that for every injective object $\mathcal{I}$ in $\mathcal{C}_1$, the object $G(\mathcal{I})$ in $\mathcal{C}_2$ if F-acyclic. In this case, for every object $\mathcal{A}$ in $\mathcal{C}_1$, we have a spectral sequence*

$$E_2^{p,q}=R^pF\big(R^qG(\mathcal{A})\big)\Rightarrow_p R^{p+q}(F\circ G)(\mathcal{A}).$$

The proof of the theorem will make use of the notion of *Cartan-Eilenberg resolution*, that we now introduce. Given a complex of objects in $\mathcal{C}$:

$$C^\bullet:\quad \ldots\to C^p\xrightarrow{d_C^p} C^{p+1}\xrightarrow{d_C^{p+1}}\ldots$$

that is *bounded below* (that is, we have $C^p=0$ for $p\ll 0$), a *Cartan-Eilenberg resolution* of $C^\bullet$ is given by a double complex $A^{\bullet,\bullet}$, together with a morphism of complexes $C^\bullet\to A^{\bullet,0}$, with the following properties:

i) There is p_0 such that $A^{p,q}=0$ for all $p\leq p_0$ and all q; we also have $A^{p,q}=0$ for all $q<0$ and all p.
ii) The morphism $C^p\to A^{p,\bullet}$ is an injective resolution for all p.
iii) The induced morphism $\ker(d_C^p)\to\ker(d_1^{p,\bullet})$ is an injective resolution for all p.
iv) The induced morphism $\mathrm{Im}(d_C^p)\to\mathrm{Im}(d_1^{p,\bullet})$ is an injective resolution for all p.
v) For every p, we have an injective resolution of $\mathcal{H}^p(C^\bullet)$ given by
$$\mathcal{H}^p(C^\bullet)\to\mathcal{H}^p(A^{\bullet,0})\to\mathcal{H}^p(A^{\bullet,1})\to\ldots.$$

We begin by proving the existence of such resolutions.

LEMMA 10.5.12. *Any bounded below complex $C^\bullet$ has a Cartain-Eilenberg resolution.*

PROOF. Let p_0 be such that $C^p=0$ for $p\leq p_0$. For every p, consider injective resolutions $\mathcal{H}^p(C^\bullet)\to U^{p,\bullet}$ and $\mathrm{Im}(d_C^{p-1})\to V^{p,\bullet}$, with $U^{p,\bullet}=0=V^{p,\bullet}$ if $p\leq p_0$. Note that we have an exact sequence

$$0\to\mathrm{Im}(d_C^{p-1})\to\ker(d_C^p)\to\mathcal{H}^p(C^\bullet)\to 0$$

for every p and using Lemma 10.1.12, we obtain an injective resolution $\ker(d_C^p)\to W^{p,\bullet}$ such that we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & V^{p,\bullet} \arrow[r] & W^{p,\bullet} \arrow[r] & U^{p,\bullet} \arrow[r] & 0\\
0 \arrow[r] & \mathrm{Im}(d_C^{p-1}) \arrow[r] \arrow[u] & \ker(d_C^p) \arrow[r] \arrow[u] & \mathcal{H}^p(C^\bullet) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Similarly, for every p we have an exact sequence

$$0 \to \ker(d_C^p) \to C^p \to \operatorname{Im}(d_C^p) \to 0$$

and using Lemma 10.1.12, we obtain an injective resolution $C^p \to A^{p,\bullet}$ such that we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & W^{p,\bullet} \arrow[r] & A^{p,\bullet} \arrow[r] & V^{p+1,\bullet} \arrow[r] & 0 \\
0 \arrow[r] & \ker(d_C^p) \arrow[r] \arrow[u] & C^p \arrow[r] \arrow[u] & \operatorname{Im}(d_C^p) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

For every p and q we have a morphism $d_2^{p,q} \colon A^{p,q} \to A^{p,q+1}$ defined by the complex $A^{p,\bullet}$ and a morphism $d_1^{p,q} \to A^{p,q} \to A^{p+1,q}$ given as the composition

$$A^{p,q} \to V^{p+1,q} \to W^{p+1,q} \to A^{p+1,q}.$$

We thus obtain a double complex $A^{\bullet,\bullet}$, together a morphism of complexes $C^\bullet \to A^{\bullet,0}$, and it is straightforward to check that this satisfies conditions i)-v) in the definition of a Cartan-Eilenberg resolution. $\square$

The next lemma explain the usefulness of Cartain-Eilenberg resolutions.

LEMMA 10.5.13. *If $G \colon \mathcal{C} \to \mathcal{C}'$ is a left exact functor*[6]*, $C^\bullet$ is a bounded below complex in $\mathcal{C}$, and $A^{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of $C^\bullet$, then we have two spectral sequences abutting to the cohomology of the total complex of $G(A^{\bullet,\bullet})$:*

$${}'E_1^{p,q} = R^qG(C^p) \Rightarrow_p \mathcal{H}^{p+q}\big(\operatorname{Tot}(G(A^{\bullet,\bullet}))\big),$$

where the map

$${}'E_1^{p,q} = R^q(C^p) \to R^q(C^{p+1}) = {}'E_1^{p+1,q}$$

is induced by $C^p \to C^{p+1}$, hence ${}'E_2^{p,q} = \mathcal{H}^p\big(R^qG(C^\bullet)$, and

$${}''E_2^{p,q} = R^pG\big(\mathcal{H}^q(C^\bullet)\big) \Rightarrow_p \mathcal{H}^{p+q}\big(\operatorname{Tot}(G(A^{\bullet,\bullet}))\big).$$

In particular, if all C^p are G-acyclic, then we have a spectral sequence

$$E_2^{p,q} = R^pG\big(\mathcal{H}^q(C^\bullet)\big) \Rightarrow_p \mathcal{H}^{p+q}\big(G(C^\bullet)\big).$$

PROOF. We consider the two spectral sequences associated to the double complex $G(A^{\bullet,\bullet})$. Note that the condition i) in the definition of a Cartan-Eilenberg resolution implies that both filtrations on $\operatorname{Tot}(A^{\bullet,\bullet})$ are pointwise finite, and thus both spectral sequences abut to the cohomology of this complex. We have seen that

$${}'E_1^{p,q} = \mathcal{H}^q\big(G(A^{p,\bullet})\big) \simeq R^qG(C^p),$$

where the isomorphism follows from the fact that $C^p \to A^{p,\bullet}$ is an injective resolution. Moreover, the map ${}'E_1^{p,q} \to {}'E_1^{p+1,q}$ is induced by the morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$, and thus is obtained by applying R^qG to the map $C^p \to C^{p+1}$.

In order to describe the second spectral sequence, note for every p and q, we have short exact sequences

$$0 \to \ker(d_1^{p,q}) \to A^{p,q} \to \operatorname{Im}(d_1^{p,q}) \to 0$$

and

$$0 \to \operatorname{Im}(d_1^{p-1,q}) \to \ker(d_1^{p,q}) \to \mathcal{H}^p(A^{\bullet,q}) \to 0.$$

[6]As usual, we assume that $\mathcal{C}$ and $\mathcal{C}'$ are categories of sheaves of modules on two ringed spaces. More generally, the same holds if $\mathcal{C}$ and $\mathcal{C}'$ are Abelian categories, with $\mathcal{C}$ having enough injectives.

By the definition of a Cartan-Eilenberg resolution, all objects that appear in these exact sequences are injective, hence the sequences are split. This implies that after applying G, the sequences remain exact. It is then straightforward to deduce that we have canonical isomorphisms

$$''E_1^{p,q} = \mathcal{H}^p\big(G(A^{\bullet,q})\big) \simeq G\big(\mathcal{H}^p(A^{\bullet,q})\big).$$

Using again the fact that $A^{\bullet,\bullet}$ is a Cartan-Eilenberg resolution, we get an isomorphism

$$''E_2^{p,q} \simeq R^qG\big(\mathcal{H}^p(C^\bullet)\big).$$

Note now that if every C^p is G-acyclic, then $'E_1^{p,q} = 0$ for $q \neq 0$ and $'E_1^{p,0} \simeq G(C^p)$. We thus conclude that $'E_2^{p,q} = 0$ for $q \neq 0$ and $'E_2^{p,0} \simeq \mathcal{H}^p\big(G(C^\bullet)\big)$. This implies that we have

$$\mathcal{H}^n\big(\mathrm{Tot}(G(A^{\bullet,\bullet}))\big) \simeq \mathcal{H}^n\big(G(C^\bullet)\big)$$

and we obtain the last assertion in the lemma. $\square$

Proof of Theorem 10.5.11. Consider an injective resolution $\mathcal{A} \to \mathcal{I}^\bullet$ in $\mathcal{C}_1$. By Lemma 10.5.12, we have a Cartan-Eilenberg resolution $A^{\bullet,\bullet}$ of $F(\mathcal{I}^\bullet)$. Note that by assumption, each $F(\mathcal{I}^p)$ is G-acyclic, hence the last assertion in Lemma 10.5.13 gives a spectral sequence

$$E_2^{p,q} = R^pG\big(\mathcal{H}^q(F(\mathcal{I}^\bullet))\big) \Rightarrow_p \mathcal{H}^{p+q}\big(G(F(\mathcal{I}^\bullet))\big).$$

Since

$$\mathcal{H}^q\big(F(\mathcal{I}^\bullet)\big) \simeq R^qF(\mathcal{A}) \quad \text{and} \quad \mathcal{H}^{p+q}\big(G(F(\mathcal{I}^\bullet))\big) \simeq R^{p+q}(G \circ F)(\mathcal{A}),$$

we obtain the assertion in the theorem. $\square$

Remark 10.5.14. Under the assumptions in Theorem 10.5.11, note that since $E_2^{p,q} = 0$ if $p < 0$ or $q < 0$, then we get canonical morphisms

$$R^nF\big(G(\mathcal{A})\big) \to R^n(F \circ G)(\mathcal{A}) \to F\big(R^nG(\mathcal{A})\big)$$

for every object $\mathcal{A}$ in $\mathcal{C}_1$ and $n \geq 0$.

Example 10.5.15. Let $g\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $f\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. We consider the left exact functors $G = g_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$ and $F = f_*\colon \mathcal{O}_Y\text{-mod} \to \mathcal{O}_Z\text{-mod}$. Note that $F \circ G = (f \circ g)_*$. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then it is flasque (see Lemma 10.2.7), hence $G(\mathcal{I})$ is flasque, and thus F-acyclic (see Corollary 10.2.16). We can thus apply Theorem 10.5.11 to deduce that for every $\mathcal{O}_X$-module $\mathcal{M}$, we have a spectral sequence

$$E_2^{p,q} = R^pf_*\big(R^qg_*(\mathcal{M})\big) \Rightarrow_p R^{p+q}(f \circ g)_*(\mathcal{M}).$$

This is known as the *Leray spectral sequence*. This induces canonical morphisms

$$R^nf_*\big(g_*(\mathcal{M})\big) \to R^n(f \circ g)_*(\mathcal{M}) \to f_*\big(R^ng_*(\mathcal{M})\big).$$

In particular, if we take Z to consist of a point, we see that for every morphism of ringed spaces $g\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and for every $\mathcal{O}_X$-module $\mathcal{M}$, we have a spectral sequence

$$E_2^{p,q} = H^p\big(Y, R^qg_*(\mathcal{M})\big) \Rightarrow_p H^{p+q}(X, \mathcal{M}).$$

In particular, we have canonical morphisms

$$H^n\big(Y, g_*(\mathcal{M})\big) \to H^n(X, \mathcal{M}) \to \Gamma\big(Y, R^ng_*(\mathcal{M})\big).$$

EXAMPLE 10.5.16. If $f\colon X \to Y$ is an affine morphism of algebraic varieties, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $R^p f_*(\mathcal{F}) = 0$ for all $p \geq 1$. Indeed, it follows from Proposition 10.2.20 that for every affine open subset U of Y, we have

$$\Gamma\big(U, R^p f_*(\mathcal{F})\big) \simeq H^p\big(f^{-1}(U), \mathcal{F}\big),$$

and the right hand side vanishes by Theorem 10.4.1 since $\mathcal{F}$ is quasi-coherent and $f^{-1}(U)$ is affine by the assumption on f. We thus deduce from the Leray spectral sequence an isomorphism

$$H^i\big(Y, f_*(\mathcal{F})\big) \simeq H^i(X, \mathcal{F}) \quad \text{for every} \quad i \geq 0.$$

This applies, in particular, when f is a closed immersion, in which case, for a quasi-coherent sheaf $\mathcal{F}$ on X, we often identify $H^i(X, \mathcal{F})$ with $H^i(Y, \mathcal{F})$ (recall that we also identify $\mathcal{F}$ with its push-forward $f_*(\mathcal{F})$).

EXAMPLE 10.5.17. More generally, given two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$ and a quasi-coherent sheaf $\mathcal{F}$ on X, if f is an affine morphism, then it follows from the Leray spectral sequence that we have isomorphisms

$$R^p g_*\big(f_*(\mathcal{F})\big) \simeq R^p(g \circ f)_*(\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

If we assume instead that g is affine, then the Leray spectral sequence gives isomorphisms

$$g_*\big(R^p f_*(\mathcal{F})\big) \simeq R^p(g \circ f)_*(\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

REMARK 10.5.18. The assertions in Examples 10.5.16 and 10.5.17 can also be proved without spectral sequences. Indeed, suppose that we have morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ and that $\mathcal{F}$ is a quasi-coherent sheaf on X. By Proposition 10.2.17, we have a resolution $\mathcal{F} \to \mathcal{I}^\bullet$, with each $\mathcal{I}^m$ flasque and quasi-coherent. If f is affine, then f_* is exact on the category of quasi-coherent $\mathcal{O}_X$-modules (see Remark 8.4.8). In this case $f_*(\mathcal{F}) \to f_*(\mathcal{I}^\bullet)$ is a flasque resolution of $f_*(\mathcal{F})$, hence we have isomorphisms

$$R^i g_*\big(f_*(\mathcal{F})\big) \simeq \mathcal{H}^i\big(g_*(f_*(\mathcal{I}^\bullet))\big) \simeq R^i(g \circ f)_*(\mathcal{F}).$$

On the other hand, if g is affine, then g_* is exact on the category of quasi-coherent $\mathcal{O}_Y$-modules, and thus we have isomorphisms

$$g_*\big(R^i f_*(\mathcal{F})\big) \simeq g_*\big(\mathcal{H}^i(f_*(\mathcal{I}^\bullet))\big) \simeq \mathcal{H}^i\big(g_*(f_*(\mathcal{I}^\bullet))\big) \simeq R^i(g \circ f)_*(\mathcal{F}).$$

REMARK 10.5.19. If $\mathcal{F}$ is a coherent sheaf on the algebraic variety X and U is an open subset of X such that $Z = \mathrm{Supp}(\mathcal{F}) \subseteq U$, then the canonical maps $H^p(X, \mathcal{F}) \to H^p(U, \mathcal{F})$ are isomorphisms. Indeed, note first that if we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and if we know the assertion for both $\mathcal{F}'$ and $\mathcal{F}''$, then the assertion for $\mathcal{F}$ follows from the long exact sequence in cohomology, using the 5-Lemma. Since we have a finite filtration of $\mathcal{F}$ by coherent sheaves, such that all successive quotients are of the form $i_*(\mathcal{G})$, where $\mathcal{G}$ is a coherent sheaf on Z and $i\colon Z \hookrightarrow X$ is the inclusion (see Remark 8.4.21), it follows that it is enough to prove our assertion when $\mathcal{F} = i_*(\mathcal{G})$. However, in this case both $H^p(X, \mathcal{F})$ and $H^p(U, \mathcal{F})$ are compatibly isomorphic to $H^p(Z, \mathcal{G})$ by Example 10.5.16, hence our assertion follows.

EXERCISE 10.5.20. Show that if X is an algebraic variety with irreducible components $X_1, \ldots, X_r$, then X is affine if and only if X_i is affine for $1 \leq i \leq r$.

EXERCISE 10.5.21. Prove the following result of Chevalley: if $f\colon X \to Y$ is a finite surjective morphism of algebraic varieties, and X is affine, then Y is affine. Hint: use the following steps:

i) Reduce to the case when both X and Y are irreducible.
ii) Show that if X and Y are irreducible, then there is a coherent sheaf $\mathcal{F}$ on X, and a morphism of sheaves $\phi\colon \mathcal{O}_Y^{\oplus r} \to f_*(\mathcal{F})$ for some $r \geq 1$, such that ϕ is an isomorphism over an open subset of Y .
iii) Use ii) and the functor $f^!$ defined in Exercise 8.4.9 to show that if X and Y are irreducible, then given a coherent sheaf $\mathcal{N}$ on Y , there is a coherent sheaf $\mathcal{M}$ on X and a morphism $f_*(\mathcal{M}) \to \mathcal{N}^{\oplus r}$ that is an isomorphism over an open subset of Y.
iv) Deduce Chevalley's result by Noetherian induction.

10.6. Čech cohomology

The main tool for computing cohomology of quasi-coherent sheaves on algebraic varieties is provided by Čech cohomology, that we now introduce. Let X be an algebraic variety and consider a finite open cover $\mathcal{U} = (U_i)_{i\in I}$ of X. For every subset J of I, we put $U_J = \bigcap_{i\in J} U_i$. Given an $\mathcal{O}_X$-module $\mathcal{F}$ on X (or, more generally, a sheaf of Abelian groups), we put

$$C^p(\mathcal{U},\mathcal{F}) := \bigoplus_{\#J=p+1} \mathcal{F}(U_J) \quad \text{for} \quad p \geq 0.$$

We choose a total order on I and define a map $d^p\colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ by

$$d^p\big((s_J)_J\big) = (s_{J'})_{J'},$$

where if the elements of J' are $i_0 < \ldots < i_{p+1}$, then

$$s_{J'} = \sum_{q=0}^{p+1} (-1)^q s_{J'\smallsetminus\{i_q\}}|_{U_{J'}}.$$

It is easy to check that $d^{p+1} \circ d^p = 0$ for all $p \geq 0$, hence we have a complex $C^\bullet(\mathcal{U},\mathcal{F})$, the *Čech complex* of $\mathcal{F}$, with respect to the cover $\mathcal{U}$. The p^{th} *Čech cohomology* of $\mathcal{F}$ with respect to $\mathcal{U}$ is given by

$$H^p(\mathcal{U},\mathcal{F}) := \mathcal{H}^p\big(C^\bullet(\mathcal{U},\mathcal{F})\big).$$

It is clear that the construction is functorial: given a morphism of sheaves $\mathcal{F} \to \mathcal{G}$, we have a morphism of complexes $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G})$ and thus morphisms $H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{U},\mathcal{G})$.

Note that $H^0(\mathcal{U},\mathcal{F})$ is the kernel of $d^0\colon C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})$, which is canonically isomorphic by the sheaf axiom to $\Gamma(X,\mathcal{F})$. The main result of this section says that if $\mathcal{F}$ is quasi-coherent and $\mathcal{U}$ is an affine cover, then also the higher cohomology groups of the Čech complex compute the higher cohomology groups of $\mathcal{F}$. Here is one key place where we make use of the fact that algebraic varieties are separated.

THEOREM 10.6.1. *If $\mathcal{F}$ is a quasi-coherent sheaf on the algebraic variety X and $\mathcal{U}$ is a finite affine open cover of X, then we have functorial isomorphisms of k-vector spaces (in fact, of $\mathcal{O}_X(X)$-modules)*

$$H^p(X,\mathcal{F}) \simeq \mathcal{H}^p(\mathcal{U},\mathcal{F}) \quad \textit{for all} \quad p \geq 0.$$

The theorem will follow from a couple of lemmas. We begin with a result concerning a sheafy version of the Čech complex. Given a subset J of I, let $\alpha_J\colon U_J \hookrightarrow X$ be the inclusion map. For an $\mathcal{O}_X$-module $\mathcal{F}$ on X (or just a sheaf of Abelian groups), we put $\mathcal{F}_J := (\alpha_J)_*(\mathcal{F}|_{U_J})$ and

$$\mathcal{C}^p = \mathcal{C}^p(\mathcal{U},\mathcal{F}) := \bigoplus_{|J|=p+1} \mathcal{F}_J \quad \text{for} \quad p \geq -1$$

(we make the convention that $U_\emptyset = X$ and $\mathcal{F}_\emptyset = \mathcal{F}$). For every open subset $U \subseteq X$ and every $p \geq -1$, we have a map

$$\bigoplus_{|J|=p+1} \mathcal{F}(U \cap U_J) \to \bigoplus_{|J'|=p+2} \mathcal{F}(U \cap U_{J'}), \quad (s_J)_J \to (s_{J'})_{J'},$$

where if the elements of J' are $i_0 < \ldots < i_{p+1}$, then

$$s_{J'} = \sum_{q=0}^{p+1} (-1)^q s_{J' \smallsetminus \{i_q\}}|_{U \cap U_{J'}}.$$

It is straightforward to check that this is a complex and that by letting U vary, we obtain a complex of sheaves

$$0 \to \mathcal{F} = \mathcal{C}^{-1} \to \mathcal{C}^0 \to \ldots, \tag{10.6.1}$$

such that by taking global sections, we have the complex

$$0 \to \mathcal{F}(X) \to C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F}) \to \ldots.$$

LEMMA 10.6.2. *For every $\mathcal{F}$, the complex (10.6.1) is exact.*

PROOF. It is enough to show that for every $x \in X$, the corresponding complex of stalks at x is exact. Choose $i_0 \in I$ such that $x \in U_{i_0}$. It is enough to show that for every open neighborhood U of x, with $U \subseteq U_{i_0}$, on the complex

$$0 \to \mathcal{F}(U) \to \mathcal{C}^0(U) \to \ldots, \tag{10.6.2}$$

the identity map is homotopic to 0. For every $J' \subseteq I$, with $|J'| = p$, we consider the map

$$\mathcal{C}^p(U) = \bigoplus_{\#J=p+1} \mathcal{F}(U \cap U_J) \to \mathcal{F}(U \cap U_{J'}),$$

taking $(s_J)_J$ to 0 if $i_0 \in J'$, and to $(-1)^q s_{J' \cup \{i_0\}}$ if $i_0 \notin J'$ and J' contains precisely q elements $< i_0$ (note that in this case we have $U \cap U_{J'} = U \cap U_{J' \cup \{i_0\}}$). By letting J' vary, we obtain a map $\theta^p\colon \mathcal{C}^p(U) \to \mathcal{C}^{p-1}(U)$, and a straightforward computation shows that the maps $(\theta^p)_{p\geq 0}$ give a homotopy on the complex (10.6.2) between the identity and the 0 map. □

LEMMA 10.6.3. *If U is an affine open subset of an algebraic variety X, with $j\colon U \hookrightarrow X$ the inclusion map, then for every quasi-coherent sheaf $\mathcal{F}$ on U, we have*

$$H^p\big(X, j_*(\mathcal{F})\big) = 0 \quad \textit{for all} \quad p \geq 1.$$

PROOF. Since X is separated, the morphism j is affine. We thus deduce using Example 10.5.16 that

$$H^p\big(X, j_*(\mathcal{F})\big) \simeq H^p(U,\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

On the other hand, since $\mathcal{F}$ is quasi-coherent and U is affine, we have $H^p(U,\mathcal{F}) = 0$ for all $p \geq 1$ by Theorem 10.4.1. □

We can now prove that for quasi-coherent sheaves, cohomology is computed by Čech cohomology with respect to affine covers.

Proof of Theorem 10.6.1. Consider the resolution

$$0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \ldots$$

given by Lemma 10.6.2. Since $\mathcal{F}$ is quasi-coherent, the restriction of $\mathcal{F}$ to each open subset U_J is quasi-coherent by Proposition 8.4.5. Moreover, since X is separated and all U_i are affine open subsets, it follows that U_J is affine for every non-empty $J \subseteq I$. We thus conclude from Lemma 10.6.3 that each $(\alpha_J)_*(\mathcal{F}|_{U_J})$ is $\Gamma(X,-)$ acyclic, hence each $\mathcal{C}^p$, with $p \geq 0$, is $\Gamma(X,-)$-acyclic. We deduce that $H^p(\mathcal{U},\mathcal{F}) = \mathcal{H}^p\big(\Gamma(X,\mathcal{C}^\bullet\big)$ is isomorphic to $H^p(X,\mathcal{F})$ for every $p \geq 0$, by Proposition 10.1.24. $\square$

Exercise 10.6.4. Show that if $U = \mathbf{A}^2 \smallsetminus \{(0,0)\}$, then we have an isomorphism

$$H^1(U,\mathcal{O}_U) \simeq \bigoplus_{i,j<0} kx^iy^j.$$

In particular, $H^1(U,\mathcal{O}_U)$ has infinite dimension over k.

We give a couple of easy consequences of Theorem 10.6.1 to vanishing results.

Corollary 10.6.5. *For every algebraic variety X, there is a positive integer d such that $H^i(X,\mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i \geq d$.*

Proof. If X has an affine open cover $\mathcal{U}$ by d subsets, then $C^i(\mathcal{U},\mathcal{F}) = 0$ for every $i \geq d$, hence $H^i(X,\mathcal{F}) = 0$ for every $i \geq d$ and every quasi-coherent sheaf $\mathcal{F}$ on X by Theorem 10.6.1 $\square$

Corollary 10.6.6. *If $f\colon X \to Y$ is a morphism of algebraic varieties, then there is a positive integer d such that $R^if_*(\mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i \geq d$.*

Proof. Consider a finite affine open cover $Y = V_1 \cup \ldots \cup V_s$. If $\mathcal{F}$ is a quasi-coherent sheaf on X, then each $R^if_*(\mathcal{F})$ is quasi-coherent, hence it vanishes if and only if

$$\Gamma\big(V_j, R^if_*(\mathcal{F})\big) \simeq H^i\big(f^{-1}(V_j),\mathcal{F}\big)$$

vanishes for all j. By Corollary 10.6.5, for every j we can find a positive integer d_j such that $H^i\big(f^{-1}(V_j),\mathcal{G}\big) = 0$ for every quasi-coherent sheaf $\mathcal{G}$ on $f^{-1}(V_j)$ and every $i \geq d_j$. This implies that if $d = \max_j\{d_j\}$, then d satisfies the condition in the corollary. $\square$

For varieties that are projective over an affine variety, we can be more precise than in Corollary 10.6.5.

Corollary 10.6.7. *If $X = \mathrm{MaxProj}(S)$ and $\dim(X) = n$, then $H^i(X,\mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > n$. Similarly, if $\mathcal{F}$ is a coherent sheaf on X, with $\dim\big(\mathrm{Supp}(\mathcal{F})\big) = r$, then $H^i(X,\mathcal{F}) = 0$ for all $i > r$.*

Proof. We first show that if Z is a closed subset of X of dimension r, then we can find affine open subsets $U_1,\ldots,U_{r+1}$ of X such that $Z \subseteq \bigcup_{i=1}^{r+1} U_i$. In order to see this, consider a graded surjective S_0-algebra homomorphism $S[x_0,\ldots,x_N] \to S$, inducing a closed immersion $j\colon X \hookrightarrow \mathbf{P}^N \times Y$, where $Y = \mathrm{MaxSpec}(S_0)$. Let $f\colon X \to \mathbf{P}^N$ be the composition of the projection onto the first component with j. Note that $\dim\big(\overline{f(Z)}\big) \leq r$. We can thus find hyperplanes $H_1,\ldots,H_{r+1}$ in $\mathbf{P}^N$ such that $f(Z)\cap H_1\cap\ldots\cap H_{r+1}=\emptyset$. If H_i is defined by $g_i\in S_0[x_0,\ldots,x_N]_1$ and $\overline{g}_i\in S$ is the image of g_i, then $V(\overline{g}_i)=f^{-1}\big(V(g_i)\big)$, hence $Z\cap\bigcap_{i=1}^{r+1}V(\overline{g_i})=\emptyset$. If we take $U_i=D_X^+(\overline{g_i})$, this is an affine open subset of X, and we have $Z\subseteq\bigcup_{i=1}^{r+1}U_i$.

If we take $Z=X$, we obtain the first assertion in the corollary: if $\mathcal{U}$ is the cover given by $U_1,\ldots,U_{n+1}$, then $C^i(\mathcal{U},\mathcal{F})=0$ for all $i>n$, and thus $H^i(X,\mathcal{F})=0$ for $i>n$ by Theorem 10.6.1.

Given a coherent sheaf $\mathcal{F}$, we take $Z=\mathrm{Supp}(\mathcal{F})$. If $U_1,\ldots,U_{r+1}$ are open subsets as above, we consider the affine open cover $\mathcal{U}$ of X given by the U_i and by the sets in an affine open cover of $X\smallsetminus Z$. Note that by definition we have $\Gamma(V,\mathcal{F})=0$ whenever $V\subseteq X\smallsetminus Z$. We thus again conclude that $C^i(\mathcal{U},\mathcal{F})=0$ for $i>r$, and thus $H^i(X,\mathcal{F})=0$ for $i>r$ by Theorem 10.6.1. □

REMARK 10.6.8. For every positive integer m, there is an irreducible variety of dimension 3 that can't be covered by $\leq m$ affine open subsets (see [**RV04**, §4.9]). Therefore the method in Corollary 10.6.7 can't be used to prove the same vanishing result on arbitrary algebraic varieties. However, we point out that there is a general vanishing result, due to Grothendieck, which implies that for every n-dimensional algebraic variety and for every sheaf of Abelian groups $\mathcal{F}$ on X, we have $H^i(X,\mathcal{F})=0$ for all $i>n$. For a proof in a very general setting, see [**God73**, Théorème 4.15.2]. We do not give the proof, which is rather intricate, since we will not make use of this result.

REMARK 10.6.9. If $(\mathcal{F}_j)_{j\in J}$ is a family of quasi-coherent sheaves on the algebraic variety X, by computing the cohomology as Čech cohomology with respect to a suitable finite affine open cover, we see that we have a functorial isomorphism

$$H^q\big(X,\oplus_{j\in J}\mathcal{F}_j\big)\simeq\oplus_{j\in J}H^q(X,\mathcal{F}_j).$$

10.7. Ext and Tor

In this section we discuss the functors Ext and Tor for modules over a commutative ring and give some applications to the study of projective dimension and flat modules. We also discuss the local and global Ext functors for $\mathcal{O}_X$-modules on a ringed space.

10.7.1. The Ext functor for modules. Let R be a commutative ring and consider the category R-mod of all R-modules (this is a trivial example of the category $\mathcal{O}_X$-mod, when X consists of one point and $\mathcal{O}(X)=R$). Given an R-module M, we consider the left exact functor $\mathrm{Hom}_R(M,-)$ from R-mod to itself. By applying the formalism of right derived functors discussed in §10.1, we have the corresponding right derived functors, denoted $\{\mathrm{Ext}_R^i(M,-)\}_{i\geq 0}$.

We note that these are in fact bifunctors (contravariant in the first variable and covariant in the second variable), in the sense that for every morphism of R-modules $u\colon M\to M'$, we have corresponding natural transformation

$$\mathrm{Ext}^i_R(M',-)\to\mathrm{Ext}^i_R(M,-).$$

Indeed, this follows by applying Theorem 10.1.19 to extend to a morphism of δ-functors the natural transformation

$$\mathrm{Hom}_R(M',-)\to\mathrm{Hom}_R(M,-)$$

given by composition with u.

On the other hand, for every R-module N, we may consider the left exact contravariant functor $\mathrm{Hom}_R(-,N)$ from R-mod to itself. If we consider this as a left-exact covariant functor from the dual category $(R\text{-mod})^\circ$ to R-mod, we may apply the considerations in §10.1. Note that $(R\text{-mod})^\circ$ is an Abelian category with enough injectives since R-mod has enough projectives. We thus obtain derived functors that we temporarily denote $\{\overline{\mathrm{Ext}}^i_R(-,N)\}_{i\geq 0}$, that we still consider as contravariant functors $R\text{-mod} \to R\text{-mod}$. We now translate the definition in our context.

Definition 10.7.1. Given an R-module M, a *projective* (*free*) resolution of M is a complex of projective (respectively, free) modules

$$F_\bullet : \ldots \to F_m \to \ldots \to F_1 \to F_0,$$

together with a morphism $F_0 \to M$ such that we get a morphism of complexes $F_\bullet \to M$ (where we consider M as a complex concentrated in degree 0) that induces an isomorphism in cohomology. Note that we use lower indexing for projective resolutions.

It follows from the definition of derived functors that the following hold:

i) We have a functorial isomorphism $\overline{\mathrm{Ext}}^0_R(-,N) \simeq \mathrm{Hom}_R(-,N)$.
ii) For every projective R-module P, we have $\overline{\mathrm{Ext}}^i_R(P,N) = 0$ for $i \geq 1$.
iii) Given a short exact sequence of R-modules:

$$0 \to M' \to M \to M'' \to 0,$$

we have an associated functorial long exact sequence:

$$\ldots \to \overline{\mathrm{Ext}}^i_R(M'',N) \to \overline{\mathrm{Ext}}^i_R(M,N) \to \overline{\mathrm{Ext}}^i_R(M',N) \to \overline{\mathrm{Ext}}^{i+1}_R(M'',N) \to \ldots.$$

These derived functors are computed as follows: given a projective resolution $F_\bullet \to M$, we have

$$\overline{\mathrm{Ext}}^i(M,N) \simeq \mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet,N).$$

(note that for $\mathrm{Hom}_R(F_\bullet,N)$ we use the upper indexing and that in the i^{th} spot we have $\mathrm{Hom}_R(F_i,N)$). Again, a morphism of R-modules $N \to N'$ induces a natural transformation

$$\mathrm{Hom}_R(-,N) \to \mathrm{Hom}_R(-,N')$$

which then extends to natural transformations

$$\mathrm{Ext}^i_R(-,N) \to \mathrm{Ext}^i_R(-,N')$$

(in fact, to a morphism of δ-functors).

We first show that the two constructions of derived functors are canonically isomorphic. In what follows we will freely use this identification.

Proposition 10.7.2. *For every $i \geq 0$, we have an isomorphism*

$$\mathrm{Ext}^i_R(M,N) \simeq \overline{\mathrm{Ext}}^i_R(M,N)$$

which is functorial in both entries.

Proof. We need to show that if $F_\bullet \to M$ is a projective resolution, then we have a functorial isomorphism

$$\mathrm{Ext}^i_R(M,N) \simeq \mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet,N).$$

Let us consider the functors $\big(\mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet,-)\big)_{i\geq 0}$. Given a short exact sequence of R-modules

$$0 \to N' \to N \to N'' \to 0,$$

we have a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F_\bullet, N') \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N'') \to 0$$

(note that since each F_i is projective, $\mathrm{Hom}_R(F_i,-)$ is an exact functor). By taking the long exact sequence of cohomology groups, it follows that the functors $\big(\mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet,-)\big)_{i\geq 0}$ form a δ-functor.

By left exactness of $\mathrm{Hom}_R(-,N)$, we see that we have a functorial isomorphism

$$\mathcal{H}^0\big(\mathrm{Hom}_R(F_\bullet, N) \simeq \mathrm{Hom}_R(M,N).$$

Note also that if N is injective, then the functor $\mathrm{Hom}_R(-,N)$ is exact, hence

$$\mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet, N) = 0 \quad \text{for all} \quad i \geq 1.$$

We thus deduce that the δ-functors $\big(\mathcal{H}^i\big(\mathrm{Hom}_R(F_\bullet,-)\big)_{i\geq 0}$ and $\big(\mathrm{Ext}^i_R(M,-)\big)_{i\geq 0}$ are isomorphic. $\square$

Corollary 10.7.3. *If R is Noetherian and M and N are finitely generated R-modules, then $\mathrm{Ext}^i_R(M,N)$ is finitely generated for every $i \geq 0$. Moreover, for every multiplicative system S in R and every $i \geq 0$, we have a functorial isomorphism*

$$S^{-1}\mathrm{Ext}^i_R(M,N) \simeq \mathrm{Hom}^i_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Proof. Indeed, since M is finitely generated over a Noetherian ring, we can construct a free resolution $F_\bullet$ of M, with all F_i finitely generated. In this case, every R-module $\mathrm{Hom}_R(F_i, N)$ is finitely generated and thus every cohomology group of $\mathrm{Hom}_R(F_\bullet, N)$ is a finitely generated R-module. This implies the first assertion by the proposition.

Moreover, using the fact that $S^{-1}R$ is a flat R-module and all F_i are finitely generated R-modules, we see that we have isomorphisms

$$S^{-1}\mathcal{H}^i\big(\mathrm{Hom}(F_\bullet, N)\big) \simeq \mathcal{H}^i\big(S^{-1}\mathrm{Hom}_R(F_\bullet, N)\big) \simeq \mathcal{H}^i\big(\mathrm{Hom}_{S^{-1}R}(S^{-1}F_\bullet, S^{-1}N)\big).$$

Since $S^{-1}F_\bullet$ is a free resolution of the $S^{-1}R$-module $S^{-1}M$, we obtain the second assertion in the proposition. $\square$

Example 10.7.4. If $\phi\colon N \to N$ is given by multiplication with an element $a \in R$, then for every R-module M, the induced map $\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N)$ is given again by multiplication by a. Indeed, if $N \to I^\bullet$ is an injective resolution of N, then we have a commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\phi"] & I^\bullet \arrow[d, "\psi"'] \\ N \arrow[r] & I^\bullet, \end{tikzcd}$$

where ψ is given by multiplication by a. It follows that the induced map

$$\mathcal{H}^i\big(\mathrm{Hom}_R(M, I^\bullet)\big) \to \mathcal{H}^i\big(\mathrm{Hom}_R(M, I^\bullet)\big)$$

is again given by multiplication by a.

Similarly, if $\phi\colon M \to M$ is given by multiplication by $a \in R$, it follows from Proposition 10.7.2, using an analogous argument with the one above, that for every R-module N, the induced morphism $\mathrm{Ext}^i(M,N) \to \mathrm{Ext}^i_R(M,N)$ is given by multiplication by a.

DEFINITION 10.7.5. If M is an R-module, the *projective dimension* of M, denoted $\mathrm{pd}_R(M)$ is the smallest non-negative integer n such that M has a projective resolution

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$$

(with the convention that if no such resolution exists, then $\mathrm{pd}_R(M) = \infty$). Note that we have $\mathrm{pd}_R(M) = 0$ if and only if M is projective.

PROPOSITION 10.7.6. *For every R-module M, the following are equivalent:*

i) $\mathrm{pd}_R(M) \leq d$.
ii) *We have* $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i > d$ *and all R-modules N.*
iii) *We have* $\mathrm{Ext}^{d+1}_R(M,N) = 0$ *for all R-modules N.*
iv) *For every exact complex*

$$F_{d-1} \xrightarrow{\phi} F_{d-2} \to \ldots \to F_1 \to F_0 \to M \to 0,$$

with $F_0, \ldots, F_{d-1}$ projective, the R-module $\ker(\phi)$ *is projective.*

PROOF. Consider an exact complex as in iv) and let $Q = \ker(\phi)$. Note that by Proposition 10.7.2, we have $\mathrm{Ext}^i_R(F,N) = 0$ for every $i \geq 1$, and every R-modules F and N, with F projective. By breaking the complex in iv) into short exact sequences and using the long exact sequences for the functors $\mathrm{Ext}^i_R(-,N)$, we see that for every $i \geq d+1$ and every R-module N, we have

$$\mathrm{Ext}^i_R(M,N) \simeq \mathrm{Ext}^{i-d}_R(Q,N).$$

If i) holds, then we have such a complex with Q projective, and we get ii).

Since the implications ii)$\Rightarrow$iii) and iv)$\Rightarrow$i) are trivial, we only need to show iii)$\Rightarrow$iv). By the above, it is enough to show that if $\mathrm{Ext}^1_R(Q,N) = 0$ for all R-modules N, then Q is projective. Consider a short exact sequence

$$0 \to G \to F \to Q \to 0, \tag{10.7.1}$$

with F projective. We obtain an exact sequence

$$0 \to \mathrm{Hom}_R(Q,G) \to \mathrm{Hom}_R(F,G) \to \mathrm{Hom}_R(G,G) \to \mathrm{Ext}^1_R(Q,G) = 0,$$

which implies that (10.7.1) is split, hence Q is projective. □

PROPOSITION 10.7.7. *Given a short exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

the following hold:

i) $\mathrm{pd}_R(M) \leq \max\{\mathrm{pd}_R(M'), \mathrm{pd}_R(M'')\}$.
ii) $\mathrm{pd}_R(M') \leq \max\{\mathrm{pd}_R(M), \mathrm{pd}_R(M'') - 1\}$.
iii) $\mathrm{pd}_R(M'') \leq \max\{\mathrm{pd}_R(M), \mathrm{pd}_R(M') + 1\}$.

PROOF. All assertions follow immediately from Proposition 10.7.6 by considering the long exact sequence for the functors $\mathrm{Ext}^i_R(-,N)$. □

DEFINITION 10.7.8. The *global projective dimension* gl-dim(R) of R is equal to $\sup_M \mathrm{pd}_R(M)$, where M runs over all finitely generated[7] R-modules. By Proposition 10.7.6, this is the smallest integer d such that $\mathrm{Ext}^i_R(M,N)$ for all $i > d$ and all R-modules M and N, with M finitely generated (it is equal to ∞ if no such d exists).

10.7.2. Local Ext and Global Ext for sheaves. Suppose now that $(X, \mathcal{O}_X)$ is a ringed space. Given any $\mathcal{O}_X$-module $\mathcal{F}$, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$, defined on the category of $\mathcal{O}_X$-modules, with values in the category of $\Gamma(X, \mathcal{O}_X)$-modules, is left exact. Its right derived functors are denoted by $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$, for $i \geq 0$. Similarly, we have the left exact functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ from the category of $\mathcal{O}_X$-modules to itself. Its right derived functors are denoted by $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)$. Note that every morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{F}'$ induces natural transformations

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', -) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) \quad \text{and} \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}', -) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$$

and thus, via Theorem 10.1.19, natural transformations

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}', -) \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -) \quad \text{and} \quad \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}', -) \to \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)$$

for all $i \geq 0$. It is common to refer to $\mathrm{Ext}^i_{\mathcal{O}_X}(-,-)$ as the *global Ext* functors and to $\mathcal{E}xt^i(-,-)$ as the *local Ext* functors.

EXAMPLE 10.7.9. For every $\mathcal{O}_X$-module $\mathcal{G}$, we have functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq \Gamma(X, \mathcal{G}) \quad \text{and} \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq \mathcal{G}.$$

It follows that for every $i \geq 0$, we have an isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq H^i(X, \mathcal{G}),$$

while $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ for $i \geq 1$.

EXAMPLE 10.7.10. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then for every $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have functorial isomorphisms

(10.7.2)
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})\big) \simeq \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}\big),$$

where the first isomorphism is given by Exercise 8.2.15 and the second one by Remark 8.5.16. Note also that we have a functorial morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \tag{10.7.3}$$

and this is an isomorphism since $\mathcal{E}$ is locally free.

By taking global sections in (10.7.2), we get a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}\big). \tag{10.7.4}$$

Suppose now that $\mathcal{G} \to \mathcal{I}^\bullet$ is an injective resolution. Since tensoring with a vector bundle is an exact functor, we deduce using (10.7.4) that each $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^j$ is an injective $\mathcal{O}_X$-module and then that $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$ is an injective resolution. Using the isomorphisms in (10.7.2), (10.7.3), and (10.7.4), we see that we have functorial isomorphisms

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \simeq \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) \simeq \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$$

[7] One can drop the finite generation requirement and the invariant does not change, but we will not need this fact.

and

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{G})\simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{E}^\vee\otimes_{\mathcal{O}_X}\mathcal{G}).$$

PROPOSITION 10.7.11. *For every $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, every open subset U of X, and every $i\geq 0$, we have a functorial isomorphism*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U\simeq \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U).$$

This will follow from the following lemma.

LEMMA 10.7.12. *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module and $U\subseteq X$ is an open subset of X, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*

PROOF. Let $i\colon U\hookrightarrow X$ be the inclusion map. We make use of the functor $i_!$ in Exercise 10.2.6. The assertion follows from the fact that for every $\mathcal{O}_U$-module we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G},\mathcal{I}|_U)\simeq \mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{G}),\mathcal{I}\big)$$

and the fact that $i_!$ is exact. $\square$

PROOF OF PROPOSITION 10.7.11. If $\mathcal{G}\to\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{G}$, then it follows from the lemma that $\mathcal{G}|_U\to\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{G}|_U$. Since by definition we have

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}^\bullet)|_U\simeq \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{I}^\bullet|_U),$$

the assertion in the proposition is an immediate consequence. $\square$

PROPOSITION 10.7.13. *For every $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have a spectral sequence*

$$E_2^{p,q}=H^p\big(X,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\big)\Rightarrow_p \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}).$$

PROOF. Note that by definition we have

$$\Gamma(-)\circ\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-)=\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-).$$

The assertion in the proposition thus follows from Theorem 10.5.11 if we show that for an injective $\mathcal{O}_X$-module $\mathcal{I}$, the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})$ is flasque. We thus need to show that for every open subset U of X, the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})\to\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{I}|_U)$$

induced by restriction is surjective. We make again use of the functor $i_!$, where $i\colon U\hookrightarrow X$ is the inclusion map (see Exercise 10.2.6). Given a morphism $\phi\colon\mathcal{F}|_U\to\mathcal{I}|_U$, using the fact that $\mathcal{I}$ is injective, we obtain a morphism $\psi\colon\mathcal{F}\to\mathcal{I}$ that makes the diagram

$$\begin{tikzcd} i_!(\mathcal{F}|_U) \arrow[r, "i_!(\phi)"] \arrow[d, "\alpha"] & i_!(\mathcal{I}|_U) \arrow[d, "\beta"'] \\ \mathcal{F} \arrow[r, "\psi"] & \mathcal{I} \end{tikzcd}$$

commutative (note that the vertical maps α and β are injective). It is then clear that $\psi|_U=\phi$. $\square$

We next show that as in the case of modules over a ring, we have long exact sequences for the Ext and $\mathcal{E}xt$ functors with respect to the first argument.

PROPOSITION 10.7.14. *Given a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

for every $\mathcal{O}_X$-module $\mathcal{G}$, we have long exact sequences

$$\ldots \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^i(\mathcal{F}', \mathcal{G}) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \ldots$$

and

$$\ldots \to \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^i(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^{i+1}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \ldots.$$

PROOF. Let $\mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$. Note that since each $\mathcal{I}^j$ is an injective $\mathcal{O}_X$-module, we obtain a short exact sequence of complexes of $\mathcal{O}_X(X)$-modules:

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^\bullet) \to 0.$$

Similarly, we have a short exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^\bullet) \to 0$$

(exactness here follows by taking sections over open subsets U of X and using the fact that by Lemma 10.7.12, each $\mathcal{I}^j|_U$ is an injective $\mathcal{O}_U$-module). By taking the long exact sequences in cohomology for the above short exact sequences of complexes, we obtain the assertion in the proposition. □

Unlike the category of modules over a ring, the category of $\mathcal{O}_X$-modules does not have enough projective objects. However, in many examples, there are locally free resolutions: such a resolution of the coherent sheaf $\mathcal{F}$ is a (possibly infinite) complex

$$\mathcal{E}_\bullet : \quad \ldots \to \mathcal{E}_m \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0,$$

where each $\mathcal{E}_i$ is a locally free sheaf on X, together with a morphism $\mathcal{E}_0 \to \mathcal{F}$ such that we have an exact complex

$$\ldots \to \mathcal{E}_m \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

For example, we will see in Remark 11.6.11 that such resolutions exist on quasi-projective varieties. The next proposition shows that we can use such resolutions to compute the local Ext functors.

PROPOSITION 10.7.15. *If the coherent sheaf $\mathcal{F}$ has a locally free resolution $\mathcal{E}_\bullet$, then for every $\mathcal{O}_X$-module $\mathcal{G}$ and every $i \geq 0$ we have a functorial isomorphism*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}^i\big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_\bullet, \mathcal{G})\big).$$

PROOF. The proof is entirely analogous to that of Proposition 10.7.2, so we leave it as an exercise for the reader. □

We now specialize to the case when X is an algebraic variety.

PROPOSITION 10.7.16. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on the algebraic variety X, then the $\mathcal{O}_X$-modules $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent. Moreover, if U is an affine open subset in X, then*

$$\Gamma\big(U, \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big) \simeq \mathrm{Ext}^i_{\mathcal{O}_X(U)}\big(\mathcal{F}(U), \mathcal{G}(U)\big)$$

and for every $x \in X$, we have

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \simeq \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

PROOF. By Proposition 10.7.11, we may assume that X is an affine variety, with $R = \mathcal{O}_X(U)$. By assumption, we have $\mathcal{F} \simeq \widetilde{M}$ and $\mathcal{G} \simeq \widetilde{N}$, for some finitely generated R-modules M and N. If we consider a free resolution $F_\bullet \to M$, with all F_i finitely generated R-modules, then we have a corresponding complex of $\mathcal{O}_X$-modules $\widetilde{F}_\bullet \to \widetilde{M}$. It is now straightforward to deduce from Propositions 10.7.2 and 10.7.15 that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is isomorphic to the coherent $\mathcal{O}_X$-module associated to $\mathrm{Ext}^i_R(M, N)$. Finally, the last assertion in the statement follows from Corollary 10.7.3. □

REMARK 10.7.17. If X is an affine algebraic variety and $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then for every $n \geq 0$, we have a canonical isomorphism

$$\mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^n_{\mathcal{O}_X(X)}\big(\mathcal{F}(X), \mathcal{G}(X)\big). \tag{10.7.5}$$

Indeed, since X is affine and the sheaves $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent by the previous proposition, the spectral sequence

$$E_2^{p,q} = H^p\big(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})\big) \Rightarrow_p \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

given by Proposition 10.7.13 satisfies $E_2^{p,q} = 0$ unless $p = 0$. We thus obtain

$$\mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \Gamma\big(X, \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big) \quad \text{for all} \quad n \geq 0,$$

and (10.7.5) follows from the previous proposition.

10.7.3. Tor modules. Suppose now that R is a commutative ring. For every R-module M, the functor $M \otimes_R -$ is right exact. Since the category of R-modules has enough projectives, we can construct the left derived functors of this functor via a dual version of Theorem 10.1.19 (in other words, we apply the theorem for the corresponding left exact functor between the dual Abelian categories). The i^{th} left derived functor is denoted $\mathrm{Tor}_i^R(M, -)$. By construction, if $F_\bullet \to N$ is a projective resolution of N, then

$$\mathrm{Tor}_i^R(M, -) \simeq \mathcal{H}_i(M \otimes_R F_\bullet).$$

It follows from the definition of derived functors that the following hold:

i) We have a functorial isomorphism $\mathrm{Tor}_0^R(M, -) \simeq M \otimes_R -$.
ii) For every projective R-module P, we have $\mathrm{Tor}_i^R(M, P) = 0$ for $i \geq 1$.
iii) Given a short exact sequence of R-modules

$$0 \to N' \to N \to N'' \to 0,$$

we have an associated functorial long exact sequence:

$$\ldots \to \mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^R(M, N'') \to \mathrm{Tor}_{i-1}^R(M', N) \to \ldots.$$

Given a morphism $M \to M'$, we have a functorial transformation $M \otimes_R - \to M' \otimes_R -$. Theorem 10.1.19 thus implies that we have natural transformations

$$\mathrm{Tor}_i^R(M, -) \to \mathrm{Tor}_i^R(M', -) \quad \text{for all} \quad i \geq 0.$$

As in the case of the Ext module, we could proceed alternatively by constructing the derived functors with respect to the second variable. The next proposition shows that we obtain the same result. In what follows we will freely use this result.

PROPOSITION 10.7.18. *For every R-modules M and N and every $i \geq 0$, we have a functorial isomorphism, in both variables*

$$\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^R(N, M);$$

equivalently, if $G_\bullet \to M$ is a projective resolution of M, then we have functorial isomorphisms

$$\mathrm{Tor}_i^R(M,N) \simeq \mathcal{H}_i(N \otimes_R G_\bullet).$$

PROOF. The argument is entirely analogous to that of Proposition 10.7.2, so we leave it as an exercise for the reader. $\square$

EXAMPLE 10.7.19. Arguing as in Example 10.7.4, we see that if the morphism $\phi\colon N \to N$ is given by multiplication with an element $a \in R$, then for every R-module M, the induced morphism $\mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M,N)$ is also given by multiplication with a. A similar assertion holds if we start with multiplication by a on M.

PROPOSITION 10.7.20. *Let M and N be R-modules.*

i) *For every multiplicative system S in R, we have functorial isomorphisms*

$$\mathrm{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N) \simeq S^{-1}\mathrm{Tor}_i^R(M,N).$$

ii) *If R is Noetherian and M and N are finitely generated R-modules, then $\mathrm{Tor}_i^R(M,N)$ is a finitely generated R-module for all $i \geq 0$.*

PROOF. Let $F_\bullet \to N$ be a free resolution of N. Using the fact that $S^{-1}(R)$ is a flat R-module, we see that $S^{-1}F_\bullet \to S^{-1}N$ is a free resolution of $S^{-1}N$ over $S^{-1}R$. The fact that tensor product commutes with localization gives, for every $i \geq 0$, a functorial isomorphism

$$\mathrm{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N) \simeq \mathcal{H}_i(S^{-1}M \otimes_{S^{-1}R} S^{-1}F_\bullet)$$

$$\simeq \mathcal{H}_i\big(S^{-1}R \otimes_R (M \otimes_R F_\bullet)\big) \simeq S^{-1}\mathcal{H}_i(M \otimes_R F_\bullet) \simeq S^{-1}\mathrm{Tor}_i^R(M,N).$$

Under the assumptions in ii), we may assume that F_j is a finitely generated R-module for all j. In this case every $M \otimes_R F_j$ is a finitely generated R-module, and since R is Noetherian, we conclude that

$$\mathrm{Tor}_i^R(M,N) \simeq \mathcal{H}_i(M \otimes_R F_\bullet)$$

is a finitely generated R-module. $\square$

The Tor modules can be used to characterize flatness, as follows.

PROPOSITION 10.7.21. *Given an R-module M, the following are equivalent:*

i) *M is a flat R-module.*
ii) *We have $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$ and all R-modules N.*
iii) *We have $\mathrm{Tor}_1^R(M,N) = 0$ for all R-modules N.*

PROOF. Suppose first that M is flat over R. Given an R-module N, is $F_\bullet$ is a free resolution of N, then $\mathcal{H}_i(M \otimes_R F_\bullet) = 0$ for all $i \geq 1$ by the flatness of M. We thus have i)$\Rightarrow$ii).

Since ii)$\Rightarrow$iii) is trivial, in order to finish the proof, it is enough to show iii)$\Rightarrow$i). Given a short exact sequence of R-modules,

$$0 \to N' \to N \to N'' \to 0,$$

the corresponding long exact sequence gives

$$0 = \mathrm{Tor}_1^R(M,N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0.$$

This implies that M is flat over R. $\square$

One can use the above characterization of flat modules to prove some basic properties of flat modules. We only give two examples.

COROLLARY 10.7.22. *Given a short exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

the following hold:

i) *If M' and M'' are flat, then M is flat.*
ii) *If M and M'' are flat, then M' is flat.*

PROOF. Given an R-module N, the long exact sequence for Tor modules gives an exact complex

$$\mathrm{Tor}_2^R(M'', N) \to \mathrm{Tor}_1^R(M', N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N).$$

The assertions in the proposition now follow from the characterization of flatness in Proposition 10.7.21. $\square$

COROLLARY 10.7.23. *Given a short exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

with M'' flat, for every R-module N, the sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact.

PROOF. The long exact sequence for the Tor modules gives an exact complex

$$\mathrm{Tor}_1^R(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

Since M'' is flat, it follows from Proposition 10.7.21 that $\mathrm{Tor}_1^R(M'', N) = 0$, which gives the assertion in the corollary. $\square$

COROLLARY 10.7.24. *If $(R, \mathfrak{m})$ is a Noetherian local ring, then every finitely generated flat R-module M is free.*

PROOF. The argument is the same as the one in the proof of Proposition C.2.1, which treats the case when M is projective. We have a short exact sequence

$$0 \to N \to F \to M \to 0,$$

where F is a finitely generated free R-module and $N \subseteq \mathfrak{m}F$. If we tensor this with $R/\mathfrak{m}$, it follows from Corollary 10.7.23 that we have a short exact sequence

$$0 \to N/\mathfrak{m}N \to F/\mathfrak{m}F \to M/\mathfrak{m}M \to 0.$$

Since $N \subseteq \mathfrak{m}F$, this implies that $N = \mathfrak{m}N$, hence $N = 0$ by Nakayama's lemma. This shows that $M \simeq F$ is free. $\square$

PROPOSITION 10.7.25. *If N is an R-module with $\mathrm{pd}_R(N) = q$, then we have $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > q$ and all R-modules M.*

PROOF. The assertion follows by computing $\mathrm{Tor}_i^R(M, N)$ using a projective resolution of N of length q. $\square$

There is a version of the Tor construction for $\mathcal{O}_X$-modules, but we do not discuss it here, since we will not need it.

CHAPTER 11

Coherent sheaves and cohomology on projective varieties

In this chapter we prove the fundamental finiteness results about the cohomology of projective and, more generally, complete varieties. In the first section we describe the quasi-coherent sheaves on a projective variety in terms of graded modules over the corresponding graded ring. In the second section we compute the cohomology of line bundles on the projective space and use this this to deduce general properties of cohomology on projective varieties. Building on this, in the third section we show that the higher direct images of coherent sheaves via proper maps are coherent. The next section treats an important invariant of coherent sheaves on projective spaces: the Hilbert polynomial. The fifth and the sixth section are devoted to morphisms to the projective space and to the connection between ample line bundles and very ample line bundles over affine varieties. Finally, in the last section we discuss the relative version of ampleness.

11.1. Coherent sheaves on projective varieties

Our goal in this section is to describe quasi-coherent sheaves on projective varieties in terms of graded modules over the homogeneous coordinate ring. In fact, it is convenient to work in the more general setting of varieties that are projective over an affine variety.

We fix the following notation. Let $S=\bigoplus_{i\geq 0}S_i$ be a reduced $\mathbf{N}$-graded k-algebra. We assume that S_0 is a finitely generated k-algebra and that S is generated as an S_0-algebra by finitely many elements in S_1 (hence, in particular, S is a finitely generated k-algebra). We put $A=S_0$, $Y=\mathrm{MaxSpec}(A)$ and $X=\mathrm{MaxProj}(S)$, so that we have a canonical morphism $\pi\colon X\to Y$. By assumption, there is a graded surjective k-algebra homomorphism $A[x_0,\ldots,x_n]\to S$, that induces a closed immersion $j\colon X\hookrightarrow \mathbf{P}^n_Y=Y\times\mathbf{P}^n$. We will be describing the quasi-coherent sheaves on X in terms of graded modules over S.

Definition 11.1.1. A *graded module* over S is a module M over S, together with a decomposition

$$M=\bigoplus_{i\in\mathbf{Z}}M_i,$$

such that $S_i\cdot M_j\subseteq M_{i+j}$ for every $i,j\in\mathbf{Z}$. An element $u\in M_i$ is *homogeneous of degree* i. If M and N are graded modules over S, a morphism of graded modules $\phi\colon M\to N$ is a morphism of S-modules such that $\phi(M_i)\subseteq N_i$ for all $i\in\mathbf{Z}$. We denote the Abelian group of graded such morphisms by $\mathrm{Hom}_{S-\mathrm{gr}}(M,N)$.

Definition 11.1.2. If M is a graded S-module, a *graded submodule* of S is a submodule N of M that is generated by homogeneous elements of M; equivalently, the decomposition of M induces a decomposition

$$N = \bigoplus_{i \in \mathbf{Z}} (N \cap M_i).$$

In particular, if we put $N_i = N \cap M_i$, then N is a graded module such that the inclusion map $N \to M$ is a morphism of graded modules. It is clear that in this case we get a decomposition

$$M/N = \bigoplus_{i \in \mathbf{Z}} M_i/N_i,$$

which makes M/N a graded module such that the projection map $M \to M/N$ is a morphism of graded modules.

REMARK 11.1.3. Note that a graded submodule of S is the same as a homogeneous ideal.

REMARK 11.1.4. It is clear that the composition of morphisms of graded modules is again a morphism of graded modules. In this way, graded S-modules form a category. It is easy to check that if $\phi\colon M \to N$ is a morphism of graded modules, then $\ker(\phi) \subseteq M$ and $\operatorname{Im}(\phi) \subseteq N$ are graded submodules, the kernel and image in the category of graded S modules, which is an Abelian category.

EXAMPLE 11.1.5. If M is a graded S-module, then for every $j \in \mathbf{Z}$, we define a new graded S-module $M(j)$, which is equal to M, as an S-module, but such that $M(j)_i = M_{i+j}$ for all $i \in \mathbf{Z}$.

We now define a functor from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules. Recall that a basis of open subsets of X is given by those of the form $D_X^+(f)$, where $f \in S$ is homogeneous, with $\deg(f) > 0$. Each such subset is affine and

$$\Gamma\big(D_X^+(f), \mathcal{O}_X\big) \simeq S_{(f)}$$

(see Propositions 4.3.15 and 4.3.16). Given any such f, say of degree d, we consider the graded S_f-module M_f, where

$$(M_f)_i = \left\{ \frac{u}{f^q} \mid q \geq 0, u \in M_{qd} \right\}.$$

It is clear that $M_{(f)} := (M_f)_0$ is a module over $S_{(f)}$. Note that if $D_X^+(f) = D_X^+(g)$ for homogeneous elements $f, g \in S$, of positive degree, then the ideals (f) and (g) have the same radical (see Proposition 4.3.8), and in this case it is easy to see that we have a graded isomorphism $M_f \simeq M_g$ compatible with the isomorphism of S-algebras $S_f \simeq S_g$.

More generally, if $D_X^+(f) \subseteq D_X^+(g)$, then Proposition 4.3.8 implies that f lies in the radical of (g), in which case we have a morphism of graded S-modules $M_g \to M_f$. This induces a morphism $M_{(g)} \to M_{(f)}$ compatible with the restriction homomorphisms

$$\Gamma\big(D_X^+(g), \mathcal{O}_X\big) \simeq S_{(g)} \to S_{(f)} \simeq \Gamma\big(D_X^+(f), S_{(f)}\big).$$

LEMMA 11.1.6. *Given homogeneous elements of positive degree $f, f_1, \ldots, f_r \in S$, such that $D_X^+(f) = \bigcup_{i=1}^r D_X^+(f_i)$, for every graded S-module M, the localization* *maps induce an exact sequence:*

$$0 \to M_{(f)} \to \bigoplus_{i=1}^{r} M_{(f_i)} \to \bigoplus_{i,j=1}^{r} M_{(f_if_j)}.$$

PROOF. The following is a graded variant of the argument in the proof of Lemma 8.3.2 (cf. also the proof of Proposition 4.3.16). Note first that the condition on $f, f_1, \ldots, f_r$ is equivalent to the fact that the ideals (f) and $(f_1, \ldots, f_r)$ have the same radical. Let $d = \deg(f) > 0$ and $d_i = \deg(f_i) > 0$.

We first show that the morphism $M_{(f)} \to \bigoplus_{i=1}^{r} M_{(f_i)}$ is injective. If $\frac{u}{f^j}$ lies in the kernel, then for $N \gg 0$, we have $f_j^N \cdot u = 0$ for all j. Since f lies in the radical of $(f_1, \ldots, f_r)$, it follows that $f^{N'}u = 0$ for some N', hence $\frac{u}{f^j} = 0$.

We next show that if we have $\frac{u_i}{f_i^{m_i}} \in M_{(f_i)}$ (hence $u_i \in M_{m_id_i}$) such that for all i and j, we have

$$\frac{u_i}{f_i^{m_i}} = \frac{u_j}{f_j^{m_j}} \in M_{(f_if_j)},$$

then there is $\frac{u}{f^m} \in M_{(f)}$ such that $\frac{u}{f^m} = \frac{u_i}{f_i^{m_i}}$ for all i. The hypothesis implies that if $N \gg 0$, then

$$(f_if_j)^N (f_j^{m_j} u_i - f_i^{m_i} u_j) = 0 \quad \text{for all} \quad i, j.$$

If we replace each $\frac{u_i}{f_i^{m_i}}$ by $\frac{u_i f_i^N}{f_i^{m_i+N}}$, then we see that we may assume that $f_j^{m_j} u_i = f_i^{m_i} u_j$ for all i and j. Moreover, we may replace each f_j by $f_j^{m_j}$ and thus assume that $m_i = 1$ for all i. By hypothesis, f lies in the radical of $(f_1, \ldots, f_r)$, hence we can write

$$f^s = \sum_{i=1}^{r} h_i f_i$$

for some $s \geq 1$ and some $h_i \in S_{sd-d_i}$.

Note that

$$\deg(u_i) + \deg(h_i) = d_i + (sd - d_i) = sd \quad \text{for} \quad 1 \leq i \leq r.$$

In order to complete the proof, it is enough to show that if we put $u = \sum_{i=1}^{r} h_i u_i \in M_{sd}$, then $\frac{u}{f^s} = \frac{u_j}{f_j}$ in $M_{(f_j)}$ for all j. Indeed, we have

$$f_j u = \sum_{i=1}^{r} h_i f_j u_i = \sum_{i=1}^{r} h_i f_i u_j = f^s u_j.$$

$\square$

By applying Proposition 8.3.1, the above lemma implies that we have an $\mathcal{O}_X$-module $\widetilde{M}$ on X, unique up to a canonical isomorphism, such that over each $D_X^+(f)$, where f is homogeneous, of positive degree, we have an isomorphism

$$\Gamma\big(D_X^+(f), \widetilde{M}\big) \simeq M_{(f)},$$

and these isomorphisms are compatible with the restriction maps.

This construction is functorial: if $M \to N$ is a morphism of graded S-modules, then for every f as above, we have a morphism of $S_{(f)}$-modules $M_{(f)} \to N_{(f)}$, and these morphisms are compatible with the restriction morphisms. We thus obtain a morphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$ via Proposition 8.3.1. We get in this way a functor from the category of graded S-modules to the category of $\mathcal{O}_X$-modules.

The functor that maps M to $\widetilde{M}$ is exact. Indeed, since the open subsets $D_X^+(f)$ form a basis of open subsets, it is enough to show that the functor that maps a graded S-module M to $M_{(f)}$ is exact: this is clear.

We also note that the functor commutes with arbitrary direct sums: given a family $(M_i)_{i\in I}$ of graded S-modules, the $\mathcal{O}_X$-module associated to $\bigoplus_{i\in I} M_i$ is canonically isomorphic to $\bigoplus_{i\in I} \widetilde{M_i}$. This is an immediate consequence of the fact that localization commutes with arbitrary direct sums.

EXAMPLE 11.1.7. We have $\widetilde{S} \simeq \mathcal{O}_X$.

EXAMPLE 11.1.8. For every $m \in \mathbf{Z}$, we put

$$\mathcal{O}_X(m) := \widetilde{S(m)}.$$

This is a line bundle. In order to see this, we use the fact that X is a union of affine open subsets of the form $D_X^+(f)$, where f is homogeneous, of degree 1 (we recall that this is a consequence of the fact that S is generated by S_1 as an S_0-algebra). Note that we have a basis of open subsets of $D_X^+(f)$ given by subsets of the form $D_X^+(f) \cap D_X^+(g) = D_X^+(fg)$, where $g \in S$ is homogeneous, and for every such f and g, we have an isomorphism

$$S(m)_{(fg)} = (S_{fg})_m \simeq S_{(fg)}, \quad u \to f^{-m} \cdot u.$$

Moreover, when we vary g, these isomorphisms are compatible with the restriction maps, giving an isomorphism

$$\mathcal{O}_X(m)|_{D_X^+(f)} \simeq \mathcal{O}_{D_X^+(f)}.$$

For every $\mathcal{O}_X$-module $\mathcal{F}$, we put $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$.

REMARK 11.1.9. When $X = \mathbf{P}^n$, the definition of $\mathcal{O}_{\mathbf{P}^n}(m)$ given in this section agrees with the one in Example 8.6.14. Indeed, if $U_i = D_{\mathbf{P}^n}^+(x_i)$, then we have seen in Example 11.1.8 that the isomorphism

$$\phi_i \colon \mathcal{O}_{\mathbf{P}^n}(m)|_{U_i} \to \mathcal{O}_{U_i}$$

is given by multiplication by x_i^{-m}. Therefore $\phi_i \circ \phi_j^{-1}$ is given by $\left(\frac{x_j}{x_i}\right)^m$, hence the transition functions are the same as for the line bundle $\mathcal{O}_{\mathbf{P}^n}(m)$ in Example 8.6.14.

PROPOSITION 11.1.10. *For every graded S-module M, the $\mathcal{O}_X$-module $\widetilde{M}$ is quasi-coherent. Moreover, if M is finitely generated, then $\widetilde{M}$ is coherent.*

PROOF. We can choose a set of homogeneous generators $(u_i)_{i\in I}$ of M, with $\deg(u_i) = a_i$, and consider the surjective morphism of graded S-modules

$$\phi\colon \bigoplus_{i\in I}^{r} S(-a_i) \to M, \quad e_i \to u_i.$$

We thus obtain a surjective morphism of $\mathcal{O}_X$-modules $\bigoplus_{i\in I} \mathcal{O}_X(-a_i) \to \widetilde{M}$. Applying the same argument for $\ker(\phi)$, we get an exact sequence

$$\bigoplus_{j\in J} \mathcal{O}_X(-b_j) \to \bigoplus_{i\in I} \mathcal{O}_X(-a_i) \to \widetilde{M} \to 0.$$

Therefore $\widetilde{M}$ is quasi-coherent, as the cokernel of a morphism of quasi-coherent sheaves. Moreover, if M is finitely generated, then we may choose I and J to be finite sets, hence $\widetilde{M}$ is coherent. $\square$

EXAMPLE 11.1.11. If $I \subseteq S$ is a homogeneous radical ideal, then $\widetilde{I}$ is a coherent ideal in $\mathcal{O}_X$; it is the radical ideal corresponding to the closed subset $V(I)$.

EXAMPLE 11.1.12. We note that for every graded S-module M and every $m \in \mathbf{Z}$, we have a canonical isomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \widetilde{M(m)}.$$

Indeed, if $U = D_X^+(f)$, where f is homogeneous, of degree 1, then the restriction $\big(\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)\big)|_U$ is the quasi-coherent sheaf associated to

$$(M_f)_0 \otimes_{(S_f)_0} (S_f)_m \simeq (M_f)_m, \quad u \otimes \frac{a}{f^s} \to \frac{au}{f^s}.$$

It is clear that these maps are compatible and thus give the desired isomorphism.

In particular, we see that for every $m, n \in \mathbf{Z}$, we have $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$. Therefore $\mathcal{O}_X(-1) \simeq \mathcal{O}_X(1)^{-1}$ and $\mathcal{O}_X(m) \simeq \mathcal{O}_X(1)^{\otimes m}$ for all $m > 0$.

REMARK 11.1.13. Recall that if $S^{(d)} = \bigoplus_{i \geq 0} S_{id}$, then we have a canonical isomorphism $g\colon \mathrm{MaxProj}(S) \to \mathrm{MaxProj}(S^{(d)})$ (see Exercise 4.3.22). It follows from the definition that we have a canonical isomorphism $g^*\big(\mathcal{O}(1)\big) \simeq \mathcal{O}(d)$.

REMARK 11.1.14. Let $p\colon T \to S$ be a surjective, graded k-algebra homomorphism, with kernel I, and consider the corresponding closed immersion $i\colon X \hookrightarrow Y = \mathrm{MaxProj}(T)$. It is easy to see that if N is a graded T-module and we take $M = N \otimes_T S \simeq N/IN$ (which is naturally a graded S-module), then we have an isomorphism

$$\widetilde{M} \simeq i^*(\widetilde{N}).$$

In particular, if $N = T(m)$, we have $M = S(m)$, and we get an isomorphism $i^*\big(\mathcal{O}_Y(m)\big) \simeq \mathcal{O}_X(m)$ for all $m \in \mathbf{Z}$.

This justifies the fact that if $X = \mathrm{MaxProj}(S)$ and Z is a closed subset of X, then we will denote $\mathcal{O}_X(m)|_Z$ by $\mathcal{O}_Z(m)$.

We now describe the modules that induce the zero sheaf. Recall that $S_+ = \bigoplus_{i>0} S_i$.

PROPOSITION 11.1.15. *If M is a graded S-module, then $\widetilde{M} = 0$ if and only if for every $u \in M$, there is N such that $S_+^N \cdot u = 0$.*

PROOF. Let $y_1, \ldots, y_r \in S_1$ generate S as an S_0-algebra. In this case, we have $S_+ = (y_1, \ldots, y_r)$ and $X = \bigcup_{i=1}^r D_X^+(y_i)$. Since $\widetilde{M}$ is quasi-coherent, we have $\widetilde{M} = 0$ if and only if $M_{(y_i)} = 0$ for all i. For every $m \in \mathbf{Z}$, multiplication by y_i^m gives an isomorphism $(M_{y_i})_0 \simeq (M_{y_i})_m$, hence $M_{(y_i)} = 0$ if and only if $M_{y_i} = 0$. This condition holds if and only if for every $u \in M$, we have $y_i^q \cdot u = 0$ for $q \gg 0$. The assertion in the proposition now follows from the fact that S_+ is generated as an ideal by $y_1, \ldots, y_r$. □

REMARK 11.1.16. Suppose that M is a finitely generated graded S-module. In this case the proposition immediately implies that $\widetilde{M} = 0$ if and only if there is q such that $S_+^q \cdot M = 0$. We claim that this is the case if and only if there is j_0 such that $M_j = 0$ for $j \geq j_0$. Indeed, if the latter condition holds, then it is clear that for every $u \in M_i$, if $q \geq \max\{0, j_0 - i\}$, then $S_+^q \cdot u = 0$. Conversely, suppose that $S_+^q \cdot M = 0$ and let $u_1, \ldots, u_r \in M$ be a system of homogeneous generators, with $\deg(u_i) = d_i$ for all i. Given any $u \in M_j$, we can write $u = \sum_{i=1}^r a_i u_i$, with $a_i \in S_{j-d_i}$. It follows that if $j \geq q + \max_i\{d_i\}$, then $M_j = 0$.

We now construct a functor going in the opposite direction, from $\mathcal{O}_X$-modules to graded S-modules. This maps $\mathcal{F}$ to

$$\Gamma_*(\mathcal{F}) := \bigoplus_{m\in\mathbf{Z}} \Gamma\big(X, \mathcal{F}(m)\big).$$

For now, this is a graded Abelian group. The definition is functorial: given a morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$, we get morphisms $\mathcal{F}(m) \to \mathcal{G}(m)$ for all $m \in \mathbf{Z}$, and thus a morphism of graded Abelian groups $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$.

For every graded S-module M, we have a functorial map

$$\Phi_M \colon M \to \Gamma_*(\widetilde{M})$$

that takes M_i to $\Gamma\big(X, \widetilde{M}(i)\big) = \Gamma\big(X, \widetilde{M(i)}\big)$, defined as follows. Given $u \in M_i$ and a homogeneous element f in S of positive degree, we consider the element of $\Gamma\big(D_X^+(f), \widetilde{M(i)}\big) = (M_f)_i$ given by $\frac{u}{1}$. It is straightforward to check that these sections glue to give $\Phi_M(u) \in \Gamma\big(X, \widetilde{M}(i)\big)$. In particular, we have a map

$$\phi_S \colon S \to \bigoplus_{m\in\mathbf{Z}} \Gamma\big(X, \mathcal{O}_X(m)\big).$$

For every $\mathcal{O}_X$-module $\mathcal{F}$, we have maps

$$\Gamma\big(X, \mathcal{O}_X(i)\big) \otimes_k \Gamma\big(X, \mathcal{F}(j)\big) \to \Gamma\big(X, \mathcal{F}(i+j)\big)$$

induced by tensor product of sections. For $\mathcal{F} = \mathcal{O}_X$, this makes $\Gamma_*(\mathcal{O}_X)$ a graded ring such that Φ_S is a morphism of graded rings. Moreover, for every $\mathcal{F}$, this makes $\Gamma_*(\mathcal{F})$ a graded module over $\Gamma_*(\mathcal{O}_X)$, and thus via Φ_S, a graded S-module. Note that Φ_M is then a morphism of graded S-modules. We thus see that $\Gamma_*(-)$ is a functor from the category of $\mathcal{O}_X$-modules to the category of graded S-modules.

Remark 11.1.17. Given $f \in S_m$, we have a corresponding section $s = \Phi_S(f) \in \Gamma\big(\mathcal{O}_X(m)\big)$. Note that a point $x \in X$ corresponding to the homogeneous prime ideal $\mathfrak{q}$ of S lies in the zero-locus of s if and only if $f \in \mathfrak{q}$, which is the case precisely when $x \in V(f)$.

Our next goal is to relate the two functors that we defined. As we will see, these give an "almost equivalence" between the category of graded S-modules and that of quasi-coherent $\mathcal{O}_X$-modules. Recall that $S_+ = \bigoplus_{i>0} S_i$.

Proposition 11.1.18. *If $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, then we have a functorial isomorphism*

$$\Psi_{\mathcal{F}} \colon \widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}.$$

We begin with the following general lemma, which describes the sections of coherent sheaves over complements of zero-loci of sections of line bundles. We will apply it in order to describe the sections of quasi-coherent sheaves on open subsets of the form $D_X^+(f)$.

Lemma 11.1.19. *Let Y be an arbitrary variety, $\mathcal{L}$ a line bundle on Y, and $s \in \Gamma(Y, \mathcal{L})$ a global section of $\mathcal{L}$. If $U = Y \smallsetminus V(s)$, then for every quasi-coherent sheaf $\mathcal{F}$ on X, the following hold:*

i) *If $t \in \Gamma(Y, \mathcal{F})$ is such that $t|_U = 0$, then there is $N \geq 0$ such that $s^N \cdot t = 0$ in $\Gamma(Y, \mathcal{F} \otimes \mathcal{L}^N)$.*
ii) *For every $t \in \Gamma(U, \mathcal{F})$, there is $q \geq 0$ such that $s^q|_U \cdot t$ is the restriction of a section in $\Gamma(Y, \mathcal{F} \otimes \mathcal{L}^q)$.*

Note that when $\mathcal{L} = \mathcal{O}_X$, this is precisely the assertion in Exercise 8.4.30.

Proof of Lemma 11.1.19. Let $Y = \bigcup_{i=1}^r U_i$ be an affine open cover of Y such that for every i, we have an isomorphism $\phi_i \colon \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$. We put $f_i = \phi_i(s|_{U_i})$. Since $\mathcal{F}$ is quasi-coherent, for every i, we have an isomorphism

$$\Gamma(U_i, \mathcal{F})_{f_i} \to \Gamma\big(U_i \cap U, \mathcal{F}\big). \tag{11.1.1}$$

Let us first prove i). Since $t|_U = 0$, it follows from (11.1.1) that for every i we can find N_i such that $f_i^{N_i} \cdot t|_{U_i} = 0$. We conclude that if $N \geq \max_i N_i$, then $(s^N \cdot t)|_{U_i} \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^N)$ is 0, and thus $s^N \cdot t = 0$.

We now prove ii). By (11.1.1), for every i, there is q_i such that $f_i^{q_i} \cdot t|_{U \cap U_i}$ is the restriction of a section in $\Gamma(U_i, \mathcal{F})$. By taking $q \geq \max_i q_i$, we thus see that for every i, the section

$$s^q|_{U \cap U_i} \cdot t|_{U \cap U_i} \in \Gamma(U \cap U_i, \mathcal{F} \otimes \mathcal{L}^q)$$

is the restriction of a section $v_i \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^q)$. For every i and j, the restriction of $v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j}$ to $U \cap U_i \cap U_j$ vanishes. Since we only have finitely many such intersections to consider, using the assertion in i), we see that after possibly replacing q by a larger value, we may assume that $v_i|_{U_i \cap U_j} = v_j|_{U_i \cap U_j}$ for all i and j. Therefore there is $v \in \Gamma(Y, \mathcal{F} \otimes \mathcal{L}^q)$ such that $v|_{U_i} = v_i$ for all i, and thus $s^q|_U \cdot t = v|_U$ since we have equality after restricting to each U_i. $\square$

Proof of Proposition 11.1.18. Given $\mathcal{F}$, we construct $\Psi = \Psi_{\mathcal{F}} \colon \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ by describing it on the affine open subsets of the form $D_X^+(f)$, where f is homogeneous, with $d = \deg(f) > 0$. We thus need to define

$$\Gamma_*(\mathcal{F})_{(f)} \to \Gamma\big(D_X^+(f), \mathcal{F}\big).$$

Given $\frac{s}{f^m}$, with $s \in \Gamma\big(X, \mathcal{F}(md)\big)$, we consider $\frac{1}{f^m}$ as a section in $\Gamma\big(D_X(f), \mathcal{O}_X(-md)\big)$ and define $\Psi\left(\frac{s}{f^m}\right)$ to be the section $\frac{1}{f^m} \cdot s|_{D_+(f)} \in \Gamma\big(D_X^+(f), \mathcal{F}\big)$. It is straightforward to see that this is well-defined and that the morphisms thus defined on each $D_X^+(f)$ glue to a morphism of sheaves $\Psi \colon \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$. Moreover, it is clear that this is functorial in $\mathcal{F}$.

In order to show that Ψ is an isomorphism of sheaves, we use the lemma. Again, it is enough to show that α is an isomorphism over every open subset $D_X^+(f) = X \smallsetminus V(f)$, where $f \in S$ is homogeneous, with $\deg(f) = d > 0$. If $\alpha\left(\frac{s}{f^m}\right) = 0$, since $\frac{1}{f^m} \in \Gamma\big(D_X^+(f), \mathcal{O}_X(-m)\big)$ does not vanish anywhere, it follows that $s|_{D_X^+(f)} = 0$. By the lemma, we can find N such that $f^N \cdot s = 0$ in $\Gamma\big(X, \mathcal{F}(md + Nd)\big)$. In this case we have $\frac{s}{f^m} = \frac{f^N \cdot s}{f^{m+N}} = 0$. We thus see that Ψ is injective.

For surjectivity, suppose that $t \in \Gamma\big(D_X^+(f), \mathcal{F}\big)$. It follows from the lemma that we can find $q \geq 0$ such that $f^q \cdot t$ is the restriction of a section $t' \in \Gamma\big(X, \mathcal{F}(qd)\big)$, in which case $t = \Psi\left(\frac{t'}{f^q}\right)$. This completes the proof of the theorem. $\square$

We now derive some consequences of the above proposition.

COROLLARY 11.1.20. *If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then there is a finitely generated graded S-module M such that $\mathcal{F} \simeq \widetilde{M}$.*

PROOF. It follows from the proposition that if $N = \Gamma_*(\mathcal{F})$, then $\widetilde{N} \simeq \mathcal{F}$. Let $y_1, \ldots, y_r$ be homogeneous elements of positive degree such that $X = \bigcup_{i=1}^r D_X(y_i)$. Since $\mathcal{F}$ is coherent, it follows that $N_{(y_i)}$ is a finitely generated $S_{(y_i)}$-module for every i. By choosing generators and letting M to be the S-submodule of N generated by the numerators that appear in all these generators, for all i, we see that M is a finitely generated graded submodule of N such that $M_{(y_i)} = N_{(y_i)}$ for all i. We thus have $\widetilde{M} \simeq \widetilde{N} \simeq \mathcal{F}$. $\square$

REMARK 11.1.21. For every graded S-module M, the morphism $\Phi_M \colon M \to \Gamma_*(\widetilde{M})$ induces a morphism of sheaves

$$\widetilde{\Phi_M} \colon \widetilde{M} \to \widetilde{\Gamma_*(\widetilde{M})}.$$

It is easy to check that $\Psi_{\widetilde{M}} \circ \widetilde{\Phi_M}$ is equal to $\mathrm{id}_{\widetilde{M}}$, hence $\widetilde{\Phi_M}$ is an isomorphism. Therefore the sheaves associated to $\ker(\Phi_M)$ and $\mathrm{coker}(\Phi_M)$ are 0. If M is finitely generated, it follows that $\ker(\Phi_M)$ is finitely generated, and Remark 11.1.16 implies that if $j \gg 0$, then the morphism

$$M_j \to \Gamma\big(X, \widetilde{M}(j)\big)$$

induced by Φ_M is injective. In fact, this is also surjective, but we postpone the proof of this fact until the next section (see Corollary 11.2.3 below).

EXERCISE 11.1.22. Let $X = \mathbf{P}^n$ and $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbf{P}^n$, with irrelevant ideal $\mathfrak{m} = (x_0, \ldots, x_n)$. A homogeneous ideal J in S is *saturated* if for every element $u \in S$ such that $u \cdot \mathfrak{m} \subseteq J$, we have $u \in J$.

i) Show that for every homogeneous ideal J in S, there is a unique saturated ideal J^{sat} in S such that $J \subseteq J^{\mathrm{sat}}$ and there is r such that $\mathfrak{m}^r \cdot J^{\mathrm{sat}} \subseteq J$.
ii) Show that if J_1 and J_2 are homogeneous ideals in S, then $\widetilde{J_1} = \widetilde{J_2}$ if and only if $J_1^{\mathrm{sat}} = J_2^{\mathrm{sat}}$.
iii) In particular, given any coherent ideal sheaf $\mathcal{J}$ on $\mathbf{P}^n$, there is a unique saturated ideal J in S such that $\widetilde{J} = \mathcal{J}$. Show that this is the unique largest homogeneous ideal I such that $\widetilde{I} = \mathcal{J}$, and it is equal to
$$\bigoplus_{m \geq 0} \Gamma\big(\mathbf{P}^n, \mathcal{J}(m)\big) \subseteq S = \bigoplus_{m \geq 0} \Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big).$$
iv) Show that if J is a homogeneous, radical ideal different from $\mathfrak{m}$, then J is saturated. Show also that if $\mathcal{J}$ is a radical coherent ideal on $\mathbf{P}^n$, then the saturated ideal J such that $\widetilde{J} = \mathcal{J}$ is radical.

We end this section by introducing two important definitions. If Y is an arbitrary algebraic variety and $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then we have a canonical morphism

$$\Gamma(Y, \mathcal{F}) \otimes_k \mathcal{O}_Y \to \mathcal{F} \tag{11.1.2}$$

that maps $\sum_{i=1}^r s_i \otimes f_i$, where $s_i \in \Gamma(Y, \mathcal{F})$ and $f_i \in \mathcal{O}_Y(U)$ for some open subset U of Y, to $\sum_{i=1}^r f_i \cdot s_i|_U$.

DEFINITION 11.1.23. An $\mathcal{O}_Y$-module $\mathcal{F}$ is *globally generated* if the canonical morphism (11.1.2) is surjective. Equivalently, for every $y \in Y$, the stalk $\mathcal{F}_y$ is generated as an $\mathcal{O}_{Y,y}$-module by $\{s_y \mid s \in \Gamma(Y, \mathcal{F})\}$.

EXERCISE 11.1.24. Show that if $\mathcal{F}$ is a coherent sheaf on an algebraic variety X, then $\mathcal{F}$ is globally generated if and only if there is a surjective morphism $\mathcal{O}_X^{\oplus N} \to \mathcal{F}$, for some non-negative integer N.

DEFINITION 11.1.25. A line bundle $\mathcal{L}$ on Y is *ample* if for every coherent sheaf $\mathcal{F}$ on Y, there is n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$. If D is a Cartier divisor on an irreducible variety Y, then D is *ample* if the line bundle $\mathcal{O}_X(D)$ is ample.

EXAMPLE 11.1.26. Note that if Y is an affine variety, then every quasi-coherent sheaf on Y is globally generated. It follows that every line bundle on Y is ample.

We will discuss ample line bundles in detail in §11.6. For now, we note the following consequence of the results discussed in this section.

PROPOSITION 11.1.27. *If* $X = \operatorname{MaxProj}(S)$ *as above, then the line bundle* $\mathcal{O}_X(1)$ *is ample.*

PROOF. Given a coherent sheaf $\mathcal{F}$ on X, it follows from Corollary 11.1.20 that there is a finitely generated graded module M such that $\mathcal{F} \simeq \widetilde{M}$. Let $u_1, \ldots, u_n$ be homogeneous generators of M, with $\deg(u_i) = d_i$. Since S_+ is generated by elements of degree 1, we see that if $d \geq \max_i\{d_i\}$ and if we take $T = \bigoplus_{i=1}^n S_+^{d-d_i} \cdot u_i$, then T is a graded submodule of M generated by elements of degree d. This implies that we have a surjective morphism $S(-d)^{\oplus q} \to T$, for some q, which induces a surjective morphism $\mathcal{O}_X(-d)^{\oplus q} \to \widetilde{T}$. This implies that $\widetilde{T} \otimes \mathcal{O}_X(m)$ is the image of $\mathcal{O}_X^{\oplus q}$, hence it is finitely generated. On the other hand, every element in M/T is annihilated by some power of S_+, hence $\widetilde{M/T} = 0$, and thus $\widetilde{T} \simeq \widetilde{M} \simeq \mathcal{F}$. $\square$

The line bundle $\mathcal{O}(1)$ that played a key role in this section admits a version on the total space of an arbitrary projective morphism. More generally we have a functor from graded quasi-coherent sheaves on the base to quasi-coherent sheaves on the total space, as follows. Suppose that Y is an arbitrary variety and $\mathcal{S} = \bigoplus_{m \in \mathbf{N}} \mathcal{S}_m$ is an $\mathbf{N}$-*graded* $\mathcal{O}_Y$-*algebra*, which is reduced, quasi-coherent, and generated over $\mathcal{S}_0$ by $\mathcal{S}_1$; moreover, both $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent. Consider the morphism

$$f \colon X = \mathcal{M}axProj(\mathcal{S}) \to Y.$$

Suppose that $\mathcal{M} = \bigoplus_{i \in \mathbf{Z}} \mathcal{M}_i$ is an $\mathcal{S}$-module, which is quasi-coherent as an $\mathcal{O}_Y$-module. For every affine open subset U of Y, we have a canonical isomorphism $f^{-1}(U) \simeq \operatorname{MaxProj}(\mathcal{S}(U))$. Note that $\mathcal{M}(U)$ is a graded $\mathcal{S}(U)$-module, and we get a corresponding quasi-coherent sheaf $\widetilde{\mathcal{M}(U)}$ on $f^{-1}(U)$. It is straightforward to check that if $V \subseteq U$ is another affine open subset, then we have a canonical isomorphism

$$\widetilde{\mathcal{M}(U)}|_{f^{-1}(V)} \simeq \widetilde{\mathcal{M}(V)}.$$

We thus obtain a quasi-coherent sheaf $\widetilde{\mathcal{M}}$ on X. It is clear that if $\mathcal{M}$ is locally finitely generated over $\mathcal{S}$ (in the sense that $\mathcal{M}(U)$ is a finitely generated $\mathcal{S}(U)$-module for every affine open subset U of Y), then $\widetilde{\mathcal{M}}$ is a coherent sheaf on X. An important example is provided by the line bundle $\mathcal{O}_X(1) := \widetilde{\mathcal{S}(1)}$.

Note that for every m and every affine open subset U of Y, we have a canonical morphism

$$\Gamma(U, \mathcal{S}_m) \to \Gamma\big(f^{-1}(U), \mathcal{O}_{f^{-1}(U)}(m)\big).$$

These induce for every m a morphism of $\mathcal{O}_Y$-modules

$$\mathcal{S}_m \to f_*\big(\mathcal{O}_X(m)\big).$$

EXAMPLE 11.1.28. Suppose that $\mathcal{I}$ is a non-zero coherent sheaf of ideals on the irreducible variety Y and let $\pi\colon \widetilde{Y} \to Y$ be the blow-up along $\mathcal{I}$. Recall that in this case, we have $\mathcal{I}\cdot\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y(-E)$, for an effective Cartier divisor E on $\widetilde{Y}$ (see Example 9.4.21). Since $\widetilde{Y} = \mathcal{M}axProj(\mathcal{S})$, where $\mathcal{S} = \bigoplus_{m\geq 0}\mathcal{I}^m$, it is easy to see that we have $\mathcal{O}_{\widetilde{Y}}(1) \simeq \mathcal{O}_Y(-E)$.

EXAMPLE 11.1.29. An important example is that when

$$X = \mathcal{M}axProj(\mathcal{O}_Y[x_0,\ldots,x_n]) = \mathbf{P}^n \times Y =: \mathbf{P}^n_Y.$$

In this case, the canonical morphism $f\colon \mathbf{P}^n_Y \to Y$ is the projection onto the second component. If $q\colon \mathbf{P}^n_Y \to \mathbf{P}^n$ is the projection onto the first component, then it is easy to see that $\mathcal{O}_{\mathbf{P}^n_Y}(1) \simeq q^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$.

11.2. Cohomology of coherent sheaves on projective varieties

We keep the notation in the previous section. In particular, we consider $X = \mathrm{MaxProj}(S)$. Our goal is to prove the following fundamental result about the cohomology of sheaves on X.

THEOREM 11.2.1. *If $X = \mathrm{MaxProj}(S)$, then for every coherent sheaf $\mathcal{F}$ on X, the following hold:*

i) *For every $i \geq 0$, the S_0-module $H^i(X,\mathcal{F})$ is finitely generated.*
ii) *If $m \gg 0$, then $H^i\big(X,\mathcal{F}(m)\big) = 0$ for all $i \geq 1$.*

The main ingredient in the proof of the theorem is the following result concerning the cohomology of the line bundles $\mathcal{O}_X(m)$ when $S = A[x_0,\ldots,x_n]$. Note that in this case we have $X = \mathbf{P}^n_Y$, where $Y = \mathrm{MaxSpec}(A)$.

THEOREM 11.2.2. *If $S = A[x_0,\ldots,x_n]$ and $X = \mathrm{MaxProj}(S)$, then the following hold:*

i) *The canonical morphism $\Phi\colon S \to \bigoplus_{j\geq 0} H^0\big(X,\mathcal{O}_X(j)\big)$ is an isomorphism.*
ii) *If $0 \leq p \leq n-1$, then $H^p\big(X,\mathcal{O}_X(j)\big) = 0$ for all j.*
iii) *We have $H^n\big(X,\mathcal{O}_X(-n-1)\big) \simeq A$ and for every j, the canonical multiplication map*

$$H^0\big(X,\mathcal{O}_X(j)\big) \times H^n\big(X,\mathcal{O}_X(-n-1-j)\big) \to H^n\big(X,\mathcal{O}_X(-n-1)\big)$$

is a perfect pairing of finitely generated, free A-modules.

PROOF. We compute the cohomology of the sheaves $\mathcal{O}_X(j)$ using the affine cover $X = \bigcup_{i=0}^n D_X(x_i)$. For every $J \subseteq \{0,\ldots,n\}$, we put $U_J = \bigcap_{i\in J} D_X(x_i)$ and $x_J = \prod_{i\in J} x_i$. Note that by definition, we have

$$\Gamma\big(U_J,\mathcal{O}_X(j)\big) = (S_{x_J})_j.$$

The Čech complex for this cover and for the sheaf $\mathcal{O}_X(j)$ is thus given by

$$0 \to C^0 \to C^1 \to \ldots \to C^n \to 0,$$

where

$$C^p = \bigoplus_{|J|=p+1} (S_{x_J})_j$$

and where the maps are given, up to a sign, by the natural inclusion maps. We also put $C^{-1} = S_j$. The map $\Gamma\big(X, \mathcal{O}_X(j)\big) \to C^0$ induces a morphism $C^{-1} \to C^0$ and we denote by $C^\bullet$ the resulting complex. Note that by Theorem 10.6.1, the first two assertions in the theorem are equivalent with $\mathcal{H}^i(C^\bullet) = 0$ for $i \leq n-1$.

For every $u = (u_0, \ldots, u_n) \in \mathbf{Z}^{n+1}$, we write x^u for the Laurent monomial $x_0^{u_0} \cdots x_n^{u_n}$. The top cohomology of $C^\bullet$ is easy to compute. Indeed, the A-module

$$\mathcal{H}^n(C^\bullet) = \operatorname{coker}\left(\bigoplus_{i=0}^n (S_{x_0 \cdots \widehat{x_i} \cdots x_n})_j \to (S_{x_0 \cdots x_n})_j\right)$$

is finitely generated and free, with a basis given by the classes of the Laurent monomials x^a, with $a_i \leq -1$ for all i and such that $\sum_{i=0}^n a_i = j$. This shows that $H^n\big(X, \mathcal{O}_X(j)\big) = 0$ if $j > -n-1$ and $H^n\big(X, \mathcal{O}_X(-n-1)\big) \simeq A$.

Note also that every $s \in \Gamma\big(X, \mathcal{O}_X(j)\big)$ induces a morphism $\mathcal{O}_X(-n-1-j) \to \mathcal{O}_X(-n-1)$ given by tensoring with s. Via the map $S_j \to \Gamma\big(X, \mathcal{O}_X(j)\big)$, by applying $H^n(-)$, we thus get a bilinear map

$$S_j \times H^n\big(X, \mathcal{O}_X(-n-1-j)\big) \to H^n\big(X, \mathcal{O}_X(-n-1)\big) \simeq A$$

which maps (x^u, x^v) to the generator x^{u+v} if $u_i + v_i = -1$ for all i, and to 0, otherwise. It is then clear that this is a perfect pairing of finitely generated, free A-modules.

In order to complete the proof of the theorem, it is thus enough to prove that $\mathcal{H}^i(C^\bullet) = 0$ for $i < n$. Given $J \subseteq \{0, \ldots, n\}$, with $|J| = p$, we write e_J for the unit in the summand of C^{p-1} corresponding to J. We thus see that

$$C^{p-1} = \bigoplus_{u \in \mathbf{Z}^{n+1}} \bigoplus_{J(u) \subseteq J, |J| = p} A x^u e_J,$$

where $J(u) = \{i \mid u_i < 0\}$. It is clear that the complex $C^\bullet$ decomposes as $\bigoplus_{u \in \mathbf{Z}^{n+1}} C_u^\bullet$, where

$$C_u^{p-1} = \bigoplus_{J(u) \subseteq J, |J| = p} A x^u e_J.$$

If $J(u) = \{0, \ldots, n\}$, then $C_u^p = 0$ for $p \leq n-1$. Therefore it is enough to show that for every $u \in \mathbf{Z}^{n+1}$, with $J(u) \neq \{0, \ldots, n\}$, the identity map on $C_u^\bullet$ is homotopic to 0. This complex is, up to a shift, the complex that computes the reduced simplicial cohomology with coefficients in A for the full simplicial complex on the set $\{0, \ldots, n\} \smallsetminus J(u)$; our assertion is then well-known, but we recall the argument. If $0 \leq i_0 \leq n$ is such that $i_0 \not\in J(u)$, and $0 \leq p \leq n$, we define $\theta^p \colon C_u^p \to C_u^{p-1}$ by $\theta^p(x^u e_J) = 0$ if $i_0 \not\in J$ and

$$\theta^p(x^u e_J) = (-1)^{\ell - 1} x^u e_{J \smallsetminus \{i_0\}}$$

if the elements of J are $j_1 < \ldots < j_\ell = i_0 < \ldots < j_{p+1}$. A straightforward computation shows that the maps $(\theta^p)_{0 \leq p \leq n}$ give a homotopy between the identity map and 0 on $C_u^\bullet$. This completes the proof of the theorem. $\square$

We can now deduce the result about the cohomology of arbitrary coherent sheaves on $\operatorname{MaxProj}(S)$.

Proof of Theorem 11.2.1. Let us consider a surjective, graded homomorphism $T = S_0[x_0, \ldots, x_n] \to S$ and the corresponding closed immersion $\iota \colon X \hookrightarrow$

$Y = \mathrm{MaxProj}(T)$. For every coherent sheaf $\mathcal{F}$ on X, the sheaf $\iota_*(\mathcal{F})$ on Y is coherent. Moreover, by Remark 11.1.14, we have $\iota^*\big(\mathcal{O}_Y(m)\big) \simeq \mathcal{O}_X(m)$ for all $m \in \mathbf{Z}$, hence the projection formula gives

$$\iota_*\big(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)\big) \simeq \iota_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m).$$

By Example 10.5.16, we thus obtain isomorphisms

$$H^i\big(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)\big) \simeq H^i\big(Y, i_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m)\big)$$

for all $m \in \mathbf{Z}$ and $i \geq 0$. We thus see that we may and will assume $S = A[x_0, \ldots, x_n]$, with the standard grading. Note that in this case, if $\mathcal{F} = \mathcal{O}_X(m)$, for some $m \in \mathbf{Z}$, then both assertions in the theorem follow from Theorem 11.2.2.

We show by descending induction on i that for every coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F})$ finitely generated as an A-module, and $H^i\big(X, \mathcal{F}(m)\big) = 0$ for $m \gg 0$. If $i > \dim(X)$, then both assertions are trivially true since $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf by Corollary 10.6.7. We now suppose that both assertions hold for $i+1$ and deduce them for i. Given a coherent sheaf $\mathcal{F}$ on X, it follows from Corollary 11.1.20 that there is a finitely generated graded S-module M such that $\mathcal{F} \simeq \widetilde{M}$. If $u_1, \ldots, u_r \in M$ form a system of homogeneous generators, with $\deg(u_i) = a_i$, we have a surjective morphism $\bigoplus_{j=1}^r S(-a_j) \to M$. By passing to the associated sheaves and taking the kernel, we obtain a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus_{j=1}^{r} \mathcal{O}_X(-a_j) \to \mathcal{F} \to 0.$$

By tensoring this with $\mathcal{O}_X(m)$ and taking the long exact sequence in cohomology, we obtain an exact sequence

$$\bigoplus_{j=1}^{r} H^i\big(X, \mathcal{O}_X(m - a_j)\big) \to H^i\big(X, \mathcal{F}(m)\big) \to H^{i+1}\big(X, \mathcal{G}(m)\big).$$

Suppose first that $m = 0$. The third term is a finitely generated module over A by the inductive assumption, while the first term is a finitely generated A-module by Theorem 11.2.2. We thus see that $H^i(X, \mathcal{F})$ is a finitely generated A-module as well.

We also see that for $m \gg 0$, the third term in the above exact sequence vanishes by the inductive assumption, while the first term vanishes by Theorem 11.2.2. We thus see that $H^i\big(X, \mathcal{F}(m)\big) = 0$ for all $m \gg 0$. This completes the proof of the induction step.

We note that regarding ii), we have shown for every i, we have $H^i\big(X, \mathcal{F}(m)\big) = 0$ for $m \gg 0$. This implies that there is m_0 such that all these vanish for $m \geq m_0$, since we only need to consider the cohomology groups for $i \leq \dim(X)$, the others being automatically 0. This completes the proof of the theorem. □

COROLLARY 11.2.3. *If $X = \mathrm{MaxProj}(S)$ and M is a finitely generated S-module, then for $j \gg 0$, the canonical morphism*

$$M_j \to \Gamma\big(X, \widetilde{M}(j)\big)$$

induced by Φ_M is an isomorphism.

PROOF. We have already seen that this is injective (see Remark 11.1.21), but we reprove this now as well. Note that if $S' \to S$ is a surjective graded S_0-algebra homomorphism, inducing the closed immersion $X \hookrightarrow \mathrm{MaxProj}(S')$, then M is a graded S'-module and the assertion over S is equivalent to the assertion over S'. We thus may and will assume that S is a polynomial ring over S_0. In this case, if $M = S(m)$ for some $m \in \mathbf{Z}$, then Φ_M is an isomorphism by Theorem 11.2.2. In general, after choosing finitely many homogeneous generators of M, we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d, "\Phi_Q"'] & P \arrow[r] \arrow[d, "\Phi_P"'] & M \arrow[r] \arrow[d, "\Phi_M"'] & 0 \\ 0 \arrow[r] & \Gamma_*(\widetilde{Q}) \arrow[r] & \Gamma_*(\widetilde{P}) \arrow[r, "\alpha"] & \Gamma_*(\widetilde{M}), & \end{tikzcd}$$

where $P = \bigoplus_{i=1}^{r} S(-m_j)$. Note that $(\Phi_P)_j$ is an isomorphism for all j. On the other hand, it follows from Theorem 11.2.1 that $H^1\big(X, \widetilde{Q}(j)\big) = 0$ for $j \gg 0$, hence α_j is surjective. We thus conclude from the above diagram that $(\Phi_M)_j$ is surjective for $j \gg 0$.

Moreover, it follows from the above diagram and the Snake lemma that for every j, we have an exact sequence

$$0 = \ker(\Phi_P)_j \to \ker(\Phi_M)_j \to \operatorname{coker}(\Phi_Q)_j.$$

Applying what we have already seen for Q, we obtain that for $j \gg 0$ we have $\operatorname{coker}(\Phi_Q)_j = 0$, hence $\ker(\Phi_M)_j = 0$. □

Given a graded S-module M and an integer q, we put

$$M_{\geq q} := \bigoplus_{m \geq q} M_m.$$

Corollary 11.2.4. *If $\mathcal{F}$ is a coherent sheaf on $X = \mathrm{MaxProj}(S)$, then for every $m_0 \in \mathbf{Z}$, the S-module $\Gamma_*(\mathcal{F})_{\geq m_0}$ is finitely generated.*

Proof. By Corollary 11.1.20, we have a finitely generated S-module M such that $\mathcal{F} \simeq \widetilde{M}$. Corollary 11.2.3 then implies that we have $m_1 \geq m_0$ such that the canonical morphism

$$M_{\geq m_1} \to \Gamma_*(\mathcal{F})_{\geq m_1}$$

is an isomorphism. Since M is finitely generated, it is easy to see that the left-hand side is finitely generated, hence the right-hand side is finitely generated as well. Since each $\Gamma\big(X, \mathcal{F}(m)\big)$, with $m_0 \leq m < m_1$ is finitely generated over S_0 by Theorem 11.2.1, we obtain the assertion in the corollary. □

Remark 11.2.5. Suppose that X is a closed subvariety of $\mathbf{P}^n_Y$, where $Y = \mathrm{MaxSpec}(A)$. Let $I \subseteq S = A[x_0, \ldots, x_n]$ be a radical homogeneous ideal such that $Y = V(I)$. The coherent ideal $\widetilde{I}$ is the radical ideal sheaf $\mathcal{I}_X$ corresponding to X, so that if $S_X = S/I$, we have $\widetilde{S_X} \simeq \mathcal{O}_X$. It follows that for every $m \geq 0$, we have a canonical morphism

$$(S_X)_m \to \Gamma\big(X, \mathcal{O}_X(m)\big)$$

and by Corollary 11.2.3, there is m_0 such that the above morphism is an isomorphism for $m \geq m_0$.

It follows that for $m \geq m_0$, the canonical map

$$\Gamma\big(X, \mathcal{O}_X(m)\big) \otimes_k \Gamma\big(X, \mathcal{O}_X(1)\big) \to \Gamma\big(X, \mathcal{O}_X(m+1)\big)$$

given by multiplication of sections is surjective. In particular, the graded k-algebra

$$S\big(\mathcal{O}_X(m)\big) := \bigoplus_{j\geq 0} \Gamma\big(X, \mathcal{O}_X(jm)\big)$$

is generated by its degree 1 part. It is easy to see that this algebra is reduced and we can thus consider $\mathrm{MaxProj}\big(S(\mathcal{O}_X(m))\big)$. In fact, this is isomorphic to X. Indeed, we have a graded homomorphism $S_X^{(m)} \to S\big(\mathcal{O}_X(m)\big)$ which is an isomorphism in all positive degrees, so that the assertion follows from Exercises 4.3.21 and 4.3.22.

EXAMPLE 11.2.6. Let $n \geq 1$ and let D be an effective Cartier divisor on $\mathbf{P}^n$. Recall that we have an isomorphism $\mathrm{Pic}(\mathbf{P}^n) \simeq \mathbf{Z}$ that maps $\mathcal{O}_{\mathbf{P}^n}(m)$ to m (see Example 9.3.4). Suppose that D *has degree* d, that is, $\mathcal{O}_{\mathbf{P}^n}(D) \simeq \mathcal{O}_{\mathbf{P}^n}(d)$; equivalently, if $D = \sum_{i=1}^r a_i D_i$, where D_i is an irreducible hypersurface in $\mathbf{P}^n$ of degree d_i, then $d = \sum_{i=1}^r a_i d_i$.

Recall that by Proposition 9.4.24, effective Cartier divisors of degree d on $\mathbf{P}^n$ are in bijection with sections of $\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)\big) \simeq S_d$, up to multiplication by non-zero elements of k, where $S = k[x_0, \ldots, x_n]$. We write $f_D \in S_d$ for such a polynomial corresponding to D. Note that if D is a hypersurface in $\mathbf{P}^n$ (that is, it is a reduced divisor, that we identify with its support), then f_D is a generator for the principal radical ideal corresponding to this hypersurface. In general, if $D = \sum_{i=1}^r a_i D_i$, where D_i is an irreducible hypersurface in $\mathbf{P}^n$, with corresponding radical ideal generated by f_i, then we can take $f_D = \prod_{i=1}^r f_i^{a_i}$.

Suppose now that $n \geq 2$ and let us compute $H^i\big(\mathbf{P}^n, \mathcal{O}_D(m)\big)$ for all m. Note that since $\mathcal{O}_X(-D) \simeq \mathcal{O}_{\mathbf{P}^n}(-d)$, we have a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-d) \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_D \to 0.$$

By tensoring with $\mathcal{O}_{\mathbf{P}^n}(m)$ and taking the long exact sequence in cohomology, we obtain for every $i \geq 0$, an exact sequence

$$H^i\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m-d)\big) \to H^i\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) \to H^i\big(\mathbf{P}^n, \mathcal{O}_D(m)\big) \to$$

$$\to H^{i+1}\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m-d)\big) \to H^{i+1}\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big).$$

We first deduce from Theorem 11.2.2 that

$$H^i\big(\mathbf{P}^n, \mathcal{O}_D(m)\big) = 0 \quad \text{for} \quad 1 \leq i \leq n-2, m \in \mathbf{Z}.$$

We also see that

$$H^0\big(\mathbf{P}^n, \mathcal{O}_D(m)\big) \simeq (S/S \cdot f_D)_m.$$

In particular, we have

$$\dim_k H^0\big(\mathbf{P}^n, \mathcal{O}_D(m)\big) = \binom{m+n}{n} - \binom{m+n-d}{n} \quad \text{for} \quad m \geq 0,$$

with the convention that the second binomial coefficient is 0 for $m < d$.

Finally, we see that for every m, we have an exact sequence

$$0 \to H^{n-1}\big(\mathbf{P}^n, \mathcal{O}_D(m)\big) \to H^n\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m-d)\big) \to H^n\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) \to 0.$$

EXAMPLE 11.2.7. Suppose that $\mathcal{E}$ is a locally free sheaf on T and

$$f\colon X = \mathbf{P}(\mathcal{E}) = \mathcal{M}axProj\big(\mathrm{Sym}^\bullet(\mathcal{E})\big) \to T$$

is the corresponding projective bundle. Arguing locally on T, we deduce from Theorem 11.2.2 that for every $m < 0$ we have $f_*\big(\mathcal{O}_X(m)\big) = 0$ and for $m \geq 0$, the canonical morphism

$$\mathrm{Sym}^m(\mathcal{E}) \to f_*\big(\mathcal{O}_X(m)\big)$$

is an isomorphism. We also see that if $\mathrm{rk}(\mathcal{E}) = n+1$, then $R^i f_*\big(\mathcal{O}_X(m)\big) = 0$ for $1 \leq i \leq n-1$ and all m, or for $i = n$ and $m \geq -n$.

In particular, we see that we have a canonical isomorphism $\mathcal{E} \to f_*\big(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)\big)$. Since f^* is the left adjoint of f_*, this morphism corresponds to a morphism on $\mathbf{P}(\mathcal{E})$

$$f^*(\mathcal{E}) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1).$$

This is a surjective morphism. Indeed, in order to check this, after considering a suitable affine open cover of T, we may assume that T is affine and $\mathcal{E} = \mathcal{O}_T^{\oplus(n+1)}$. In this case, the surjectivity comes down to the fact that on the variety $\mathcal{M}axProj\big(\mathcal{O}(T)[x_0, \ldots, x_n]\big)$, the line bundle $\mathcal{O}(1)$ is generated by its global sections, which holds since $\bigcap_{i=0}^n V(x_i) = \emptyset$.

11.3. Coherence of higher direct images for proper morphisms

In this section we prove an important finiteness result for proper morphisms, and use a special case to define some numerical invariants of smooth, complete varieties. We end the section by introducing the Grothendieck groups of vector bundles and of coherent sheaves on an algebraic variety.

11.3.1. Higher direct images for proper morphisms. The results in the previous section imply that each cohomology group of a coherent sheaf on a projective variety is finite-dimensional over the ground field. We want to extend this to the case of arbitrary complete varieties. More generally, we prove the following result concerning the higher direct images of a coherent sheaf by a proper morphism.

THEOREM 11.3.1. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties, then for every coherent sheaf $\mathcal{F}$ on X and every $p \geq 0$, the sheaf $R^p f_*(\mathcal{F})$ on Y is coherent.*

PROOF. The proof of the theorem is based on "dévissage", a technique introduced by Grothendieck. We recall that the quasi-coherence of $R^p f_*(\mathcal{F})$ follows from Proposition 10.2.20 and thus we only need to prove that if $U \subseteq Y$ is an affine open subset, then $H^p\big(f^{-1}(U), \mathcal{F}\big)$ is a finitely generated $\mathcal{O}_Y(U)$-module.

We note that the theorem holds if f admits a factorization $X \xrightarrow{i} Y \times \mathbf{P}^n \xrightarrow{\pi} Y$, for some $n \geq 0$, where i is a closed immersion and π is the projection onto the first component. Indeed, if U is an affine open subset of Y, then

$$H^p\big(f^{-1}(U), \mathcal{F}\big) \simeq H^p\big(U \times \mathbf{P}^n, i_*(\mathcal{F})\big)$$

and the right-hand side is a finitely generated $\mathcal{O}_Y(U)$-module by Theorem 11.2.1.

In order to handle the general case, we make use of Chow's lemms. Arguing by Noetherian induction on X, we may assume that for every closed subvariety Z of X, different from X, the composition $Z \overset{j}{\hookrightarrow} X \longrightarrow Y$ satisfies the conclusion of the theorem with respect to any coherent sheaf on Z. If $\mathcal{G}$ is a coherent sheaf on X such that $\mathcal{I}_Z \cdot \mathcal{G} = 0$, where $\mathcal{I}_Z$ is the radical ideal sheaf defining Z, then we have a coherent sheaf $\mathcal{G}_Z$ on Z such that $\mathcal{G} = j_*(\mathcal{G}_Z)$ and thus

$$R^p f_*(\mathcal{G}) \simeq R^p(f \circ j)_*(\mathcal{G}_Z)$$

is coherent for every $p \geq 0$.

We see that if $\mathcal{G}$ is a coherent sheaf on X such that $\mathrm{Supp}(\mathcal{G}) \neq X$, then $R^p f_*(\mathcal{G})$ is coherent for every p. Indeed, let $\mathcal{I}$ be the radical ideal defining $\mathrm{Supp}(\mathcal{G})$ and let $d \geq 1$ be such that $\mathcal{I}^d \cdot \mathcal{G} = 0$. As we have seen, for every $j \geq 0$ and every p, the sheaf $R^p f_F(\mathcal{I}^j\mathcal{G}/\mathcal{I}^{j+1}\mathcal{G})$ is coherent. The long exact sequence for higher direct images corresponding to the short exact sequence

$$0 \to \mathcal{I}^{j+1}\mathcal{G} \to \mathcal{I}^j\mathcal{G} \to \mathcal{I}^j\mathcal{G}/\mathcal{I}^{j+1}\mathcal{G} \to 0$$

gives an exact sequence

$$R^p f_*(\mathcal{I}^{j+1}\mathcal{G}) \to R^p f_*(\mathcal{I}^j\mathcal{G}) \to R^p f_*(\mathcal{I}^j\mathcal{G}/\mathcal{I}^{j+1}\mathcal{G}).$$

We thus deduce by descending induction on j, starting with $j = d$, that $R^p f_*(\mathcal{I}^j\mathcal{G})$ is coherent. By taking $j = 0$, we conclude that $R^p f_*(\mathcal{G})$ is coherent.

This implies that if $\phi\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a morphism such that the supports of $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are proper subsets of X, then for every q we have that $R^p f_*(\mathcal{G}_1)$ is coherent if and only if $R^p f_*(\mathcal{G}_2)$ is coherent. Indeed, using the long exact sequences for higher direct images associated to the short exact sequences

$$0 \to \ker(\phi) \to \mathcal{G}_1 \to \mathrm{Im}(\phi) \to 0$$

and

$$0 \to \mathrm{Im}(\phi) \to \mathcal{G}_2 \to \mathrm{coker}(\phi) \to 0,$$

we obtain the exact sequences

$$R^p f_*\big(\ker(\phi)\big) \to R^p f_*(\mathcal{G}_1) \to R^p f_*\big(\mathrm{Im}(\phi)\big) \to R^{p+1} f_*\big(\ker(\phi)\big)$$

and

$$R^p f_*\big(\mathrm{Im}(\phi)\big) \to R^p f_*(\mathcal{G}_2) \to R^p f_*\big(\mathrm{coker}(\phi)\big) \to R^{p+1} f_*\big(\mathrm{Im}(\phi)\big).$$

Since we know that $R^j f_*\big(\ker(\phi)\big)$ and $R^j f_*\big(\mathrm{coker}(\phi)\big)$ are coherent for all j, the first exact sequence shows that $R^p f_*(\mathcal{G}_1)$ is coherent if and only if $R^p f_*\big(\mathrm{Im}(\phi)\big)$ is coherent and the second exact sequence shows that $R^p f_*\big(\mathrm{Im}(\phi)\big)$ is coherent if and only if $R^p f_*(\mathcal{G}_2)$ is coherent.

By Chow's lemma (see Theorem 5.2.2), we have a proper morphism $g\colon W \to X$ that satisfies the following two properties:

i) We have dense open subsets $U \subseteq X$ and $V \subseteq W$ such that g induces an isomorphism $V \simeq U$.
ii) The composition $h = f \circ g$ factors as $W \overset{i}{\hookrightarrow} Y \times \mathbf{P}^n \xrightarrow{\pi} Y$, where i is a closed immersion and π is the projection onto the first components.

Moreover, in this case g admits such a factorization, too (see Remark 5.2.3). As we have noted, the conclusion of the theorem thus holds for both g and $f \circ g$.

We may and will assume that $V = g^{-1}(U)$. Indeed, for this it is enough to replace U by $U \smallsetminus g(W \smallsetminus V)$, which is open (since g is closed) and dense in X. Note that if this last property fails, then there is an irreducible component Z of $W \smallsetminus V$ such that $g(Z)$ is an irreducible component of X. However, any irreducible component of W containing Z meets V, and thus is birational to $g(Z)$, hence $\dim(Z) < \dim\big(g(Z)\big)$, a contradiction.

Consider the morphism $\phi\colon \mathcal{F} \to g_*\big(g^*(\mathcal{F})\big)$. Since the sheaf $g^*(\mathcal{F})$ is coherent and since the theorem holds for g, the sheaf $g_*\big(g^*(\mathcal{F})\big)$ is coherent. Moreover, since ϕ is clearly an isomorphism on U, it follows that both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ have the support contained in $X \smallsetminus U$. We thus see that it is enough to show that for every p, the sheaf $R^pf_*\big(g_*(\mathcal{G})\big)$ is coherent, where $\mathcal{G} = g^*(\mathcal{F})$.

Consider the Leray spectral sequence for $\mathcal{G}$:

$$E_2^{p,q} = R^pf_*\big(R^qg_*(\mathcal{G})\big) \Rightarrow_p R^{p+q}h_*(\mathcal{G}).$$

We need to show that $E_2^{p,0}$ is coherent for all p. It is clear that if $q \geq 1$, then $R^qg_*(\mathcal{G})|_U = 0$, hence $\mathrm{Supp}\big(R^qg_*(\mathcal{G})\big) \subseteq X \smallsetminus U$. By the inductive assumption, we see that in this case $E_2^{p,q}$ is a coherent sheaf on Y, and thus $E_r^{p,q}$ is coherent for every $r \geq 2$ if $q \geq 1$. On the other hand, for every p and q, $E_\infty^{p,q}$ is a subquotient of $R^{p+q}h_*(\mathcal{G})$, hence it is coherent. Since $E_\infty^{p,q} = E_r^{p,q}$ for $r \gg 0$, in order to conclude that $E_2^{p,0}$ is coherent, it is enough to show that for every $r \geq 2$, if $E_{r+1}^{p,0}$ is coherent, then $E_r^{p,0}$ is coherent. Recall that we have morphisms

$$E_r^{p-r,r-1} \xrightarrow{\alpha} E_r^{p,0} \xrightarrow{\beta} E_r^{p+r,1-r} = 0$$

such that

$$E_{r+1}^{p,0} = \ker(\beta)/\mathrm{Im}(\alpha) = \mathrm{coker}(\alpha).$$

Since $r - 1 \geq 1$, we know that $E_r^{p-r,r-1}$ is coherent, and thus $\mathrm{Im}(\alpha)$ is coherent (recall that these are all quasi-coherent sheaves). Since also $\mathrm{coker}(\alpha)$ is coherent, we conclude that $E_r^{p,0}$ is coherent. This completes the proof. $\square$

By taking Y to be a point, we obtain the following

Corollary 11.3.2. *If X is a complete algebraic variety over k, then for every coherent sheaf $\mathcal{F}$ on X, we have $\dim_k H^i(X, \mathcal{F}) < \infty$.*

Corollary 11.3.3. *If X is a complete algebraic variety over k, then for every coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we have $\dim_k \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) < \infty$ for all $i \geq 0$.*

Proof. By Proposition 10.7.13, we have a spectral sequence

$$E_2^{p,q} = H^p\big(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big) \Rightarrow_p \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Since each $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent sheaf by Proposition 10.7.16, it follows from the previous corollary that $\dim_k E_2^{p,q} < \infty$, and thus $\dim_k E_\infty^{p,q} < \infty$ for every p and q. Moreover, by Corollary 10.6.5, there is a positive integer d such that $H^i(X, \mathcal{M}) = 0$ for all $i > d$ and all quasi-coherent sheaves $\mathcal{M}$ on X. In particular, we have $E_2^{p,q} = 0 = E_\infty^{p,q}$ unless $0 \leq p \leq d$. Therefore $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has a finite filtration such that each successive quotient is finite-dimensional over k, hence $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is finite-dimensional. $\square$

We also have the slightly stronger version of the finiteness result in the theorem:

Corollary 11.3.4. *If $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{F}$ is a coherent sheaf of X such that $\mathrm{Supp}(\mathcal{F})$ is proper over Y, then for every $p \geq 0$, the $\mathcal{O}_Y$-module $R^pf_*(\mathcal{F})$ is coherent.*

Proof. Let $Z = \mathrm{Supp}(\mathcal{F})$ and $i\colon Z \hookrightarrow X$ be the inclusion map. If $\mathcal{F} = i_*(\mathcal{G})$, for some coherent sheaf $\mathcal{G}$ on Z, then the assertion follows from the theorem, since

$$R^pf_*(\mathcal{F}) \simeq R^p(f \circ i)_*(\mathcal{G})$$

by Example 10.5.17. In general, we have a finite filtration of $\mathcal{F}$, with successive quotients of the form $i_*(\mathcal{G})$, for some coherent sheaf $\mathcal{G}$ on Z (see Remark 8.4.21). We finally note that if we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

on X, such that $R^p f_*(\mathcal{F})$ and $R^p f_*(\mathcal{F}'')$ are coherent for all p, then it follows from the long exact sequence for higher direct images that $R^p f_*(\mathcal{F})$ is coherent for all p. We thus obtain the assertion in the corollary. $\square$

Given a coherent sheaf $\mathcal{F}$ on the complete variety X, we will put

$$h^i(\mathcal{F}) = h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F}).$$

The *Euler-Poincaré characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \chi(X, \mathcal{F}) := \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F}).$$

Note that this is well-defined since $h^i(X, \mathcal{F}) = 0$ for $i \gg 0$ by Corollary 10.6.5.

EXERCISE 11.3.5. Show that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of coherent sheaves on the complete variety X, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

11.3.2. The geometric genus. One can use the finite-dimensionality of cohomology on complete varieties to define numerical invariants.

DEFINITION 11.3.6. For every complete variety X, the *arithmetic genus* of X is the number $p_a(X) := (-1)^n \big(\chi(\mathcal{O}_X) - 1\big)$, where $n = \dim(X)$. For example, if X is connected, of dimension 1, then $h^0(\mathcal{O}_X) = 1$, so that $p_1(X) = h^1(\mathcal{O}_X)$.

DEFINITION 11.3.7. If X is a smooth, connected, complete variety, of dimension n, the *Hodge numbers* of X are given by

$$h^{p,q}(X) = h^q(X, \Omega_X^q).$$

In particular, we have the *geometric genus* of X, given by $p_g(X) := h^{n,0}(X) = h^0(X, \omega_X)$. More generally, for $m \geq 1$, the m^{th} *plurigenus* of X is given by $p_m(X) := h^0(X, \omega_X^{\otimes m})$.

We now show that the geometric genus (and some of the other invariants) are birational invariants.

THEOREM 11.3.8. *If X and Y are smooth, connected, complete varieties, with X and Y birational, then $p_g(X) = p_g(Y)$. More generally, we have*

$$h^{q,0}(X) = h^{q,0}(X) \quad \textit{and} \quad p_m(X) = p_m(Y) \quad \textit{for all} \quad q, m.$$

PROOF. We give the argument for the geometric genus: the one for the other invariants is entirely similar. By assumption, we have a birational map $\phi \colon X \dashrightarrow Y$. Let $U \subseteq X$ be the domain of ϕ and $f \colon U \to Y$ the corresponding morphism. Since X is smooth in codimension 1 and Y is complete, it follows from Proposition 9.1.6 that $\operatorname{codim}_X(X \smallsetminus U) \geq 2$. Since ω_X is a locally free sheaf and X is normal, it follows that restriction of sections gives an isomorphism

$$\Gamma(X, \omega_X) \simeq \Gamma(U, \omega_X) \tag{11.3.1}$$

(see Proposition 9.1.4).

By assumption, f is birational, hence we can find open subsets $Y_0 \subseteq Y$ and $U_0 \subseteq U$ such that f induces an isomorphism $U_0 \simeq Y_0$. Recall now that we can pull-back differential forms via a morphism (see Remark 8.7.22). We consider the commutative diagram

$$\begin{tikzcd} \Gamma(Y, \omega_Y) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \Gamma(U, \omega_U) \arrow[d, "\gamma"'] \\ \Gamma(Y_0, \omega_{Y_0}) \arrow[r, "\delta"] & \Gamma(U_0, \omega_{U_0}) \end{tikzcd}$$

in which all maps are given by pull-back of top differential forms; in particular, the vertical maps are given by restriction. Note that δ is an isomorphism and α is an injective since Y is irreducible and ω_Y is locally free (see Exercise 8.5.26). It follows from the comuttaive diagram that β is injective. By combining this with (11.3.1), we conclude that

$$p_g(X) = h^0(X, \omega_X) \geq h^0(Y, \omega_Y) = p_g(Y).$$

Since the opposite inequality follows by symmetry, this completes the proof of the theorem. $\square$

EXAMPLE 11.3.9. It follows from Example 8.7.29 that for $n \geq 1$, we have $\omega_{\mathbf{P}^n} \simeq \mathcal{O}_{\mathbf{P}^n}(-n-1)$, hence $p_g(\mathbf{P}^n) = 0$. More generally, we see that $p_m(\mathbf{P}^n) = 0$ for all $m \geq 1$.

EXAMPLE 11.3.10. Suppose that $n \geq 2$ and $Y \subseteq \mathbf{P}^n$ is a smooth hypersurface of degree $d \geq 1$. Note that $N_{Y/X} = \mathcal{O}_X(d)|_Y = \mathcal{O}_Y(d)$ and Corollary 8.7.27 and Example 8.7.29 give

$$\omega_Y \simeq \omega_{\mathbf{P}^n}|_Y \otimes_{\mathcal{O}_Y} N_{Y/X} \simeq \mathcal{O}_Y(d-n-1).$$

We thus see, using Example 11.2.6, that if $d \leq n$, then $p_g(Y) = 0$ and if $d \geq n+1$, then $p_g(Y) = \binom{d-1}{n}$. For example, if $n = 2$, we see that $p_g(Y) = \frac{(d-1)(d-2)}{2}$.

DEFINITION 11.3.11. An irreducible n-dimensional algebraic variety X is *rational* if it is birational to $\mathbf{P}^n$.

REMARK 11.3.12. Showing that certain varieties are not rational is a classical problem that has seen a lot of recent progress (see [**Bea16**] for an overview of classical and recent results). An easy criterion is provided by Theorem 11.3.8 and Example 11.3.9: if a smooth, irreducible, complete variety X has $p_g(X) \neq 0$ (or, more generally, some $p_m(X) \neq 0$), then X is not rational. For example, it follows from Example 11.3.10 that if $X \subseteq \mathbf{P}^n$, with $n \geq 2$, is a smooth hypersurface of degree $d \geq n+1$, then X is not rational.

11.3.3. Grothendieck groups. We end this section by introducing two invariants of algebraic varieties: the Grothendieck group of vector bundles and that of coherent sheaves on the given variety.

DEFINITION 11.3.13. Given an algebraic variety X, the *Grothendieck group* $K^0(X)$ *of vector bundles* on X is the quotient of the free Abelian group on the set of isomorphism classes of locally free sheaves on X, by the subgroup generated by relations of the form $[\mathcal{E}] - [\mathcal{E}'] - [\mathcal{E}'']$, where

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally free sheaves on X. We denote by $[\mathcal{E}]$ the image in $K^0(X)$ of the isomorphism class of $\mathcal{E}$.

Given an exact sequence of vector bundles as in the definition, if $\mathcal{F}$ is another vector bundle, then we have an exact sequence

$$0 \to \mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}'' \otimes_{\mathcal{O}_X} \mathcal{F} \to 0.$$

This easily implies that we get a bilinear map

$$K^0(X) \times K^0(X) \to K^0(X), \quad \big([\mathcal{E}], [\mathcal{F}]\big) \to [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}].$$

This is clearly associative and commutative and has a unit element given by $[\mathcal{O}_X]$. We thus see that $K^0(X)$ is a commutative ring.

If $f\colon X \to Y$ is a morphism and we have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of locally free sheaves on Y, since this is locally split, the induced sequence

$$0 \to f^*(\mathcal{E}') \to f^*(\mathcal{E}) \to f^*(\mathcal{E}'') \to 0$$

is exact. We thus see that we get a morphism of Abelian groups

$$f^*\colon K^0(Y) \to K^0(X), \quad [\mathcal{E}] \to [f^*(\mathcal{E})].$$

Since $f^*(\mathcal{E}_1 \otimes_{\mathcal{O}_Y} \mathcal{E}_2) \simeq f^*(\mathcal{E}_1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E}_2)$ for every vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on Y, it follows that f^* is a ring homomorphism. It is clear from definition that if $g\colon Y \to Z$ is another morphism, then $f^* \circ g^* = (g \circ f)^*$ on $K^0(Z)$.

Remark 11.3.14. Recall that if

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally free sheaves on X, then we have an isomorphism

$$\det(\mathcal{E}) \simeq \det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'').$$

This implies that we get a morphism of Abelian groups

$$K^0(X) \to \operatorname{Pic}(X), \quad [\mathcal{E}] \to [\det(\mathcal{E})].$$

Remark 11.3.15. If X is connected, then we have a ring homomorphism

$$K^0(X) \to \mathbf{Z}, \quad [\mathcal{E}] \to \operatorname{rank}(\mathcal{E}).$$

Definition 11.3.16. For an algebraic variety X, the *Grothendieck group* $K_0(X)$ *of coherent sheaves* on X is the quotient of the free Abelian group on isomorphism classes of coherent sheaves on X by the subgroup generated by relations of the form $[\mathcal{F}] - [\mathcal{F}'] - [\mathcal{F}'']$, where

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of coherent sheaves on X.

Remark 11.3.17. If we have an exact complex of coherent sheaves

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_r \to 0,$$

by breaking this into short exact sequences and using the relations in $K_0(X)$, we see that in $K_0(X)$ we have

$$\sum_{i=0}^{r} (-1)^i [\mathcal{F}_i] = 0.$$

A similar fact for exact complexes of locally free sheaves holds in $K^0(X)$.

Given a *proper* morphism $f\colon X \to Y$ of algebraic varieties, we define a morphism of Abelian groups $f_*\colon K_0(X) \to K_0(Y)$, as follows. For every coherent sheaf $\mathcal{F}$ on X, we put

$$f_*\big([\mathcal{F}]\big) = \sum_{i\geq 0}(-1)^i[R^if_*(\mathcal{F})].$$

Note first that by Theorem 11.3.1, each sheaf $R^if_*(\mathcal{F})$ is coherent. Moreover, it follows from Corollary10.6.6 that $R^if_*(\mathcal{F}) = 0$ for $i \gg 0$. Therefore $f_*\big([\mathcal{F}]\big)$ is well-defined. We claim that we get in this way a morphism of Abelian groups $f_*\colon K_0(X) \to K_0(Y)$. In order to see this, it is enough to note that given a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on X, the long exact sequence for higher direct images gives, via Remark 11.3.17 the equality

$$f_*\big([\mathcal{F}']\big) - f_*\big([\mathcal{F}]\big) + f_*\big([\mathcal{F}]\big) = 0.$$

EXERCISE 11.3.18. Use the Leray spectral sequence to deduce that if $f\colon X \to Y$ and $g\colon Y \to Z$ are proper morphisms, then $(g \circ f)_* = g_* \circ f_*$ as morphisms $K_0(X) \to K_0(Z)$.

EXAMPLE 11.3.19. If $X = \mathrm{MaxSpec}(k)$ is a point, then a coherent sheaf on X is just a finite-dimensional vector space over k. It is then clear that we have $K^0(X) = K_0(X)$ and the rank map that takes $[V]$ to $\dim_k(V)$ gives an isomorphism $K_0(X) \simeq \mathbf{Z}$. If W is a complete variety and we consider the morphism to X, then the morphism f_* gets identified with the map $K_0(W) \to \mathbf{Z}$ that takes $[\mathcal{F}]$ to $\chi(\mathcal{F})$.

Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and a locally free sheaf $\mathcal{E}$, using the fact that a free module is flat, we see that the sequence

$$0 \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}' \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}'' \to 0$$

is exact, too. We can thus define an operation

$$K^0(X) \times K_0(X) \xrightarrow{-\cap-} K_0(X), \quad \big([\mathcal{E}],[\mathcal{F}]\big) \to [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}]$$

that makes $K_0(X)$ a module over $K^0(X)$. Note that the projection formula in Exercise 10.2.23 induces the following version in this setting: if $f\colon X \to Y$ is a proper morphism, then for every $\alpha \in K^0(Y)$ and $\beta \in K_0(X)$, we have

$$f_*\big(f^*(\alpha) \cap \beta\big) = \alpha \cap f_*(\beta).$$

It is clear that we have a morphism of $K^0(X)$-modules $K^0(X) \to K_0(X)$ given by $\alpha \to \alpha \cap [\mathcal{O}_X]$. In other words, this maps $[\mathcal{E}] \in K^0(X)$ to $[\mathcal{E}] \in K_0(X)$.

EXAMPLE 11.3.20. It is easy to see that the structure theorem for finitely generated modules over PIDs implies that the canonical morphisms

$$K^0(\mathbf{A}^1) \to K_0(\mathbf{A}^1) \quad \text{and} \quad K^0(\mathbf{A}^1) \xrightarrow{\mathrm{rk}(-)} \mathbf{Z}$$

are isomorphisms.

REMARK 11.3.21. The reader might note that for an algebraic variety X, the pair $\big(K^0(X), K_0(X)\big)$ behaves formally like the pair $\big(H^*(Y,\mathbf{Z}), H_*(Y,\mathbf{Z})\big)$ in the case of a topological space Y. One can also note that in this case we also have a "fundamental class" $[\mathcal{O}_X]\in K_0(X)$ such that the morphism $K^0(X)\to K_0(X)$ is the analogue of the Poincaré isomorphism (in fact, we will see later that this is an isomorphism when X is a smooth quasi-projective variety).

11.4. Hilbert polynomials

In this section we discuss an important invariant for coherent sheaves on a projective space.

THEOREM 11.4.1. *If $\mathcal{F}$ is a coherent sheaf on a projective space $\mathbf{P}^n$, for a positive integer n, then there is a polynomial $P_{\mathcal{F}}\in\mathbf{Q}[t]$ such that*

$$P_{\mathcal{F}}(m)=\chi\big(\mathcal{F}(m)\big)\quad \textit{for all}\quad m\in\mathbf{Z}.$$

Moreover, if $\mathcal{F}\neq 0$, then $\deg(P_{\mathcal{F}})=\dim\big(\mathrm{Supp}(\mathcal{F})\big)$.

The polynomial $P_{\mathcal{F}}$ attached to $\mathcal{F}$ is the *Hilbert polynomial of* $\mathcal{F}$. If $\mathcal{F}=\mathcal{O}_X$ for a closed subvariety X in $\mathbf{P}^n$, we write P_X for $\mathcal{P}_{\mathcal{O}_X}$; the is the *Hilbert polynomial of* X.

Before giving the proof of the theorem, we make some preparations. For every non-negative integer m, consider the polynomial

$$Q_d=\binom{t+d}{d}:=\frac{(t+1)\cdots(t+d)}{d!}$$

(with the convention $Q_0=1$). It is clear that $\deg(Q_d)=d$ and that $Q_0,Q_1,\ldots,Q_d$ give a basis for the $\mathbf{Q}$-vector space of polynomials in $\mathbf{Q}[t]$ of degree $\leq d$. It is straightforward to check that

$$Q_d(t)-Q_d(t-1)=Q_{d-1}(t)\quad\text{for all}\quad d\geq 1. \tag{11.4.1}$$

LEMMA 11.4.2. *Let $\phi\colon\mathbf{Z}\to\mathbf{Q}$ be a function with the property that there is a polynomial Q of degree d such that $\phi(m)-\phi(m-1)=Q(m)$ for all $m\in\mathbf{Z}$. In this case, there is a polynomial P of degree $d+1$ such that $\phi(m)=P(m)$ for all $m\in\mathbf{Z}$. Moreover, if $Q=\sum_{i=0}^d a_iQ_i$, then $P=\Big(\phi(0)-\sum_{i=0}^d a_i\Big)+\sum_{i=1}^{d+1}a_{i-1}Q_i$.*

PROOF. If P is given as in the statement, using the fact that $Q_i(0)=1$ for all i, we see that $P(0)=\phi(0)$. Moreover, using (11.4.1) and the hypothesis on Q, we see that $P(m)-P(m-1)=\phi(m)-\phi(m-1)$ for all $m\in\mathbf{Z}$. The fact that $\phi(m)=P(m)$ for all $m\in\mathbf{Z}$ follows by induction for $m\geq 0$ and by descending induction for $m\leq 0$. $\square$

Given a coherent sheaf $\mathcal{F}$ on the algebraic variety X, we define the set $\mathrm{Ass}(\mathcal{F})$ of *associated subvarieties* of $\mathcal{F}$ to consist of those closed irreducible subvarieties $Y\subseteq X$ with the property that if $U\subseteq X$ is an affine open subset with $U\cap Y\neq\emptyset$, then the prime ideal in $\mathcal{O}_X(U)$ corresponding to $U\cap Y$ lies in $\mathrm{Ass}_{\mathcal{O}(U)}\big(\mathcal{F}(U)\big)$. Note that the condition is independent of the choice of U: if U and V are affine open subsets of X intersecting Y, then it follows from Lemma 5.3.3 that there is W which is a principal affine open subset in both U and V and such that $W\cap Y\neq\emptyset$; we can now use the fact that if M is an A-module, $\mathfrak{p}$ is a prime ideal in A, and $f\in A\smallsetminus\mathfrak{p}$, then $\mathfrak{p}\in\mathrm{Ass}_R(M)$ if and only if $\mathfrak{p}R_f\in\mathrm{Ass}_{R_f}(M_f)$ (see Remark E.3.6).

We can thus describe $\mathrm{Ass}(\mathcal{F})$ as follows: given a finite affine open cover $X = U_1 \cup \ldots \cup U_r$, if $A_i = \mathcal{O}_X(U_i)$ and $M_i = \mathcal{F}(U_i)$, then

$$\mathrm{Ass}(\mathcal{F}) = \bigcup_{i=1}^{r} \bigcup_{\mathfrak{p}_i \in \mathrm{Ass}_{A_i}(M_i)} \overline{V(\mathfrak{p}_i)}.$$

In particular, we see that $\mathrm{Ass}(\mathcal{F})$ is a finite set. Note that every irreducible component of $\mathrm{Supp}(\mathcal{F})$ is in $\mathrm{Ass}(\mathcal{F})$ (see Remark E.3.7).

Example 11.4.3. Suppose that X is an irreducible variety. A coherent sheaf $\mathcal{F}$ on X is *torsion-free* if for some (any) affine open cover $X = \bigcup_{i=1}^r U_i$, each $\mathcal{F}(U_i)$ is a torsion-free $\mathcal{O}_X(U_i)$-module (that is, every non-zero element of $\mathcal{O}_X(U_i)$ is a non-zero-divisor on $\mathcal{F}(U_i)$). In this case, the only associated variety of $\mathcal{F}$ is X.

This notion is often applied as follows. Suppose that X is irreducible, $\mathcal{F}$ is a coherent sheaf on X, and D is an effective Cartier divisor on X. If $\mathrm{Supp}(D)$ does not contain any associated subvariety of $\mathcal{F}$, then by tensoring with $\mathcal{F}$ the canonical short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

we obtain a short exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \xrightarrow{\alpha} \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_D \to 0.$$

Indeed, we only need to check the injectivity of α. If $U \subseteq X$ is an affine open subset such that D is described on U by $f \in \mathcal{O}_X(U)$, then α gets identified to the morphism of sheaves associated to the map $\mathcal{F}(U) \to \mathcal{F}(U)$ given by multiplication by f. This is an injective map since f does not lie in any associated prime of $\mathcal{F}(U)$ (see assertion iii) in Proposition E.3.2).

Proof of Theorem 11.4.1. We may assume that $\mathcal{F} \neq 0$, since the case $\mathcal{F} = 0$ is trivial. We argue by induction on $r = \dim\big(\mathrm{Supp}(\mathcal{F})\big)$. If $r = 0$, then the support of $\mathcal{F}$ consists of finitely many points $p_1, \ldots, p_s$. In this case, for every $m \in \mathbf{Z}$ we have

$$\chi\big(\mathcal{F}(m)\big) = h^0\big(\mathcal{F}(m)\big) = h^0(\mathcal{F}) = \sum_{i=1}^{s} \dim_k \mathcal{F}_{p_i}.$$

It follows that we can find $P_{\mathcal{F}}$ as in the theorem and this is a non-zero constant.

Suppose now that $r \geq 1$ and that we know the result for sheaves whose support has dimension $r-1$. We choose a general hyperplane H in $\mathbf{P}^n$ such that H does not contain any subvariety in $\mathrm{Ass}(\mathcal{F})$. As we have seen, in this case we have a short exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(-H) \to \mathcal{F} \to \mathcal{F}_H = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_H \to 0.$$

By tensoring with $\mathcal{O}_{\mathbf{P}^n}(m)$ and taking the Euler-Poincaré characteristic, we obtain

$$\chi\big(\mathcal{F}(m)\big) - \chi\big(\mathcal{F}(m-1)\big) = \chi\big(\mathcal{F}_H(m)\big).$$

Note that $\mathrm{Supp}(\mathcal{F}_H) = \mathrm{Supp}(\mathcal{F}) \cap H$ and our assumption on H implies, in particular, that it does not contain any irreducible component of $\mathrm{Supp}(\mathcal{F})$. It follows that $\dim\big(\mathrm{Supp}(\mathcal{F}|_H)\big) = r-1$, and by the inductive assumption, we have a polynomial Q of degree $r-1$ such that $\chi\big(\mathcal{F}_H(m)\big) = Q(m)$ for all $m \in \mathbf{Z}$. We then conclude by applying Lemma 11.4.2. $\square$

REMARK 11.4.4. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$, it follows from Theorem 11.2.1 that for $m \gg 0$, we have $h^i\big(\mathbf{P}^n, \mathcal{F}(m)\big) = 0$. We thus conclude that

$$P_{\mathcal{F}}(m) = h^0\big(\mathcal{F}(m)\big) \quad \text{for} \quad m \gg 0.$$

EXAMPLE 11.4.5. If $X = \mathbf{P}^n$, then it follows from Theorem 11.2.2 that for $m \geq 0$ we have

$$\chi\big(\mathcal{O}_{\mathbf{P}^n}(m)\big) = h^0\big(\mathcal{O}_{\mathbf{P}^n}(m)\big) = \binom{m+n}{n},$$

hence $P_X(t) = \binom{t+n}{n}$. Similarly, if $L \subseteq \mathbf{P}^n$ is an r-dimensional linear subspace, then $P_L(t) = \binom{t+r}{r}$.

EXAMPLE 11.4.6. If D is an effective Cartier divisor in $\mathbf{P}^n$ of degree d, then it follows from Example 11.2.6 that for $m \geq d$, we have

$$\chi\big(\mathcal{O}_D(m)\big) = h^0\big(\mathcal{O}_D(m)\big) = \binom{m+n}{n} - \binom{m-d+n}{n},$$

hence the Hilbert polynomial of $\mathcal{O}_D$ is $P(t) = \binom{t+n}{n} - \binom{t+n-d}{n}$.

REMARK 11.4.7. We emphasize that the Hilbert polynomial P_X does not depend just on the variety X, but also on the embedding in $\mathbf{P}^n$. For example, it follows from Example 11.4.5 that if we embed $\mathbf{P}^1$ as a line in $\mathbf{P}^2$, then the Hilbert polynomial is $t+1$, while if we embed it as a smooth conic, via the Veronese embedding, the Hilbert polynomial is $2t+1$. More generally, if $C \subseteq \mathbf{P}^n$ is a rational normal curve (recall that this is projectively equivalent to the image of the Veronese embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^n$), then $\mathcal{O}_{\mathbf{P}^n}(1)|_C \simeq \mathcal{O}_{\mathbf{P}^1}(n)$, hence the Hilbert polynomial is $nt+1$.

Given a finitely generated, graded S-module M, where $S = k[x_0, \ldots, x_n]$, we define the *Hilbert polynomial of* M to be the Hilbert polynomial of $\widetilde{M}$. We denote it by P_M. Recall that by Corollary 11.2.3, we have $M_i \simeq \Gamma\big(\mathbf{P}^n, \widetilde{M(i)}\big)$ for $i \gg 0$, so that the Hilbert polynomial of M is the unique polynomial P_M such that

$$P_M(i) = \dim_k M_i \quad \text{for} \quad i \gg 0.$$

REMARK 11.4.8. If $P \in \mathbf{Q}[t]$ is a polynomial, then $P(m)$ is an integer for all $m \in \mathbf{Z}$ if and only if P lies in the Abelian subgroup generated by $\{Q_i \mid i \geq 0\}$. Indeed, it is well-known that $Q_i(m) \in \mathbf{Z}$ for all m. The converse follows by induction on degree: if the polynomial $P(m) - P(m-1)$ lies in the subgroup generated by the Q_i and if $P(0) \in \mathbf{Z}$, then it follows from the last assertion in Lemma 11.4.2 that P also lies in the subgroup generated by the Q_i.

It follows from the above remark that for every non-zero coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$, the top degree term of $P_{\mathcal{F}}$ is $\frac{e}{r!}t^r$, where $r = \dim\big(\mathrm{Supp}(\mathcal{F})\big)$ and $e \in \mathbf{Z} \smallsetminus \{0\}$. Since $P(m) \geq 0$ for $m \gg 0$ (see Remark 11.4.4), it follows that e is a positive integer. This is the *degree* of $\mathcal{F}$, denoted $\deg(\mathcal{F})$. If X is a closed subvariety of $\mathbf{P}^n$, then the *degree* $\deg(X)$ is the degree of $\mathcal{O}_X$.

EXAMPLE 11.4.9. It follows from Example 11.4.5 that if $L \subseteq \mathbf{P}^n$ is a linear subspace, then $\deg(L) = 1$. Similarly, it follows from Example 11.4.6 that if X is a hypersurface in $\mathbf{P}^n$, then the above definition for the degree of X coincides with our old one.

EXAMPLE 11.4.10. If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^n$ with $r = \dim\big(\mathrm{Supp}(\mathcal{F})\big)$ and $\deg(\mathcal{F}) = d$, and if D is an effective Cartier divisor on $\mathbf{P}^n$, of degree e, whose support does not contain any associated subvariety of $\mathcal{F}$, then $\mathcal{F}_D := \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_D$ has $\dim(\mathrm{Supp}(\mathcal{F}_D)\big) = r-1$ and $\deg(\mathcal{F}_Y) = de$. Indeed, we have an exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(-D) \to \mathcal{F} \to \mathcal{F}_D \to 0,$$

which implies that

$$P_{\mathcal{F}_D}(m) = P_{\mathcal{F}}(m) - P_{\mathcal{F}}(m-d),$$

hence the top degree term of $P_{\mathcal{F}_Y}$ is $\frac{de}{(r-1)!}t^{r-1}$.

EXAMPLE 11.4.11. A closed subset $Y \subseteq \mathbf{P}^n$ is *non-degenerate* if there is no hyperplane in $\mathbf{P}^n$ containing Y. Let us show that if $C \subseteq \mathbf{P}^n$ is an irreducible, non-degenerate curve, where $n \geq 2$, then $d = \deg(C) \geq n$. Note that since C is irreducible, the only associated subvariety of $\mathcal{O}_C$ is C itself. By assumption, for every hyperplane H in $\mathbf{P}^n$, we have $C \not\subseteq H$, hence by Example 11.4.10 we have $\deg(C) = \deg(\mathcal{O}_C) = \deg(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_H)$. Note that $\mathcal{F} := \mathcal{O}_C \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_H$ is a coherent sheaf, with support the finite set $C \cap H$, hence

$$d = \deg(\mathcal{F}) = \sum_{p \in C \cap H} \ell_{\mathcal{O}_{\mathbf{P}^n,p}}(\mathcal{F}_p).$$

In particular, we see that $C \cap H$ consists of at most d points.

Suppose now that H is a general hyperplane. Since C has finitely many singular points, it follows from Bertini's theorem that H is transversal to C, meeting C only at smooth points of C (see Remark 6.4.2). In this case, it follows from Proposition 6.3.26 that the radical ideal sheaf corresponding to $C \cap H$ is equal to $\mathcal{I}_C + \mathcal{O}_{\mathbf{P}^n}(-H)$, where $\mathcal{I}_C$ is the radical ideal sheaf corresponding to C. In other words, in this case we have $\mathcal{F} = \mathcal{O}_{C\cap H}$, and thus $C \cap H$ consists precisely of d points.

It is now easy to see that for such general H, the intersection $C \cap H$ is non-degenerate in $H \simeq \mathbf{P}^{n-1}$. Indeed, if $C \cap H$ is contained in a codimension 1 linear subspace $\Lambda \subseteq H$, and if $p \in C \smallsetminus H$, then the linear span H' of Λ and p is a hyperplane in $\mathbf{P}^n$ that meets C in at least $(d+1)$ points. We have seen that this is not possible, and thus $C \cap H$ is non-degenerate in H. Since any $(n-1)$ points in $\mathbf{P}^{n-1}$ are contained in a hyperplane, we conclude that $d \geq n$, as claimed.

11.5. Morphisms to projective spaces

Our goal in this section is to describe morphisms to $\mathbf{P}^n$. Let $S = k[x_0, \ldots, x_n]$ and recall that on $\mathbf{P}^n = \mathrm{MaxProj}(S)$ we have the line bundle $\mathcal{O}_{\mathbf{P}^n}(1) = \widetilde{S(1)}$ such that we have a canonical isomorphism

$$S_1 \simeq \Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big).$$

We put $V = S_1$. Note that the line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ is globally generated: we have $\bigcap_{i=0}^n V(x_i) = \emptyset$.

Given an arbitrary variety Y and a morphism $f\colon Y \to \mathbf{P}^n$, we have a line bundle $\mathcal{L}$ on Y given by $\mathcal{L} = f^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$. Moreover, by pulling back the canonical surjective morphism on $\mathbf{P}^n$:

$$V \otimes_k \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1),$$

we obtain a surjective morphism on Y:

$$\phi\colon V \otimes_k \mathcal{O}_Y \to \mathcal{L}.$$

By considering the basis of $x_0, \ldots, x_n$ of V, we see that giving ϕ is equivalent to giving the sections $s_i = \phi(x_i \otimes 1) \in \Gamma(Y, \mathcal{L})$ and the surjectivity of ϕ is equivalent with the fact that $\bigcap_{i=0}^n V(s_i) = \emptyset$. We will say that two pairs $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ as above are *isomorphic* if there is an isomorphism $\alpha\colon \mathcal{L} \to \mathcal{L}'$ such that $\alpha \circ \phi = \phi'$.

PROPOSITION 11.5.1. *For every algebraic variety Y, if $V = S_1$, we have a natural bijection between morphisms $f\colon Y \to \mathbf{P}^n$ and isomorphism classes of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a line bundle on Y and $\phi\colon V \otimes_k \mathcal{O}_Y \to \mathcal{L}$ is a surjective morphism of $\mathcal{O}_Y$-modules.*

PROOF. We have a map that associates to every morphism $f\colon Y \to \mathbf{P}^n$ the isomorphism class of the pair $(\mathcal{L}, \phi)$, where $\mathcal{L} = f^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ and ϕ is the pull-back of the canonical morphism $V \otimes_k \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)$. We now define a map in the opposite direction.

Given a line bundle $\mathcal{L}$ on Y and a surjective morphism $\phi\colon V \otimes_k \mathcal{O}_Y \to \mathcal{L}$, let $s_i = \phi(x_i \otimes 1)$. We want to define $f\colon Y \to \mathbf{P}^n$ by $f(y) = [s_0(y), \ldots, s_n(y)]$, so we need to make sense of this expression. For every i with $0 \leq i \leq n$, let $V_i = Y \smallsetminus V(s_i)$. Note that since ϕ is surjective, we have $\bigcup_{i=0}^n V_i = Y$. Given i with $0 \leq i \leq n$, for every j there is a unique $a_{i,j} \in \mathcal{O}_X(V_i)$ such that $s_j|_{V_i} = a_{i,j} s_i|_{V_i}$. We define a morphism

$$f_i\colon V_i \to U_i = D^+_{\mathbf{P}^n}(x_i), \quad f_i(x) = [a_{i,0}(x), \ldots, a_{i,n}(x)].$$

The uniqueness of the functions $a_{i,j}$ implies that if $y \in V_{i_1} \cap V_{i_2}$, then $a_{i_1,j}(y) a_{i_2,i_1}(y) = a_{i_2,j}(y)$ for all j and $a_{i_2,i_1}(y) \neq 0$. This implies that $f_{i_1}|_{V_{i_1} \cap V_{i_2}} = f_{i_2}|_{V_{i_1} \cap V_{i_2}}$ for every i_1 and i_2, hence there is a unique $f\colon Y \to \mathbf{P}^n$ such that $f|_{V_i} = f_i$ for all i. It is clear from definition that the morphism f only depends on the isomorphism class of $(\mathcal{L}, \phi)$.

Checking the fact that the two maps we defined are mutual inverses is now a straightforward exercise. $\square$

EXERCISE 11.5.2. If we denote by $\mathrm{Aut}(\mathbf{P}^n)$ the group of automorphisms of $\mathbf{P}^n$, we have seen that we have a group homomorphism

$$PGL_{n+1}(k) \to \mathrm{Aut}(\mathbf{P}^n).$$

Show that this is an isomorphism.

REMARK 11.5.3. There is a way to make precise the "naturality" in the statement of the proposition, as follows. For every variety Y, let $\mathcal{P}^n(Y)$ denote the set of isomorphism classes of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a line bundle on Y and $\phi\colon V \otimes_k \mathcal{O}_X \to \mathcal{L}$ is a surjective morphism. We have a contravariant functor $\mathcal{P}^n$ from the category of varieties to the category of sets, which associates to Y the set $\mathcal{P}^n(Y)$; if $g\colon Z \to Y$ is a morphism, then $\mathcal{P}^n(g)$ maps $(\mathcal{L}, \phi) \in \mathcal{P}^n(Y)$ to $\big(g^*(\mathcal{L}), g^*(\phi)\big) \in \mathcal{P}^n(Z)$. The map we defined in the proof of the proposition is a natural transformation

$$\mathrm{Hom}_{\mathrm{Var}/k}(-, \mathbf{P}^n) \to \mathcal{P}^n$$

and we showed that this is in fact an isomorphism. In other words, $\mathbf{P}^n$ represents the functor $\mathcal{P}^n$.

REMARK 11.5.4. It is sometimes convenient to formulate the assertion in the above proposition in a slightly more functorial way, as follows. Given a finite-dimensional vector space V over k, let $S = \mathrm{Sym}^\bullet(V)$, so that we have a canonical isomorphism $V \simeq S_1$. Of course, if $\dim_k(V) = n+1$, then we have an isomorphism of graded k-algebras $S \simeq k[x_0, \ldots, x_n]$, but this depends on the choice of a basis on V, which we prefer to avoid now. If $X = \mathrm{MaxProj}(S)$, then we have an isomorphism $V \simeq \Gamma\big(X, \mathcal{O}_X(1)\big)$, which induces a surjective morphism of locally free sheaves

$$V \otimes_k \mathcal{O}_X \to \mathcal{O}_X(1)$$

and the proposition implies that the set of morphisms $Y \to X$ is in natural bijection with the set of isomorphism classes of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a line bundle on Y and $\phi\colon V \otimes_k \mathcal{O}_Y \to \mathcal{L}$ is a surjective morphism on Y.

In particular, by taking $Y = \mathrm{MaxSpec}(k)$, we see that the set of morphisms $Y \to X$ (which can be, of course, identified with the set underlying X) is in bijection with isomorphism classes of surjections $V \to k$, that is, with hyperplanes in V. In other words, X is identified naturally with the projective space $\mathbf{P}(V)$ parametrizing the hyperplanes in V.

Note now that for every variety Y and every pair $(\mathcal{L}, \phi)$, with $\mathcal{L}$ a line bundle on Y and $\phi\colon V \otimes_k \mathcal{O}_X \to \mathcal{L}$ surjective, we can describe the corresponding map $f\colon Y \to \mathbf{P}(V)$ by

$$f(y) = \ker\big(V \to \mathcal{L}_{(y)}\big).$$

REMARK 11.5.5. Given a variety X, a line bundle $\mathcal{L}$ on X, and a finite-dimensional vector space V over k, giving a morphism

$$V \otimes_k \mathcal{O}_X \to \mathcal{L}$$

is equivalent to giving a k-linear map $V \to \Gamma(X, \mathcal{L})$.

EXAMPLE 11.5.6. If V, W are finite-dimensional vector spaces over k and we have a surjective morphism $\phi\colon V \to W$, then on $\mathbf{P}(W)$ we have the surjective morphism

$$V \otimes_k \mathcal{O}_{\mathbf{P}(W)} \to W \otimes_k \mathcal{O}_{\mathbf{P}(W)} \to \mathcal{O}_{\mathbf{P}(W)}(1),$$

where the second map is the canonical one and the first map is induced by ϕ. We obtain a corresponding morphism $\mathbf{P}(W) \to \mathbf{P}(V)$, which is a closed embedding of $\mathbf{P}(W)$ as a linear subspace of $\mathbf{P}(V)$, mapping a hyperplane in W to its inverse image in V.

EXAMPLE 11.5.7. If V is a finite-dimensional vector space over k and W is a linear subspace, then by the previous example, we have a closed immersion

$$\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V).$$

If U is the complement of the image of this map, then on U we have a surjective morphism

$$W \otimes_k \mathcal{O}_U \hookrightarrow V \otimes_k \mathcal{O}_U \to \mathcal{O}_{\mathbf{P}(V)}(1)|_U.$$

The induced morphism $\pi\colon U \to \mathbf{P}(W)$ is the *projection onto* $\mathbf{P}(W)$, *with center* $\mathbf{P}(V/W)$. Note that if we choose a splitting of the short exact sequence

$$0 \to W \to V \to V/W \to 0,$$

then we also obtain a closed immersion $\mathbf{P}(W) \hookrightarrow \mathbf{P}(V)$ and the projection gets identified with the morphism described in Example 5.3.9.

EXAMPLE 11.5.8. Given a projective space $\mathbf{P} = \mathbf{P}(V)$, the canonical morphism $V \otimes_k \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(1)$ induces for every $d \geq 1$ a surjective morphism

$$\mathrm{Sym}^d(V) \otimes_k \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(d).$$

The induced morphism $\mathbf{P}(V) \to \mathbf{P}\big(\mathrm{Sym}^d(V)\big)$ is the d^{th} Veronese embedding of $\mathbf{P}(V)$ (cf. Exercise 4.2.22).

EXAMPLE 11.5.9. Given two projective spaces $\mathbf{P}(V)$ and $\mathbf{P}(W)$, we have canonical morphisms

$$V \otimes_k \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(1) \quad \text{and} \quad W \otimes_k \mathcal{O}_{\mathbf{P}(W)} \to \mathcal{O}_{\mathbf{P}(W)}(1).$$

By pulling these back to $\mathbf{P}(V) \times \mathbf{P}(W)$ via the two projections, and taking the tensor product, we obtain a surjective morphism

$$V \otimes_k W \otimes_k \mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(W)} \to \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \mathrm{pr}_1^*\big(\mathcal{O}_{\mathbf{P}(V)}(1)\big) \otimes \mathrm{pr}_2^*\big(\mathcal{O}_{\mathbf{P}(W)}(1)\big).$$

The corresponding morphism $f\colon \mathbf{P}(V) \times \mathbf{P}(W) \to \mathbf{P}(V \otimes_k W)$ is the Segre embedding of $\mathbf{P}(V) \times \mathbf{P}(W)$ (cf. Exercise 4.2.21).

DEFINITION 11.5.10. Recall that a closed subvariety X in $\mathbf{P}^n$ is *non-degenerate* if there is no hyperplane H in $\mathbf{P}^n$ that contains X. Equivalently, the canonical map induced by restriction

$$\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) \to \Gamma\big(X, \mathcal{O}_X(1)\big) \tag{11.5.1}$$

is injective.

A closed subvariety X of $\mathbf{P}^n$ is *linearly normal* if the canonical map (11.5.1) is surjective. We say that X is *projectively normal* if for every $m > 0$, the canonical map

$$\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) \to \Gamma\big(X, \mathcal{O}_X(m)\big)$$

is surjective.

REMARK 11.5.11. Let X be a variety and $f\colon X \to \mathbf{P}(V)$ be a morphism corresponding to the surjective morphism of sheaves on X

$$\phi\colon V \otimes_k \mathcal{O}_X \to \mathcal{L}.$$

Note that a hyperplane in $\mathbf{P}(V)$ defined by $v \in V \smallsetminus \{0\}$ contains $f(X)$ if and only if the image of v in $\Gamma(X, \mathcal{L})$ is 0. Therefore the closure of $f(X)$ is non-degenerate if and only if the induced k-linear map $V \to \Gamma(X, \mathcal{L})$ is injective. In general, if we put $W = \mathrm{Im}\big(V \to \Gamma(X, \mathcal{L})\big)$, then the morphism f factors as

$$X \xrightarrow{g} \mathbf{P}(W) \overset{\iota}{\hookrightarrow} \mathbf{P}(V),$$

where ι is a linear embedding and $\overline{g(X)}$ is non-degenerate in $\mathbf{P}(W)$.

REMARK 11.5.12. Let X be a complete variety, $\mathcal{L}$ a line bundle on X, and W a linear subspace of $V = \Gamma(X, \mathcal{L})$. If the composition

$$W \otimes_k \mathcal{O}_X \hookrightarrow V \otimes_k \mathcal{O}_X \to \mathcal{L}$$

is surjective, then we get a morphism $f\colon X \to \mathbf{P}(W)$, whose image Y is a non-degenerate subvariety of $\mathbf{P}(V)$. Note that we also have a morphism $g\colon X \to \mathbf{P}(V)$ such that if $\pi\colon \mathbf{P}(V) \dashrightarrow \mathbf{P}(W)$ is the projection corresponding to the inclusion $W \hookrightarrow V$, we have $f = \pi \circ g$.

If $V = W$, then the composition

$$V = \Gamma\big(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)\big) \xrightarrow{\phi} \Gamma\big(Y, \mathcal{O}_Y(1)\big) \xrightarrow{\psi} \Gamma(X, \mathcal{L}) = V$$

is the identity. Since the morphism $X \to Y$ is surjective, the map ψ is injective, and thus is an isomorphism. We thus see that in this case Y is a linearly normal subvariety of $\mathbf{P}(V)$. We also see that if $W \neq V$ and f is a closed immersion, then its image in $\mathbf{P}(W)$ is not linearly normal.

REMARK 11.5.13. If $\mathcal{E}$ is a locally free sheaf on the variety T and $g\colon T' \to T$ is an arbitrary morphism, then it is easy to check that we have a Cartesian diagram

$$\begin{tikzcd}
\mathbf{P}\big(g^*(\mathcal{E})\big) \arrow[r] \arrow[d] & \mathbf{P}(\mathcal{E}) \arrow[d, "f"'] \\
T' \arrow[r, "g"] & T.
\end{tikzcd}$$

In particular, we see that the fiber of f over a point $t \in T$ is canonically isomorphic to $\mathbf{P}\big(\mathcal{E}_{(t)}\big)$, that is, to the projective space of hyperplanes in $\mathcal{E}_{(t)}$.

Recall that we have a surjective morphism

$$f^*(\mathcal{E}) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

(see Example 11.2.7). Given a point $t \in T$ and $u \in f^{-1}(t)$, the induced k-linear map between the fibers at u

$$f^*(\mathcal{E})_{(u)} = \mathcal{E}_{(t)} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)_{(u)}$$

is the quotient map by the hyperplane in $\mathcal{E}_{(t)}$ corresponding to u.

EXERCISE 11.5.14. Prove the following variant of Proposition 11.5.1 for projective bundles. Let T be an algebraic variety and $\mathcal{E}$ a locally free sheaf on T. To a variety $g\colon Y \to T$ over T we associate the set $F(Y)$ of isomorphism classes of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a line bundle on Y and $\phi\colon g^*(\mathcal{E}) \to \mathcal{L}$ is a surjective morphism (two such pairs $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are *isomorphic* if there is an isomorphism $\psi\colon \mathcal{L} \to \mathcal{L}'$ such that $\psi \circ \phi = \phi'$). If $h\colon Z \to Y$ is a morphism of varieties over T, then $F(h)$ maps $(\mathcal{L}, \phi)$ to $\big(h^*(\mathcal{L}), h^*(\phi)\big)$. Show that $\mathbf{P}(\mathcal{E})$ represents the functor F in the category of varieties over T.

REMARK 11.5.15. Given a variety X, a line bundle $\mathcal{L}$ on X, a finite-dimensional vector space W over k, and a morphism

$$\phi\colon W \otimes_k \mathcal{O}_X \to \mathcal{L}$$

corresponding to the k-linear map $\psi\colon W \to \Gamma(X, \mathcal{L})$, the image of ϕ is a coherent subsheaf of $\mathcal{L}$. Therefore this is of the form $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}$, for some coherent ideal sheaf $\mathcal{I}$ on X. The closed subvariety $V(\mathcal{I})$ is equal to $\bigcap_{w \in W} V\big(\psi(w)\big)$. If $U = X \smallsetminus V(\mathcal{I})$, then U is non-empty if and only if the map $W \to \Gamma(X, \mathcal{L})$ is nonzero; in this case, the restriction of ϕ to U is surjective and it induces a morphism $U \to \mathbf{P}(W)$.

Let X be an irreducible complete variety and $\mathcal{L}$ a line bundle on X. Recall that the linear system $|\mathcal{L}|$ consists of the effective Cartier divisors D on X that such that $\mathcal{O}_X(D) \simeq \mathcal{L}$ (see Remark 9.4.25). We thus have a canonical bijection between $|\mathcal{L}|$ and $\mathbf{P}\big(\Gamma(X, \mathcal{L})^\vee\big)$.

DEFINITION 11.5.16. More generally, a *linear system* corresponding to $\mathcal{L}$ is a linear subspace of the projective space $\mathbf{P}\big(\Gamma(X,\mathcal{L})^\vee\big)$. The *dimension* of the linear system is the dimension of this linear space. Such a linear system corresponds to a linear subspace V of $\Gamma(X,\mathcal{L})$ and the linear system corresponding to V is denoted by $|V|$. A linear system is *complete* if it is equal to $|\mathcal{L}|$.

Given a non-empty linear system $|V|$, its *base locus* is the closed subset

$$\mathrm{Bs}\big(|V|\big) := \bigcap_{s\in V} V(s).$$

A linear system $|V|$ is *base-point free* if $\mathrm{Bs}\big(|V|\big)=\emptyset$. In general, the restriction of the canonical morphism $V\otimes_k \mathcal{O}_X\to\mathcal{L}$ to $X\smallsetminus \mathrm{Bs}\big(|V|\big)$ is surjective, and we thus have a corresponding morphism

$$f\colon X\smallsetminus \mathrm{Bs}\big(|V|\big)\longrightarrow \mathbf{P}(V)$$

such that the closure of the image of f is non-degenerate in X.

Note that, by definition, we have a bijection between the hyperplanes in $\mathbf{P}(V)$ and the effective divisors in $|V|$ such that if the hyperplane $H\subseteq \mathbf{P}(V)$ corresponds to D, then $f^*(H)=D|_U$.

Our next goal is to give a criterion to determine when a morphism $f\colon Y\to \mathbf{P}(W)$, described by a surjective morphism $\phi\colon W\otimes_k\mathcal{O}_Y\to\mathcal{L}$, with Y complete, is a closed immersion. Note that by Remark 11.5.11, we may replace W by its image in $\Gamma(Y,\mathcal{L})$, in order to assume that $W\subseteq\Gamma(Y,\mathcal{L})$.

DEFINITION 11.5.17. Given a variety Y, a line bundle $\mathcal{L}$ on Y, and a finite-dimensional linear subspace $W\subseteq\Gamma(Y,\mathcal{L})$, we make the following definitions:

i) We say that W *generates* $\mathcal{L}$ if the induced morphism $W\otimes_k\mathcal{O}_Y\to\mathcal{L}$ is surjective.
ii) We say that W *separates points* if for every points $p\neq q$ in Y, there is $s\in W$ such that[1] $s(p)=0$, but $s(q)\neq 0$.
iii) We say that W *separates tangent directions* if for every $p\in Y$ and every non-zero $v\in T_pY$, there is $s\in W$ such that $s(p)=0$, but v does not vanish on the image of s_p in $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p\simeq\mathfrak{m}_p/\mathfrak{m}_p^2$ (if the ideal $\mathcal{I}(s)$ is radical, then this condition is equivalent to saying that $v\not\in T_pV(s)$).

PROPOSITION 11.5.18. *Let Y be a complete variety, $\mathcal{L}$ a line bundle on Y, and $W\subseteq\Gamma(Y,\mathcal{L})$ a subspace that generates $\mathcal{L}$. If $f\colon Y\to\mathbf{P}(W)$ is the morphism corresponding to $\mathcal{L}$ and W, then f is a closed immersion if and only if W separates points and tangent directions.*

PROOF. Note that every non-zero $s\in W$ defines a hyperplane H_s in $\mathbf{P}(W)$ and all hyperplanes arise in this way. It is clear that for a point $p\in Y$, we have $s(p)=0$ if and only if $f(p)\in H_s$. Since for two different points in $\mathbf{P}(W)$, there is a hyperplane that contains the first one, but not the second one, it follows that W separates points if and only if f is injective. Similarly, it is easy to see that if $s(p)=0$ and $v\in T_pY$, then v does not vanish of the image of s_p in $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p\simeq\mathfrak{m}_p/\mathfrak{m}_p^2$ if and only if $df_p(v)\not\in T_{f(p)}H_s$. Since for every non-zero $w\in T_{f(p)}\mathbf{P}(W)$, there is a hyperplane H containing $f(p)$ and with $w\not\in T_{f(p)}H$, it follows that W separates

[1] Recall that $s(p)$ is the image of s_p in $\mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p$, where $\mathfrak{m}_p$ is the ideal defining p.

tangent directions if and only if for every $p \in Y$, the map $df_p\colon T_pY \to T_{f(p)}\mathbf{P}(W)$ is injective (in other words, f is an *immersion*).

It is clear that if f is a closed immersion, then it is injective and it is an immersion. Conversely, let us suppose that these two conditions hold. Note first that since Y is complete, f is proper, hence closed (see Remark 5.1.8). Since f is injective and closed, it gives a homeomorphism $g\colon Y \to Z$ onto a closed subset Z of $\mathbf{P}^n$. If we show that the canonical morphism $\mathcal{O}_{\mathbf{P}^n} \to f_*(\mathcal{O}_Y)$ is surjective, we get an isomorphism $\mathcal{O}_Z \simeq g_*(\mathcal{O}_Y)$, hence g is an isomorphism, and thus f is a closed immersion.

Since f is a homeomorphism onto its image, for every $p \in Y$, we have

$$f_*(\mathcal{O}_Y)_{f(p)} \simeq \mathcal{O}_{Y,p},$$

such that the morphism $\mathcal{O}_{\mathbf{P}^n,f(p)} \to f_*(\mathcal{O}_Y)_{f(p)}$ gets identified with the canonical morphism $\alpha\colon A = \mathcal{O}_{\mathbf{P}^n,f(p)} \to \mathcal{O}_{Y,p} = B$. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ denote the maximal ideals of A and B, respectively. Note that since f is proper, it follows from Theorem 11.3.1 that $f_*(\mathcal{O}_Y)$ is a coherent sheaf on $\mathbf{P}^n$, hence B is a finitely generated A-module. The condition that $T_pY \to T_{f(p)}\mathbf{P}^n$ is injective is equivalent to the fact that the induced k-linear map

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$$

is surjective. This implies by Nakayama's lemma (see Corollary C.1.2) that we have $\mathfrak{m}_B = \mathfrak{m}_A \cdot B$. On the other hand, the induced morphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism (since both quotient rings are isomorphic to k), hence

$$B = \alpha(A) + \mathfrak{m}_B = \alpha(A) + \mathfrak{m}_A \cdot B.$$

Since B is a finitely generated A-module, another application of Nakayama's lemma gives $B = \alpha(A)$, hence α is surjective. This hold for every $p \in Y$, hence the morphism $\mathcal{O}_{\mathbf{P}^n} \to f_*(\mathcal{O}_Y)$ is surjective. This completes the proof of the proposition. □

REMARK 11.5.19. For future reference, we note the following variant of the assertion in Proposition 11.5.18. Suppose that Y is a complete variety and $f\colon Y \to \mathbf{P}(W)$ is the morphism defined by a linear subspace $W \subseteq H^0(Y,\mathcal{L})$. If U is an open subset in $\mathbf{P}(W)$ such that W separates the points in $f^{-1}(U)$ and the tangent directions at the points in $f^{-1}(U)$, then f induces a closed immersion $f^{-1}(U) \to U$. Indeed, since f is proper, the induced morphism $f^{-1}(U) \to U$ is proper and then the argument in the proof of the proposition applies verbatim for this morphism.

11.6. Ample and very ample line bundles

In this section we discuss the connection between ample line bundles and embeddings in projective space.

11.6.1. Ample line bundles. We begin with a discussion of general properties of ample line bundles.

PROPOSITION 11.6.1. *Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a variety X.*

i) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are globally generated $\mathcal{O}_X$-modules, then $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is globally generated.*
ii) *If $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.*
iii) *If m is a positive integer, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is ample.*
iv) *If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.*

v) *If $\mathcal{L}$ is ample, then there is q_0 such that $\mathcal{L}^q \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample for all $q \geq q_0$.*

PROOF. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are globally generated, the following canonical k-linear maps are surjective:

$$\alpha\colon \Gamma(X, \mathcal{F}_1) \otimes_k \mathcal{O}_X \to \mathcal{F}_1 \quad \text{and} \quad \beta\colon \Gamma(X, \mathcal{F}_2) \otimes_k \mathcal{O}_X \to \mathcal{F}_2.$$

Therefore the map $\alpha \otimes_{\mathcal{O}_X} \beta$ is surjective, and since this factors as

$$\Gamma(X, \mathcal{F}_1) \otimes_k \Gamma(X, \mathcal{F}_2) \otimes_k \mathcal{O}_X \xrightarrow{\delta} \Gamma(X, \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2) \otimes_k \mathcal{O}_X \xrightarrow{\gamma} \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2,$$

we conclude that the canonical map γ is surjective, hence $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is globally generated.

If $\mathcal{L}$ and $\mathcal{M}$ are ample, then for every coherent sheaf $\mathcal{F}$ on X, we can find q_1 and q_2 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^q$ is globally generated for $q \geq q_1$ and $\mathcal{M}^q$ is globally generated for $q \geq q_2$. We deduce using i) that if $q \geq \max\{q_1, q_2\}$, then $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M})^q$ is globally generated. Therefore $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.

If $\mathcal{L}$ is ample, by applying ii), we deduce by induction on $m \geq 1$ that $\mathcal{L}^m$ is ample. Conversely, suppose that $\mathcal{L}^m$ is ample. For every coherent sheaf $\mathcal{F}$, applying the definition for each of the sheaves $\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}, \ldots, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{m-1}$, we conclude that there is q_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^i \otimes_{\mathcal{O}_X} (\mathcal{L}^m)^q$ is globally generated for all $0 \leq i \leq m-1$ and $q \geq q_0$. In this case $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^q$ is globally generated for all $q \geq mq_0$. We thus see that $\mathcal{L}$ is ample.

Suppose now that $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated. Given any coherent sheaf $\mathcal{F}$, there is $q_0 \geq 1$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^q$ is globally generated for all $q \geq q_0$. Since $\mathcal{M}$ is globally generated, we deduce using i) that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^q \otimes_{\mathcal{O}_X} \mathcal{M}^q \simeq \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M})^q$$

is globally generated for all $q \geq q_0$. Therefore $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.

Let us prove v). Since $\mathcal{L}$ is ample, there is q_0 such that $\mathcal{L}^q \otimes_{\mathcal{O}_X} \mathcal{M}$ is globally generated for all $q \geq q_0$. In this case, it follows from iv) that $\mathcal{L}^q \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample for all $q \geq q_0 + 1$. □

PROPOSITION 11.6.2. *Let X be an algebraic variety.*

i) *If $\mathcal{M}$ is a globally generated $\mathcal{O}_X$-module, then for every morphism $f\colon Y \to X$, the $\mathcal{O}_Y$-module $f^*(\mathcal{F})$ is globally generated.*
ii) *If $\mathcal{L}$ is an ample line bundle on the algebraic variety X and Z is a locally closed subset of X, then $\mathcal{L}|_X$ is ample on Z.*

PROOF. By definition, the canonical map

$$\alpha\colon \Gamma(X, \mathcal{M}) \otimes_k \mathcal{O}_X \to \mathcal{M}$$

is surjective. Since $f^*(-)$ is a right-exact functor, we deduce that

$$f^*(\alpha)\colon \Gamma(X, \mathcal{M}) \otimes_k \mathcal{O}_Y \to f^*(\mathcal{M})$$

is surjective. Since this factors as

$$\Gamma(X, \mathcal{M}) \otimes_k \mathcal{O}_Y \to \Gamma\big(Y, f^*(\mathcal{M})\big) \otimes_k \mathcal{O}_Y \xrightarrow{\beta} f^*(\mathcal{M}),$$

it follows that β is surjective, hence $f^*(\mathcal{M})$ is globally generated.

In order to prove ii), it is enough to show that given a coherent sheaf $\mathcal{F}$ on Z, we can find a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \simeq \mathcal{G}|_Z$. Indeed, in this case we can find q_0 such that $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^q$ is globally generated for all $q \geq q_0$. By applying i) for the inclusion map $i\colon Z \hookrightarrow X$, we then conclude that $\mathcal{F} \otimes_{\mathcal{O}_Z} (\mathcal{L}|_Z)^q$ is globally generated for all $q \geq q_0$.

In order to prove the existence of $\mathcal{G}$, since Z is the intersection of an open and a closed subvariety, it is enough to treat separately the two cases. If Z is a closed subvariety, the assertion is trivial: we can simply take $\mathcal{G} = i_*(\mathcal{F})$. When Z is an open subvariety, the assertion is more subtle. We treat it separately in the proposition below. □

PROPOSITION 11.6.3. *If U is an open subvariety of the algebraic variety X and $\mathcal{F}$ is a coherent sheaf on U, then the following hold:*

i) *There is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U \simeq \mathcal{F}$.*
ii) *Moreover, if $\mathcal{M}$ is a quasi-coherent sheaf on X such that $\mathcal{F}$ is a subsheaf of $\mathcal{M}|_U$, then there is a coherent subsheaf $\mathcal{G}$ of $\mathcal{M}$ such that we have $\mathcal{F} = \mathcal{G}|_U$.*

PROOF. Let $\alpha\colon U \hookrightarrow X$ be the inclusion map. We first note that it is enough to prove the assertion in ii), since the sheaf $\alpha_*(\mathcal{F})$ on X is quasi-coherent by Proposition 8.4.5 and we have an isomorphism $\mathcal{F} \simeq \alpha_*(\mathcal{F})|_U$. We thus focus on ii) and divide the proof into 2 steps.

Step 1. We first consider the case when X is affine and let $R = \mathcal{O}(X)$. We may assume that $\mathcal{M} = \widetilde{M}$ for an R-module M and choose a finite cover $U = \bigcup_{i=1}^r D_X(f_i)$ by principal affine open subsets in X. For every i, since $\mathcal{F}$ is coherent, $\Gamma\big(D_X(f_i), \mathcal{F}\big)$ is a finitely generated R_{f_i}-module and we have

$$\Gamma\big(D_X(f_i), \mathcal{F}\big) \subseteq \Gamma\big(D_X(f_i), \mathcal{M}\big) = M_{f_i}.$$

Therefore we have elements $u_{i,j} \in M$ for j in a finite set Λ_i such that $\Gamma\big(D_X(f_i), \mathcal{F}\big)$ is generated by the elements $\frac{u_{i,j}}{1} \in M_{f_i}$ for $j \in \Lambda_i$.

Let N be the R-submodule of M generated by the $u_{i,j}$, for $1 \leq i \leq r$ and $j \in \Lambda_i$, so that the subsheaf $\mathcal{N} = \widetilde{N}$ of $\mathcal{M}$ is coherent. We see that for every i, we have $\mathcal{F}|_{D_X(f_i)} \subseteq \mathcal{N}|_{D_X(f_i)}$, and thus $\mathcal{F} \subseteq \mathcal{N}|_U$.

Consider now the canonical morphism $\phi\colon \mathcal{M} \to \alpha_*(\mathcal{M}|_U)$ and let $\mathcal{G} = \mathcal{N} \cap \phi^{-1}\big(\alpha_*(\mathcal{F})\big)$. Therefore $\mathcal{G}$ is quasi-coherent, and being an $\mathcal{O}_X$-submodule of $\mathcal{N}$, it is coherent. Moreover, we have $\mathcal{G}|_U = \mathcal{N}|_U \cap \mathcal{F} = \mathcal{F}$. We are thus done when X is affine.

Step 2. We now deduce the assertion in ii) in the general case. Consider now pairs $(V, \mathcal{G})$, where V is an open subset of X with $U \subseteq V$, and $\mathcal{G} \subseteq \mathcal{M}|_V$ is a coherent subsheaf such that $\mathcal{F} = \mathcal{G}|_U$. For example, $(U, \mathcal{F})$ is such a pair. Since every open subset of X is quasi-compact, it follows that that there is such a pair $(V, \mathcal{G})$ with V maximal. We will show that if $V \neq X$, then we have a contradiction.

If $V \neq X$, then we can choose an affine open subset V' of X such that $V' \not\subseteq V$. Applying Step 1 for the open subset $V \cap V' \subseteq V'$ and $\mathcal{G}|_{V\cap V'} \subseteq \mathcal{M}|_{V \cap V'}$, we find a coherent subsheaf $\mathcal{G}' \subseteq \mathcal{M}|_{V'}$ such that $\mathcal{G}'|_{V\cap V'} = \mathcal{G}|_{V \cap V'}$. In this case, on $V'' = V \cup V'$ we have an $\mathcal{O}_{V''}$-submodule $\mathcal{G}'' \subseteq \mathcal{M}|_{V''}$ such that $\mathcal{G}''|_V = \mathcal{G}$ (hence, in particular, we have $\mathcal{G}''|_U = \mathcal{F}$) and $\mathcal{G}''|_{V'} = \mathcal{G}'$. Since $\mathcal{G}$ and $\mathcal{G}'$ are coherent, it follows that $\mathcal{G}''$ is coherent. This contradicts the maximality of V, completing the proof of the proposition. □

EXAMPLE 11.6.4. Since the structure sheaf of an affine variety is ample, it follows from Proposition 11.6.2 that for every quasi-affine variety X, the structure sheaf $\mathcal{O}_X$ is ample.

REMARK 11.6.5. If X is a complete variety, then $\mathcal{O}_X$ is ample if and only if X is a finite set. The "if" part is clear and for the "only if" part, we may and will assume that X is irreducible (if $X_1, \ldots, X_r$ are the irreducible components of X, since $\mathcal{O}_X$ is ample, it follows that each $\mathcal{O}_{X_i}$ is ample on X_i, and it is enough to show that each X_i consists of one point). Note that $\mathcal{O}_X$ is ample if and only if every coherent sheaf on X is globally generated. Given a point $x \in X$, with corresponding radical ideal sheaf $\mathfrak{m}_x$, we have $\Gamma(X, \mathfrak{m}_x) \subseteq \Gamma(X, \mathcal{O}_X) = k$ and since the sections of $\mathfrak{m}_x$ vanish at x, it follows that $\Gamma(X, \mathfrak{m}_x) = 0$. This implies that $\mathfrak{m}_x = 0$, and thus $X = \{x\}$.

11.6.2. Very ample line bundles over affine varieties.

DEFINITION 11.6.6. If $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{L}$ is a line bundle on X, we say that $\mathcal{L}$ is *f-very ample* (or *very ample over Y*) if there is a locally closed immersion $j\colon X \hookrightarrow \mathbf{P}^n_Y$ over Y such that $\mathcal{L} \simeq j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big)$. We will be especially interested in the case of a proper morphism f, in which case j is automatically a closed immersion. If Y is a point, then we simply say that $\mathcal{L}$ is *very ample*. If D is a Cartier divisor on an irreducible variety X, then D is *very ample* if the line bundle $\mathcal{O}_X(D)$ is very ample.

We begin with a proposition giving some general properties of very ample line bundles.

PROPOSITION 11.6.7. *Let $f\colon X \to Y$ be a morphism of algebraic varieties and $\mathcal{L}, \mathcal{M}$ line bundles on X.*

i) *If $\mathcal{L}$ is very ample over Y, then $\mathcal{L}$ is globally generated.*
ii) *If $\mathcal{L}$ is very ample over Y and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is very ample over Y.*
iii) *If $\mathcal{L}$ is very ample over Y, then $\mathcal{L}^d$ is very ample over Y for all $d > 0$.*

PROOF. If $\mathcal{L}$ is very ample over Y, then there is a locally closed immersion $j\colon X \hookrightarrow \mathbf{P}^n_Y$ such that $j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big) \simeq \mathcal{L}$. Since $\mathcal{O}_{\mathbf{P}^n}(1)$ is globally generated, its pull-back $\mathcal{O}_{\mathbf{P}^n_Y}(1)$ to $\mathbf{P}^n_Y$ is globally generated, and thus the restriction $\mathcal{L}$ of this line bundle to X has the same property. This proves i).

Suppose now that $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated. We thus have a locally closed immersion $j\colon X \hookrightarrow \mathbf{P}^n_Y$ such that $j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big) \simeq \mathcal{L}$. On the other hand, since $\mathcal{M}$ is globally generated, we have a morphism $g\colon X \to \mathbf{P}^m$ such that $g^*\big(\mathcal{O}_{\mathbf{P}^m}(1)\big) \simeq \mathcal{M}$. Consider now the morphism $h\colon X \to \mathbf{P}^n_Y \times \mathbf{P}^m = Y \times \mathbf{P}^n \times \mathbf{P}^m$ over Y given by (j, g). If $\pi\colon \mathbf{P}^n_Y \times \mathbf{P}^m \to \mathbf{P}^n_Y$ is the first projection, the composition $\pi \circ h = j$ is a locally closed immersion, and thus h is a locally closed immersion (see Remark 5.1.8). If $\iota\colon Y \times \mathbf{P}^n \times \mathbf{P}^m \to Y \times \mathbf{P}^N$ maps (y, u, v) to $\big(y, \phi(u, v)\big)$, where $\phi\colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ is the Segre embedding, then $\iota \circ h$ is a locally closed immersion such that $(\iota \circ h)^*\big(\mathcal{O}_{\mathbf{P}^N_Y}(1)\big) \simeq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$. Therefore $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is very ample, completing the proof of ii).

If $\mathcal{L}$ is very ample over Y, it follows from i) that it is globally generated, hence the assertion in iii) follows by induction on d from the assertion in ii). $\square$

Our first goal is to relate the notion of ampleness with that of very ampleness in the case of varieties over an affine variety. Note that while *ampleness* is an absolute notion, *very ampleness* is a relative one.

THEOREM 11.6.8. *Let $f\colon X \to Y$ be a morphism of algebraic varieties, with Y affine. For any line bundle $\mathcal{L}$ on X, the following are equivalent:*

i) *$\mathcal{L}$ is ample.*
ii) *There is a positive integer m such that $\mathcal{L}^m$ is very ample over Y.*

PROOF. The implication ii)⇒i) is easy. If $\mathcal{L}^m$ is very ample over Y, then we have a locally closed immersion $j\colon X \hookrightarrow \mathbf{P}^n_Y$ such that $\mathcal{L}^m \simeq \mathcal{O}_{\mathbf{P}^n_Y}(1)|_X$. Since $\mathcal{O}_{\mathbf{P}^n_Y}(1)$ is ample on $\mathbf{P}^n_Y$ by Proposition 11.1.27, its restriction $\mathcal{L}^m$ to X is ample by assertion ii) in Proposition 11.6.2. Finally, assertion iii) in Proposition 11.6.1 allows us to deduce from $\mathcal{L}^m$ being ample, that $\mathcal{L}$ is ample as well.

We now prove the converse. Suppose that $\mathcal{L}$ is ample. Given a point $x \in X$, choose an affine open neighborhood W of x such that $\mathcal{L}|_W \simeq \mathcal{O}_W$, and let $Z = X \smallsetminus W$. If $\mathcal{I}_Z$ is the radical ideal sheaf corresponding to the closed set Z, using the fact that $\mathcal{L}$ is ample, we obtain that for $m \gg 0$, the sheaf $\mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{L}^m$ is globally generated. Since $\mathcal{I}_{Z,x} = \mathcal{O}_{X,x}$, it follows that there is $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^m)$ such that $s(x) \in \mathcal{L}^m_{(x)}$ is non-zero. Let $U = X \smallsetminus V(s)$. Note that $x \in U \subseteq W$, and thus $U = W \smallsetminus V(s|_W)$, and since $\mathcal{L}|_W$ is trivial, it follows that U is a principal affine open subset of W, hence it is affine. We also note that if we replace s by $s^q \in \Gamma(X, \mathcal{L}^{mq})$, then the set U does not change.

When we let x vary over the points of X, the corresponding open subsets U cover X. Since X is quasi-compact, we deduce that we have finitely many affine open subsets $U_1, \ldots, U_r$ that cover X and for each U_i a section $s_i \in \Gamma(X, \mathcal{L}^{m_i})$ such that $U_i = X \smallsetminus V(s_i)$. Furthermore, after replacing each s_i by a suitable tensor power, we may assume that $m_i = m$ for all i.

Each $\mathcal{O}_X(U_i)$ is a finitely generated k-algebra, hence a finitely generated R-algebra, where $R = \mathcal{O}(Y)$. Let us choose generators $a_{i,1}, \ldots, a_{i,q_i}$ for $\mathcal{O}_X(U_i)$ as an R-algebra. By Lemma 11.1.19, for every i and j, there is m' such that $s_i^{m'} \cdot a_{i,j}$ extends to a section $t_{i,j} \in \Gamma(X, \mathcal{L}^{mm'})$. Moreover, we may clearly replace any m' by a larger value and thus assume that the same m' works for all i and j.

Consider on X the map

$$k^{N+1} \otimes_k \mathcal{O}_X \to \mathcal{L}^{mm'},$$

where $N+1 = r + \sum_{i=1}^r q_i$, that maps the elements of the standard basis of k^{N+1} to the sections $s_i^{m'}$ and $t_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq q_i$. This is a surjective morphism since $\bigcap_{i=1}^r V(s_i^{m'}) = \bigcap_{i=1}^r V(s_i) = \emptyset$. Therefore it induces via Proposition 11.5.1 a morphism

$$g\colon X \to \mathbf{P}^N = \mathrm{Proj}\big(k[x_i, y_{i,j}]\big) \quad \text{with} \quad g^*\big(\mathcal{O}_{\mathbf{P}^N}(1)\big) \simeq \mathcal{L}^{mm'}.$$

If $\alpha = (f,g)\colon X \to \mathbf{P}^N_Y = Y \times \mathbf{P}^N$, then α is a morphism over Y such that $\alpha^*\big(\mathcal{O}_{\mathbf{P}^N_Y}(1)\big) \simeq \mathcal{L}^{mm'}$. In order to conclude the proof, it is enough to show that α is a locally closed immersion.

It is clear from definition that the image of α lies in the open subset $V = \bigcup_{i=1}^r D^+(x_i)$ of $\mathbf{P}^N_Y$. Moreover, the induced morphism $\beta\colon X \to V$ is a closed immersion. This follows from Proposition 2.3.10, since each $\beta^{-1}\big(D^+(x_i)\big) = U_i$ is affine and the induced homomorphism

$$\mathcal{O}\big(D^+(x_i)\big) \to \mathcal{O}_X(U_i)$$

is surjective, since each $\frac{y_{i,j}}{x_i}$ maps to $a_{i,j}$, and $a_{i,1}, \ldots, a_{i,q_i}$ generate $\mathcal{O}_X(U_i)$. Therefore α is a locally closed immersion, completing the proof of the theorem. □

COROLLARY 11.6.9. *A variety X is quasi-projective if and only if it has an ample line bundle.*

PROOF. The assertion follows from the theorem by taking Y to be a point, since it is clear that X is quasi-projective if and only if it has a very ample line bundle. $\square$

REMARK 11.6.10. We can make a stronger assertion than the one in the theorem: if $f\colon X \to Y$ is a morphism of algebraic varieties, with Y affine, and $\mathcal{M}$ and $\mathcal{L}$ are line bundles on X, with $\mathcal{L}$ ample, then there is $d_0 \geq 1$ such that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^d$ is very ample over Y for every $d \geq d_0$. Indeed, it follows from the theorem that there is m such that $\mathcal{L}^m$ is very ample over Y. On the other hand, since $\mathcal{L}$ is ample, it follows that there is q such that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^j$ is globally generated for every $j \geq q$. Using assertion ii) in Proposition 11.6.7, we conclude that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^d$ is very ample over Y for all $d \geq m+q$.

This observation implies that if $f\colon X \to Y$ is a morphism, with Y affine, and if there is an ample line bundle $\mathcal{L}$ on X, then every $\mathcal{M} \in \mathrm{Pic}(X)$ can be written as $\mathcal{M} \simeq \mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2^{-1}$, where both $\mathcal{M}_1$ and $\mathcal{M}_2$ are very ample over Y. Indeed, we may take $\mathcal{M}_1 = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^d$ and $\mathcal{M}_2 = \mathcal{L}^d$, for $d \gg 0$.

REMARK 11.6.11. A useful consequence of the existence of an ample line bundle on a variety X is that every coherent sheaf $\mathcal{F}$ on X has a locally free resolution. Indeed, if $\mathcal{L}$ is an ample line bundle on X, then there is m such that there is a surjective morphism

$$\mathcal{O}_X^{\oplus N_0} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m.$$

By tensoring this with $\mathcal{L}^{-m}$, we obtain a surjective morphism $\phi_0\colon \mathcal{E}_0 \to \mathcal{F}$, where $\mathcal{E}_0 = (\mathcal{L}^{-m})^{\oplus N_0}$. We can now repeat the construction with $\mathcal{F}$ replaced by $\ker(\phi_0)$ in order to get an exact complex

$$\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

with $\mathcal{E}_1$ locally free as well. By repeating this argument, we obtain the locally free resolution of $\mathcal{F}$. In particular, it follows from Corollary 11.6.9 that on a quasi-projective variety, every coherent sheaf has a locally free resolution.

COROLLARY 11.6.12. *Let $f\colon X \to Y$ be a proper morphism, with Y affine. If $\mathcal{L}$ is an ample line bundle on X, then for d divisible enough, the following hold:*

i) *For every $j \geq 1$, the canonical multiplication map*

$$S^j_{\mathcal{O}(Y)}\Gamma(X, \mathcal{L}^d) \to \Gamma(X, \mathcal{L}^{dj})$$

is surjective.

ii) *If $S(\mathcal{L}^d) = \bigoplus_{j\geq 0} \Gamma(X, \mathcal{L}^{dj})$, then we have an isomorphism*

$$X \simeq \mathrm{MaxProj}\big(S(\mathcal{L}^d)\big).$$

PROOF. It follows from Theorem 11.6.8 that we can find $m \geq 1$ such that $\mathcal{L}^m$ is very ample. We thus have a closed immersion $j\colon X \hookrightarrow \mathbf{P}^n_Y$ such that $j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big) \simeq \mathcal{L}^m$. The assertion in the corollary now follows from Remark 11.2.5. $\square$

We will mostly be interested in the case when X is proper over Y. In this case (still assuming that Y is affine), we have the following cohomological characterization of ampleness:

THEOREM 11.6.13. *Let $f\colon X\to Y$ be a proper morphism of algebraic varieties, with Y affine. For any line bundle $\mathcal{L}$ on X, the following are equivalent:*

i) *$\mathcal{L}$ is ample.*
ii) *For every coherent sheaf $\mathcal{F}$ on X, there is m_0 such that*

$$H^i(X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m)=0\quad\textit{for all}\quad i\geq 1\quad\textit{and}\quad m\geq m_0.$$

PROOF. In fact, we will show that the condition in ii) is equivalent to the very ampleness over Y for some power of $\mathcal{L}$ and then use the previous theorem. Suppose first that $\mathcal{L}^q$ is very ample, for some positive integer q, and let $j\colon X\hookrightarrow \mathbf{P}^n_Y$ be a closed embedding over Y such that $j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big)\simeq\mathcal{L}^q$. If $\mathcal{F}$ is a coherent sheaf on X, then it follows from assertion ii) in Theorem 11.2.1 that there is m_1 such that

$$H^i\big(\mathbf{P}^n_Y,j_*(\mathcal{F})\otimes\mathcal{O}_{\mathbf{P}^n_Y}(m)\big)=0\quad\text{for all}\quad i\geq 1\quad\text{and}\quad m\geq m_1.$$

Since j is a closed immersion, using the projection formula, we obtain

$$H^i\big(\mathbf{P}^n_Y,j_*(\mathcal{F})\otimes\mathcal{O}_{\mathbf{P}^n_Y}(m)\big)\simeq H^i\big(\mathbf{P}^n_Y,j_*(\mathcal{F}\otimes j^*(\mathcal{O}_{\mathbf{P}^n_Y}(m)))\big)\simeq H^i(X,\mathcal{F}\otimes\mathcal{L}^{mq}).$$

We thus conclude that $H^i(X,\mathcal{F}\otimes\mathcal{L}^{mq})=0$ for all $m\geq m_1$ and $i\geq 1$. Applying this for $\mathcal{F}\otimes\mathcal{L}^j$, with $1\leq j\leq q-1$, we get m_{j+1} such that $H^i(X,\mathcal{F}\otimes\mathcal{L}^{mq+j})=0$ for all $m\geq m_{j+1}$ and $i\geq 1$. We thus see that if $m_0=q\cdot\max\{m_j\mid 1\leq j\leq q\}$, then $H^i(X,\mathcal{L}^m)=0$ for all $m\geq m_0$ and all $i\geq 1$.

Conversely, we show that if the condition in ii) holds, then $\mathcal{L}$ is very ample over Y. By running the same argument as in the proof of Theorem 11.6.8, we see that it is enough to show that for every $x\in X$, there is a positive integer m and $s\in\Gamma(X,\mathcal{L}^m)$ such that $X\smallsetminus V(s)$ is an affine open neighborhood of x. The argument is similar to the one in the proof of Theorem 10.4.1: let W be an affine open neighborhood of x such that $\mathcal{L}|_W\simeq\mathcal{O}_W$, put $Z=(X\smallsetminus W)\cup\{x\}$, and let $\mathcal{I}_Z$ be the radical sheaf of ideals defining Z. By tensoring the short exact sequence

$$0\to\mathcal{I}_Z\to\mathcal{O}_X\to\mathcal{O}_Z\to 0$$

with $\mathcal{L}^m$, for $m\gg 0$, and taking the long exact sequence in cohomology, we obtain an exact sequence

$$\Gamma(X,\mathcal{L}^m)\xrightarrow{\phi}\Gamma(Z,\mathcal{L}^m|_Z)\to H^1(X,\mathcal{I}_Z\otimes\mathcal{L}^m)=0.$$

Since ϕ is surjective and since there is a section of $\mathcal{L}^m|_Y$ that is 0 on $X\smallsetminus U$ and is non-zero at x, we obtain a section $s\in\Gamma(X,\mathcal{L}^m)$ such that $s(x)\neq 0$ and $s|_{X\smallsetminus U}=0$. In this case $X\smallsetminus V(s)=W\smallsetminus V(s|_W)$ is a principal affine open subset of W, and thus it is affine. Since it contains x, this completes the proof of the theorem. $\square$

COROLLARY 11.6.14. *Let $f\colon X\to Y$ be a proper morphism, with Y affine, and $\mathcal{L}$ an ample line bundle on X. For every exact complex of coherent sheaves on X:*

$$\mathcal{F}_1\to\mathcal{F}_2\to\ldots\to\mathcal{F}_m,$$

the induced complex of A-modules

$$\Gamma(X,\mathcal{F}_1\otimes_{\mathcal{O}_X}\mathcal{L}^q)\to\Gamma(X,\mathcal{F}_2\otimes_{\mathcal{O}_X}\mathcal{L}^q)\to\ldots\to\Gamma(X,\mathcal{F}_m\otimes_{\mathcal{O}_X}\mathcal{L}^q)$$

is exact for all $q\gg 0$.

PROOF. Of course, it is enough to treat the case when $m=3$. If

$$\mathcal{A}=\ker(\mathcal{F}_1\to\mathcal{F}_2),\quad\mathcal{B}=\mathrm{Im}(\mathcal{F}_1\to\mathcal{F}_2),\quad\text{and}\quad\mathcal{C}=\mathrm{Im}(\mathcal{F}_2\to\mathcal{F}_3),$$

then we have short exact sequences

$$0 \to \mathcal{A} \to \mathcal{F}_1 \to \mathcal{B} \to 0$$

and

$$0 \to \mathcal{B} \to \mathcal{F}_2 \to \mathcal{C} \to 0.$$

Since $\mathcal{L}$ is ample, it follows from Theorem 11.6.13 that

$$H^1(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^q) = 0 = H^1(X, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \quad \text{for} \quad q \gg 0.$$

For such q, the long exact sequences in cohomology give short exact sequences

$$0 \to \Gamma(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to 0$$

and

$$0 \to \Gamma(X, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to 0.$$

Since the morphism

$$\Gamma(X, \mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{L}^q)$$

is injective for all q, we deduce that the sequence

$$\Gamma(X, \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{L}^q) \to \Gamma(X, \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{L}^q)$$

is exact for all $q \gg 0$. □

REMARK 11.6.15. One can make a slightly stronger vanishing statement than the one in the theorem, and this is sometimes useful: if $f \colon X \to Y$ is a morphism of algebraic varieties, with Y affine, $\mathcal{F}$ is a coherent sheaf on X, and $\mathcal{L}$ is a line bundle on X such that $\mathcal{L}|_{\mathrm{Supp}(\mathcal{F})}$ is ample and $\mathrm{Supp}(\mathcal{F}) \to Y$ is proper, then there is m_0 such that

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad m \geq m_0.$$

Indeed, it follows Remark 8.4.21 that if $\mathcal{I}$ is the radical ideal sheaf defining $\mathrm{Supp}(\mathcal{F})$, then $\mathcal{F}$ has a finite filtration

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_r = \mathcal{F}$$

such that $\mathcal{F}_j/\mathcal{F}_{j-1}$ is annihilated by $\mathcal{I}$ for $1 \leq j \leq r$. By Theorem 11.6.13, we can find m_0 such that $H^i\big(X, (\mathcal{F}_j/\mathcal{F}_{j-1}) \otimes_{\mathcal{O}_X} \mathcal{L}^m\big) = 0$ for all $i \geq 1$, $1 \leq j \leq r$, and $m \geq m_0$. Using the long exact sequence in cohomology for

$$0 \to \mathcal{F}_{j-1} \to \mathcal{F}_j \to \mathcal{F}_j/\mathcal{F}_{j-1} \to 0,$$

we see by induction on $j \geq 1$ that $H^i(X, \mathcal{F}_j \otimes_{\mathcal{O}_X} \mathcal{L}^m) = 0$ for all $i \geq 1$ and $m \geq m_0$. For $j = r$, we obtain our assertion.

EXERCISE 11.6.16. Let $f \colon X \to Y$ be a morphism of algebraic varieties, with Y affine, and let $\mathcal{F}$ be a coherent sheaf on X such that $\mathrm{Supp}(\mathcal{F}) \to Y$ is proper. Show that if $\mathcal{L}$ is a line bundle on X such that $\mathcal{L}|_{\mathrm{Supp}(\mathcal{F})}$ is ample, then there is m_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m$ is globally generated for all $m \geq m_0$.

The cohomological characterization of ample line bundles Theorem 11.6.13 is very useful for proving permanence properties of ample line bundles. We give a few applications in this direction.

COROLLARY 11.6.17. *Let X be a variety that is proper over an affine variety (for example, a complete variety) and $\mathcal{L}$ a line bundle on X. If $f \colon W \to X$ is a finite morphism, then $f^*(\mathcal{L})$ is ample.*

PROOF. Given a coherent sheaf $\mathcal{F}$ on X, by Example 10.5.16 and the projection formula, we have

$$H^i\big(W, \mathcal{F}\otimes f^*(\mathcal{L}^m)\big) \simeq H^i\big(X, f_*(\mathcal{F}\otimes f^*(\mathcal{L}^m))\big) \simeq H^i\big(X, f_*(\mathcal{F})\otimes \mathcal{L}^m\big) \tag{11.6.1}$$

for all $i\geq 0$ and $m\in \mathbf{Z}$. Since $f_*(\mathcal{F})$ is coherent and $\mathcal{L}$ is ample, the right-most term in (11.6.1) vanishes for all $i\geq 1$ if $m\gg 0$. The characterization in the theorem thus implies that $f^*(\mathcal{L})$ is ample. $\square$

REMARK 11.6.18. In particular, the above corollary implies that if $f\colon W\to X$ is a finite morphism of algebraic varieties and X is projective, then W is projective. For example, this implies that the normalization of any projective variety is again a projective variety.

EXERCISE 11.6.19. Show that if X is a complete variety, with irreducible components $X_1,\ldots,X_r$, and $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}$ is ample if and only if $\mathcal{L}|_{X_i}$ is ample for every i.

EXERCISE 11.6.20. Using the approach in Exercise 10.5.21, prove the following converse to Corollary 11.6.17: if X is a variety that is proper over an affine variety, $f\colon W\to X$ is a finite, surjective morphism, and $\mathcal{L}$ is a line bundle on X such that $f^*(\mathcal{L})$ is ample, then $\mathcal{L}$ is ample.

EXERCISE 11.6.21. Let X and Y be algebraic varieties and $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the two projections. Show that if $\mathcal{L}\in \operatorname{Pic}(X)$ and $\mathcal{M}\in\operatorname{Pic}(Y)$ are very ample (ample), then $p^*(\mathcal{L})\otimes_{\mathcal{O}_{X\times Y}} q^*(\mathcal{M})$ is very ample (respectively ample).

EXERCISE 11.6.22. Recall that if $Y=\mathbf{P}^m\times\mathbf{P}^n$, then every line bundle $\mathcal{L}$ on Y is isomorphic to $\mathrm{pr}_1^*\big(\mathcal{O}_{\mathbf{P}^m}(a)\big)\otimes_{\mathcal{O}_Y}\mathrm{pr}_2^*\big(\mathcal{O}_{\mathbf{P}^n}(b)\big)$ for a unique $(a,b)\in\mathbf{Z}^2$ (see Example 9.4.29); in this case, we say that $\mathcal{L}$ has type (a,b). Show that if $\mathcal{L}$ has type (a,b), then the following are equivalent:

i) $\mathcal{L}$ is very ample.
ii) $\mathcal{L}$ is ample.
iii) We have $a>0$ and $b>0$.

11.7. Relatively ample line bundles and projective morphisms

In this section we introduce a relative version of ampleness, for proper morphisms, and use it to study projective morphisms.

DEFINITION 11.7.1. Given a proper morphism $f\colon X\to Y$ of algebraic varieties, a line bundle $\mathcal{L}$ on X is *f-ample* (or *ample over* Y) if for every affine open subset $V\subseteq Y$, the line bundle $\mathcal{L}|_{f^{-1}(V)}$ on $f^{-1}(V)$ is ample.

EXAMPLE 11.7.2. If $f\colon X\to Y$ is a finite morphism, then every line bundle on X is f-ample.

EXAMPLE 11.7.3. If $f\colon X\to Y$ is a proper morphism, then every f-very ample line bundle on X is f-ample.

EXAMPLE 11.7.4. If Y is an algebraic variety, $\mathcal{S}$ is an $\mathcal{O}_Y$-algebra as in §8.6.3, and $f\colon X=\mathcal{P}roj(\mathcal{S})\to Y$ is the corresponding projective morphism, then the line bundle $\mathcal{O}_X(1)$ is ample over Y.

REMARK 11.7.5. It follows from definition and assertion iii) in Proposition 11.6.1 that if $f\colon X\to Y$ is a proper morphism and m is a positive integer, then a line bundle $\mathcal{L}$ on X is f-ample if and only if $\mathcal{L}^m$ is f-ample.

PROPOSITION 11.7.6. *Given a proper morphism of algebraic varieties $f\colon X\to Y$ and a line bundle $\mathcal{L}$ on X, the following are equivalent:*

i) *$\mathcal{L}$ is f-ample.*

ii) *There is a finite affine open cover $Y=\bigcup_{j\in J}V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is ample for all $j\in J$.*

iii) *For every coherent sheaf $\mathcal{F}$ on X, there is m_0 such that*
$$R^if_*(X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m)=0\quad \textit{for all}\quad m\geq m_0\quad \textit{and}\quad i\geq 1.$$

iv) *For every coherent sheaf $\mathcal{F}$ on X, there is m_0 such that the canonical morphism*
$$f^*f_*\big(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m\big)\to\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m$$
induced by the adjunction is surjective for all $m\geq m_0$.

PROOF. Given a finite affine open cover $Y=\bigcup_{j\in J}V_j$, note that for a given m and i, we have
$$R^if_*(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m)=0\quad\text{if and only if}\quad H^i\big(f^{-1}(V_j),\mathcal{F}\otimes\mathcal{L}^m\big)=0\quad\text{for all}\quad j\in J.$$
Similarly, we see that the morphism
$$f^*f_*\big(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m\big)\to\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m$$
is surjective if and only if for every $j\in J$, the restriction of this morphism to $f^{-1}(V_j)$, which gets identified to the morphism
$$\Gamma\big(f^{-1}(V_j),\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m\big)\otimes_k\mathcal{O}_{f^{-1}(V_j)}\to\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^m|_{f^{-1}(V_j)}$$
is surjective. We also note that by Proposition 11.6.3, for every coherent sheaf $\mathcal{G}$ on some $f^{-1}(V_j)$, we can write $\mathcal{G}\simeq\mathcal{F}|_{f^{-1}(V_j)}$ for some coherent sheaf $\mathcal{F}$ on X. We thus conclude that each restriction $\mathcal{L}|_{f^{-1}(V_j)}$ is ample if and only if condition iv) holds, if and only if condition iii) holds (by Theorem 11.6.13). Since the conditions in iii) and iv) are independent of the given cover, we see that also conditions ii) and i) are equivalent. □

REMARK 11.7.7. It follows from the above proposition that if X is a proper variety over Y, with Y affine, then a line bundle $\mathcal{L}$ on X is ample over Y if and only if it is ample (in the usual sense).

REMARK 11.7.8. Given a proper morphism $f\colon X\to Y$ and a line bundle $\mathcal{L}$ on X, for every line bundle $\mathcal{M}$ on Y, we have $\mathcal{L}$ ample over Y if and only if $\mathcal{L}\otimes_{\mathcal{O}_X}f^*(\mathcal{M})$ is ample over Y. For this, it is enough to consider a finite affine open cover $Y=\bigcup_{j\in J}V_j$ such that each $\mathcal{M}|_{V_j}$ is trivial, and use the characterization ii) in Proposition 11.7.6.

PROPOSITION 11.7.9. *A proper morphism $f\colon X\to Y$ is projective if and only if there is an f-ample line bundle on X.*

PROOF. The "only if" part follows from Example 11.7.4. Conversely, suppose that $\mathcal{L}$ is an f-ample line bundle on X. Consider a finite affine open cover $Y=\bigcup_{i\in I}U_i$. By Theorem 11.6.8, we can find m such that for every $i\in I$, the line bundle $\mathcal{L}^m|_{f^{-1}(U_i)}$ is very ample; in order to have the same m for all i, we use the fact that the power of a very ample line bundle is very ample (see assertion iii) in Proposition 11.6.7). Moreover, after replacing m by a suitable multiple, we may assume that for every $j \geq 0$, the canonical multiplication map

$$\mathrm{Sym}^j_{\mathcal{O}_Y(U_i)}\Gamma\big(f^{-1}(U_i), \mathcal{L}^m\big) \to \Gamma\big(f^{-1}(U_i), \mathcal{L}^{jm}\big)$$

is surjective and we have a canonical isomorphism

$$f^{-1}(U_i) \simeq \mathrm{MaxProj}\left(\bigoplus_{j\geq 0}\Gamma\big(f^{-1}(U_i), \mathcal{L}^{jm}\big)\right)$$

(see Remark 11.2.5). The $\mathcal{O}_Y$-algebra

$$\mathcal{S} = \bigoplus_{j\geq 0} f_*(\mathcal{L}^{jm})$$

is quasi-coherent and reduced, and generated over $\mathcal{S}_0$ by $\mathcal{S}_1$. Moreover, $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent, and we can thus consider $Z = \mathcal{M}axProj(\mathcal{S}) \to Y$. The above isomorphisms glue together to an isomorphism $X \simeq Z$ of varieties over Y. Therefore f is a projective morphism. $\square$

Proposition 11.7.10. *Let $f\colon X \to Y$ be a proper morphism. If $\mathcal{L}$ is an f-ample line bundle one X and $\mathcal{M}$ is an ample line bundle on Y, then there is $m > 0$ such that the line bundle $\mathcal{L}^m \otimes f^*(\mathcal{M}^d)$ is f-very ample for every $d \gg 0$.*

Proof. We choose an integer $m > 0$ and the corresponding $\mathcal{O}_Y$-algebra $\mathcal{S} = \bigoplus_{j\geq 0} f_*(\mathcal{L}^{jm})$ as in the proof of Proposition 11.7.9. Note that if

$$g\colon Z = \mathcal{M}axProj(\mathcal{S}) \to Y$$

is the canonical morphism, then we have an isomorphism $h\colon X \to Z$ over Y such that $\mathcal{L}^m \simeq h^*\big(\mathcal{O}_Z(1)\big)$. Since $\mathcal{M}$ is ample, it follows that $\mathcal{S}_1 \otimes_{\mathcal{O}_Y} \mathcal{M}^d$ is globally generated for all $d \gg 0$. For every such d, consider the $\mathcal{O}_Y$-algebra $\mathcal{S}' = \bigoplus_{j\geq 0}(\mathcal{S}_j \otimes_{\mathcal{O}_Y} \mathcal{M}^{dj})$. Note that for every open subset U of Y with $\mathcal{L}|_U \simeq \mathcal{O}_U$, we have $\mathcal{S}'|_U \simeq \mathcal{S}|_U$. We may thus define $g'\colon Z' = \mathcal{M}axProj(\mathcal{S}')$ and for every U as above, we have an isomorphism $(g')^{-1}(U) \simeq g^{-1}(U)$. These isomorphisms glue together to an isomorphism $\alpha\colon Z \to Z'$ over Y. It is straightforward to see that $\alpha^*\big(\mathcal{O}_{Z'}(1)\big) \simeq \mathcal{O}_Z(1) \otimes_{\mathcal{O}_Z} g^*(\mathcal{M}^d)$. On the other hand, since $\mathcal{S}'_1$ is globally generated, it follows that we have a surjective morphism $\mathcal{O}_Y[x_0, \ldots, x_N] \to \mathcal{S}'$ which induces a closed immersion $j\colon Z' \hookrightarrow \mathbf{P}^N_Y$ over Y such that $j^*\big(\mathcal{O}_{\mathbf{P}^N_Y}(1)\big) \simeq \mathcal{O}_{Z'}(1)$. By putting these together, we conclude that for every such d, the line bundle $\mathcal{L}^m \otimes f^*(\mathcal{M}^d)$ on X is f-ample. $\square$

Proposition 11.7.11. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be proper morphisms, $\mathcal{L}$ a line bundle on X and $\mathcal{M}$ a line bundle on Y.*

i) *If $\mathcal{L}$ is f-very ample and $\mathcal{M}$ is g-very ample, then $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{M})$ is $(g \circ f)$-very ample.*

ii) *If $\mathcal{L}$ is f-ample and $\mathcal{M}$ is g-ample, then $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{M}^d)$ is $(g \circ f)$-ample for $d \gg 0$*

Proof. In order to prove i), consider closed immersions

$$i = (\alpha, f)\colon X \hookrightarrow \mathbf{P}^n_Y = \mathbf{P}^n \times Y \quad \text{and} \quad j = (\beta, g)\colon Y \hookrightarrow \mathbf{P}^m_Z = \mathbf{P}^m \times Z$$

such that $i^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big) \simeq \mathcal{L}$ and $j^*\big(\mathcal{O}_{\mathbf{P}^m_Z}(1)\big) \simeq \mathcal{M}$. In this case we have a closed immersion ϕ given by the composition

$$X \xrightarrow{i} \mathbf{P}^n \times Y \xrightarrow{\mathrm{id}_{\mathbf{P}^n}\times j} \mathbf{P}^n \times \mathbf{P}^m \times Z \xrightarrow{\tau\times \mathrm{id}_Z} \mathbf{P}^N \times Z,$$

where $\tau\colon \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N$ is the Segre embedding. Since

$$\phi^*\big(\mathcal{O}_{\mathbf{P}^N_Z}(1)\big) \simeq \mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{M}),$$

it follows that $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{M})$ is $(g\circ f)$-very ample.

We now prove ii). Since we can cover Z by finitely many affine open subsets, we see that it is enough to prove ii) when Z is an affine variety, in which case $\mathcal{M}$ is ample. Applying Proposition 11.7.10 for f, we can find positive integers d_1 and m such that $\mathcal{L}^m \otimes_{\mathcal{O}_X} f^*(\mathcal{M}^d)$ is f-very ample for all $d \geq d_1$. Similarly, applying Proposition 11.7.10 for g, we can find a positive integer d_2 such that $\mathcal{M}^{d_2}$ is g-very ample. In this case, it follows from i) that $\mathcal{L}^m \otimes_{\mathcal{O}_X} f^*(\mathcal{M}^d)$ is $(g\circ f)$-very ample for all $d \geq d_1 + d_2$, and thus $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{M}^d)$ is $(g\circ f)$-ample if $d \geq \frac{d_1+d_2}{m}$. $\square$

By combining Proposition 11.7.9 with assertion ii) in Proposition 11.7.11, we obtain the following corollary.

COROLLARY 11.7.12. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are projective morphisms, then $g \circ f$ is a projective morphism.*

REMARK 11.7.13. If $f\colon X \to Y$ is a proper morphism and $\mathcal{L}$ is an f-ample line bundle on X, then for every Cartesian diagram

$$\begin{tikzcd} X\times_Y Z \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[d, "u"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

then $v^*(\mathcal{L})$ is g-ample. Indeed, after covering Y by affine open subsets V_i and covering each $u^{-1}(V_i)$ by affine open subsets $U_{i,j}$, we reduce the statement to the case when both Y and Z are affine. In this case $\mathcal{L}$ is ample and by Theorem 11.6.8 there is a positive integer m and a closed immersion $j = (\alpha, f)\colon X \hookrightarrow \mathbf{P}^n_Y = \mathbf{P}^n \times Y$ such that $\mathcal{L}^m \simeq j^*\big(\mathcal{O}_{\mathbf{P}^n_Y}(1)\big)$. Consider now the commutative diagram

$$\begin{tikzcd} X\times_Y Z \arrow[r, "j_Z"] \arrow[d, "v"] & \mathbf{P}^n_Z \arrow[r] \arrow[d, "w"] & Z \arrow[d, "u"] \\ X \arrow[r, "j"] & \mathbf{P}^n_Y \arrow[r] & Y, \end{tikzcd}$$

where $j_Z = (\alpha\circ v, g)$. Since the right square is Cartesian and the big rectangle is Cartesian by hypothesis, it follows that the left square is Cartesian. In particular, since j is a closed immersion, it follows that j_Z is a closed immersion. Since

$$v^*(\mathcal{L}^m) \simeq v^*\big(j^*(\mathcal{O}_{\mathbf{P}^n_Y}(1))\big) \simeq j_Z^*\big(w^*(\mathcal{O}_{\mathbf{P}^n_Z}(1))\big) \simeq j_Z^*\big(\mathcal{O}_{\mathbf{P}^n_Z}(1)\big),$$

we conclude that $v^*(\mathcal{L})$ is g-ample.

In particular, we see that for every $y \in Y$, the restriction $\mathcal{L}|_{X_y}$ to the fiber $X_y = f^{-1}(y)$ is ample. In Corollary 11.7.15 below we will see that conversely, if the restriction of $\mathcal{L}$ to each fiber of f is ample, then $\mathcal{L}$ is f-ample.

We will not make use of the following result in what follows, but we include it for the sake of completeness. Its proof makes use of the fact that proper morphisms with finite fibers are finite; we thus put it off until Chapter 14.

THEOREM 11.7.14. *Let $f\colon X \to Y$ be a proper morphism and $\mathcal{L}$ a line bundle on X. If $y \in Y$ is such that the restriction of $\mathcal{L}$ to the fiber $X_y = f^{-1}(y)$ is an ample line bundle, then there is an affine open neighborhood U of y such that $\mathcal{L}|_{f^{-1}(U)}$ is ample. In particular, the set*

$$\{z \in Y \mid \mathcal{L}|_{X_z} \textit{ is ample}\}$$

is open in Y.

COROLLARY 11.7.15. *If $f\colon X \to Y$ is a proper morphism and $\mathcal{L}$ is a line bundle on X such that the restriction of $\mathcal{L}$ to each fiber of f is ample, then $\mathcal{L}$ is f-ample.*

PROOF. Using the theorem, we obtain an affine open cover $Y = \bigcup_{i \in I} U_i$ such that $\mathcal{L}|_{U_i}$ is ample for every $i \in I$. In this case $\mathcal{L}$ is f-ample by Proposition 11.7.6. $\square$

CHAPTER 12

Depth and Cohen-Macaulay rings

In this chapter we discuss some more advanced topics in local algebra. We assume some basic results in commutative algebra as covered in Appendices A-H and the facts about Ext and Tor modules covered in §10.7. For the applications in the geometric setting, we only need the case when the rings we deal with are algebras of finite type over an algebraically closed field k, as well as the localizations of such rings. However, it is more natural to work in the general setting.

In the first section we discuss the notion of depth and characterize it in terms of the vanishing of certain Ext modules. In particular, we prove an important result of Auslander-Buchsbaum relating depth to projective dimension. We also give here a normality criterion, due to Serre, involving the notion of depth. In the second section we discuss the Koszul complex and its relationship to depth. Finally, in the last section we introduce Cohen-Macaulay rings and modules; for the sake of simplicity, we restrict to the case of algebraic varieties and coherent sheaves on them. For a more in-depth treatment of the topics discussed in this chapter, we refer to [**BH93**].

12.1. Depth

12.1.1. Regular sequences and depth. The following is a key notion for this chapter.

Definition 12.1.1. Given a finitely generated module M over the Noetherian ring R, a sequence of elements $x_1, \ldots, x_n \in R$ is an *M-regular sequence* (or a *regular sequence for M*) if the following conditions hold:

i) We have $(x_1, \ldots, x_n)M \neq M$.
ii) For every i with $1 \leq i \leq n$, the element x_i is a non-zero-divisor on the R-module $M/(x_1, \ldots, x_{i-1})M$.

If $M = R$, we simply say that $x_1, \ldots, x_n$ form a *regular sequence.*

Remark 12.1.2. If $x_1, \ldots, x_n$ is an M-regular sequence and S is a multiplicative system in R such that $S^{-1}M/(x_1, \ldots, x_n)S^{-1}M \neq 0$, then it is clear that $\frac{x_1}{1}, \ldots, \frac{x_n}{1} \in S^{-1}R$ is an $S^{-1}M$-regular sequence.

As we will see in the next section, regular sequences tend to behave better when we work in a local ring.

Remark 12.1.3. Suppose that M is a finitely generated module over a Noetherian ring R. If $x_1, \ldots, x_n$ is an M-regular sequence, then

$$(x_1)M \subsetneq (x_1, x_2)M \subsetneq \ldots \subsetneq (x_1, \ldots, x_n)M.$$

Indeed, if $1 \leq i \leq n$ is such that $x_iM \subseteq (x_1, \ldots, x_{i-1})M$, since x_i is a non-zero-divisor on $M/(x_1, \ldots, x_{i-1})M$, it follows that $(x_1, \ldots, x_{i-1})M = M$ contradicting condition i) in the definition of an M-regular sequence.

Since M is a Noetherian module, it follows that given any ideal I in M, every M-regular sequence of elements in I can be completed to a *maximal* such sequence. We note that if $IM \neq M$, an M-regular sequence $x_1, \ldots, x_n$ of elements in I is maximal among such sequences if and only if I is contained in the set of zero-divisors of $M/(x_1, \ldots, x_n)M$. By Remark E.3.5, this is the case if and only if there is $u \in M \smallsetminus (x_1, \ldots, x_n)M$ such that $I \cdot u \subseteq (x_1, \ldots, x_n)M$.

DEFINITION 12.1.4. Let M be a finitely generated module over a Noetherian ring R. If I is an ideal in R, we put

$$\operatorname{depth}(I, M) := \min\{i \geq 0 \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Note that if the set on the right-hand side is empty, then we follow the convention tht $\operatorname{depth}(I, M) = \infty$. If R is a local ring and $\mathfrak{m}$ is the maximal ideal, then we write $\operatorname{depth}(M)$ for $\operatorname{depth}(\mathfrak{m}, M)$.

The following result makes the connection with regular sequences and motivates the above definition.

THEOREM 12.1.5. *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal in R.*

i) *If $IM = M$, then* $\operatorname{depth}(I, M) = \infty$.
ii) *If $IM \neq M$, then* $\operatorname{depth}(I, M)$ *is equal to the length of every maximal M-regular sequence of elements of I.*

PROOF. Suppose first that $IM = M$. In order to show that $\operatorname{depth}(I, M) = \infty$ it is enough to show that for every prime ideal $\mathfrak{p}$ in R, we have $\operatorname{Ext}^i_R(R/I, M)_{\mathfrak{p}} = 0$ for all $i \geq 0$. Note that by Corollary 10.7.3, we have

$$\operatorname{Ext}^i_R(R/I, M)_{\mathfrak{p}} \simeq \operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

If $I \subseteq \mathfrak{p}$, then the hypothesis together with Nakayama's lemma implies $M_{\mathfrak{p}} = 0$, hence $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$. On the other hand, if $I \not\subseteq \mathfrak{p}$, then $R_{\mathfrak{p}} = IR_{\mathfrak{p}}$, and again we have $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$. This completes the proof of i).

Suppose now that $IM \neq M$ and let $x_1, \ldots, x_n$ be a maximal M-regular sequence in I. We show that $\operatorname{depth}(I, M) = n$ arguing by induction on n. If $n = 0$, then there is no non-zero-divisor on M in I (we use here that $IM \neq M$). It follows that we have $u \in M \smallsetminus \{0\}$ such that $I \cdot u = 0$, and thus a non-zero morphism $R/I \to M$ that maps the image of 1 to u. This shows that $\operatorname{Hom}(R/I, M) \neq 0$, hence $\operatorname{depth}(I, M) = 0$.

Suppose now that we know the assertion when we have a maximal M-regular sequence of length $n-1$. Since x_1 is a non-zero-divisor on M, we have a short exact sequence

$$0 \to M \xrightarrow{\cdot x_1} M \to M/x_1M \to 0.$$

Note that multiplication by x_1 on each $\operatorname{Ext}^i_R(R/I, M)$ is 0 since $x_1 \in I$. The long exact sequence for Ext modules thus breaks into short exact sequences

$$0 \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, M/x_1M) \to \operatorname{Ext}^{i+1}_R(R/I, M) \to 0.$$

This immediately implies that $\operatorname{depth}(I, M/x_1M) = \operatorname{depth}(I, M) - 1$. On the other hand, it is clear that $x_2, \ldots, x_n$ is a maximal M/x_1M-regular sequence in I. Since $I \cdot (M/x_1M) \neq M/x_1M$, we conclude using the induction hypothesis that $n - 1 = \operatorname{depth}(I, M) - 1$. This completes the proof of the induction step and thus the proof of the theorem. $\square$

REMARK 12.1.6. It follows from the above proof that if $J = \mathrm{rad}(I)$, then

$$\mathrm{depth}(J, M) = \mathrm{depth}(I, M).$$

Note first that we have $IM = M$ if and only if $JM = M$. If this is not the case, then it is enough to show that $\mathrm{depth}(J, M)$ is equal to the length of any maximal M-regular sequence contained in I. The above proof carries through with one modification: we need to note that if there is a non-zero $u \in M$ such that $I \cdot u = 0$, then there is also a non-zero $v \in M$ with $J \cdot v = 0$ (this is straightforward to check), and thus $\mathrm{Hom}_R(R/J, M) \neq 0$.

REMARK 12.1.7. It follows from the theorem that if $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a} \cdot M = 0$, then $\mathrm{depth}(I, M) = \mathrm{depth}\big((I + \mathfrak{a})/\mathfrak{a}, M\big)$, where in the second depth, we consider M as an $R/\mathfrak{a}$-module.

COROLLARY 12.1.8. *If R, M, and I are as in the theorem, and $x \in I$ is a non-zero-divisor on M, then*

$$\mathrm{depth}(I, M/xM) = \mathrm{depth}(I, M) - 1. \tag{12.1.1}$$

PROOF. We have already noticed this equality in the proof of Theorem 12.1.5. □

EXAMPLE 12.1.9. If M is a finitely generated module over a Noetherian ring R and I is an ideal generated by an M-regular sequence $x_1, \ldots, x_n$, then $\mathrm{depth}(I, M) = n$. Indeed, a repeated application of the previous corollary reduces the assertion to showing that if $I = 0$ and M is non-zero, then $\mathrm{depth}(I, M) = 0$, which is obvious.

COROLLARY 12.1.10. *If R, M, and I are as in the theorem and J is an ideal containing I, then*

$$\mathrm{depth}(I, M) \leq \mathrm{depth}(J, M).$$

PROOF. The inequality follows immediately from the theorem. □

COROLLARY 12.1.11. *If R, M, and I are as in the theorem, then*

$$\mathrm{depth}(I, M) = \min\{\mathrm{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \supseteq I\},$$

where the minimum is over the prime ideals $\mathfrak{p}$ containing I. In particular, if $\mathfrak{m}$ is a maximal ideal in R, then $\mathrm{depth}(\mathfrak{m}, M) = \mathrm{depth}(M_{\mathfrak{m}})$.

PROOF. For every prime ideal $\mathfrak{p}$ and every i, we have

$$\mathrm{Ext}^i_R(R/I, M)_{\mathfrak{p}} \simeq \mathrm{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})$$

by Proposition 10.7.3. It follows from the description in the theorem (see also Remark 12.1.2) that for every prime ideal $\mathfrak{p}$ containing I, we have

$$\mathrm{depth}(I, M) \leq \mathrm{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \leq \mathrm{depth}(M_{\mathfrak{p}}),$$

where the second inequality follows from the previous corollary.

If $IM = M$, then we are done. Suppose now that $IM \neq M$ and let $x_1, \ldots, x_n$ be a maximal M-regular sequence contained in I. Since every element of I is a zero-divisor on $M/(x_1, \ldots, x_n)M$, it follows that there is $\mathfrak{p} \in \mathrm{Ass}_R\big(M/(x_1, \ldots, x_n)M\big)$ such that $I \subseteq \mathfrak{p}$ (see Remark E.3.5). Since $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}}\big(M_{\mathfrak{p}}/(x_1, \ldots, x_n)M_{\mathfrak{p}}\big)$, it follows that $\frac{x_1}{1}, \ldots, \frac{x_n}{1}$ is a maximal $M_{\mathfrak{p}}$-regular sequence in $\mathfrak{p}R_{\mathfrak{p}}$, and thus

$$\mathrm{depth}(I, M) = \mathrm{depth}(M_{\mathfrak{p}}).$$

□

PROPOSITION 12.1.12. *Given a short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of finitely generated modules over the Noetherian ring R, the following hold:

i) $\mathrm{depth}(I, M) \geq \min\{\mathrm{depth}(I, M'), \mathrm{depth}(I, M'')\}$.
ii) $\mathrm{depth}(I, M') \geq \min\{\mathrm{depth}(I, M), \mathrm{depth}(I, M'') + 1\}$.
iii) $\mathrm{depth}(I, M'') \geq \min\{\mathrm{depth}(I, M), \mathrm{depth}(I, M') - 1\}$.

PROOF. All assertions follows directly from definition and the long exact sequence for the Ext modules:

$$\ldots \to \mathrm{Ext}_R^{i-1}(R/I, M'') \to \mathrm{Ext}_R^{i}(R/I, M') \to \mathrm{Ext}_R^{i}(R/I, M)$$
$$\to \mathrm{Ext}_R^{i}(R/I, M'') \to \mathrm{Ext}_R^{i+1}(R/I, M') \to \ldots.$$

□

Recall that for a non-zero finitely generated R-module M, one defines

$$\dim(M) = \dim\big(R/\mathrm{Ann}_R(M)\big).$$

PROPOSITION 12.1.13. *If M is a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$, then for every $\mathfrak{p} \in \mathrm{Ass}_R(M)$, we have*

$$\mathrm{depth}(M) \leq \dim(R/\mathfrak{p}).$$

In particular, we have $\mathrm{depth}(M) \leq \dim(M)$.

PROOF. Let $\mathfrak{p} \in \mathrm{Ass}_R(M)$. In particular, we see that $M \neq 0$, and thus $\mathfrak{m}M \neq M$ by Nakayama's lemma. We argue by induction on $n = \mathrm{depth}(M)$. If $n = 0$, then there is nothing to prove. Otherwise, let $x \in \mathfrak{m}$ be a non-zero-divisor on M. By Corollary 12.1.8, we have $\mathrm{depth}(M/xM) = n - 1$. On the other hand, by hypothesis, there is $u \in M$ such that $\mathfrak{p} = \mathrm{Ann}_R(u)$. By Krull's Intersection theorem (see Theorem C.4.1), we have $\bigcap_{j \geq 0} x^j M = 0$, hence there is $\ell \geq 0$ such that $u = x^\ell v$ and $v \notin xM$. Since x is a non-zero-divisor on M, it follows that $\mathfrak{p} = \mathrm{Ann}_R(v)$ and thus $\mathfrak{p}$ annihilates the non-zero element $\overline{v} \in M/xM$. It follows from Remark E.3.5 that there is $\mathfrak{q} \in \mathrm{Ass}_R(M/xM)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Note that $x \in \mathrm{Ann}_R(M/xM) \subseteq \mathfrak{q}$, while $x \notin \mathfrak{p}$, since x is a non-zero-divisor on M. We thus have $\dim(R/\mathfrak{p}) \geq \dim(R/\mathfrak{q}) + 1$ and we conclude using the induction hypothesis. □

REMARK 12.1.14. A related inequality says that if $\mathfrak{a} \subsetneq R$ is an ideal in a Noetherian ring R, then

$$\mathrm{depth}(\mathfrak{a}, R) \leq \mathrm{codim}(\mathfrak{a})$$

(recall that $\mathrm{codim}(\mathfrak{a}) = \min_{\mathfrak{p}}\{\mathrm{codim}(\mathfrak{p})\}$, where the minimum is over the minimal primes containing $\mathfrak{a}$ or, equivalently, over all primes containing $\mathfrak{a}$). The above inequality follows easily by induction on $\mathrm{depth}(\mathfrak{a}, R) = r$, noting that if $a \in R$ is a non-zero-divisor, then it does not lie in any minimal prime ideal of R, and thus for every prime ideal $\mathfrak{p}$ containing a, we have $\mathrm{codim}(\mathfrak{p}) \geq \mathrm{codim}\big(\mathfrak{p}/(a)\big) + 1$.

EXAMPLE 12.1.15. Let X be an algebraic variety and $x \in X$ a smooth point, with $\dim(\mathcal{O}_{X,x}) = n$. It follows from Proposition 6.3.8 that $\mathcal{O}_{X,x}$ is a domain, hence every non-zero element in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$ forms an $\mathcal{O}_{X,x}$-regular sequence. Moreover, if $x_1, \ldots, x_n$ is a minimal system of generators of $\mathfrak{m}$, then it follows from Proposition 6.3.20 that for every $i \geq 1$, the ring $\mathcal{O}_{X,x}/(x_1, \ldots, x_i)$ is the local ring of a smooth variety; in particular, it is a domain. We thus see that $x_1, \ldots, x_n$ is an $\mathcal{O}_{X,x}$-regular sequence, and thus $\operatorname{depth}(\mathcal{O}_{X,x}) = n$.

In fact, the same holds for any regular local ring, see [**BH93**, §2.2]. We have restricted to the case of local rings of smooth points on algebraic varieties since we only proved in this setting the basic properties of regular local rings.

12.1.2. Depth and projective dimension. Our next goal is to prove a result due to Auslander and Buchsbaum, relating depth and projective dimension. We first need some preparations regarding minimal free resolutions over Noetherian local rings.

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module. If $u_1, \ldots, u_n \in M$ give a minimal system of generators and if $\phi_0 \colon F_0 = R^n \to M$ is given by $\phi_0(e_i) = u_i$, then $\ker(\phi_0) \subseteq \mathfrak{m}F_0$ (this follows from the fact that $\overline{u_1}, \ldots, \overline{u_n} \in M/\mathfrak{m}M$ are linearly independent over k). If we choose a minimal system of generators of $\ker(\phi_0)$, we obtain a morphism $\phi_1 \colon F_1 \to F_0$ such that $\operatorname{Im}(\phi_1) = \ker(\phi_0) \subseteq \mathfrak{m}F_0$. Continuing in this way we obtain a free resolution

$$F_\bullet : \ldots \to F_m \xrightarrow{\phi_m} \ldots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

of M such that $\phi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all $i \geq 1$. A free resolution with this property is called *minimal*.

Proposition 12.1.16. *If $F_\bullet \to M$ is a minimal free resolution of M, then*

$$\operatorname{rank}(F_i) = \dim_k \operatorname{Tor}_i^R(k, M).$$

Proof. Note first that since $\operatorname{Tor}_i^R(k, M)$ is annihilated by $\mathfrak{m}$, it follows that indeed, $\operatorname{Tor}_i^R(k, M)$ is a k-vector space (finitely generated by Proposition 10.7.20). Since $F_\bullet \to M$ is a free resolution, we have

$$\operatorname{Tor}_i^R(k, M) \simeq \mathcal{H}_i(k \otimes_R F_\bullet).$$

On the other hand, with respect to suitable bases of F_i and F_{i-1}, the matrix of ϕ_i has entries in $\mathfrak{m}$; therefore all maps in $k \otimes_R F_\bullet$ are 0, hence

$$\operatorname{Tor}_i^R(k, M) \simeq k \otimes_R F_i.$$

This gives the assertion in the proposition. $\square$

Remark 12.1.17. The above proposition shows that the ranks of the free modules in a minimal free resolution do not depend on the resolution. In fact, any two minimal free resolutions of M are isomorphic (though the isomorphism is not unique). Indeed, suppose that $F_\bullet \to M$ and $G_\bullet \to M$ are two minimal free resolutions. It follows from the analogue of Proposition 10.1.11 for projective resolutions that we have a morphism of complexes $F_\bullet \to G_\bullet$ that lifts the identity on M. By tensoring with k, we see that each morphism $k \otimes_R F_i \to k \otimes_R G_i$ is an isomorphism. Since R is local, it follows that each map $F_i \to G_i$ is an isomorphism, proving our assertion. Because of this, one often talks about *the* minimal free resolution of M.

Corollary 12.1.18. *If M is a finitely generated module over the Noetherian local ring R and q is a non-negative integer, then the following are equivalent:*

i) $\operatorname{pd}_R(M) \leq q$.
ii) $\operatorname{Tor}_i^R(N, M) = 0$ *for all $i \geq q + 1$ and all R-modules N.*
iii) $\operatorname{Tor}_{q+1}^R(k, M) = 0$.
iv) *If $F_\bullet \to M$ is the minimal free resolution of M, then $F_{q+1} = 0$.*

Proof. The implication i)⇒ii) follows from Proposition 10.7.25, ii)⇒iii) is trivial, iii)⇒iv) follows from the proposition, and iv)⇒i) is clear. □

Corollary 12.1.19. *If $(R, \mathfrak{m}, k)$ is a Noetherian local ring, then the global dimension of R is equal to $\mathrm{pd}_R(k)$.*

Proof. The fact that gl-dim(R) is at least as large as $\mathrm{pd}_R(k)$ follows from the definition of global dimension. On the other hand, if $\mathrm{pd}_R(k) = n$, then it follows from Proposition 10.7.25, that $\mathrm{Tor}_i^R(N, k)$ for every $i > n$ and every R-module N. Applying the previous corollary, we see that for every finitely generated R-module N, we have $\mathrm{pd}_R(N) \leq n$. □

Lemma 12.1.20. *Given an R-module M, if $x \in R$ is a non-zero divisor on both R and M, we have $\mathrm{Tor}_R^i\big(M, R/(x)\big) = 0$ for $i \geq 1$.*

Proof. Since x is a non-zero-divisor on R, we have the following free resolution of $R/(x)$:

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0.$$

It is then clear that $\mathrm{Tor}_i^R\big(M, R/(x)\big) = 0$ for $i \geq 2$, while

$$\mathrm{Tor}_1^R\big(M, R/(x)\big) \simeq \ker\big(M \xrightarrow{\cdot x} M\big) = 0.$$

□

We can now prove the connection between depth and projective dimension.

Theorem 12.1.21 (Auslander-Buchsbaum). *If $(R, \mathfrak{m})$ is a Noetherian local ring and M is a non-zero R-module with $\mathrm{pd}_R(M) < \infty$, then*

$$\mathrm{depth}(R) = \mathrm{depth}(M) + \mathrm{pd}_R(M). \tag{12.1.2}$$

Proof. We argue by induction on depth(R) and treat separately different cases. Since $\mathrm{pd}_R(M) < \infty$, it follows from Corollary 12.1.18 that the minimal free resolution of M is finite:

$$0 \to F_n \xrightarrow{\phi} F_{n-1} \to \ldots \to F_0 \to M \to 0.$$

Case 1. Suppose first that depth$(R) = 0$. Therefore $\mathfrak{m} \in \mathrm{Ass}_R(R)$, hence there is a non-zero $u \in R$ such that $\mathfrak{m} \cdot u = 0$. If $n > 0$, then $\phi(F_n) \subseteq \mathfrak{m}F_{n-1}$, hence $\phi(u \cdot F_n) = 0$, contradicting the injectivity of ϕ. Therefore $n = 0$, that is, M is free, in which case it is clear that depth(M) = depth$(R) = 0$. This proves (12.1.2) in this case.

Case 2. Suppose now that depth$(R) > 0$ and depth$(M) > 0$. In this case there is $x \in \mathfrak{m}$ which is a non-zero-divisor on both R and M. The hypothesis on x implies, using the lemma, that $F_\bullet \otimes_R R/(x)$ is an exact complex, hence a minimal free resolution of M/xM. By Nakayama's lemma, we have $F_i = 0$ if and only if $F_i/xF_i = 0$, and we deduce using Corollary 12.1.18 that $\mathrm{pd}_{R/(x)}(M/xM) = \mathrm{pd}_R(M)$. On the other hand, using Corollary 12.1.8, we have

$$\mathrm{depth}(M/xM) = \mathrm{depth}(M) - 1 \quad \text{and} \quad \mathrm{depth}\big(R/(x)\big) = \mathrm{depth}(R) - 1.$$

We deduce the equality in (12.1.2) from the induction hypothesis.

Case 3. Suppose that depth$(R) > 0$ and depth$(M) = 0$. Note that in this case M can't be free, hence $N = \ker(F_0 \to M)$ is non-zero. Note that $\mathrm{pd}_R(N) = \mathrm{pd}_R(M) - 1$. On the other hand, we have depth$(N) = 1$ by Proposition 12.1.12. We thus obtain the equality in (12.1.2) by applying Case 2 to N. □

12.1.3. Serre's normality criterion. We now use the notion of depth to prove a normality criterion due to Serre.

DEFINITION 12.1.22. Given a Noetherian ring R, we say that R satisfies Serre's condition (R_i) if for every prime ideal $\mathfrak{p}$ in R, with $\operatorname{codim}(R_{\mathfrak{p}}) \leq i$, the local ring $R_{\mathfrak{p}}$ is regular.

EXAMPLE 12.1.23. If X is an affine variety and $A = \mathcal{O}(X)$, then A satisfies property R_i if and only if $\operatorname{codim}_X(X_{\mathrm{sing}}) \geq i+1$.

DEFINITION 12.1.24. We say that a Noetherian ring R satisfies Serre's condition (S_i) if for every prime ideal $\mathfrak{p}$ in R, we have

$$\operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{\dim(R_{\mathfrak{p}}), i\}.$$

EXAMPLE 12.1.25. A Noetherian ring R satisfies (S_1) if and only if every associated prime of R is minimal. It satisfies both (R_0) and (S_1) if and only if for every associated prime $\mathfrak{p}$ of R, we have $\mathfrak{p}R_{\mathfrak{p}} = 0$. It is clear that this holds if R is reduced. The converse also holds: if $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ is a minimal primary decomposition, then conditions (R_0) and (S_1) imply that if $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$, then each $\mathfrak{p}_i$ is a minimal prime ideal and $\mathfrak{q}_i R_{\mathfrak{p}_i} \subseteq \mathfrak{p}_i R_{\mathfrak{p}_i} = 0$; since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, it follows that $\mathfrak{q}_i = \mathfrak{p}_i$ for all i, hence R is reduced.

As in the geometric setting, we say that an arbitrary Noetherian ring R is *normal* if $R_{\mathfrak{p}}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ in R (or, equivalently, for every maximal ideal $\mathfrak{p}$ in R).

REMARK 12.1.26. We note that a normal ring is isomorphic to a product of normal domains. Indeed, if R is normal and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of R, then $\mathfrak{p}_i + \mathfrak{p}_j = R$ for every $i \neq j$ (this is due to the fact that $R_{\mathfrak{p}}$ is a domain for every maximal ideal $\mathfrak{p}$ in R). Moreover, since al localizations of R are reduced, it follows that R is reduced, hence $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r = 0$. We thus conclude from the Chinese Remainder theorem that the canonical morphism

$$R \to R/\mathfrak{p}_1 \times \ldots \times R/\mathfrak{p}_r$$

is an isomorphism. Furthermore, for every prime ideal $\mathfrak{q}$ containing $\mathfrak{p}_i$, the localization $R_{\mathfrak{q}}$ is a normal domain, hence $(R/\mathfrak{p}_i)_{\mathfrak{q}} = R_{\mathfrak{q}}$ is normal. We thus deduce that each $R/\mathfrak{p}_i$ is a normal domain.

THEOREM 12.1.27 (Serre). *A Noetherian ring R is normal if and only if it satisfies conditions (R_1) and (S_2).*

PROOF. After localizing, we may assume that $(R, \mathfrak{m})$ is a local ring. It is straightforward to see that if R is a domain, then having (R_1) + (S_2) is just a reformulation of conditions i) + ii) in Proposition E.5.1. In particular, the "only if" assertion in the theorem is clear. For the "if" part, the subtlety is that we don't know *a priori* that R is a domain.

Suppose now that R satisfies conditions (R_1) and (S_2). In particular, it satisfies (R_0) + (S_1), and thus R is reduced by Example 12.1.25. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of R, and let $S = R \smallsetminus \bigcup_{i=1}^r \mathfrak{p}_i$ be the set of non-zero-divisors in R. Consider the inclusion map $\phi\colon R \hookrightarrow K = S^{-1}R$. The Chinese Remainder theorem gives an isomorphism $K \simeq \prod_{i=1}^r K_i$, where $K_i = \operatorname{Frac}(R/\mathfrak{p}_i) = R_{\mathfrak{p}_i}$. If we can show that $r = 1$, then R is a domain, in which case we are done. We follow the proof of Proposition E.5.1 to show that R is integrally closed in K. If we know this, and $e_i \in K$ is the idempotent corresponding to $1 \in K_i$, then $e_i^2 = e_i$ implies that e_i lies in R. Since R is local, the only idempotents it has are 0 and 1, and these are mapped by ϕ to 0 and 1, respectively, in K. We thus see that $r = 1$.

Suppose that $\frac{b}{a} \in K$ is a non-zero element that is integral over R (note that a is a non-zero-divisor). Consider a minimal primary decomposition

$$(a) = \mathfrak{q}_1 \cap \ldots \cap q_s.$$

If $\widetilde{\mathfrak{q}}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\widetilde{\mathfrak{q}}_i \in \mathrm{Ass}\big(R/(a)\big)$ by Remark E.3.13. Condition (S_2) implies that $\mathrm{codim}(\widetilde{\mathfrak{q}_j}) = 1$, and condition (R_1) implies that $R_{\widetilde{\mathfrak{q}}_j}$ is a DVR. Let j be fixed and consider i such that $\mathfrak{p}_i \subseteq \widetilde{\mathfrak{q}}_j$. Since $\frac{b}{a}$ is integral over R, its image in K_i is integral over R, and since $R_{\widetilde{\mathfrak{q}}_j} \subseteq K_i$ is a DVR, hence integrally closed, we conclude that there is $s \in R \smallsetminus \widetilde{\mathfrak{q}}_j$ such that $sb \in (a)$. Since $\mathfrak{q}_j$ is a primary ideal, it follows that $b \in \mathfrak{q}_j$. Since this holds for every j, we conclude that $b \in (a)$ and thus $\frac{b}{a} \in R$. This completes the proof of the theorem. $\square$

12.2. The Koszul complex

In this section we discuss a complex that can be used to compute the depth of a module. We begin by considering the algebraic context and then we describe how the construction can be globalized in the geometric setting. We end this chapter with an application of the Koszul complex to proving the basic properties of Castelnuovo-Mumford regularity of a sheaf on the projective space.

12.2.1. The Koszul complex: definition and first properties. Let R be a commutative ring, E an R-module, and $\phi\colon E \to R$ a morphism of R-modules. We define a complex $K(\phi) = K(\phi)_\bullet$, as follows. For every $p \geq 1$, we put $K(\phi)_p = \wedge^p E$ and define $d_p\colon \wedge^p E \to \wedge^{p-1} E$ by

$$d_p(e_1 \wedge \ldots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i-1} \phi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_p.$$

Note that $d_1 = \phi$. It is straightforward to check that $d_p \circ d_{p+1} = 0$ for all $p \geq 1$, hence $K(\phi)$ is a complex: the *Koszul complex* associated to ϕ. We will mostly be interested in the case when E is a free R-module. In this case, if $\mathrm{rank}(E) = n$, then $K(\phi)_p = 0$ for $p > n$. If M is an R-module, then we put $K(\phi; M) := K(\phi) \otimes_R M$.

If $\underline{x}$ is a sequence $x_1, \ldots, x_n$ of elements of R, then we write $K(x_1, \ldots, x_n)$ or $K(\underline{x})$ for the complex $K(\phi)$, where $\phi\colon R^n \to R$ is given by $\phi(e_i) = x_i$, and similarly, $K(x_1, \ldots, x_n; M)$ or $K(\underline{x}; M)$ for $K(\underline{x}) \otimes_R M$. We also write $\mathcal{H}_i(x_1, \ldots, x_n; M)$ or $\mathcal{H}_i(\underline{x}; M)$ for $\mathcal{H}_i\big(K(\underline{x}; M)\big)$.

EXAMPLE 12.2.1. If $n = 1$, then the Koszul complex corresponding to the element $x \in R$ consists of

$$0 \to R \xrightarrow{\cdot x} R \to 0.$$

REMARK 12.2.2. Note that for every $\phi\colon E \to R$, we have

$$\mathcal{H}_0\big(K(\phi; M)\big) \simeq \mathrm{coker}(\phi) \otimes_R M.$$

REMARK 12.2.3. Given a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "u"'] & R \arrow[d, "\mathrm{id}_R"] \\ F \arrow[r, "\psi"] & R, \end{tikzcd}$$

we obtain a morphism of complexes $K(\phi) \to K(\psi)$ given for every $p \geq 0$ by $\wedge^p u\colon \wedge^p E \to \wedge^p F$.

In particular, we see that if $u\colon E \to F$ is an isomorphism, then we have an induced isomorphism $K(\psi \circ u) \simeq K(\psi)$ for every $\psi\colon F \to R$. For example, given $x_1, \ldots, x_n \in R$, if σ is a permutation of $\{1, \ldots, n\}$, we see that $K(x_1, \ldots, x_n) \simeq K\big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big)$.

EXAMPLE 12.2.4. If $(R, \mathfrak{m})$ is a Noetherian local ring, $\mathfrak{a}$ is an ideal of R, and $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_n$ are minimal systems of generators of $\mathfrak{a}$, then the Koszul complexes $K(\underline{x})$ and $K(\underline{y})$ are isomorphic. Indeed, we can write $y_i = \sum_{j=1}^n a_{i,j} x_j$ for all i and we have a commutative diagram

$$\begin{tikzcd} R^n \arrow[r, "\phi"] \arrow[d, "u"'] & R \arrow[d, "\mathrm{id}_R"] \\ R^n \arrow[r, "\psi"] & R, \end{tikzcd}$$

where $\phi(e_i) = y_i$, $\psi(e_i) = x_i$, and $u(e_i) = \sum_{j=1}^n a_{i,j} e_j$. Since both $\underline{x}$ and $\underline{y}$ are minimal systems of generators of $\mathfrak{a}$, it follows that $\det(a_{i,j}) \not\in \mathfrak{m}$, hence u is an isomorphism. Our assertion then follows from Remark 12.2.3.

PROPOSITION 12.2.5. *Given a morphism $\phi\colon E \to R$, then multiplication by $a \in \phi(E)$ on $K(\phi)$ is homotopic to* 0. *In particular, if ϕ is a surjective map, then for every R-module M, the complex $K(\phi; M)$ is exact.*

PROOF. If $a = \phi(e)$, then we define for every $p \geq 0$ a morphism $\theta_p\colon K(\phi)_p \to K(\phi)_{p+1}$ given by

$$\theta_p(e_1, \ldots, e_p) = e \wedge e_1 \wedge \ldots \wedge e_p.$$

It is straightforward to see that $d_{p+1} \circ \theta_p + \theta_{p-1} \circ d_p = a \cdot \mathrm{id}_{K(\phi)_p}$, hence $(\theta_p)_{p \geq 0}$ give a homotopy between $a \cdot \mathrm{id}_{K(\phi)}$ and 0. □

Our next goal is to relate the Koszul complex associated to n elements to the Koszul complex associated to $(n-1)$ of these elements. We do this more generally, as follows.

Let $\phi\colon E \to R$ be a linear map and consider $\psi\colon F = E \oplus R \to R$ given by $\psi(u, \lambda) = \phi(u) + \lambda a$, for some $a \in R$. Note that for every $p \geq 1$, we have a decomposition

$$\wedge^p F \simeq \wedge^p E \oplus \wedge^{p-1} E,$$

where the injective map $\wedge^p E \hookrightarrow \wedge^p F$ is induced by the injective map $E \hookrightarrow F$ and the injective map $\wedge^{p-1} E \hookrightarrow \wedge^p F$ takes $e_1 \wedge \ldots \wedge e_{p-1}$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{p-1}$, with $e_0 = (0, 1) \in F$. The decomposition also holds for $p = 0$ if we make the convention $\wedge^{-1} E = 0$.

It is clear that we have an injective morphism of complexes $K(\phi) \hookrightarrow K(\psi)$ and let $C_\bullet$ be the quotient complex. Note that we have $C_p \simeq K(\phi)_{p-1}$ and the map

$d_C\colon C_p \to C_{p-1}$ is equal to $-d_{K(\phi)}$. In particular, for every p we have $\mathcal{H}_p(C_\bullet) \simeq \mathcal{H}_{p-1}\big(K(\phi)\big)$. Moreover, the long exact sequence associated to the short exact sequence of complexes

$$0 \to K(\phi) \to K(\psi) \to C_\bullet \to 0$$

looks as follows:

$$\mathcal{H}_p\big(K(\phi)\big) \to \mathcal{H}_p\big(K(\psi)\big) \to \mathcal{H}_{p-1}\big(K(\phi)\big) \xrightarrow{\tau_{p-1}} \mathcal{H}_{p-1}\big(K(\phi)\big) \to \mathcal{H}_{p-1}\big(K(\psi)\big) \to \ldots.$$

A straightforward computation using the maps giving $K(\psi)$ shows that the maps τ_{p-1} is given by multiplication by $a \in R$. This applies, in particular, if $\phi\colon R^{n-1} \to R$ is the map corresponding to a sequence $x_1, \ldots, x_{n-1}$ and ψ is the map corresponding to $x_1, \ldots, x_{n-1}, a$.

A similar picture holds if we tensor with an R-module M. Indeed, each exact sequence

$$0 \to K(\phi)_p \to K(\psi)_p \to C_p \to 0$$

is split, hence by tensoring with M we obtain an exact sequence of complexes

$$0 \to K(\phi, M) \to K(\psi, M) \to C_\bullet \otimes_R M \to 0$$

such that the corresponding long exact sequence looks as follows:

$$\ldots \to \mathcal{H}_p\big(K(\phi, M)\big) \to \mathcal{H}_p\big(K(\psi, M)\big) \to \mathcal{H}_{p-1}\big(K(\phi, M)\big) \xrightarrow{\cdot a} \mathcal{H}_{p-1}\big(K(\phi, M)\big) \to \ldots.$$

EXAMPLE 12.2.6. Suppose that $x_1, \ldots, x_{n-1} \in R$ and $x_n \in (x_1, \ldots, x_{n-1})$. We can thus write $x_n = \sum_{i=1}^{n-1} a_i x_i$, for suitable $a_i \in R$, and we have a commutative diagram

$$\begin{tikzcd} R^n \arrow[r, "\phi"] \arrow[d, "u"'] & R \arrow[d, "\mathrm{id}_R"] \\ R^n \arrow[r, "\psi"] & R, \end{tikzcd}$$

where $\phi(e_i) = x_i$ for $1 \leq i \leq n$, $\psi(e_i) = x_i$ for $1 \leq i \leq n-1$ and $\psi(e_n) = 0$, while $u(e_i) = e_i$ for $1 \leq i \leq n-1$ and $u(e_n) = e_n + \sum_{i=1}^{n-1} a_i e_i$. Since u is an isomorphism, it follows from Remark 12.2.3 that

$$K(x_1, \ldots, x_{n-1}, x_n) \simeq K(x_1, \ldots, x_{n-1}, 0).$$

On the other hand, it follows from the above discussion that for every R-module M and every i, we have a short exact sequence

$$0 \to \mathcal{H}_i(x_1, \ldots, x_{n-1}; M) \to \mathcal{H}_i(x_1, \ldots, x_{n-1}, 0; M) \to \mathcal{H}_{i-1}(x_1, \ldots, x_{i-1}; M) \to 0.$$

12.2.2. The Koszul complex and depth. We begin by discussing the connection between the exactness of the Koszul complex and regular sequences.

PROPOSITION 12.2.7. *Let R be a Noetherian ring, M a finitely generated R-module, and $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R.*

i) *If $x_1, \ldots, x_n$ is an M-regular sequence, then $\mathcal{H}_i(\underline{x}; M) = 0$ for $i \geq 1$, while $\mathcal{H}_0(\underline{x}; M) = M/(x_1, \ldots, x_n)M$.*
ii) *Conversely, if $(R, \mathfrak{m})$ is regular, $x_1, \ldots, x_n \in \mathfrak{m}$, $M \neq 0$, and $\mathcal{H}_i(\underline{x}; M) = 0$ for $i \geq 1$, then $x_1, \ldots, x_n$ is an M-regular sequence.*

PROOF. We prove i) by induction on n. Note that the equality $\mathcal{H}_0(\underline{x}; M) = M/(x_1, \ldots, x_n)M$ holds for all $\underline{x}$ by Remark 12.2.2 If $n = 1$, then the complex $K(x_1) \otimes_R M$ consists of

$$0 \to M \xrightarrow{\cdot x_1} M \to 0.$$

The map is injective since x_1 is a non-zero-divisor on M, hence $\mathcal{H}_i(x_1; M) = 0$ for $i \neq 0$.

Suppose now that $n \geq 2$ and we know the assertion for $n - 1$. We have seen that if $\underline{x}'$ consists of $x_1, \ldots, x_{n-1}$, then we have a long exact sequence

$$\to \mathcal{H}_i(\underline{x}'; M) \xrightarrow{\cdot x_n} \mathcal{H}_i(\underline{x}'; M) \to \mathcal{H}_i(\underline{x}, M) \to \mathcal{H}_{i-1}(\underline{x}'; M) \to \ldots.$$

By induction, we know that $\mathcal{H}_i(\underline{x}'; M) = 0$ for $i \geq 1$, which immediately implies $\mathcal{H}_i(\underline{x}; M) = 0$ for $i \geq 2$. Moreover, we have an exact sequence

$$0 \to \mathcal{H}_1(\underline{x}; M) \to \mathcal{H}_0(\underline{x}'; M) \xrightarrow{\cdot x_n} \mathcal{H}_i(\underline{x}'; M).$$

Since $\mathcal{H}_0(\underline{x}'; M) = M/(x_1, \ldots, x_{n-1})M$ and x_n is a non-zero-divisor on the R-module $M/(x_1, \ldots, x_{n-1})M$, we conclude that $\mathcal{H}_1(\underline{x}; M) = 0$. This completes the proof of i).

Suppose now that we are under the assumptions of ii). Note that by Nakayama's lemma, we have $M/(x_1, \ldots, x_n)M \neq 0$, hence we only need to show that x_i is a non-zero-divisor on $M/(x_1, \ldots, x_{i-1})M$ for $1 \leq i \leq n$. We argue again by induction on n. If $n = 1$, then

$$0 = \mathcal{H}_1\big(K(x_1; M)\big) = \ker\left(M \xrightarrow{\cdot x_1} M\right),$$

hence x_1 is a non-zero-divisor on M.

For the induction step we use the exact sequence

$$\mathcal{H}_i(\underline{x}'; M) \xrightarrow{\cdot x_n} \mathcal{H}_i(\underline{x}'; M) \to \mathcal{H}_i(\underline{x}; M) = 0,$$

where $i \geq 1$ and $\underline{x}' = x_1, \ldots, x_{n-1}$. Since $\mathcal{H}_i(\underline{x}'; M)$ is a finitely generated R-module, it follows from Nakayama's lemma that $\mathcal{H}_i(\underline{x}'; M) = 0$ for all $i \geq 1$, hence by induction $x_1, \ldots, x_{n-1}$ is an M-regular sequence. Moreover, we have an exact sequence

$$0 = \mathcal{H}_1(\underline{x}; M) \to \mathcal{H}_0(\underline{x}'; M) \xrightarrow{\cdot x_n} \mathcal{H}_0(\underline{x}'; M).$$

Therefore x_n is a non-zero-divisor on $\mathcal{H}_0(\underline{x}'; M) = M/(x_1, \ldots, x_{n-1})M$, and thus $x_1, \ldots, x_n$ is an M-regular sequence. □

COROLLARY 12.2.8. *If M is a finitely generated module over the Noetherian local ring $(R, \mathfrak{m})$ and $x_1, \ldots, x_n \in \mathfrak{m}$ form an M-regular sequence, then every permutation of this sequence is still M-regular.*

PROOF. The assertion follows from the proposition and the fact that the Koszul complex for a permutation of a sequence is isomorphic to the Koszul complex for the sequence (see Remark 12.2.3). □

REMARK 12.2.9. If R, M, and $x_1, \ldots, x_n$ are as in the above corollary, then $x_1, \ldots, x_n$ form a minimal system of generators for the ideal $(x_1, \ldots, x_n)$. Indeed, if this is not the case, then there is i such that $x_i \in \mathfrak{b} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. However, it follows from the corollary that x_i is a non-zero-divisor on the non-zero R-module $M/\mathfrak{b}M$, a contradiction.

REMARK 12.2.10. If $(R, \mathfrak{m})$ is a local Noetherian ring, $\mathfrak{a} \subseteq \mathfrak{m}$ is an ideal, and M is a finitely generated, non-zero R-module, then we can choose a maximal M-regular sequence in $\mathfrak{a}$ that is part of a minimal system of generators of $\mathfrak{a}$. Indeed, suppose that we have chosen $x_1, \ldots, x_r \in \mathfrak{a}$ that form an M-regular sequence and which are part of a minimal system of generators of $\mathfrak{a}$. If $\operatorname{depth}(\mathfrak{a}, M) > r$, then $\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$, where $\mathcal{P}$ is the set of associated primes of $M/(x_1, \ldots, x_r)M$. Since there is $y \in \mathfrak{a}$ such that $x_1, \ldots, x_r, y$ is an M-regular sequence, it follows that $\mathfrak{a} \neq (x_1, \ldots, x_r)$, and thus, by Nakayama's lemma, $\mathfrak{a} \not\subseteq \mathfrak{b} := \mathfrak{m}\mathfrak{a} + (x_1, \ldots, x_r)$. We thus conclude using Lemma E.1.1 that $\mathfrak{a} \not\subseteq \mathfrak{b} \cup \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$, and therefore we can choose $x_{r+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_{r+1}$ is an M-regular sequence and $x_1, \ldots, x_{r+1}$ are part of a minimal generating system for $\mathfrak{a}$.

COROLLARY 12.2.11. *If $(R, \mathfrak{m})$ is a Noetherian local ring and I is an ideal generated by a regular sequence $x_1, \ldots, x_r \in \mathfrak{m}$, then $\operatorname{pd}_R(R/I) = r$.*

PROOF. It follows from the proposition that the Koszul complex $K(x_1, \ldots, x_r)$ gives a free resolution of R/I and it is clear from the definition of the Koszul complex that this is a minimal resolution. $\square$

When the ring is local, we can use the Koszul complex to compute the depth, as follows.

PROPOSITION 12.2.12. *If R is a Noetherian local ring, M is a non-zero finitely generated R-module, and $x_1, \ldots, x_n$ generate an ideal $\mathfrak{a}$ contained in the maximal ideal, then $r = \operatorname{depth}(\mathfrak{a}, M) \leq n$ and*

$$\mathcal{H}_i\big(K(\underline{x}; M)\big) = 0 \quad \textit{for} \quad i > n - r, \quad \textit{while} \quad \mathcal{H}_{n-r}\big(K(\underline{x}, M)\big) \neq 0.$$

PROOF. Note first that we may assume that $x_1, \ldots, x_n$ is a minimal system of generators of $\mathfrak{a}$. Indeed, if for example $x_n \in (x_1, \ldots, x_{n-1})$ and we put $\underline{x}' = x_1, \ldots, x_{n-1}$, then it follows from Example 12.2.6 that

$$n - \max\{i \mid \mathcal{H}_i(\underline{x}, M) \neq 0\} = n - 1 - \max\{i \mid \mathcal{H}_i(\underline{x}', M) \neq 0\}.$$

We may thus replace $\underline{x}$ by $\underline{x}'$ and after repeating this argument several times, we reduce to the case when $x_1, \ldots, x_n$ form a minimal generating set of $\mathfrak{a}$.

By Remark 12.2.10, there is a maximal M-regular sequence $y_1, \ldots, y_r$ in $\mathfrak{a}$, that is part of a minimal system of generators of $\mathfrak{a}$. In particular, we have $r \leq n$. We choose $y_{r+1}, \ldots, y_n$ such that $\underline{y} = y_1, \ldots, y_n$ is a minimal system of generators of $\mathfrak{a}$. Since $\mathcal{H}_i(\underline{y}, M) \simeq \mathcal{H}_i(\underline{x}, M)$ for all i by Remark 12.2.4, we may replace $\underline{x}$ by $\underline{y}$ and thus assume that $x_1, \ldots, x_r$ is a maximal M-regular sequence in $\mathfrak{a}$.

Let $M' = M/(x_1, \ldots, x_r)M$. We show by induction on j, with $r \leq j \leq n$, that

$$\mathcal{H}_i(x_1, \ldots, x_j; M) = 0 \quad \text{for} \quad i \geq j - r + 1 \quad \text{and} \tag{12.2.1}$$

$$\mathcal{H}_{j-r}(x_1, \ldots, x_j; M) \simeq \{u \in M' \mid x_{r+1}u = \ldots = x_j u = 0\}.$$

For $j = n$, this implies the assertion in the proposition, since $x_1, \ldots, x_r$ being a maximal M-regular sequence in $\mathfrak{a}$ implies that there is a nonzero $u \in M'$ such that $\mathfrak{a} \cdot u = 0$.

For $j = r$, (12.2.1) follows from Proposition 12.2.7. Suppose now that we know the assertion for j and let us prove it for $j + 1$. Consider the exact complex

$$\ldots \to \mathcal{H}_i(x_1, \ldots, x_j; M) \to \mathcal{H}_i(x_1, \ldots, x_{j+1}; M) \to \mathcal{H}_{i-1}(x_1, \ldots, x_j; M)$$
$$\xrightarrow{\cdot x_{j+1}} \mathcal{H}_{i-1}(x_1, \ldots, x_j; M) \to \ldots.$$

We deduce that $\mathcal{H}_i(x_1,\ldots,x_{j+1};M)=0$ for $i\geq j-r+2$ using the inductive vanishing assumption in (12.2.1). Moreover, we see that

$$\mathcal{H}_{j-r+1}(x_1,\ldots,x_{j+1};M)=\ker\Big(H_{j-r}(x_1,\ldots,x_j;M)\xrightarrow{\cdot x_{j+1}}H_{j-r}(x_1,\ldots,x_j;M)\Big)$$

and using the inductive formula for $H_{j-r}(x_1,\ldots,x_j;M)$, we deduce that

$$\mathcal{H}_{j-r+1}(x_1,\ldots,x_{j+1};M)\simeq\{u\in M'\mid x_{r+1}u=\ldots=x_{j+1}u=0\}.$$

This completes the proof of the induction step for (12.2.1) and thus that of the proposition. □

REMARK 12.2.13. If we drop the assumption that R is local in Proposition 12.2.12, we can still say that if I is generated by $x_1,\ldots,x_n$ and $\operatorname{depth}(I,M)=r$, then $\mathcal{H}_i\big(K(\underline{x};M)\big)=0$ for $i>n-r$. Indeed, it is enough to show that $\mathcal{H}_i\big(K(\underline{x};M)\big)_{\mathfrak{m}}=0$ for every maximal ideal $\mathfrak{m}$ in R and every $i>n-r$. Since we clearly have

$$\mathcal{H}_i\big(K(x_1,\ldots,x_n;M)\big)_{\mathfrak{m}}\simeq\mathcal{H}_i\big(K(x_1/1,\ldots,x_n/1;M_{\mathfrak{m}}\big),$$

the required vanishing follows from Proposition 12.2.5 if $I\not\subseteq\mathfrak{m}$ and it follows from Proposition 12.2.12 if $I\subseteq\mathfrak{m}$, since $\operatorname{depth}(I,M)\leq\operatorname{depth}(IR_{\mathfrak{m}},M_{\mathfrak{m}})$ (see Corollary 12.1.11).

COROLLARY 12.2.14. *If $(R,\mathfrak{m})$ is a Noetherian local ring, M is a finitely generated non-zero R-module, and $\mathfrak{a}=(x_1,\ldots,x_n)\subseteq\mathfrak{m}$, then* $\operatorname{depth}(\mathfrak{a},M)=n$ *if and only if $x_1,\ldots,x_n$ form an M-regular sequence.*

PROOF. Both implications follow by combining Propositions12.2.7 and 12.2.12. □

COROLLARY 12.2.15. *If $(A,\mathfrak{m})$ is a Noetherian local ring and $\phi\colon A\to B$ is a finite homomorphism, then for every ideal $\mathfrak{a}\subseteq\mathfrak{m}$ and every finitely generated B-module M, we have*

$$\operatorname{depth}(\mathfrak{a},M)=\operatorname{depth}(\mathfrak{a}B,M).$$

PROOF. Of course, we may assume that M is non-zero. Let $x_1,\ldots,x_r\in\mathfrak{a}$ be a system of generators. By Proposition 12.2.12, we have

$$\operatorname{depth}(\mathfrak{a},M)=r-\max\big\{i\mid\mathcal{H}_i\big(K(\underline{x};M)\big)\neq0\big\}.$$

On the other hand, if $\mathfrak{n}_1,\ldots,\mathfrak{n}_s$ are the maximal ideals of B (note that since B is finite over A, these are precisely the prime ideals containing $\mathfrak{m}B$), then

$$\mathcal{H}_i\big(K(\underline{x};M)\big)=0\quad\text{if and only if}\quad\mathcal{H}_i\big(K(\underline{x};M)\big)_{\mathfrak{n}_j}=0\text{ for }1\leq j\leq s.$$

Since the complex $K(\underline{x};M)_{\mathfrak{n}_j}$ is isomorphic to the Kozsul complex on the images of $x_1,\ldots,x_r$ in $B_{\mathfrak{n}_j}$, with respect to $M_{\mathfrak{n}_j}$, applying one more time Proposition 12.2.12, we conclude that

$$\operatorname{depth}(\mathfrak{a},M)=\min_j\operatorname{depth}(\mathfrak{a}B_{\mathfrak{n}_j},M_{\mathfrak{n}_j}).\tag{12.2.2}$$

On the other hand, the right-hand side of (12.2.2) is equal to $\operatorname{depth}(\mathfrak{a}B,M)$. Indeed, the inequality $\operatorname{depth}(\mathfrak{a}B,M)\leq\min_j\operatorname{depth}(\mathfrak{a}B_{\mathfrak{n}_j},M_{\mathfrak{n}_j})$ follows from Remark 12.1.2, while the opposite inequality follows from Proposition 12.1.11. □

12.2.3. Finite locally free resolutions on smooth quasi-projective varieties. We begin with the following consequence of the exactness of the Koszul complex for a regular sequence:

PROPOSITION 12.2.16. *Let X be an algebraic variety and $x \in X$ a smooth point. If $R = \mathcal{O}_{X,x}$, then the global dimension of R is equal to* $\dim(R)$.

PROOF. We have seen in Example 12.1.15 that if $\dim(R) = n$ and $x_1, \ldots, x_n$ form a minimal system of generators for the maximal ideal $\mathfrak{m}$ in R, then $x_1, \ldots, x_n$ form an R-regular sequence, and we conclude using the previous corollary that $\mathrm{pd}_R(R/\mathfrak{m}) = n$. The assertion then follows from Corollary 12.1.19. ☐

REMARK 12.2.17. As we have mentioned in Example 12.1.15, the fact that a minimal system of generators for the maximal ideal forms a regular sequence is shared by all regular local rings. It follows that for every regular local ring, its global dimension is equal to the dimension. Conversely, if a Noetherian local ring has finite global dimension, then the ring is regular: this is a result proved independently by Auslander-Buchsbaum and Serre (see [**BH93**, Theorem 2.2.7]).

REMARK 12.2.18. Suppose that X is a smooth algebraic variety of dimension n, $\mathcal{F}$ is a coherent sheaf on X, and we have an exact complex

$$\mathcal{E}_m \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

with all $\mathcal{E}_i$ locally free $\mathcal{O}_X$-modules. If $m \geq n-1$ and $\mathcal{G} = \ker(\mathcal{E}_m \to \mathcal{E}_{m-1})$, then $\mathcal{G}$ is locally free. Indeed, it follows from Corollary 12.2.16 that for every $x \in X$, we have $\mathrm{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \leq n$ and thus $\mathcal{G}_x$ is free by Proposition 10.7.6.

In particular, by combining this observation with Remark 11.6.11, we see that on a smooth quasi-projective variety, every coherent sheaf has a finite resolution by locally free sheaves.

We can now show that on a smooth, quasi-projective variety, the two flavors of Grothendieck groups that we introduced are isomorphic.

PROPOSITION 12.2.19. *If X is a smooth, quasi-projective algebraic variety, then the canonical morphism of Abelian groups $K^0(X) \to K_0(X)$ is an isomorphism.*

PROOF. If $\mathcal{F}$ is a coherent sheaf on X, it follows from Remark 12.2.18 that we have an exact complex

$$0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

with all $\mathcal{E}_i$ locally free. Since $[\mathcal{F}] = \sum_{i=0}^n (-1)^i [\mathcal{E}_i]$ in $K_0(X)$, it follows that the canonical morphism $\alpha\colon K^0(X) \to K_0(X)$ that maps $[\mathcal{E}]$ to $[\mathcal{E}]$, is surjective.

The argument for the injectivity of α is more involved. In fact, we will construct an inverse map $\beta\colon K_0(X) \to K^0(X)$, proceeding as above. Given a coherent sheaf $\mathcal{F}$, we choose a finite locally free resolution $\mathcal{E}_\bullet$ as above and define $\beta\big([\mathcal{F}]\big) = \sum_{i\geq 0} (-1)^i [\mathcal{E}_i]$. The first step is to show that the definition is independent of the choice of locally free resolution. In order to do this, we recall the following property of fiber products of sheaves: given a Cartesian diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd} \mathcal{G} \arrow[r, "v_1"] \arrow[d, "v_2"'] & \mathcal{M}_1 \arrow[d, "u_1"] \\ \mathcal{M}_2 \arrow[r, "u_2"'] & \mathcal{M}, \end{tikzcd}$$

if u_1 is surjective, then v_2 is surjective, and we have an induced isomorphism $\ker(v_2) \to \ker(u_1)$. Note that $\mathcal{G} = \mathcal{M}_1 \times_{\mathcal{M}} \mathcal{M}_2$ is the subsheaf of $\mathcal{M}_1 \times \mathcal{M}_2$ equal to the kernel of $u_1 \circ \mathrm{pr}_1 - u_2 \circ \mathrm{pr}_2$, with v_1 and v_2 being induced by the two projections. By passing to stalks, both assertions about the diagram are reduced to the case of modules over a ring, in which case they are easy to check.

Suppose now that we have finite locally free resolutions $\mathcal{E}'_\bullet \to \mathcal{F}$ and $\mathcal{E}''_\bullet \to \mathcal{F}$. Note first that we are done if there is a surjective morphism of complexes $u\colon \mathcal{E}'_\bullet \to \mathcal{E}''_\bullet$ that induces the identity on $\mathcal{F}$. Indeed, if $\mathcal{G}_\bullet = \ker(u)$, then $\mathcal{G}_\bullet$ is a complex of locally free sheaves having only finitely many non-zero terms. By looking at the short exact sequence of complexes

$$0 \to \mathcal{G}_\bullet \to \mathcal{E}'_\bullet \to \mathcal{E}''_\bullet \to 0$$

in each degree, we see that

$$[\mathcal{E}'_i] - [\mathcal{E}''_i] = [\mathcal{G}_i] \quad \text{in} \quad K^0(X) \quad \text{for all} \quad i \geq 0.$$

On the other hand, the long exact sequence in cohomology for the above short exact sequence of complexes implies that $\mathcal{G}_\bullet$ is an exact complex, and thus $\sum_{i\geq 0}(-1)^i[\mathcal{G}_i] = 0$ in $K^0(X)$, implying that

$$\sum_{i\geq 0}(-1)^i[\mathcal{E}'_i] = \sum_{i\geq 0}(-1)^i[\mathcal{E}''_i] \quad \text{in} \quad K^0(X).$$

It follows that given arbitrary $\mathcal{E}'_\bullet$ and $\mathcal{E}''_\bullet$ as above, it is enough to construct another finite locally free resolution $\mathcal{E}_\bullet \to \mathcal{F}$ together with surjective morphisms of complexes $\mathcal{E}_\bullet \to \mathcal{E}'_\bullet$ and $\mathcal{E}_\bullet \to \mathcal{E}''_\bullet$ that induce the identity on $\mathcal{F}$. Constructing $\mathcal{E}_\bullet$ recursively, we see that it is enough to prove the following: given surjective morphisms of coherent sheaves:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "u'"] & \mathcal{P}' \\ & \mathcal{P} \arrow[u] \arrow[d] \\ \mathcal{E}'' \arrow[r, "u''"] & \mathcal{P}'', \end{tikzcd}$$

we can find surjective morphisms $u\colon \mathcal{E} \to \mathcal{P}$, $\mathcal{E} \to \mathcal{E}'$ and $\mathcal{E} \to \mathcal{E}''$, with $\mathcal{E}$ locally free, such that the resulting squares are commutative and the induced morphisms $\ker(u) \to \ker(u')$ and $\ker(u) \to \ker(u'')$ are surjective. In order to see this, let $\mathcal{G}' = \mathcal{P} \times_{\mathcal{P}'} \mathcal{E}'$ and $\mathcal{G}'' = \mathcal{P} \times_{\mathcal{P}''} \mathcal{E}''$. We then take $\mathcal{G} = \mathcal{G}' \times_{\mathcal{P}} \mathcal{G}''$ and consider a surjective morphism $\mathcal{E} \to \mathcal{G}$, with $\mathcal{E}$ locally free. If the morphisms $\mathcal{E} \to \mathcal{P}$, $\mathcal{E} \to \mathcal{E}'$, and $\mathcal{E} \to \mathcal{E}''$ are given by the obvious compositions, it is easy to see, using the above property of fibered products, that these satisfy the required properties.

We next show that by mapping $[\mathcal{F}]$ to $\beta\big([\mathcal{F}]\big)$, we have a well-defined group homomorphism $K_0(X) \to K^0(X)$. In other words, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, we need to show that

$$\beta\big([\mathcal{F}']\big) - \beta\big([\mathcal{F}]\big) + \beta\big([\mathcal{F}'']\big) = 0. \tag{12.2.3}$$

Let us consider a finite locally free resolution $\mathcal{E}''_\bullet \to \mathcal{F}''$ of $\mathcal{F}''$. A similar argument to the one used above shows that there is a finite locally free resolution $\mathcal{E}_\bullet \to \mathcal{F}$, together with a surjective morphism $u\colon \mathcal{E}_\bullet \to \mathcal{E}''_\bullet$ inducing the given morphism $\mathcal{F} \to \mathcal{F}''$. If $\mathcal{E}'_\bullet = \ker(u)$, then this is a complex with finitely many terms, all of them locally free. Moreover, using the long exact sequence in cohomology for the exact sequence of complexes

$$0 \to \mathcal{E}'_\bullet \to \mathcal{E}_\bullet \to \mathcal{E}''_\bullet \to 0,$$

we see that $\mathcal{E}'_\bullet$ is a resolution of $\mathcal{F}'$. It follows from the above exact sequence of complexes that we have $[\mathcal{E}'_i] - [\mathcal{E}_i] + [\mathcal{E}''_i] = 0$ in $K^0(X)$ for every $i \geq 0$, hence by computing $\beta\big([\mathcal{F}']\big)$, $\beta\big([\mathcal{F}]\big)$, and $\beta\big([\mathcal{F}'']\big)$ using $\mathcal{E}'_\bullet$, $\mathcal{E}_\bullet$, and $\mathcal{E}''_\bullet$, respectively, we obtain (12.2.3).

We thus have a group homomorphism $\beta\colon K_0(X) \to K^0(X)$. The composition $\beta \circ \alpha$ is the identity on $K^0(X)$ since for a locally free sheaf $\mathcal{E}$, we can compute $\beta\big([\mathcal{E}]\big)$ using the resolution

$$0 \to \mathcal{E} \to \mathcal{E} \to 0.$$

On the other hand, if $\mathcal{F}$ is a coherent sheaf and $\mathcal{E}_\bullet \to \mathcal{F}$ is a locally free resolution, we have $[\mathcal{F}] = \sum_{i\geq 0}(-1)^i[\mathcal{E}_i]$ in $K_0(X)$, hence $\alpha \circ \beta$ is the identity on $K_0(X)$. Therefore β is the inverse of α, completing the proof of the fact that α is an isomorphism. $\square$

12.2.4. The Koszul complex in the global setting. We now consider the Koszul complex in a global geometric setting. Let X be an algebraic variety and $\mathcal{E}$ a locally free sheaf on X, of rank r. Given a section $s \in \Gamma(X, \mathcal{E}^\vee) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, we obtain a Koszul complex:

$$0 \to \wedge^r\mathcal{E} \to \ldots \to \wedge^2\mathcal{E} \to \mathcal{E} \to \mathcal{O}_X,$$

where the map $\wedge^p\mathcal{E} \to \wedge^{p-1}\mathcal{E}$ is given by

$$u_1 \wedge \ldots \wedge u_p \to \sum_{i=1}^{p}(-1)^{i-1}s(u_i)u_1 \wedge \ldots \wedge \widehat{u_i} \wedge \ldots \wedge u_n.$$

It is clear that if we restrict to an affine open subset U of X, this is the complex of coherent sheaves associated to the Koszul complex corresponding to $s|_U\colon \mathcal{E}(U) \to \mathcal{O}_X(U)$.

EXAMPLE 12.2.20. Let V be k-vector space, with $\dim_k(V) = n+1$. On the projective space $\mathbf{P}(V)$, we have the canonical surjective morphism

$$V \otimes_k \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(1).$$

By tensoring this with $\mathcal{O}_{\mathbf{P}(V)}(-1)$, we obtain a surjective morphism

$$\phi\colon V \otimes_k \mathcal{O}_{\mathbf{P}(V)}(-1) \to \mathcal{O}_{\mathbf{P}(V)}$$

and a corresponding Koszul complex

$$0 \to \wedge^{n+1}V \otimes_k \mathcal{O}_{\mathbf{P}(V)}(-n-1) \xrightarrow{\phi_{n+1}} \ldots \xrightarrow{\phi_3} \wedge^2 V \otimes_k \mathcal{O}_{\mathbf{P}(V)}(-2)$$

$$\xrightarrow{\phi_2} V \otimes_k \mathcal{O}_{\mathbf{P}(V)}(-1) \xrightarrow{\phi_1=\phi} \mathcal{O}_{\mathbf{P}(V)} \to 0.$$

This is an exact complex by Proposition 12.2.5, since ϕ is surjective.

EXERCISE 12.2.21. With the notation in the above example, show that we have an isomorphism

$$\ker(\phi_i) \simeq \Omega^i_{\mathbf{P}(V)}.$$

EXAMPLE 12.2.22. Let $\mathcal{E}$ be a locally free sheaf of rank r on a variety X, and let $s \in \Gamma(X, \mathcal{E})$. We may consider s as a morphism $\mathcal{E}^\vee \to \mathcal{O}_X$ and denote by $\mathcal{I}(s)$ the image of this morphism. Explicitly, if over an affine open subset U of X we have a trivialization $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$ such that $s|_U$ corresponds to $(f_1, \ldots, f_r)$, then $\Gamma\big(U, \mathcal{I}(s)\big)$ is generated by $f_1, \ldots, f_r$. We also denote by $V(s)$ the closed subset defined by $\mathcal{I}(s)$; this consists of those $x \in X$ such that $s(x) = 0$ in $\mathcal{E}_{(x)}$. Note that these definitions extend the ones given in §9.4.4 in the case of a line bundle.

We say that s is a *regular section* of $\mathcal{E}$ if

$$\operatorname{depth}\big(\mathcal{I}(s)_x\big) = r \quad \text{for every} \quad x \in V(s);$$

equivalently, if we choose an isomorphism $\mathcal{E}_x \simeq \mathcal{O}_{X,x}^{\oplus r}$, such that s_x corresponds to $(a_1, \ldots, a_r)$, then $a_1, \ldots, a_r$ is an $\mathcal{O}_{X,x}$-regular sequence. For example, this is always the case if $r = 1$ and X is irreducible, in which case $\mathcal{I}(s)$ is the ideal corresponding to the effective Cartier divisor $Z(s)$.

Consider the "extended" Koszul complex associated to s:

$$0 \to \wedge^r(\mathcal{E}^\vee) \to \ldots \to \wedge^2(\mathcal{E}^\vee) \to \mathcal{E}^\vee \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(s) \to 0. \tag{12.2.4}$$

If s is a regular section, then the complex is exact at every $x \in V\big(\mathcal{I}(s)\big)$ by assumption, while it is always exact at all other points by Proposition 12.2.5. We thus conclude that in this case, the Koszul complex gives a locally free resolution of $\mathcal{O}_X/\mathcal{I}(s)$.

If s is a regular section such that the ideal $\mathcal{I}(s)$ is radical, defining a closed subvariety $Y \subseteq X$, then we have an exact sequence

$$\wedge^2(\mathcal{E}^\vee) \to \mathcal{E}^\vee \to \mathcal{I}(s) \to 0.$$

By tensoring this with $\mathcal{O}_X/\mathcal{I}(s)$, we see that the conormal sheaf of Y in X is isomorphic to $\mathcal{E}^\vee|_Y$. In particular, the conormal sheaf is locally free and its dual, the normal sheaf, is isomorphic to $\mathcal{E}|_Y$.

12.2.5. Castelnuovo-Mumford regularity. We give an application of the Koszul complex on the projective space to the notion of Castelnuovo-Mumford regularity. A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ is *m-regular* if

$$H^i\big(\mathbf{P}^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(m-i)\big) = 0 \quad \text{for all} \quad i \geq 1.$$

Note that by Theorem 11.2.1, any coherent sheaf $\mathcal{F}$ is m-regular for $m \gg 0$. The *Castelnuovo-Mumford regularity* of $\mathcal{F}$ is the smallest m such that $\mathcal{F}$ is m-regular.

PROPOSITION 12.2.23. *If $\mathcal{F}$ is an m-regular coherent sheaf on $\mathbf{P}^n$, the following hold:*

i) *The sheaf $\mathcal{F}$ is also m'-regular for every $m' \geq m$.*
ii) *The canonical map*

$$H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) \otimes_k H^0\big(\mathbf{P}^n, \mathcal{F}(m)\big) \to H^0\big(\mathbf{P}^n, \mathcal{F}(m+1)\big),$$

given by multiplication of sections, is surjective.

ii) *The sheaf $\mathcal{F}(m)$ is globally generated.*

Proof. After replacing $\mathcal{F}$ by $\mathcal{F}(m)$, we may and will assume that $m = 0$. Moreover, in order to prove i), arguing by induction on m', it is clear that it is enough to prove the case $m' = 1$. Consider the Koszul complex on $\mathbf{P}^n$:

$$K_\bullet: \ 0 \to K_{n+1} \xrightarrow{\phi_{n+1}} \ldots \xrightarrow{\phi_2} K_1 \xrightarrow{\phi_1} K_0 \to 0,$$

where $K_i = \wedge^i V \otimes_k \mathcal{O}_{\mathbf{P}^n}(-i)$, with $V = H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big)$. If we put $Z_i = \ker(\phi_i)$, then the Koszul complex induces the short exact sequences of locally free sheaves

$$(S_j) \qquad 0 \to Z_j \to K_j \to Z_{j-1} \to 0$$

for $1 \le j \le n$ (note that $Z_n = K_{n+1}$ and $Z_0 = K_0$). Since (S_j) is locally split, by tensoring it with $\mathcal{F}(1-i)$ we get a short exact sequence, and we consider the following piece of the corresponding long exact sequence in cohomology:

$$H^{i+j-1}\big(\mathbf{P}^n, K_j \otimes \mathcal{F}(1-i)\big) \to H^{i+j-1}\big(\mathbf{P}^n, Z_{j-1} \otimes \mathcal{F}(1-i)\big) \to H^{i+j}\big(\mathbf{P}^n, Z_j \otimes \mathcal{F}(1-i)\big).$$

Since $\mathcal{F}$ is 0-regular, we have

$$H^{i+j-1}\big(\mathbf{P}^n, K_j \otimes \mathcal{F}(1-i)\big) \simeq H^{i+j-1}\big(\mathbf{P}^n, \mathcal{F}(1-i-j)\big)^{\bigoplus \binom{n+1}{j}} = 0.$$

For every i with $1 \le i \le n$, arguing by descending induction on j, with $1 \le j \le n+1$, we see that

$$H^{i+j-1}\big(\mathbf{P}^n, Z_{j-1} \otimes \mathcal{F}(1-i)\big) = 0$$

(note that for $j = n+1$, this follows from the fact that $H^{i+n}(\mathbf{P}^n, \mathcal{G}) = 0$ for every coherent sheaf $\mathcal{G}$ on $\mathbf{P}^n$ by Corollary 10.6.7). For $j = 1$, this gives

$$H^i\big(\mathbf{P}^n, \mathcal{F}(1-i)\big) = 0$$

for $1 \le i \le n$ (while for $i > n$, the vanishing follows again by Corollary 10.6.7). We thus see that $\mathcal{F}$ is 1-regular.

The argument for ii) is similar. By tensoring (S_1) with $\mathcal{F}(1)$, and taking the long exact sequence in cohomology, we obtain an exact sequence

$$V \otimes_k H^0(\mathbf{P}^n, \mathcal{F}) \xrightarrow{\mu} H^0\big(\mathbf{P}^n, \mathcal{F}(1)\big) \to H^1\big(\mathbf{P}^n, Z_1 \otimes \mathcal{F}(1)\big).$$

Since μ is the map given by multiplication of sections, in order to prove ii), it is enough to show that $H^1\big(\mathbf{P}^n, Z_1 \otimes \mathcal{F}(1)\big) = 0$. By tensoring (S_{j+1}) with $\mathcal{F}(1)$ for $1 \le j \le n-1$, we obtain from the long exact sequence in cohomology the exact sequence

$$H^j\big(\mathbf{P}^n, K_{j+1} \otimes \mathcal{F}(1)\big) \to H^j\big(\mathbf{P}^n, Z_j \otimes \mathcal{F}(1)\big) \to H^{j+1}(\mathbf{P}^n, Z_{j+1} \otimes \mathcal{F}(1)\big).$$

Since we have by hypothesis

$$H^j\big(\mathbf{P}^n, K_{j+1} \otimes \mathcal{F}(1)\big) = H^j\big(\mathbf{P}^n, \mathcal{F}(-j)\big)^{\bigoplus \binom{n+1}{j+1}} = 0,$$

we conclude by descending induction on j that $H^j\big(\mathbf{P}^n, Z_j \otimes \mathcal{F}(1)\big) = 0$ for $1 \le j \le n-1$; note that we have an injective map

$$H^{n-1}\big(\mathbf{P}^n, Z_{n-1} \otimes \mathcal{F}(1)\big) \hookrightarrow H^n\big(\mathbf{P}^n, \mathcal{F}(-n)\big)$$

and the right-hand side vanishes since $\mathcal{F}$ is 0-regular. For $j = 1$, we obtain $H^1\big(\mathbf{P}^n, Z_1 \otimes \mathcal{F}(1)\big) = 0$, hence ii) holds.

Since $\mathcal{O}_X(1)$ is an ample line bundle on $\mathbf{P}^n$, we know that $\mathcal{F}(q)$ is globally generated for $q \gg 0$. In order to show that $\mathcal{F}$ is globally generated, it is thus enough to show that for $q \geq 0$, if $\mathcal{F}(q+1)$ is globally generated, then $\mathcal{F}(q)$ is globally generated. Note that we have a commutative diagram

$$\begin{tikzcd} H^0\big(\mathbf{P}^n, \mathcal{F}(q)\big) \otimes_k H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big) \otimes_k \mathcal{O}_{\mathbf{P}^n} \arrow[r, "\beta"] \arrow[d, "\alpha"'] & H^0\big(\mathbf{P}^n, \mathcal{F}(q+1)\big) \otimes_k \mathcal{O}_{\mathbf{P}^n} \arrow[d, "\gamma"] \\ H^0\big(X, \mathcal{F}(q)\big) \otimes \mathcal{O}_{\mathbf{P}^n}(1) \arrow[r, "\delta"'] & \mathcal{F}(q+1). \end{tikzcd}$$

Since $\mathcal{F}(q+1)$ is globally generated, γ is surjective; on the other hand, the assertion in i) implies that $\mathcal{F}(q)$ is 0-regular, and thus ii) gives that β is surjective. We deduce from the commutative diagram that δ is surjective. By tensoring δ with $\mathcal{O}_{\mathbf{P}^n}(-1)$, we deduce that $\mathcal{F}(q)$ is globally generated. This completes the proof of iii). $\square$

We end with an application of Castelnuovo-Mumford regularity to an assertion about points in the projective plane. We begin with a definition: given a positive integer d, a finite subset $\Lambda \subseteq \mathbf{P}^n$ *imposes independent conditions on hypersurfaces of degree* d if the canonical restriction morphism

$$\Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)\big) \to \Gamma\big(\Lambda, \mathcal{O}_\Lambda(d)\big)$$

is surjective. Equivalently, for every point $p \in \Lambda$, there is an effective Cartier divisor D in $\mathbf{P}^n$ of degree d, such that $p \not\in \mathrm{Supp}(D)$ and $q \in \mathrm{Supp}(D)$ for all $q \in \Gamma$ with $q \neq p$. If $\mathcal{I}_\Lambda$ is the radical ideal sheaf corresponding to Λ, then it follows from the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{I}_\Lambda \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_\Lambda \to 0$$

that Λ imposes independent conditions on hypersurfaces of degree d if and only if $H^1\big(\mathbf{P}^n, \mathcal{I}_\Lambda(d)\big) = 0$.

Remark 12.2.24. If $\Lambda \subseteq \mathbf{P}^n$ imposes independent conditions on hypersurfaces of degree d, then it also imposes independent conditions on hypersurfaces of degree d', for every $d' \geq d$. Indeed, it is enough to show this when $d' = d+1$. For every $p \in \Lambda$, if D is an effective Cartier divisor in $\mathbf{P}^n$ of degree d such that $p \not\in \mathrm{Supp}(D)$ and $q \in \mathrm{Supp}(D)$ for every $q \in \Lambda$ with $q \neq p$, then for any hyperplane H not containing p, the divisor $D + H$ has degree $d+1$ and satisfies the same conditions.

Proposition 12.2.25. *For every $n \geq 3$, let $\mathcal{S}_n$ consist of all subsets of $\mathbf{P}^2$ with n elements and $\mathcal{S}'_n \subseteq \mathcal{S}_n$ the subset consisting of all sets that do not lie on a line.*

i) *There is N such that $h^0\big(\mathbf{P}^2, \mathcal{I}_\Lambda(n-1)\big) = N$ for all $\Lambda \in \mathcal{S}$, where $\mathcal{I}_\Lambda$ is the radical ideal sheaf corresponding to Λ.*
ii) *The map $\mathcal{S}'_n \to G$ that takes Λ to $H^0\big(\mathbf{P}^2, \mathcal{I}_\Lambda(n-1)\big)$, where G is the Grassmann variety of N-dimensional linear subspaces of the vector space $H^0\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n-1)\big)$ is injective.*

Note that if Λ is a set of n points lying on a line L in $\mathbf{P}^2$, then every polynomial of degree $n-1$ that vanishes on Λ vanishes on L, hence $H^0\big(\mathbf{P}^2, \mathcal{I}_\Lambda(n-1)\big) = H^0\big(\mathbf{P}^2, \mathcal{I}_L(n-1)\big)$. In particular, Λ is not determined by $H^0\big(\mathbf{P}^2, \mathcal{I}_\Lambda(n-1)\big)$.

Proof of Proposition 12.2.25. For every finite subset Λ of $\mathbf{P}^2$ and every $d \geq 0$, we deduce from the short exact sequence

$$0 \to \mathcal{I}_\Lambda(d) \to \mathcal{O}_{\mathbf{P}^2}(d) \to \mathcal{O}_\Lambda(d) \to 0$$

that

$$\chi\big(\mathcal{I}_\Lambda(d)\big)=\chi\big(\mathcal{O}_{\mathbf{P}^2}(d)\big)-\chi\big(\mathcal{O}_\Lambda(d)\big)=h^0\big(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(d)\big)-\#(\Lambda)=\binom{d+2}{2}-\#(\Lambda).$$

Moreover, the long exact sequence in cohomology gives an exact sequence

$$0=H^1\big(\mathbf{P}^2,\mathcal{O}_\Lambda(d)\big)\to H^2\big(\mathbf{P}^2,\mathcal{I}_\Lambda(d)\big)\to H^2\big(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(d)\big)=0,$$

hence $h^2\big(\mathbf{P}^2,\mathcal{I}_\Lambda(d)\big)=0$.

The assertion in i) thus follows if we show that if $\#(\Lambda)=n$, then Λ imposes independent conditions on hypersurfaces of degree $n-1$. This is clear: given any $p\in\Lambda$, for every $q\in\Lambda\smallsetminus\{p\}$, let L_q be a line containing q, but not containing p. The effective Cartier divisor $\sum_{q\in\Lambda\smallsetminus\{p\}}L_q$ has degree $n-1$, does not contain p in its support, but contains all other points in Λ.

In order to prove ii), it is enough to show that for every $\Lambda\in\mathcal{S}'_n$, the sheaf $\mathcal{I}_\Lambda(n-1)$ is globally generated. By Proposition 12.2.23, it is enough to show that $\mathcal{I}_\Lambda$ is $(n-1)$-regular, that is,

$$H^1\big(\mathbf{P}^2,\mathcal{I}_\Lambda(n-2)\big)=0=H^2\big(\mathbf{P}^2,\mathcal{I}_\Lambda(n-3)\big).$$

As we have seen, the second equality automatically holds, while the first one is equivalent with the fact that Λ imposes independent conditions on hypersurfaces of degree $n-2$. Given any $p\in\Lambda$, there is a line L containing two of the other points in Λ, but not containing p. Otherwise, by considering the lines joining a point $q\in\Lambda\smallsetminus\{p\}$ with the other points in $\Lambda\smallsetminus\{p,q\}$, we see that in fact all these lines coincide (if this is not the case, we would get $p=q$); moreover, this line also contains p, contradicting the hypothesis that the points of Λ do not lie on a line. For each of the remaining $n-3$ points in Λ, we choose a line passing through that point, but not containing p. The sum of the $n-2$ lines that we constructed is an effective Cartier divisor of degree $n-2$, not containing p in its support, but containing all the other points in Λ. This completes the proof of ii). $\square$

12.2.6. The Grothendieck ring of $\mathbf{P}^n$. . We end this section by computing $K_0(\mathbf{P}^n)$ and $K^0(\mathbf{P}^n)$. This will be based on a result of independent interest about resolutions of coherent sheaves on $\mathbf{P}^n$. We begin with the following graded version of Nakayama's lemma. In what follows we put $S=k[x_0,\ldots,x_n]$. The first two results hold, more generally, for graded quotient rings of S.

LEMMA 12.2.26. *Let R be a graded quotient ring of S and $\mathfrak{m}=\oplus_{i>0}R_i$. If M is a finitely generated, graded R-module and N is a graded submodule of M such that $M=\mathfrak{m}M+N$, then $N=M$.*

PROOF. After replacing M by M/N, we reduce to the case when $N=0$. Suppose that $M\neq 0$ and let $u_1,\ldots,u_r$ be non-zero homogeneous generators of M, with $\deg(u_i)=d_i$. If $d=\min_i d_i$, then using the fact that $\mathfrak{m}_i=0$ for $i\leq 0$, it follows that $(\mathfrak{m}M)_i=0$ for $i<d+1$. This contradicts that fact that we have a non-zero homogeneous element of degree d in $M=\mathfrak{m}M$. $\square$

REMARK 12.2.27. As a side remark, we note that if R and $\mathfrak{m}$ are as in the lemma and M is a finitely generated, graded R-module, then for every homogeneous elements $f_1,\ldots,f_s$ that generate an ideal $I\subseteq\mathfrak{m}$, we have $\operatorname{depth}(I,M)=s$ if and only if $f_1,\ldots,f_s$ form an M-regular sequence. The "if" part is a general fact (see Example 12.1.9). On the other hand, if $\operatorname{depth}(I,M)=s$, then it follows from

Remark 12.2.13 that $\mathcal{H}_i(\underline{x}; M) = 0$ for $i > 0$. We can now deduce that $x_1, \ldots, x_s$ is an M-regular sequence by arguing as in the proof of assertion ii) in Theorem 12.2.7: that proof made use of the local assumption in order to apply Nakayama's lemma; this is now replaced by Lemma 12.2.26.

As in the local case, Lemma 12.2.26 allows us to talk about *minimal systems of homogeneous generators* for M. Indeed, if $u_1, \ldots, u_r \in M$ are homogeneous elements, these generate M if and only if their classes in $M/\mathfrak{m}M$ generate this k-vector space. Therefore $u_1, \ldots, u_r$ form a minimal system of generators if and only if their classes in $M/\mathfrak{m}M$ form a k-basis.

A *finitely generated, free, graded R-module* is a graded R-module isomorphic to $\bigoplus_{i=1}^r R(-d_i)$ for some integers $d_1, \ldots, d_r$. Given any M as above and a minimal system of homogeneous generators $u_1, \ldots, u_r$, with $\deg(u_i) = d_i$, we have a surjective morphism of graded R-modules $\phi\colon F \to M$, with F a finitely generated, free, graded R-module. Since the classes $\overline{u_1}, \ldots, \overline{u_r} \in M/\mathfrak{m}M$ form a k-basis, it follows that if $K = \ker(\phi)$, then $K \subseteq \mathfrak{m}M$. If M is a projective R-module, then ϕ is a split surjection, and we thus have an exact sequence

$$0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \to M/\mathfrak{m}M \to 0.$$

Since $K \subseteq \mathfrak{m}F$, we conclude that $K = \mathfrak{m}K$, hence $K = 0$ by Lemma 12.2.26. This proves the following:

PROPOSITION 12.2.28. *If R is a graded quotient ring of S, then a finitely generated graded R-module M is projective if and only if it is a free graded R-module.*

By combining this with the result concerning the projective dimension of regular local rings, we obtain the following:

COROLLARY 12.2.29. *Given any finitely generated, graded module M over $S = k[x_0, \ldots, x_n]$, we have a free resolution of M of the form:*

$$0 \to F_{n+1} \to \ldots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is a finitely generated, free, graded S-module $F_i = \bigoplus_j S(-j)^{\oplus \beta_{i,j}}$. In particular, every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ has a locally free resolution of the form

$$0 \to \mathcal{E}_{n+1} \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{E}_i$ is a direct sum of finitely many line bundles on $\mathbf{P}^n$.

PROOF. By choosing finitely many homogeneous generators for M, we construct a surjective graded homomorphism

$$\phi\colon F_0 \to M, \quad \text{where} \quad F_0 = \bigoplus_j S(-j)^{\oplus \beta_{0,j}}.$$

By taking the kernel of ϕ and repeating this construction step by step, we obtain an exact complex

$$0 \to F_{n+1} \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0,$$

where $F_0, \ldots, F_n$ are finitely generated, free, graded S-modules. Since the affine variety corresponding to S is smooth, irreducible, of dimension $n+1$, it follows from Proposition 12.2.16 that for every maximal ideal $\mathfrak{m}$ of S, we have $\mathrm{pd}_{S_\mathfrak{m}}(M_\mathfrak{m}) \leq n+1$, hence $(F_{n+1})_\mathfrak{m}$ is a projective $S_\mathfrak{m}$-module. Therefore F_{n+1} is a projective S-module, and thus it is a free graded S-module by Proposition 12.2.28. This gives the first assertion in the proposition. The second one follows from the fact that every coherent sheaf on $\mathbf{P}^n$ is isomorphic to $\widetilde{M}$, for some finitely generated, graded S-module M. $\square$

REMARK 12.2.30. We emphasize that for $n \geq 2$, it is *not* true that every locally free sheaf on $\mathbf{P}^n$ is a direct sum of line bundles. For example, for $n \geq 2$, the cotangent bundle is not isomorphic to a direct sum of line bundles. Since $H^1\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) = 0$ for every m by Theorem 11.2.2, it is enough to show that $H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}) \neq 0$. Recall that we have the Euler exact sequence

$$0 \to \Omega_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n} \to 0.$$

The long exact sequence in cohomology gives an exact sequence

$$0 = \Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)}\big) \to \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n})$$
$$\to H^1\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)}\big) = 0.$$

We thus obtain $H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}) \simeq k$.

We use Corollary 12.2.29 to give the following description for the Grothendieck group of $\mathbf{P}^n$.

PROPOSITION 12.2.31. *The Grothendieck group $K_0(\mathbf{P}^n)$ of coherent sheaves on $\mathbf{P}^n$ is freely generated by $[\mathcal{O}_{\mathbf{P}^n}], [\mathcal{O}_{\mathbf{P}^n}(-1)], \ldots, [\mathcal{O}_{\mathbf{P}^n}(-n)]$. Moreover, we have a ring isomorphism between the Grothendieck ring $K^0(\mathbf{P}^n)$ of locally free sheaves on $\mathbf{P}^n$ and $\mathbf{Z}[x]/(x^{n+1})$ that maps $1 - [\mathcal{O}_{\mathbf{P}^n}(-1)]$ to $\overline{x} \in \mathbf{Z}[x]/(x^{n+1})$.*

PROOF. We deduce from Corollary 12.2.29 that $K_0(\mathbf{P}^n)$ is generated by the elements $[\mathcal{O}_{\mathbf{P}^n}(m)]$, for $m \in \mathbf{Z}$. Let A be the subgroup of $K_0(\mathbf{P}^n)$ generated by $[\mathcal{O}_{\mathbf{P}^n}], [\mathcal{O}_{\mathbf{P}^n}(-1)], \ldots, [\mathcal{O}_{\mathbf{P}^n}(-n)]$. Recall that we have the exact Koszul complex:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-n-1) \to \mathcal{O}_{\mathbf{P}^n}(-n)^{\oplus(n+1)} \to \ldots \to \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n} \to 0.$$

By tensoring this with $\mathcal{O}_{\mathbf{P}^n}(m)$, we see that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in the subgroup generated by $[\mathcal{O}_{\mathbf{P}^n}(m-i)]$ for $1 \leq i \leq n+1$. We thus deduce, by induction on $m \geq 0$, that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in A for every $m \geq 0$. Similarly, by tensoring the Koszul complex with $\mathcal{O}_{\mathbf{P}^n}(m+n+1)$, we see that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in the subgroup generated by $[\mathcal{O}_{\mathbf{P}^n}(m+i)]$, for $1 \leq i \leq n+1$. Using this, we see by decreasing induction on $m \leq -n-1$ that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in A. We thus conclude that $A = K_0(\mathbf{P}^n)$.

Note now that for every closed subvariety X of $\mathbf{P}^n$, given an exact sequence of coherent sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

by tensoring this with $\mathcal{O}_X(m)$ and taking the Euler-Poincaré characteristic, we obtain

$$\chi\big(\mathcal{F}(m)\big) = \chi\big(\mathcal{F}'(m)\big) + \chi\big(\mathcal{F}''(m)\big) \quad \text{for all} \quad m \in \mathbf{Z}.$$

We thus have $P_{\mathcal{F}} = P_{\mathcal{F}'} + P_{\mathcal{F}''}$. This implies that we have a morphism of Abelian groups $K_0(X) \to \mathbf{Q}[t]$ that maps $[\mathcal{F}]$ to the Hilbert polynomial $P_{\mathcal{F}}$.

We apply this to the case $X = \mathbf{P}^n$. The morphism $\gamma\colon K_0(\mathbf{P}^n) \to \mathbf{Q}[t]$ maps $[\mathcal{O}_{\mathbf{P}^n}(-i)]$ to the polynomial $P(t-i)$, where

$$P(t) = \frac{(t+1)\cdots(t+n)}{n!}.$$

We claim that $P(t), P(t-1), \ldots, P(t-n)$ are linearly independent over $\mathbf{Z}$. Indeed, if $\sum_{i=0}^n \lambda_i P(t-i) = 0$, where not all λ_i are 0, and if $j = \max\{i \mid \lambda_i \neq 0\}$, then by taking $t = j$, we obtain $\lambda_j = 0$, a contradiction. Since the images by γ of

$[\mathcal{O}_{\mathbf{P}^n}], [\mathcal{O}_{\mathbf{P}^n}(-1)], \ldots, [\mathcal{O}_{\mathbf{P}^n}(-n)]$ are linearly independent over $\mathbf{Z}$, we conclude that these elements freely generate $K_0(\mathbf{P}^n)$, completing the proof of the first assertion in the proposition.

Recall now that by Proposition 12.2.19, since $\mathbf{P}^n$ is smooth and carries an ample line bundle, the canonical group homomorphism $K^0(\mathbf{P}^n) \to K_0(\mathbf{P}^n)$ is an isomorphism. What we proved so far thus shows that if $h = [\mathcal{O}_{\mathbf{P}^n}(-1)]$, then $1, h, \ldots, h^n$ give a $\mathbf{Z}$-basis of $K^0(\mathbf{P}^n)$. Moreover, the Koszul complex gives the relation $\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}h^i = 0$, that is, $(1-h)^{n+1} = 0$. Since $\{(1-\overline{x})^i \mid 0 \leq i \leq n\}$ gives a basis of $\mathbf{Z}[x]/(x^{n+1})$, we obtain the second assertion in the proposition. $\square$

12.3. Cohen-Macaulay varieties and sheaves

Using the notion of depth, we introduce Cohen-Macaulay rings and modules, give some examples, and discuss some basic properties. When doing this, we restrict ourselves most of the time to the geometric setting[1]. For a treatment in arbitrary Noetherian rings, we refer to [**BH93**, §2.1].

Definition 12.3.1. If R is a Noetherian local ring and M is a finitely generated, non-zero R-module, then M is a *Cohen-Macaulay module* if $\operatorname{depth}(M) = \dim(M)$. If R is an arbitrary Noetherian ring and M is a finitely generated R-module then M is a *Cohen-Macaulay module* if $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module for all maximal ideals $\mathfrak{p} \in \operatorname{Supp}(M)$ (thus, by convention, $M = 0$ is considered Cohen-Macaulay). If $M = R$, we say instead that R is a *Cohen-Macaulay* ring.

Remark 12.3.2. It follows from Remark 12.1.7 that if M is a finitely generated module over a Noetherian ring R and $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a} \cdot M = 0$, then M is a Cohen-Macaulay module over R if and only if it is a Cohen-Macaulay module over $R/\mathfrak{a}$.

Proposition 12.3.3. *A Noetherian ring R is Cohen-Macaulay if and only if for every ideal $\mathfrak{a} \subsetneq R$, we have*

$$\operatorname{depth}(\mathfrak{a}, R) = \operatorname{codim}(\mathfrak{a}).$$

Proof. Note that by Proposition 12.1.11, we have

$$\operatorname{depth}(\mathfrak{a}, R) = \min_{\mathfrak{p}} \operatorname{depth}(R_{\mathfrak{p}}),$$

where the minimum is over all prime ideals $\mathfrak{p}$ containing $\mathfrak{a}$. It is then clear that if R is Cohen-Macaulay, then $\operatorname{depth}(\mathfrak{a}, R) = \operatorname{codim}(\mathfrak{a})$. Conversely, if this holds for all ideals $\mathfrak{a}$, then in particular it holds for all maximal ideals $\mathfrak{m}$. On the other hand, for every such $\mathfrak{m}$, we have

$$\operatorname{depth}(\mathfrak{m}, R) = \operatorname{depth}(R_{\mathfrak{m}}) \leq \dim(R_{\mathfrak{m}}) = \operatorname{codim}(\mathfrak{m}),$$

where the first equality follows from Proposition 12.1.11 and the inequality follows from Proposition 12.1.13. We thus conclude that $\operatorname{depth}(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$, hence R is Cohen-Macaulay. $\square$

Remark 12.3.4. If $\mathfrak{a}$ is an ideal in a Noetherian ring R, then for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, we have the following inequalities

$$\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{depth}(\mathfrak{a}R_{\mathfrak{m}}, R_{\mathfrak{m}}) \leq \operatorname{codim}(\mathfrak{a}R_{\mathfrak{m}}) \leq \operatorname{codim}(\mathfrak{a}),$$

[1]This is done simply because we developed the basic properties of dimension theory only in the geometric setting.

where the second one follows from Remark 12.1.14. We deduce from the proposition that if R is Cohen-Macaulay, then the above inequalities are all equalities.

EXERCISE 12.3.5. Show that if R is a reduced, Noetherian ring with $\dim(R) = 1$, then R is Cohen-Macaulay.

EXERCISE 12.3.6. Show that the subring $k[x^4, x^3y, xy^3, y^4]$ of $k[x,y]$, where k is a field, is not Cohen-Macaulay.

DEFINITION 12.3.7. A coherent sheaf $\mathcal{F}$ on an algebraic variety X is *Cohen-Macaulay at* $x \in X$ if $\mathcal{F}_x$ is a Cohen-Macaulay $\mathcal{O}_{X,x}$-module. If $\mathcal{F} = \mathcal{O}_X$, we say that X *is Cohen-Macaulay at* x. The sheaf $\mathcal{F}$ is *Cohen-Macaulay* if it is Cohen-Macaulay at every point; equivalently, for every affine open subset $U \subseteq X$, the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is Cohen-Macaulay. Similarly, X is Cohen-Macaulay if it is Cohen-Macaulay at every point.

We now give some examples of Cohen-Macaulay varieties.

EXAMPLE 12.3.8. Every smooth variety is Cohen-Macaulay. Indeed, if x is a smooth point on the variety X, then it follows from Example 12.1.15 that $\mathrm{depth}(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x})$, hence $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring. In particular, every affine space $\mathbf{A}^n$ is a Cohen-Macaulay variety.

EXAMPLE 12.3.9. If X and Y are Cohen-Macaulay varieties, then $X \times Y$ is Cohen-Macaulay. Indeed, we may assume that X and Y are affine, with $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$ and let $x \in X$ and $y \in Y$ be points corresponding to the maximal ideals $\mathfrak{m} \subseteq A$ and $\mathfrak{n} \subseteq B$. In this case $X \times Y$ is an affine variety with $\mathcal{O}(X \times Y) \simeq A \otimes_k B$ and the point (x,y) corresponds to the maximal ideal $\mathfrak{p} = \mathfrak{m} \otimes_k B + A \otimes_k \mathfrak{n}$. If $\mathrm{codim}(\mathfrak{m}) = r$ and $\mathrm{codim}(\mathfrak{n}) = s$, then $\mathrm{codim}(\mathfrak{p}) = r + s$. Moreover, by assumption we have an A-regular sequence $a_1, \ldots, a_r \in \mathfrak{m}$ and a B-regular sequence $b_1, \ldots, b_s \in \mathfrak{n}$. It is then straightforward to check that $a_1 \otimes 1, \ldots, a_r \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_s$ is an $A \otimes_k B$ regular sequence, hence $\mathrm{depth}(\mathcal{O}_{X \times Y,(x,y)}) \geq r + s$. Since the opposite inequality follows from Proposition 12.1.13, we are done.

EXAMPLE 12.3.10. If $f\colon X \to Y$ is a finite, surjective morphism of algebraic varieties, with Y smooth, then X is Cohen-Macaulay if and only if f is flat (which, by Corollary 10.7.24 is equivalent to $f_*(\mathcal{O}_X)$ being a locally free $\mathcal{O}_Y$-module). In order to check this, we may assume that both X and Y are affine. Let $A = \mathcal{O}(Y)$ and $B = \mathcal{O}(X)$ and consider the finite, injective homomorphism $\phi\colon A \to B$ corresponding to f. The key point is that if $\mathfrak{m}$ is a maximal ideal in A, then

$$\mathrm{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \mathrm{depth}(\mathfrak{m}B_{\mathfrak{m}}, B_{\mathfrak{m}}) = \min_{\mathfrak{n}} \mathrm{depth}(B_{\mathfrak{n}}), \tag{12.3.1}$$

where the minimum on the right-hand side is over the maximal ideals in B containing $\mathfrak{m}B$; note that the first equality in (12.3.1) follows from Corollary 12.2.15 and the second one from Corollary 12.1.11. Since for every $\mathfrak{n}$ as above, we have $\mathrm{depth}(B_{\mathfrak{n}}) \leq \dim(B_{\mathfrak{n}}) = \dim(A_{\mathfrak{m}})$, it follows from (12.3.1) that $B_{\mathfrak{n}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{n}$ in B, with $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$, if and only if

$$\mathrm{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}}).$$

On the other hand, $A_{\mathfrak{m}}$ is regular and thus $\mathrm{pd}_{A_{\mathfrak{m}}}(B_{\mathfrak{m}}) < \infty$ and $\mathrm{depth}(A_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$. By the Auslander-Buchsbaum formula, we conclude that

$$\mathrm{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}}) - \mathrm{pd}_{A_{\mathfrak{m}}}(B_{\mathfrak{m}}),$$

and thus $\mathrm{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$ if and only if $B_{\mathfrak{m}}$ is projective (equivalently, flat) over $A_{\mathfrak{m}}$. This proves our assertion.

EXAMPLE 12.3.11. If X is an affine toric variety which is normal, then X is Cohen-Macaulay (see [**Ful93**, Chapter 2.1]).

In what follows we give some general properties of Cohen-Macaulay varieties and Cohen-Macaulay sheaves.

PROPOSITION 12.3.12. *If $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{I}$ is a coherent sheaf of ideals on X, then for every $x \in X$ such that $\mathcal{I}_x$ is generated by an $\mathcal{F}_x$-regular sequence, $\mathcal{F}$ is Cohen-Macaulay at x if and only if $\mathcal{F}/\mathcal{I}\mathcal{F}$ is Cohen-Macaulay at x.*

PROOF. Arguing by induction of the length on the regular sequence, it is clear that it is enough to treat the case of a regular sequence of length 1. Since $\mathcal{I}_x$ is generated by a non-zero-divisor on $\mathcal{F}_x$, it follows that $V(\mathcal{I})$ does not contain any irreducible component of $\mathrm{Supp}(\mathcal{F})$ passing through x, and since $\mathrm{Supp}(\mathcal{F}/\mathcal{I}\mathcal{F}) = \mathrm{Supp}(\mathcal{F}) \cap V(\mathcal{I})$, we have

$$\dim\big((\mathcal{F}/\mathcal{I}\mathcal{F})_x\big) = \dim(\mathcal{F}_x) - 1,$$

while Corollary 12.1.8 gives

$$\mathrm{depth}\big((\mathcal{F}/\mathcal{I}\mathcal{F})_x\big) = \mathrm{depth}(\mathcal{F}_x) - 1.$$

The assertion in the proposition now follows from the definition. $\square$

PROPOSITION 12.3.13. *If $\mathcal{F}$ is a coherent sheaf on a variety X and $\mathcal{F}$ is Cohen-Macaulay at $x \in X$, then every associated variety of $\mathcal{F}$ that passes through x is an irreducible component of $\mathrm{Supp}(\mathcal{F})$. Moreover, any two irreducible components of $\mathrm{Supp}(\mathcal{F})$ that pass through x have the same dimension, equal to $\dim_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$.*

PROOF. If $U \subseteq X$ is an affine open subset containing x, and $\mathfrak{m} \subseteq R = \mathcal{O}_X(U)$ is the maximal ideal corresponding to x, it follows from Proposition 12.1.13 that for every $\mathfrak{p} \subseteq \mathfrak{m}$ in $\mathrm{Ass}_R(M)$, where $M = \mathcal{F}(U)$, we have

$$\mathrm{depth}(M_{\mathfrak{m}}) \leq \dim(R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}) \leq \dim(M_{\mathfrak{m}}).$$

Since $M_{\mathfrak{m}}$ is a Cohen-Macaulay module, it follows that the above inequalities are equalities. Therefore $\mathfrak{p}$ is a minimal prime ideal in $\mathrm{Supp}(M)$ and all minimal prime ideals in $\mathrm{Supp}(M)$ that are contained in $\mathfrak{m}$ correspond to varieties of dimension equal to $\dim(M_{\mathfrak{m}})$; we thus have both assertions in the proposition. $\square$

PROPOSITION 12.3.14. *If a coherent sheaf $\mathcal{F}$ on the algebraic variety X is Cohen-Macaulay at $x \in X$, then $\mathcal{F}_Y$ is a Cohen-Macaulay $\mathcal{O}_{X,Y}$-module for every closed, irreducible subvariety Y of X, with $x \in Y$.*

PROOF. We may assume that X is affine, with $R = \mathcal{O}(X)$, and let $\mathfrak{p}$ and $\mathfrak{m}$ be the prime ideals in R corresponding to Y and x, respectively. We also put $M = \mathcal{F}(X)$. We may assume that $M_{\mathfrak{p}} \neq 0$ and need to show that $\mathrm{depth}(M_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}})$. For this, we argue by induction on $r = \mathrm{depth}(M_{\mathfrak{p}})$. If $r = 0$, then $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Since $M_{\mathfrak{m}}$ is Cohen-Macaulay, it follows from Proposition 12.3.13 that $\mathfrak{p}$ is a minimal prime in $\mathrm{Supp}(M)$, and thus $\dim(M_{\mathfrak{p}}) = 0$.

Suppose now that $r \geq 1$. In this case $\mathfrak{p} \notin \mathrm{Ass}_R(M)$, hence there is $h \in \mathfrak{p}$ which is a non-zero-divisor on M, in which case $\mathrm{depth}(M_{\mathfrak{p}}/hM_{\mathfrak{p}}) = r-1$ by Corollary 12.1.8. Since $M_{\mathfrak{m}}/hM_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module by Proposition 12.3.12, we can apply the inductive hypothesis to conclude that $r-1=\dim(M_{\mathfrak{p}}/hM_{\mathfrak{p}})$. Since h is a non-zero-divisor on M, it does not vanish on any irreducible component of $\mathrm{Supp}(M)$, and thus if $Z=\mathrm{Supp}(M/hM)=\mathrm{Supp}(M)\cap V(h)$, we have

$$r-1=\dim(M_{\mathfrak{p}}/hM_{\mathfrak{p}})=\mathrm{codim}_Z(Y)=\mathrm{codim}_{\mathrm{Supp}(\mathcal{F})}(Y)-1=\dim(M_{\mathfrak{p}})-1.$$

This completes the proof of the induction step. □

REMARK 12.3.15. It follows from the above proposition that if X is a Cohen-Macaulay affine variety, then $\mathcal{O}(X)$ trivially satisfies Serre's condition (S_i) for every i. In particular, it follows from Theorem 12.1.27 that a Cohen-Macaulay variety is normal if and only if it is smooth in codimension 1.

DEFINITION 12.3.16. Given a variety X, a coherent ideal sheaf $\mathcal{I}$ on X is *locally a complete intersection ideal* if for every $x\in V(\mathcal{I})$, the ideal $\mathcal{I}_x\subseteq\mathcal{O}_{X,x}$ is generated by a regular sequence. We say that a closed subvariety Y of X is *regularly embedded* if the radical ideal sheaf corresponding to Y is locally a complete intersection.

REMARK 12.3.17. The conormal sheaf of a regularly embedded subvariety is locally free: this is a special case of the computation in Example 12.2.22.

REMARK 12.3.18. If X is a Cohen-Macaulay variety, then a coherent ideal sheaf $\mathcal{I}$ is locally a complete intersection ideal if and only if for every $x\in V(\mathcal{I})$, the ideal $\mathcal{I}_x$ can be generated by r elements, where $r=\mathrm{codim}(\mathcal{I}_x)$ (this follows from Proposition 12.3.3 and Corollary 12.2.14). Moreover, in this case, the $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{I}$ is Cohen-Macaulay by Proposition 12.3.12.

EXAMPLE 12.3.19. If Y is a smooth closed subvariety of the smooth variety X, it follows from Proposition 6.3.21 that Y is regularly embedded in X.

We now give an application to Bézout's theorem. Suppose that $H_1,\ldots,H_n$ are effective Cartier divisors in $\mathbf{P}^n$, with $\deg(H_i)=d_i$, such that $Z:=\bigcap_{i=1}^n\mathrm{Supp}(H_i)$ is 0-dimensional. For every point $p\in Z$, we define the *intersection multiplicity* $i_p(H_1,\ldots,H_n)$ of $H_1,\ldots,H_n$ at p, as follows. If $f_i\in\mathcal{O}_{\mathbf{P}^n,p}$ is the image of an equation of H_i in a neighborhood of p, then the quotient $\mathcal{O}_{\mathbf{P}^n,p}/(f_1,\ldots,f_n)$ is 0-dimensional and its length is $i_p(H_1,\ldots,H_n)$. With this notation, we have the following result.

PROPOSITION 12.3.20 (Bézout). *With the above notation, we have*

$$\sum_{p\in Z}i_p(H_1,\ldots,H_n)=\prod_{i=1}^n d_i.$$

PROOF. For j, with $1\leq j\leq n$, let

$$\mathcal{F}_j=\mathcal{O}_{H_1}\otimes_{\mathcal{O}_{\mathbf{P}^n}}\ldots\otimes_{\mathcal{O}_{\mathbf{P}^n}}\mathcal{O}_{H_j}\simeq\mathcal{O}_{\mathbf{P}^n}/\mathcal{I}_j.$$

Note that $\mathrm{Supp}(\mathcal{F}_j)=\bigcap_{i=1}^j\mathrm{Supp}(H_i)$ has all its irreducible components of codimension $\leq j$ in $\mathbf{P}^n$ by Corollary 3.3.7. In fact, every such irreducible component has codimension precisely j: otherwise, its intersection with $\bigcap_{i=j+1}^n\mathrm{Supp}(H_i)$ would be non-empty, of dimension ≥ 1, by Corollary 4.2.12, contradicting the hypothesis. Since $\mathcal{I}_j$ is generated at every point by j elements, we see that $\mathcal{I}_j$ is locally a complete intersection ideal, and thus $\mathcal{F}_j$ is a Cohen-Macaulay module. In particular, Proposition 12.3.13 implies that the associated subvarieties of $\mathcal{F}_j$ are precisely the irreducible components of $\mathrm{Supp}(\mathcal{F}_j)$. For $j\leq n-1$, since $\mathrm{Supp}(H_{j+1})$ does not contain any irreducible component of $\mathrm{Supp}(\mathcal{F}_j)$, it follows from Example 11.4.10 that

$$\deg(\mathcal{F}_{j+1}) = \deg(\mathcal{F}_j) \cdot d_{j+1}.$$

We thus conclude that

$$\prod_{i=1}^{n} d_i = \deg(\mathcal{F}_n) = \sum_{p \in Z} i_p(H_1, \ldots, H_n).$$

□

We end this section with a discussion of complete intersection projective varieties.

EXAMPLE 12.3.21. A closed subvariety $X \subseteq \mathbf{P}^n$, of codimension r, is a (global) *complete intersection* if its homogeneous ideal $I_X \subseteq S = k[x_0, \ldots, x_n]$ is generated by r homogeneous elements $f_1, \ldots, f_r$. Let $d_i = \deg(f_i) > 0$.

We first note that such a variety is Cohen-Macaulay. Indeed, if $\mathcal{I}$ is the radical ideal sheaf corresponding to X, by Remark 12.3.18 it is enough to show that for every $p \in X$, the ideal $\mathcal{I}_p \subseteq \mathcal{O}_{\mathbf{P}^n,p}$ is generated by $\mathrm{codim}(\mathcal{I}_p)$ elements (we use here that $\mathbf{P}^n$ is smooth, hence Cohen-Macaulay). If $p \in U_i = (x_i \neq 0)$, then $\mathcal{I}_p$ is generated by $\frac{f_1}{x_i^{d_1}}, \ldots, \frac{f_r}{x_i^{d_r}}$. In particular, we have $\mathrm{codim}(\mathcal{I}_p) \leq r$ by Corollary 3.3.7. On the other hand, we clearly have

$$\mathrm{codim}(\mathcal{I}_p) \geq \mathrm{codim}_{\mathbf{P}^n}(X) = r,$$

hence this is an equality and X is Cohen-Macaulay. Moreover, this argument shows that every irreducible component of X has pure dimension $n - r$.

We next show that the following hold:

(α) If $r \leq n-1$, then the canonical morphism $S/I_X \to \bigoplus_{m \in \mathbf{Z}} \Gamma\big(X, \mathcal{O}_X(m)\big)$ is an isomorphism.

(β) For every q, with $1 \leq q \leq \dim(X) - 1$, we have

$$H^q\big(X, \mathcal{O}_X(m)\big) = 0 \quad \text{for all} \quad m \in \mathbf{Z}.$$

In order to show this, we consider $\mathcal{F}_j = \mathcal{O}_{\mathbf{P}^n}/\mathcal{I}_j$, for $1 \leq j \leq r$, where $\mathcal{I}_j$ is the ideal sheaf corresponding to the homogeneous ideal $(f_1, \ldots, f_j)$. Arguing as above, we see that each $\mathcal{F}_j$ is a Cohen-Macaulay $\mathcal{O}_{\mathbf{P}^n}$-module. Indeed, $\mathcal{I}_j$ is locally defined by j equations. In particular, every irreducible component of $\mathrm{Supp}(\mathcal{F}_j)$ has codimension $\leq j$ in $\mathbf{P}^n$. In fact, the codimension is precisely j: otherwise, the intersection of this component with $\bigcap_{i=j+1}^{r} V(f_i)$ would be non-empty, of codimension $< r$ by Corollary 4.2.12, a contradiction with the fact $\mathrm{codim}_{\mathbf{P}^n}(X) = r$.

With the convention that $\mathcal{F}_0 = \mathcal{O}_{\mathbf{P}^n}$, we show by induction on $j \geq 0$, that the following hold:

(α_j) If $j \leq n-1$, then the canonical morphism

$$S/(f_1, \ldots, f_j) \to \bigoplus_{m \in \mathbf{Z}} \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m)\big)$$

is an isomorphism.

(β_j) For every q, with $1 \leq q \leq n - j - 1$, we have

$$H^q\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) = 0 \quad \text{for all} \quad m \in \mathbf{Z}.$$

For $j = r$, we obtain (α) and (β).

Both (α_0) and (β_0) hold by Theorem 11.2.2, hence it is enough to show that for $0 \leq j \leq r-1$, if we know the assertions for j, then we obtain them for $j+1$. Since $\mathcal{F}_j$ is Cohen-Macaulay, it follows from Proposition 12.3.13 that the associated subvarieties of $\mathcal{F}_j$ are precisely the irreducible components of $\mathrm{Supp}(\mathcal{F}_j)$. Moreover, we have seen that each such irreducible component has codimension j, and thus can't be contained in $\mathrm{Supp}(\mathcal{F}_{j+1})$. Therefore $V(f_{j+1})$ contains no associated subvariety of $\mathcal{F}_j$, and thus we have a short exact sequence

$$0 \to \mathcal{F}_j(m - d_{j+1}) \to \mathcal{F}_j(m) \to \mathcal{F}_{j+1}(m) \to 0$$

for every $m \in \mathbf{Z}$. By taking the long exact sequence in cohomology, we obtain exact sequences

$$0 \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_{j+1}(m)\big) \to H^1\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) \tag{12.3.2}$$

and

$$H^q\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) \to H^q\big(\mathbf{P}^n, \mathcal{F}_{j+1}(m)\big) \to H^{q+1}\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) \tag{12.3.3}$$

for all $q \geq 0$. It is clear that using (β_j), the sequence in (12.3.3) implies (β_{j+1}). Moreover, if $j + 1 \leq n - 1$, then (β_j) implies $H^1\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) = 0$, and thus (12.3.2) and (α_j) imply (α_{j+1}). This completes the proof of the induction step.

In particular, it follows from (α) that if $r \leq n - 1$, then $\Gamma(X, \mathcal{O}_X) = k$. We thus conclude that in this case X is connected.

It is easy to compute the normal bundle of X. Indeed, $s = (f_1, \ldots, f_r)$ is a regular section of $\bigoplus_{j=1}^r \mathcal{O}_{\mathbf{P}^n}(d_i)$ such that $\mathcal{I} = \mathcal{I}(s)$, and thus $N_{X/\mathbf{P}^n} \simeq \bigoplus_{j=1}^r \mathcal{O}_X(d_i)$ by Example 12.2.22. In particular, if X is smooth, then we conclude using Corollary 8.7.27 that

$$\omega_X \simeq \omega_{\mathbf{P}^n}|_X \otimes_{\mathcal{O}_X} \det(N_{X/\mathbf{P}^n}) \simeq \mathcal{O}_X(d_1 + \ldots + d_r - n - 1).$$

REMARK 12.3.22. Suppose that $I \subseteq S = k[x_0, \ldots, x_n]$ is an ideal generated by homogeneous elements $f_1, \ldots, f_r$ of positive degree, with $r \leq n$, such that $\mathrm{codim}_{\mathbf{P}^n}\big(V(I)\big) = r$. We claim that in this case I is saturated (see Exercise 11.1.22) and $f_1, \ldots, f_r$ form a regular sequence. In particular, if $\widetilde{I}$ is the ideal sheaf corresponding to a closed subvariety X of $\mathbf{P}^n$, then $I = I_X$.

Indeed, since S is Cohen-Macaulay, the assumption on $V(I)$ implies that

$$\mathrm{depth}(I, S) = \mathrm{codim}(I) = r.$$

We thus deduce that $f_1, \ldots, f_r$ is a regular sequence (see Remark 12.2.27). The fact that I is saturated, is equivalent with $\mathfrak{m} \not\in \mathrm{Ass}(S/I)$, where $\mathfrak{m} = (x_0, \ldots, x_n)$. Since I is generated by a regular sequence and S is Cohen-Macaulay, it follows from Proposition 12.3.12 that S/I is Cohen-Macaulay. If $\mathfrak{m} \in \mathrm{Ass}(S/I)$, we deduce from Proposition 12.3.13 that $\mathfrak{m}$ is minimal over I, contradicting the fact that I is generated by $r \leq n$ elements, while $\mathrm{codim}(\mathfrak{m}) = n + 1$. Therefore I is saturated.

CHAPTER 13

Flatness and smoothness, revisited

13.1. Flatness, revisited

In this section we discuss a few more advanced topics related to flatness. We begin by considering the constancy of invariants of fibers of flat morphisms. We then prove Grothendieck's Generic Flatness theorem, and then end by discussing the Local Flatness criterion.

13.1.1. Hilbert polynomials of flat families. We begin with an easy statement in the case of finite morphisms.

Let $f\colon X \to Y$ be a *quasi-finite* morphism, that is, f has finite fibers. Given a point $y \in Y$ corresponding to the radical ideal sheaf $\mathfrak{m}_y$, the coherent sheaf $\mathcal{O}_X/\mathfrak{m}_y\mathcal{O}_X$ is supported on the fiber $f^{-1}(y)$. In particular, its stalk at every $x \in f^{-1}(y)$ has finite length, and we put

$$\mathrm{mult}_x f^{-1}(y) := \ell(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}).$$

Suppose now that $f\colon X \to Y$ is a finite morphism of algebraic varieties. Note that f is flat if and only if $f_*(\mathcal{O}_X)$ is a flat $\mathcal{O}_Y$-module. Since this is a coherent $\mathcal{O}_Y$-module, it is flat if and only if it is locally free (see Corollary 10.7.24). If this is the case and Y is connected, then $f_*(\mathcal{O}_X)$ has a well-defined rank: this is the *degree of* f, denoted $\deg(f)$. We note that if f is finite and flat, then it is both closed and open by Proposition 5.3.6 and Theorem 5.6.15. In particular, if Y is connected, then f is surjective. It is clear that if f is a finite, flat morphism of irreducible varieties, then the above notion of degree agrees with the one for generically finite maps introduced in Definition 5.3.11.

Proposition 13.1.1. *If $f\colon X \to Y$ is a finite, flat morphism of algebraic varieties, with Y connected, then for every $y \in Y$, we have*

$$\sum_{x\in f^{-1}(y)} \mathrm{mult}_x f^{-1}(y) = \deg(f),$$

Proof. It is clear that after replacing Y by a suitable affine open neighborhood U of y and X by $f^{-1}(U)$, we may assume that Y and X are affine, and $B = \mathcal{O}(X)$ is a free module over $A = \mathcal{O}(Y)$, of rank $\deg(f)$. If we identify $\mathfrak{m}_y$ with the maximal ideal in A corresponding to y, then we have

$$\sum_{x\in f^{-1}(y)} \mathrm{mult}_x f^{-1}(y) = \ell(B/\mathfrak{m}_yB) = \dim_k(B/\mathfrak{m}_yB) = \deg(f).$$

giving our assertion. $\square$

It is convenient to consider flatness for more general quasi-coherent modules. Given a morphism of algebraic varieties $f\colon X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X is *flat over* Y if the following equivalent conditions hold:

i) For every affine open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$, the $\mathcal{O}_Y(V)$-module $\mathcal{F}(U)$ is flat.
ii) There are affine open covers $Y = \bigcup_{i \in I} V_i$ and $X = \bigcup_{i \in I} U_i$ such that for every $i \in I$, we have $f(U_i) \subseteq V_i$ and the $\mathcal{O}_Y(V_i)$-module $\mathcal{F}(U_i)$ is flat.
iii) For every $x \in X$, the $\mathcal{O}_{Y,f(x)}$-module $\mathcal{F}_x$ is flat.

The proof of the equivalence of these conditions is entirely similar to that in Proposition 5.6.8. Note that $\mathcal{O}_X$ is flat over Y precisely when f is flat. If f is the identity map of X, we say that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module. Note that by Corollary 10.7.24, a coherent sheaf $\mathcal{F}$ on X is flat if and only if it is locally free.

Given a morphism $f\colon X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a point $y \in Y$ corresponding to the radical ideal sheaf $\mathfrak{m}_y$, we put $\mathcal{F}_y := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_y\mathcal{O}_X$. If f factors as $X \hookrightarrow \mathbf{P}^n_Y \xrightarrow{g} Y$, where the first morphism is a closed immersion and g is the canonical projection, then $\mathcal{F}_y$ can be identified to a coherent sheaf on the fiber $g^{-1}(y) \simeq \mathbf{P}^n$. In particular, we may consider the Hilbert polynomial of $\mathcal{F}_y$. We now show that flatness guarantees that this invariant is constant on the connected components of Y.

THEOREM 13.1.2. *Given a connected algebraic variety Y, a closed subvariety X of $\mathbf{P}^n_Y$, and $\mathcal{F}$ a coherent sheaf on X that is flat over Y, the Hilbert polynomial of $\mathcal{F}_y$ is independent of the point $y \in Y$.*

PROOF. It is clear that we may assume that Y is affine, with $A = \mathcal{O}(Y)$. Furthermore, after replacing $\mathcal{F}$ by its push-forward to $\mathbf{P}^n_Y$, we may assume that $X = \mathbf{P}^n_Y$. We consider the Čech complex computing the cohomology of $\mathcal{F}(q)$ with respect to the cover of $\mathbf{P}^n_Y$ by the affine open subsets $D^+(x_i)$, for $0 \leq i \leq n$:
(13.1.1)
$$0 \to \Gamma\big(\mathbf{P}^n_Y, \mathcal{F}(q)\big) \to C^0 = \bigoplus_i \Gamma\big(U_i, \mathcal{F}(q)\big) \to C^1 = \bigoplus_{i<j} \Gamma\big(U_i \cap U_j, \mathcal{F}(q)\big) \to \ldots.$$

Since $\mathcal{F}$ is flat over Y, it is clear that C^p is a flat A-module for all $p \geq 0$. On the other hand, it follows from Theorem 11.2.1 that there is q_0 such that $H^p\big(X, \mathcal{F}(q)\big) = 0$ for $q \geq q_0$. This implies that the complex (13.1.1) is exact for $q \geq q_0$. By breaking the complex into short exact sequences and using assertion ii) in Corollary 10.7.22, we conclude that $\Gamma\big(\mathbf{P}^n_Y, \mathcal{F}(q)\big)$ is flat over A for all $q \geq q_0$. By Theorem 11.2.1, this is also a finitely generated A-module, hence it is projective, of a well-defined rank. In order to obtain the conclusion of the theorem, it is enough to show that for every $y \in Y$, corresponding to the maximal ideal $\mathfrak{m}_y \subseteq A$, we have an isomorphism

$$\Gamma\big(\mathbf{P}^n_Y, \mathcal{F}(q)\big) \otimes_A A/\mathfrak{m}_y \simeq \Gamma\big(\mathbf{P}^n, \mathcal{F}_y(q)\big) \quad \text{for} \quad q \gg 0. \tag{13.1.2}$$

By choosing a system of generators of $\mathfrak{m}_y$, we obtain an exact sequence

$$A^{\oplus N} \to A \to A/\mathfrak{m}_y \to 0.$$

By taking the corresponding sheaves on Y, pulling them back to $\mathbf{P}^n_Y$, and tensoring with $\mathcal{F}$, we obtain an exact sequence of coherent sheaves on $\mathbf{P}^n_Y$:

$$\mathcal{F}^{\oplus N} \to \mathcal{F} \to \mathcal{F}_y \to 0.$$

Corollary 11.6.14 implies that after tensoring with $q \gg 0$ and taking global sections, we get an exact sequence

$$\Gamma\big(\mathbf{P}^n_Y, \mathcal{F}(q)\big)^{\oplus N} \to \Gamma\big(\mathbf{P}^n_Y, \mathcal{F}(q)\big) \to \Gamma(\mathbf{P}^n, \mathcal{F}_y) \to 0.$$

Since the first map has entries in $\mathfrak{m}_y$, after tensoring with $A/\mathfrak{m}_y$, we obtain the isomorphism (13.1.2). This completes the proof.

□

REMARK 13.1.3. The converse of the above theorem also holds: if the Hilbert polynomial of $\mathcal{F}_y$ is independent of $y \in Y$, then $\mathcal{F}$ is flat over Y.

After evaluating the Hilbert polynomial at 0, we obtain the following consequence.

COROLLARY 13.1.4. *Given a projective morphism $f\colon X \to Y$, with Y connected, and a coherent sheaf $\mathcal{F}$ on X which is flat over Y, the Euler-Poincaré characteristic $\chi(\mathcal{F}_y)$ is independent of $y \in Y$.*

13.1.2. Generic Flatness. In this section we prove the following result of Grothendieck, known as the Generic Flatness theorem.

THEOREM 13.1.5. *If $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{F}$ is a coherent sheaf on X, then there is a dense open subset U of Y such that $\mathcal{F}|_{f^{-1}(U)}$ is flat over U.*

PROOF. The proof of the theorem is another instance of "dévissage". We may assume that Y is affine and irreducible. Indeed, if $Y_1, \ldots, Y_s$ are the irreducible components of Y, let V_i be an affine open subset of Y_i that does not intersect any Y_j, with $j \neq i$. If $U_i \subseteq V_i$ is a non-empty open subset such that $\mathcal{F}|_{f^{-1}(U_i)}$ is flat over U_i, then $U = \bigcup_{i=1}^{s} U_i$ satisfies the condition in the theorem.

Suppose now that Y is irreducible and affine. We may also assume that X is affine. Indeed, if $X = W_1 \cup \ldots \cup W_r$ is an affine open cover and if $U_i \subseteq Y$ is a non-empty open subset such that $\mathcal{F}|_{f^{-1}(U_i)\cap W_i}$ is flat over U_i, then $U = \bigcap_{i=1}^{r} U_i$ satisfies the conditions in the theorem. When X and Y are affine, with Y irreducible, the assertion follows from the more general and more precise lemma below. □

LEMMA 13.1.6 (Generic Freeness lemma). *If R is a Noetherian domain, S is a finitely generated R-algebra, and M is a finitely generated S-module, then there is a non-zero $a \in R$ such that M_a is a free R_a-module.*

PROOF. We first note that if we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

such that we know the theorem for M' and M'', so that we have non-zero $a', a'' \in R$ such that $M'_{a'}$ if free over $R_{a'}$ and $M''_{a''}$ is free over $R_{a''}$, then M_a is free over R_a, where $a = a'a''$. Indeed, both M'_a and M''_a are free R_a-modules; in particular, after localizing at a, the above exact sequence splits, so that $M_a \simeq M'_a \oplus M''_a$ is a free R-module. Since R is a domain, $a \neq 0$, hence M satisfies the conclusion of the lemma.

By Corollary E.3.4, we have a finite sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_q = M$$

such that $M_i/M_{i-1} \simeq S/\mathfrak{p}_i$ for $1 \leq i \leq q$, for prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_q \subseteq S$. By the previous observation, it is enough to prove the lemma when $M = S/\mathfrak{p}$, for a prime ideal $\mathfrak{p}$, and thus, after replacing S by $S/\mathfrak{p}$ and M by $S/\mathfrak{p}$, we may assume that S is a domain and $M = S$. If the structure morphism $R \to S$ is not injective, then we are trivially done: for every non-zero a in the kernel, we have $S_a = 0$, hence it is free.

From now on, we suppose that the morphism $R \to S$ is injective. Of course, at any point we may replace R by R_c and S by S_c, for some non-zero $c \in R$. Let K be the fraction field of R. We argue by induction on $n = \mathrm{trdeg}_K(S \otimes_R K) \geq 0$. By Noether Normalization (see Theorem 1.2.2), there are algebraically independent elements $x_1, \ldots, x_n \in S \otimes_R K$ such that the inclusion map $K[x_1, \ldots, x_n] \hookrightarrow S \otimes_R K$ is finite. After multiplying each x_i by a suitable element in R, we may assume that $x_i \in S$ for all i. If $u_1, \ldots, u_d \in S$ generate S as an R-algebra, then each u_i satisfies a monic polynomial with coefficients in $K[x_1, \ldots, x_n]$. After clearing the denominators we obtain a polynomial with coefficients in $R[x_1, \ldots, x_n]$. If $c_i \in R$ is the coefficient of the top degree term, and if $c = \prod_{i=1}^d c_i$, after replacing R by R_c and S by S_c, we may assume that the inclusion map $R[x_1, \ldots, x_n] \hookrightarrow S$ is finite.

In this case, we may replace S by $R[x_1, \ldots, x_n]$ (though we keep $M = S$). As before, we have a finite filtration of M by $R[x_1, \ldots, x_n]$-submodules, with each successive quotient isomorphic to $R[x_1, \ldots, x_n]/\mathfrak{q}$, for some prime ideal $\mathfrak{q}$ in $R[x_1, \ldots, x_n]$. If $\mathfrak{q} = 0$, then the corresponding quotient is free over R. On the other hand, if $\mathfrak{q} \neq 0$, then either $\mathfrak{q} \cap R \neq 0$ (as we have seen, in this case $\big(R[x_1, \ldots, x_n]/\mathfrak{q}\big)_a$ for some non-zero $a \in R$) or $\mathfrak{q} \cap R = 0$, but $\mathrm{trdeg}_K\big((R[x_1, \ldots, x_n]/\mathfrak{q}) \otimes_R K\big) < n$, in which case we are done by induction. Arguing as above, since all successive quotients satisfy the conclusion of the lemma, the same holds for M. This completes the proof. $\square$

COROLLARY 13.1.7. *Given an algebraic variety Y, a closed subvariety X of P^n_Y, and a coherent sheaves $\mathcal{F}$ on X, there is a finite set of polynomials $\mathcal{S}$ such that for every $y \in Y$, the Hilbert polynomial of $\mathcal{F}_y$ lies in $\mathcal{S}$.*

PROOF. We argue by Noetherian induction. By Generic Flatness, we have a non-empty open subset U of Y such that $\mathcal{F}|_{f^{-1}(U)}$ is flat over U. In particular, it follows from Theorem 13.1.2 that for $y \in U$, the Hilbert polynomial of $\mathcal{F}_y$ only depends on the connected component of U containing y. Since by the induction hypothesis the Hilbert polynomials of $\mathcal{F}_y$, for $y \in Y \smallsetminus U$, lie in a finite set, this completes the proof. $\square$

13.1.3. Local Flatness criterion. A finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ is flat if and only if it is free, which holds if and only if $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$ (see Corollary 12.1.18). It turns out that the latter characterization of flatness over R holds under weaker assumptions: it is enough for M to be finitely generated over a Noetherian local ring S such that we have a local homomorphism $R \to S$. In order to prove this result, we begin with the following general criterion for the flatness of a module.

LEMMA 13.1.8. *Given a commutative ring R and an R-module M, the following are equivalent:*

i) *M is flat.*
ii) *For every ideal $\mathfrak{a}$ in R, the canonical map*

$$\mathfrak{a} \otimes_R M \to M$$

is injective.

iii) *For every ideal $\mathfrak{a}$ in R, we have $\mathrm{Tor}_1^R(R/\mathfrak{a}, M) = 0$.*

PROOF. The implication i)⇒ii) is obvious, while the equivalence of ii) and iii) follows from the long exact sequence for Tor modules associated to

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0,$$

namely

$$0 = \mathrm{Tor}_1^R(R, M) \to \mathrm{Tor}_1^R(R/\mathfrak{a}, M) \to \mathfrak{a} \otimes_R M \to M \to R/\mathfrak{a} \otimes_R M \to 0.$$

In order to complete the proof, it is enough to show that iii)⇒i).

By Proposition 10.7.21, it is enough to show that $\mathrm{Tor}_1^R(N, M) = 0$ for every R-module N. Since N is the filtering direct limit of its finitely generated R-submodules, using Lemma 13.1.9 below, we see that it is enough to consider the case when N is finitely generated. Suppose that N can be generated by r elements and let us argue by induction on r. If $r = 0$, then $N = 0$ and the assertion is trivial. For the induction step, if N is generated by $u_1, \ldots, u_r$ and N' is the R-submodule generated by $u_1, \ldots, u_{r-1}$, then we have an exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

where N'' is generated by one element, hence $N'' \simeq R/\mathfrak{a}$, for some ideal $\mathfrak{a}$ in R. The long exact sequence for Tor modules gives an exact sequence

$$\mathrm{Tor}_1^R(N', M) \to \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}^1_R(N'', M).$$

The left term vanishes by induction and the right term vanishes by hypothesis, hence the middle one vanishes, too. This completes the proof. $\square$

LEMMA 13.1.9. *If $N = \varinjlim_{i\in I} N_i$, where $(I, \leq)$ is a filtering ordered set, then for every R-module M, we have a functorial isomorphism*

$$\mathrm{Tor}_j^R(N, M) \simeq \varinjlim_{i\in I} \mathrm{Tor}_j^R(N_i, M).$$

PROOF. If $F_\bullet \to M$ is a free resolution of M, then the assertion follows from the isomorphisms

$$\mathrm{Tor}_j^R(N, M) \simeq \mathcal{H}_j(N \otimes_R F_\bullet) \simeq \mathcal{H}_j\big(\varinjlim_{i\in I}(N_i \otimes_R F_\bullet)\big)$$

$$\simeq \varinjlim_{i\in I} \mathcal{H}_j(N_i \otimes_R F_\bullet) \simeq \varinjlim_{i\in I} \mathrm{Tor}_j^R(N_i, M),$$

where we used the fact that the tensor product commutes with direct limits and that filtering direct limits give an exact functor. $\square$

The following result is known as the Local Flatness criterion.

PROPOSITION 13.1.10. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings. If M is a finitely generated S-module, then M is flat over R if and only if $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$.*

PROOF. The "only if" part is obvious, so we only need to prove the "if" part. By Lemma 13.1.8, it is enough to show that the canonical map $\phi\colon \mathfrak{a} \otimes_R M \to M$ is injective for every ideal $\mathfrak{a}$ in R. Let $\mathfrak{a}$ be such an ideal and denote by Q the kernel of ϕ.

Note first that $\mathrm{Tor}_1^R(N, M) = 0$ if N is an R-module of finite length. Indeed, we argue by induction on $\ell(N)$, the case $\ell(N) = 1$ being taken care of by hypothesis. By considering a composition series of N, we obtain a short exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

with $\ell(N') = \ell(N) - 1$ and $\ell(N'') = 1$. Using the long exact sequence for Tor modules and the inductive hypothesis, we obtain our assertion.

In particular, we see that $\mathrm{Tor}_1^R\big(R/(\mathfrak{a}+\mathfrak{m}^j), M\big) = 0$ for all $j \geq 1$, or equivalently, the canonical morphism

$$\phi_j\colon (\mathfrak{a} + \mathfrak{m}^j) \otimes_R M \to M$$

is injective. Since the map ϕ factors as

$$\mathfrak{a} \otimes_R M \to (\mathfrak{a} + \mathfrak{m}^j) \otimes_R M \xrightarrow{\phi_j} M,$$

it follows that

$$Q \subseteq \ker\big(\mathfrak{a} \otimes_R M \to (\mathfrak{a} + \mathfrak{m}^j) \otimes_R M\big) \subseteq \ker\big(\mathfrak{a} \otimes_R M \to ((\mathfrak{a} + \mathfrak{m}^j)/\mathfrak{m}^j) \otimes_R M\big).$$

Using the isomorphism $(\mathfrak{a} + \mathfrak{m}^j)/\mathfrak{m}^j \simeq \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{m}^j)$, we thus obtain the inclusion

$$Q \subseteq \mathrm{Im}\big((\mathfrak{a} \cap \mathfrak{m}^j) \otimes_R M \to \mathfrak{a} \otimes_R M\big).$$

By the Artin-Rees lemma (see Lemma C.4.2), for every $\ell \geq 1$, we have $\mathfrak{a}\cap\mathfrak{m}^j \subseteq \mathfrak{m}^\ell\mathfrak{a}$ for $j \gg 0$. Given $\ell \geq 1$, by taking $j \gg 0$, we conclude that

$$Q \subseteq \mathfrak{m}^\ell \cdot (\mathfrak{a} \otimes_R M).$$

Note that $\mathfrak{a} \otimes_R M$ is in fact a finitely generated S module and we see that

$$Q \subseteq \bigcap_{\ell \geq 1} \mathfrak{n}^\ell \cdot (\mathfrak{a} \otimes_R M).$$

Since the right-hand side is 0 by Krull's Intersection theorem (see Theorem C.4.1), we conclude that $Q = 0$, completing the proof of the proposition. □

The above proposition is often used via the following corollary.

Corollary 13.1.11. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings and M a finitely generated S-module. If $x_1, \ldots, x_n \in \mathfrak{m}$ form a regular sequence on both R and M, then M is flat over R if and only if $M/(x_1, \ldots, x_n)M$ is flat over $R/(x_1, \ldots, x_n)$.*

Proof. By an obvious induction on n, it is enough to treat the case $n = 1$, when we have $x \in \mathfrak{m}$ which is a non-zero-divisor on both R and M. The "only if" part is clear: it follows from assertion i) in Proposition 5.6.6. In fact, for both implications, by the proposition, it is enough to show that we have an isomorphism

$$\mathrm{Tor}_1^R(R/\mathfrak{m}, M) \simeq \mathrm{Tor}_1^{R/(x)}(R/\mathfrak{m}, M/xM). \tag{13.1.3}$$

Note first that

$$\mathrm{Tor}_i^R\big((R/(x), M\big) = 0 \quad \text{for all} \quad i \geq 1. \tag{13.1.4}$$

Indeed, since x is a non-zero-divisor on R, we have a free resolution of $R/(x)$ given by

$$0 \to R \xrightarrow{\cdot x} R \to 0.$$

By tensoring with M and using the fact that x is a non-zero-divisor on M, we obtain (13.1.4).

Consider now a free resolution $F_\bullet$ of M over R. Note that because of (13.1.4), the complex $R/(x) \otimes_R F_\bullet$ gives a free resolution of M/xM over $R/(x)$. We thus have

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) \simeq \mathcal{H}_i(R/\mathfrak{m} \otimes_R F_\bullet)$$
$$\simeq \mathcal{H}_i\big(R/\mathfrak{m} \otimes_{R/(x)} (R/(x) \otimes_R F_\bullet)\big) \simeq \mathrm{Tor}_i^{R/(x)}(R/\mathfrak{m}, M/xM).$$

In particular, for $i = 1$, we obtain (13.1.3). □

As an application of the above corollary we show that a morphism from a Cohen-Macaulay variety to a smooth variety, having all fibers of the same dimension, is flat.

PROPOSITION 13.1.12. *Let $f\colon X \to Y$ be a morphism of algebraic varieties, with Y smooth and X Cohen-Macaulay. If Y is irreducible, of dimension r, X has pure dimension n, and every non-empty fiber of f has pure dimension $n - r$, then f is flat, of relative dimension $n - r$.*

Note that the assumptions that Y is irreducible and X has pure dimension are not really restrictive. Indeed, since Y is smooth, every connected component of Y is irreducible, and since X is Cohen-Macaulay, every connected component of X has pure dimension (see Proposition 12.3.13).

PROOF OF PROPOSITION 13.1.12. Consider a point $x \in X$ and let $y = f(x)$ and $\phi\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be the local homomorphism induced by f. Note that by assumption, we have

$$\dim(\mathcal{O}_{Y,y}) = r, \quad \dim(\mathcal{O}_{X,x}) = n, \quad \text{and} \quad \dim(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) = n - r,$$

where $\mathfrak{m}_y$ is the maximal ideal in $\mathcal{O}_{Y,y}$. Since the ring $\mathcal{O}_{Y,y}$ is regular, of dimension r, the ideal $\mathfrak{m}_y$ is generated by r-elements $a_1, \ldots, a_r$, which form a regular sequence on $\mathcal{O}_{Y,y}$ (see Example 12.1.15). Note that

$$\mathrm{codim}\big((a_1, \ldots, a_r)\mathcal{O}_{X,x}\big) = n - (n - r) = r,$$

and since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, it follows that $a_1, \ldots, a_r$ also form an $\mathcal{O}_{X,x}$-regular sequence (see Remark 12.3.18). We can thus apply Corollary 13.1.11: since $\mathcal{O}_{X,x}/(a_1, \ldots, a_r)\mathcal{O}_{X,x}$ is clearly flat over $\mathcal{O}_{Y,y}/(a_1, \ldots, a_r)$, which is a field, we conclude that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. Therefore f is flat, of relative dimension $n - r$. □

13.2. Smooth morphisms between arbitrary varieties

Our goal in this section is to discuss smooth morphisms between arbitrary algebraic varieties, extending the discussion in §6.5. However, for the sake of simplicity, we only consider such morphisms of fixed relative dimension.

DEFINITION 13.2.1. A morphism of algebraic varieties $f\colon X \to Y$ is *smooth, of relative dimension n* if the following conditions hold:

i) The morphism f is flat, of relative dimension n, and
ii) The coherent sheaf $\Omega_{X/Y}$ on X is locally free, of rank n.

An *étale* morphism is a smooth morphism, of relative dimension 0.

EXAMPLE 13.2.2. It is clear that every open immersion is étale, of relative dimension 0.

EXAMPLE 13.2.3. Let X and Y be algebraic varieties, and consider the projection onto the first component $p\colon X \times Y \to X$. This is always flat and it is flat of relative dimension n if and only if Y has pure dimension n. Moreover, it follows from Proposition 8.7.8 that $\Omega_{X\times Y/X} = q^*(\Omega_Y)$, where $q\colon X \times Y \to Y$ is the projection onto the second component. We thus deduce that p is smooth, of relative dimension n if and only if Y is smooth, of pure dimension n.

REMARK 13.2.4. It is clear from the definition that the notion of smooth morphism is local on the source: given a morphism $f\colon X \to Y$ and an open cover $X = \bigcup_{i\in I} U_i$, the morphism f is smooth, of relative dimension n if and only if each restriction $f|_{U_i}\colon U_i \to Y$ satisfies the same property.

REMARK 13.2.5. We also note that if $f\colon X \to Y$ is a morphism that factors through an open subset $V \subseteq Y$, then f is smooth, of relative dimension n if and only if the induced morphism $g\colon X \to V$ has the same property. Indeed, it is clear that f is flat, of relative dimension n, if and only if g is; moreover, it follows from the definition of the sheaf of relative differentials that $\Omega_{X/Y} = \Omega_{X/V}$.

PROPOSITION 13.2.6. *A morphism $f\colon X \to Y$ is smooth, of relative dimension n if and only if it the following conditions hold:*

i) *f is flat, and*
ii) *For every $y \in Y$, defined by by the radical ideal sheaf $\mathfrak{m}_y$, the ideal sheaf $\mathfrak{m}_y\mathcal{O}_X$ is radical and the fiber $f^{-1}(y)$ is smooth, of pure simension n.*

PROOF. We may assume that f is flat, of relative dimension n. We need to show that $\Omega_{X/Y}$ is locally free, of rank n, if and only if for every $y \in Y$, the radical ideal sheaf defining the fiber $f^{-1}(y)$ is $\mathfrak{m}_y\mathcal{O}_X$, and this fiber is smooth. Let $x \in f^{-1}(y)$ and consider the induced local homomorphism $\phi\colon A = \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} = B$. Let $\mathfrak{m}_A$ be the maximal ideal of A, which is the stalk at y of $\mathfrak{m}_y$, and $\mathfrak{m}_B$ the maximal ideal in B. Let $\overline{B} = B/\mathfrak{m}_A B$ and $\mathfrak{m}_{\overline{B}} = \mathfrak{m}_B/\mathfrak{m}_A B$. Note that by Proposition 8.7.8, we have

$$\Omega_{\overline{B}/k} \simeq \Omega_{B/A} \otimes_B \overline{B}.$$

The same computation as the one in the proof of Proposition 8.7.23 shows that

$$\Omega_{\overline{B}/k} \otimes_{\overline{B}} \overline{B}/\mathfrak{m}_{\overline{B}} \simeq \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2.$$

We deduce that if $\Omega_{X/Y}$ is locally free of rank n, then $\dim_k \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 = n = \dim(\overline{B})$. Therefore $\overline{B}$ is a regular ring, hence reduced (see Remark 6.3.11). Since this holds for every $x \in f^{-1}(y)$, we conclude that $\mathfrak{m}_y\mathcal{O}_X$ is a radical ideal and the fiber $f^{-1}(y)$ is smooth (of pure dimension n).

Conversely, if $\mathfrak{m}_y\mathcal{O}_X$ is a radical ideal sheaf and $f^{-1}(y)$ is a smooth variety (of pure dimension n), we conclude that $\dim_k(\Omega_{X/Y})_{(x)} = n$ for every $x \in X$. By Proposition 8.5.6, this implies that $\Omega_{X/Y}$ is locally free, of rank n. $\square$

LEMMA 13.2.7. *If $f\colon X \to Y$ is a morphism of algebraic varieties,, then for every $x \in X$, we have a canonical isomorphism*

$$(\Omega_{X/Y})^\vee_{(x)} \simeq \ker(df_x\colon T_xX \to T_yY),$$

where $y = f(x)$. In particular, if the fiber $f^{-1}(y)$ has an r-dimensional irreducible component containing x, then $\dim_k(\Omega_{X/Y})_{(x)} \geq r$.

Proof. Recall the we have an exact sequence

$$f^*(\Omega_Y) \to \Omega_X \to \Omega_{X/Y} \to 0.$$

By taking the fibers at $x \in X$, we get an exact sequence

$$(\Omega_Y)_{(f(x))} \to (\Omega_X)_{(x)} \to (\Omega_{X/Y})_{(x)} \to 0.$$

By taking duals and using Proposition 8.7.23, we obtain the first assertion in the lemma. Note that if Z is an irreducible component of $f^{-1}(y)$ containing x, then $T_xZ \subseteq \ker(df_x)$. Since $\dim_k T_xZ \geq \dim(Z)$, we obtain the last assertion in the lemma. $\square$

We use the above lemma to show that for morphisms between smooth algebraic varieties, the notion introduced in this chapter agrees with the one introduced in Chapter 6.5. If $f\colon X \to Y$ is a morphism between algebraic varieties, then for every connected component X' of X, the image $f(X')$ is contained in some connected component Y' of Y. We thus easily reduce to morphisms between connected varieties. In particular, when X and Y are smooth, then we reduce in this way to the case when X and Y are irreducible. For example, f is smooth in the sense of Chapter 6.5 if and only if for every X' and Y' and above, the corresponding morphism $X' \to Y'$ is smooth.

Proposition 13.2.8. *Let $f\colon X \to Y$ be a morphism between irreducible, smooth varieties. The following are equivalent:*

i) *The morphism f is smooth, of relative dimension $r = \dim(X) - \dim(Y)$.*
ii) *For every $x \in X$, the induced linear map $df_x\colon T_xX \to T_{f(x)}Y$ is surjective.*

Proof. Let $\dim(X) = m$ and $\dim(Y) = n$. Suppose first that i) holds. Since f is flat, of relative dimension r, every fiber of f has all irreducible components of dimension r. Since $\Omega_{X/Y}$ is locally free, of rank r, it follows from the lemma that the kernel of $df_x\colon T_xX \to T_{f(x)}Y$ has dimension r. Since X and Y are smooth, we have $\dim_k T_xX = m$ and $\dim_k T_{f(x)}Y = n$, and we see that df_x is surjective.

Conversely, suppose that ii) holds. Applying the lemma, we see that

$$\dim_k(\Omega_{X/Y})_{(x)} = m - n \quad \text{for every} \quad x \in X,$$

hence $\Omega_{X/Y}$ is locally free of rank $m-n$ by Proposition 8.5.6. On the other hand, it follows from Proposition 6.5.2 that every non-empty fiber of f is smooth, of pure dimension $m-n$. Since X and Y are smooth, by Proposition 13.1.12, f is flat, of relative dimension $m-n$, hence i) holds. $\square$

Remark 13.2.9. It follows from the above proof that if $f\colon X \to Y$ is a smooth morphism of smooth varieties, then the canonical morphism $f^*(\Omega_Y) \to \Omega_X$ is an injective morphism of vector bundles. In particular, if f is étale, then this is an isomorphism. In fact, for every smooth morphism $f\colon X \to Y$, the morphism $f^*(\Omega_Y) \to \Omega_X$ is injective; however, we do not give a proof here for this more general fact.

Proposition 13.2.10. *If $f\colon X \to Y$ is a smooth morphism, of relative dimension r and $g\colon Y \to Z$ is a smooth morphism, of relative dimension s, then $g \circ f$ is smooth, of relative dimension $r+s$.*

Proof. Note that $g \circ f$ is flat, of relative dimension $r+s$. Indeed, flatness follows from Remark 5.6.12. On the other hand, if X_0 is an irreducible component of X dominating an irreducible component Z_0 of Z, then there is an irreducible component Y_0 of $g^{-1}(Z_0)$ containing $f(X_0)$. In this case, Y_0 is an irreducible component of Y dominated by X_0, and which dominates Z_0; we thus have

$$\dim(X_0) = \dim(Y_0) + r = \dim(Z_0) + r + s.$$

Recall now that we have an exact sequence

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

(see Proposition 8.7.20). By taking the fiber at some $x \in X$ and using the fact that f and g are smooth, we obtain

$$\dim_k(\Omega_{X/Z})_{(x)} \leq \dim_k(\Omega_{X/Y})_{(x)} + \dim_k(\Omega_{Y/Z})_{(f(x))} = r + s.$$

On the other hand, since all fibers of $g \circ f$ have pure dimension $r+s$, it follows from Lemma 13.2.7 that $\dim_k(\Omega_{X/Z})_{(x)} \geq r+s$, hence we have equality. Proposition 8.5.6 implies that $\Omega_{X/Z}$ is locally free, of rank $r+s$. This shows that $g \circ f$ is smooth, of relative dimension $r+s$. □

We next give some equivalent descriptions of étale morphisms.

Definition 13.2.11. A morphism of algebraic varieties $f\colon X \to Y$ is *unramified at* $x \in X$ if the induced local morphism $(\mathcal{O}_{Y,f(x)}, \mathfrak{m}_{f(x)}) \to (\mathcal{O}_{X,x}, \mathfrak{m}_x)$ has the property that $\mathfrak{m}_{f(x)} \cdot \mathcal{O}_{X,x} = \mathfrak{m}_x$. The morphism is *unramified* if it is unramified at every $x \in X$.

Remark 13.2.12. Given a morphism $f\colon X \to Y$, a point $x \in X$ and $y = f(x)$, we have an induced linear map $df_x\colon T_xX \to T_yY$. If $\phi\colon (\mathcal{O}_{Y,y}, \mathfrak{m}_y) \to (\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is the local homomorphism induced by f, then df_x is the dual of the induced map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. Therefore df_x is injective if and only if $\mathfrak{m}_x = \mathfrak{m}_x^2 + \mathfrak{m}_y\mathcal{O}_{X,x}$. By Nakayama's lemma, this is equivalent to f being unramified at x.

Proposition 13.2.13. *Given a morphism of algebraic varieties $f\colon X \to Y$, the following are equivalent:*

i) *f is étale.*
ii) *f is flat and $\Omega_{X/Y} = 0$.*
iii) *f is flat and unramified.*
iv) *For every $x \in X$, the induced morphism*

$$\widehat{\mathcal{O}_{Y,y}} \to \widehat{\mathcal{O}_{X,x}},$$

where $y = f(x)$, is an isomorphism.

Proof. By Proposition 8.7.20, we have an exact sequence of coherent sheaves on X:

$$f^*(\Omega_Y) \to \Omega_X \to \Omega_{X/Y} \to 0.$$

By taking the fiber at some $x \in X$, we obtain an exact sequence

$$(\Omega_Y)_{(f(x))} \xrightarrow{df_x^\vee} (\Omega_X)_{(x)} \to (\Omega_{X/Y})_{(x)} \to 0.$$

We thus conclude that $\Omega_{X/Y} = 0$ if and only if for every $x \in X$, the induced map $df_x\colon T_xX \to T_{f(x)}Y$ is injective, condition which, by Remark 13.2.12, is equivalent to f being unramified at x. This gives the equivalence ii)⇔iii). The equivalence i)⇔ii) is also clear if we show that under the assumptions in ii) all fibers of f are

0-dimensional. Note that if Z is an irreducible, closed subvariety of dimension ≥ 1 that is mapped to $y \in Y$, and if $x \in Z$, then df_x maps $T_xZ \subseteq T_xX$ to 0. Since $\dim_k(T_xZ) \geq 1$, this contradicts the fact that df_x is injective.

We now prove the equivalence of iii) and iv). Given $x \in X$ and $y = f(x)$, consider the induced local homomorphism

$$\phi\colon A = \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} = B.$$

Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be the maximal ideals in A and B, respectively.

Suppose first that iv) holds, hence for every x as above, $\widehat{\phi}$ is an isomorphism. In particular, it is flat, hence by Lemma 13.2.14 below, ϕ is flat. Since this holds for all $x \in X$, we conclude that f is flat. Moreover, since $\widehat{\phi}$ is an isomorphism and the maximal ideals in $\widehat{A}$ and $\widehat{B}$ are $\mathfrak{m}_A\widehat{A}$ and $\mathfrak{m}_B\widehat{B}$, we conclude that

$$\mathfrak{m}_B \otimes_B \widehat{B} = \mathfrak{m}_B\widehat{B} = \mathfrak{m}_A\widehat{B} = (\mathfrak{m}_A B) \otimes_B \widehat{B}.$$

Since the local homomorphism $B \to \widehat{B}$ is flat, we deduce from the above equality, using Lemma 5.6.18, that $\mathfrak{m}_B = \mathfrak{m}_A B$. Since this holds for all $x \in X$, we conclude that f is unramified.

Conversely, suppose that iii) holds. Since ϕ is flat, it follows from Lemma 13.2.14 that $\widehat{\phi}$ is flat, too. In particular, by Lemma 5.6.18, it is injective. On the other hand, since f is unramified at x, we have $\mathfrak{m}_B = \mathfrak{m}_A B$, and thus for all $j \geq 1$, the induced map

$$\mathfrak{m}_A^j/\mathfrak{m}_A^{j+1} \to \mathfrak{m}_B^j/\mathfrak{m}_B^{j+1} \simeq \mathfrak{m}_B^j\widehat{B}/\mathfrak{m}_B^{j+1}\widehat{B}. \tag{13.2.1}$$

is surjective. This also holds for $j = 0$ since the map between the residue fields $A/\mathfrak{m}_A \to B/\mathfrak{n}_B$ is an isomorphism. Given $b \in B$, it is straightforward to see, using (13.2.1), that we can construct inductively a sequence $(a_j)_{j\geq 1}$ with $a_j \in A$, $a_j - a_{j+1} \in \mathfrak{m}_A^j$ and $b - \phi(a_j) \in \mathfrak{m}_B^j\widehat{B}$ for all $j \geq 1$. In this case $a = (a_j)_{j\geq 1}$ gives an element in $\widehat{A}$ such that $\widehat{\phi}(a) = b$. We thus conclude that the homomorphism $\widehat{\phi}\colon \widehat{\mathcal{O}_{Y,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$ is an isomorphism for every $x \in X$. $\square$

LEMMA 13.2.14. *If $\phi\colon (A,\mathfrak{m}) \to (B,\mathfrak{n})$ is a local homomorphism between Noetherian local rings, then ϕ is flat if and only if the morphism $\widehat{\phi}\colon \widehat{A} \to \widehat{B}$ is flat.*

PROOF. Since $\widehat{\phi}$ is a local morphism between Noetherian, local rings, with the maximal ideals of $\widehat{A}$ and $\widehat{B}$ being $\mathfrak{m}\widehat{A}$ and $\mathfrak{n}\widehat{B}$, respectively (see Proposition G.2.7 and Remark G.2.8,) we can apply the characterization of flatness in Proposition 13.1.10. We thus see that $\widehat{\phi}$ is flat if and only if the canonical map

$$\beta\colon \mathfrak{m}\widehat{A} \otimes_{\widehat{A}} \widehat{B} \to \widehat{B}$$

is injective. Similarly, ϕ is flat if and only if the canonical morphism

$$\alpha\colon \mathfrak{m} \otimes_A B \to B.$$

is injective. Note now that we have an isomorphism

$$\mathfrak{m}\widehat{A} \otimes_{\widehat{A}} \widehat{B} \simeq (\mathfrak{m} \otimes_A \widehat{A}) \otimes_{\widehat{A}} \widehat{B} \simeq \mathfrak{m} \otimes_A \widehat{B} \simeq (\mathfrak{m} \otimes_A B) \otimes_B \widehat{B},$$

such that β gets identified to $\alpha \otimes_B \widehat{B}$. Since the morphism $B \to \widehat{B}$ is a local, flat homomorphism (see Corollary G.2.3), it follows from Lemma 5.6.18 that α is injective if and only if β is injective. This completes the proof of the lemma. $\square$

We next turn to the Generic Smoothness theorem. We begin with a lemma concerning separable field extensions.

DEFINITION 13.2.15. A finite type field extension L/K is *separable* if there is a transcendence basis $a_1, \ldots, a_n$ of L over K such that the finite field extension $L/K(a_1, \ldots, a_n)$ is separable. Note that for finite field extensions, we recover the usual definition.

REMARK 13.2.16. Clearly, if $\mathrm{char}(K) = 0$, every finite type field extension is separable.

LEMMA 13.2.17. *Given a finite type, separable field extension L/K, we have*

$$\dim_L \Omega_{L/K} = \mathrm{trdeg}_K(L).$$

PROOF. Let $a_1, \ldots, a_n \in L$ be a transcendence basis such that the finite field extension L/K' is separable, where $K' = K(a_1, \ldots, a_n)$. In particular, by the Primitive Element theorem, we can write $L = K'[b]$ for some $b \in L$. Let $f \in K'[y]$ be the minimal polynomial of b over K', so that $L \simeq K'[y]/(f)$. Since b is separable over K', it follows that $f'(b) \neq 0$. Note that the coefficients of f are rational functions of $a_1, \ldots, a_n$. After clearing the denominators, we find a polynomial $g \in K[x_1, \ldots, x_n, y]$ such that $g(a_1, \ldots, a_n, b) = 0$ and $\frac{\partial g}{\partial y}(a_1, \ldots, a_n, b) \neq 0$. Note that L is obtained as a suitable ring of fractions of $A = K[x_1, \ldots, x_n, y]/(g)$. By Example 8.7.12, $\Omega_{A/K}$ is the quotient of the free A-module with basis $dx_1, \ldots, dx_n, dy$ by the relation

$$\frac{\partial g}{\partial y}(\overline{x_1}, \ldots, \overline{x_n}, \overline{y})dy + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\overline{x_1}, \ldots, \overline{x_n}, \overline{y})dx_i = 0,$$

where $\overline{x_1}, \ldots, \overline{x_n}, \overline{y}$ are the images of $x_1, \ldots, x_n, y$ in A. Since $\Omega_{L/K} \simeq \Omega_{A/K} \otimes_A L$ by Proposition 8.7.7, and since $\frac{\partial g}{\partial y}(a_1, \ldots, a_n, b)$ is invertible in L, we conclude that $\Omega_{L/K}$ has dimension n over L. $\square$

THEOREM 13.2.18 (Generic Smoothness). *Let $f\colon X \to Y$ be a dominant morphism of irreducible algebraic varieties, with X smooth. If $\mathrm{char}(k) = 0$, then there is an open subset V of Y such that the induced morphism $f^{-1}(V) \to V$ is smooth.*

We prove this in several steps. We begin begin by proving a weaker assertion, which sometimes also holds in positive characteristic.

LEMMA 13.2.19. *If $f\colon X \to Y$ is a dominant morphism of irreducible algebraic varieties, such that the field extension $k(X)/k(Y)$ is separable, then there is a nonempty open subset U of X such that the induced morphism $U \to Y$ is smooth.*

PROOF. After replacing both X and Y by suitable open subsets, we may assume that they are both smooth. Since the stalk V of $\Omega_{X/Y}$ at X is isomorphic to $\Omega_{k(X)/k(Y)}$, the hypothesis implies, by Lemma 13.2.17, that V is a $k(X)$-vector space of dimension equal to

$$\mathrm{trdeg}_{k(Y)} k(X) = \dim(X) - \dim(Y).$$

By Remark 8.5.2, we deduce that there is an open subset U of X such that $\Omega_{X/Y}|_U$ is locally free of rank equal to $\dim(X) - \dim(Y)$. In this case, arguing as in the proof of Proposition 13.2.8, we see that for every $x \in U$, the map $T_xX \to T_{f(x)}Y$ is surjective, which by the same proposition, implies that the map $U \to Y$ induced by f is smooth. $\square$

LEMMA 13.2.20. *If* $f\colon X \to Y$ *is a morphism of algebraic varieties over* k, *with* $\mathrm{char}(k) = 0$, *and*

$$Z_r = \{x \in X \mid \mathrm{rank}(T_xX \to T_{f(x)}Y) \leq r\},$$

then every irreducible component of $\overline{f(Z_r)}$ *has dimension* $\leq r$.

PROOF. Let W be an irreducible component of $\overline{f(Z_r)}$ and Z an irreducible component of $\overline{Z_r}$ that dominates W. Since we are in characteristic 0, we can apply Lemma 13.2.19 for the induced map $g\colon Z \to W$ to we find an open subset U of Z_{sm} mapping to W_{sm}, and such that the map $U \to W$ is smooth. For every $x \in U$, we have the following commutative diagram of linear maps

$$\begin{tikzcd}
T_xZ \arrow[r] \arrow[d, "dg_x"'] & T_xX \arrow[d, "df_x"] \\
T_{f(x)}W \arrow[r] & T_{f(x)}Y,
\end{tikzcd}$$

in which the horizontal maps are injective. Since $\mathrm{rank}(df_x) \leq r$ and dg_x is surjective, we conclude that $\dim_k T_{f(x)}W \leq r$, hence $\dim(W) \leq r$. □

We can now prove the Generic Smoothness result.

PROOF OF THEOREM 13.2.18. After replacing Y by its smooth locus, we may assume that Y is smooth, too. We apply Lemma 13.2.20 with $r = \dim(Y) - 1$ to conclude that if Z is the locus of those $x \in X$ such that df_x is not surjective, then $\overline{f(Z)}$ is a proper closed subset of Y. The open subset $V = Y \smallsetminus \overline{f(Z)}$ is thus non-empty and it satisfies the conclusion of the theorem. □

We end this chapter with two applications of the Generic Smoothness theorem. We first prove a version of Bertini's theorem, due to Kleiman.

THEOREM 13.2.21. *If* X *is a smooth, irreducible variety over an algebraically closed field of characteristic* 0, $\mathcal{L}$ *a line bundle on* X, *and* $V \subseteq \Gamma(X, \mathcal{L})$ *a finite-dimensional vector space that generates* $\mathcal{L}$, *then there is a non-empty open subset* U *of* $\mathbf{P}(V^\vee)$ *such that every effective Cartier divisor corresponding to a point in* U *is reduced, with smooth support.*

PROOF. We choose a basis $s_0, \ldots, s_n$ of V, which allows us to identify $\mathbf{P}(V^\vee)$ to $\mathbf{P}^n = \mathrm{Proj}\big(k[x_0, \ldots, x_n]\big)$. The case $n = 0$ is trivial, hence from now on we assume $n \geq 1$. By assumption, we have $\bigcap_{i=0}^n V(s_i) = 0$, hence $X = \bigcup_{i=0}^n U_i$, where $U_i = X \smallsetminus V(s_i)$. Let π_1 and π_2 be the projections of $Y = X \times \mathbf{P}^n$ onto the two factors and consider the section of $\pi_1^*(\mathcal{L}) \otimes_{\mathcal{O}_Y} \pi_2^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ given by $\sigma = \sum_{i=0}^n x_i s_i$.

We claim that the effective Cartier divisor $Z(s)$ is a prime, smooth divisor. Indeed, on U_i we can write $s_j = a_{j,i}s_i$ for $a_{j,i} \in \mathcal{O}_X(U_i)$ and we have an isomorphism

$$\pi_1^*(\mathcal{L}) \otimes_{\mathcal{O}_Y} \pi_2^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)|_{U_i \times \mathbf{P}^n} \simeq \pi_2^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)|_{U_i \times \mathbf{P}^n}$$

such that the restriction of s corresponds to the section $\phi_i = \sum_{j=0}^n a_{j,i}x_j$. Note that we have an automorphism

$$\alpha_i\colon U_i \times \mathbf{P}^n \to U_i \times \mathbf{P}^n, \big(u, [v_0, \ldots, v_n]\big) \to \big(u, [v_0, \ldots, v_{i-1}, \sum_{ji} a_{j,i}v_j, v_{i+1}, \ldots, v_n]\big)$$

such that $\alpha_i^*(H_i) = Z(s|_{U_i \times \mathbf{P}^n})$, where H_i is the effective Cartier divisor $U_i \times Z(x_i)$ on $U_i \times \mathbf{P}^n$. We thus see that that the restriction of $Z(s)$ to $U_i \times \mathbf{P}^n$ is a smooth prime divisor. The fact that $Z(s)$ is irreducible is an easy consequence of Exercise 1.3.17. While we do not need this, it is easy to deduce from the above description that as a variety over X, $Z(s)$ is a projective bundle.

Consider now the morphism $f\colon Z(s) \to \mathbf{P}^n$ induced by π_1. Note that if $v = [v_0, \ldots, v_n] \in \mathbf{P}^n$, then $f^{-1}(v)$ is the support of the effective Cartier divisor defined by $s_v := \sum_{i=0}^n v_i s_i$. In particular, if non-empty, $f^{-1}(v)$ has dimension $\dim(X) - 1$. Since $\dim\big(Z(s)\big) = \dim(X) + n - 1$, we conclude that f is dominant. Since $Z(s)$ is smooth, we can apply Theorem 13.2.18 to conclude that there is an open subset $U \subseteq \mathbf{P}^n$ such that the induced morphism $f^{-1}(U) \to U$ is smooth. We see that if $v \in U$, then $\mathrm{Supp}(Z(s_v)\big)$ is smooth. Moreover, by Proposition 13.2.6, the radical ideal defining $f^{-1}(v)$ in $Z(s)$ is $\mathfrak{m}_v \cdot \mathcal{O}_{Z(s)}$, where $\mathfrak{m}_v$; this implies that the ideal sheaf in $\mathcal{O}_X$ corresponding to s_v is a radical ideal, hence $Z(s_v)$ is a reduced divisor. This completes the proof of the theorem. $\square$

Our second application is to the study of the dual projective variety of a given projective variety. Let $X \subseteq \mathbf{P}^n$ be a closed subvariety of dimension d and consider the subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$ given by

$$I_0(X) := \{(p, H) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in X_{\mathrm{sm}}, H \supseteq \mathbf{T}_p X\}.$$

Let $I(X)$ be the closure of $I_0(X)$ in $\mathbf{P}^n \times (\mathbf{P}^n)^*$ and let $p\colon I(X) \to X$ and $q\colon I(X) \to (\mathbf{P}^n)^*$ be the maps induced by the two projections. Note that these are both proper, hence closed morphisms.

Arguing as in the proof of Theorem 6.4.1, we see that $I_0(X)$ is a closed subvariety of $X_{\mathrm{sm}} \times \mathbf{P}^n$ and each fiber of the morphism $I_0(X) \to X_{\mathrm{sm}}$ induced by the first projection is isomorphic to $\mathbf{P}^{n-d-1}$, hence $I_0(X)$ is an irreducible variety of dimension $n-1$ by Proposition 5.5.1. Therefore $I(X)$ is a closed, irreducible subvariety of $X \times (\mathbf{P}^n)^*$, of dimension $n-1$. The *dual projective variety* X^* is the image $q\big(I(X)\big) \subseteq (\mathbf{P}^n)^*$. Our goal is to prove the following reflexivity statement:

PROPOSITION 13.2.22. *If* $\mathrm{char}(k) = 0$, *then the isomorphism* $\Phi\colon \mathbf{P}^n \times (\mathbf{P}^n)^* \to (\mathbf{P}^n)^* \times \mathbf{P}^n$, $\big([v],[u]\big) \to \big([u],[v]\big)$ *maps* $I(X)$ *onto* $I(X^*)$. *In particular, we have* $(X^*)^* = X$.

We need some preparations. Let $V = \Gamma\big(\mathbf{P}^n, \mathcal{O}(1)\big)^\vee$, so that $\mathbf{P}^n$ parametrizes the lines in V and $(\mathbf{P}^n)^*$ parametrizes hyperplanes in V. We thus have surjective morphisms $\pi_V\colon V \smallsetminus \{0\} \to \mathbf{P}^n$ and $\pi_{V^\vee}\colon V^\vee \smallsetminus \{0\} \to (\mathbf{P}^n)^*$, and for $v \in V \smallsetminus \{0\}$ and $u \in V^\vee \smallsetminus \{0\}$, we write $[v]$ and $[u]$ for the images in $\mathbf{P}^n$ and, respectively, $(\mathbf{P}^n)^*$. We denote by $\langle -, - \rangle$ the canonical pairing $V^\vee \times V \to k$.

The morphism π_V is smooth (over $U_i = D^+_{\mathbf{P}^n}(x_i)$, the inverse image $\pi_V^{-1}(U_i)$ is isomorphic to $U_i \times (\mathbf{A}^1 \smallsetminus \{0\})$). Note also that for every $v \in V \smallsetminus \{0\}$, the line $\langle v \rangle$ is contained in the kernel of the induced morphism $(d\pi_V)_v\colon V = T_v V \to T_{[v]}\mathbf{P}^n$. Therefore we have a canonical isomorphism $T_{[v]}\mathbf{P}^n \simeq V/\langle v \rangle$. Similarly, $\pi_{V^\vee}$ is smooth and $T_{[u]}(\mathbf{P}^n)^* \simeq V^\vee/\langle u \rangle$ for every $u \in V^\vee \smallsetminus \{0\}$.

It is useful to interpret from this point of view the projective tangent space to a subvariety $X \subseteq \mathbf{P}^n$. Given a point $[v] \in X$, then by definition of the projective tangent space $\mathbf{T}_{[v]}X$, if $C(X) = \overline{\pi_V^{-1}(X)}$ is the affine cone over X, then

$$T_v C(X) \smallsetminus \{0\} = \pi_V^{-1}(\mathbf{T}_{[v]}X).$$

Since $C(X) \smallsetminus \{0\} = \pi_V^{-1}(X)$, we also have

$$T_v C(X) = (d\pi_V)_v^{-1}(T_v X).$$

We thus conclude that that if we write $T_{[v]}X = W/\langle v\rangle \subseteq T_{[v]}\mathbf{P}^n$, then $\mathbf{T}_{[v]}X = \pi_V(W \smallsetminus \{0\})$.

Let $Q \subseteq \mathbf{P}^n \times (\mathbf{P}^n)^\vee$ be the incidence pairing

$$Q = \{(p,H) \in \mathbf{P}^n \times (\mathbf{P}^n)^\vee \mid p \in H\}.$$

It is easy to see that this is a closed subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$ (see Exercise 4.2.18). Note that if $\widetilde{Q} \subseteq V \times V^\vee$ is the closed subset given by

$$\widetilde{Q} = \{(v,u) \in V \times V^\vee \mid \langle u,v\rangle = 0\},$$

then

$$W := \widetilde{Q} \cap \big((V \smallsetminus \{0\}) \times (V^\vee \smallsetminus \{0\})\big) = (\pi_V \times \pi_{V^\vee})^{-1}(Q).$$

Note that $\widetilde{Q}$ is the affine cone over a smooth quadric in $\mathbf{P}^{2n+1}$, hence W is smooth. We can cover Q by affine open subsets $V_{i,j}$ such that $(\pi_V \times \pi_{V^\vee})^{-1}(V_{i,j}) \simeq V_{i,j} \times (\mathbf{A}^1 \smallsetminus \{0\})^2$. It follows that Q is smooth and the morphism $W \to Q$ is smooth as well. It is easy to see that for every $(v,u) \in \widetilde{Q}$, we have

$$T_{(v,u)}\widetilde{Q} = \{(b,a) \in V \times V^\vee \mid \langle u,b\rangle + \langle a,v\rangle = 0\}.$$

We thus conclude that for every $\big([v],[u]\big) \in Q$, we have

$$T_{([v],[u])}Q = \{(\overline{b},\overline{a}) \in V/\langle v\rangle \times V^\vee/\langle u\rangle \mid \langle u,b\rangle + \langle a,v\rangle = 0\}.$$

We can now prove the reflexivity of the dual variety.

Proof of Proposition 13.2.22. It is clear from definition that $I_0(X) \subseteq Q$, hence $I(X) \subseteq Q$. By Lemma 13.2.19, we have an open subset $U \subseteq I_0(X)$ such that for every $\big([v],[u]\big) \in U$, we have $[u] \in (X^*)_{\mathrm{sm}}$ and the tangent map $T_{([v],[u])}I(X) \to T_{[u]}X^*$ is surjective. Given $\big([v],[u]\big) \in U$, since $I(X) \subseteq Q \cap \big(X \times (\mathbf{P}^n)^*\big)$, we have

$$T_{([v],[u])}I(X) \subseteq \{(\overline{b},\overline{a}) \in V/\langle v\rangle \times V^\vee/\langle u\rangle \mid \overline{b} \in T_{[v]}X, \langle u,b\rangle + \langle a,v\rangle = 0\}.$$

Note now that since $\big([v],[u]\big) \in I_0(X)$, we have $\langle u,b\rangle = 0$ for every $b \in V$ such that $\overline{b} \in T_{[v]}X$. We thus see that $\langle a,v\rangle = 0$ for every $(\overline{b},\overline{a}) \in T_{([v],[u])}I(X)$. Since the tangent map at $\big([v],[u]\big)$ is surjective, we conclude that for every $\big([v],[u]\big) \in U$ and every $\overline{a} \in T_{[u]}X^*$, we have $\langle a,v\rangle = 0$. We thus see that $\Phi(U) \subseteq I(X^*)$, hence $\Phi\big(I(X)\big) \subseteq I(X^*)$. These are both irreducible closed subvarieties of $(\mathbf{P}^n)^* \times \mathbf{P}^n$, of dimension $n-1$, hence they are equal. This gives the first assertion in the proposition, and the second one is an immediate consequence. $\square$

CHAPTER 14

The theorem on formal functions and Serre duality

In this chapter we discuss two important results about cohomology of sheaves on projective varieties: the theorem on formal functions, describing the completion of the stalk of higher direct image sheaves by proper morphisms, and Serre duality, a result providing an analogue of Poincaré duality in algebraic geometry.

14.1. The theorem on formal functions

In the classical setting, it is very often the case that one can explicitly describe the stalk of a higher direct image sheaf via a proper map as the cohomology for the restriction of the sheaf to the corresponding fiber. For example, we have the following elementary result (see, for example, [**God73**, Rémarque 4.17.1] for a proof).

THEOREM 14.1.1. *If $f\colon X \to Y$ is a proper[1], continuous map between locally compact topological spaces, then for every $y \in Y$, every sheaf of Abelian groups $\mathcal{F}$ on X, and every $q \geq 0$, we have a functorial isomorphism*

$$R^q f_*(\mathcal{F})_y \simeq H^q(X_y, \mathcal{F}|_{X_y}),$$

where $X_y = f^{-1}(y)$, and if $i\colon X_y \hookrightarrow X$ is the inclusion map, we put $\mathcal{F}|_{X_y} = i^{-1}(\mathcal{F})$.

In the algebraic setting the situation is more complicated. There are two results that handle this issue. In this section we give a general result that describes the completion of $R^q f_*(\mathcal{F})_y$ when $f\colon X \to Y$ is a proper morphism and $\mathcal{F}$ is a coherent sheaf on X. In fact, a similar result holds for the inverse image of an an arbitrary closed subset. Another important result, the *base-change theorem*, gives necessary conditions for having an isomorphism between $R^q f_*(\mathcal{F})_{(y)}$ and the cohomology of (a suitable version of) the restriction of $\mathcal{F}$ to the fiber X_y over y (for a treatment of this result, see [**Har77**, Chapter III.12]).

In order to formulate the main result, we set up some notation. Let $f\colon X \to Y$ be a proper morphism of algebraic varieties and $\mathcal{I}$ a coherent ideal on Y. Given a coherent sheaf $\mathcal{F}$ on X and $q \geq 0$, it follows from Theorem 11.3.1 that the $\mathcal{O}_Y$-module $R^q f_*(\mathcal{F})$ is coherent. For every $i \geq 0$, we consider the coherent sheaf $\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^{i+1}\mathcal{O}_X$, whose support is contained in $f^{-1}\big(V(\mathcal{I})\big)$. The surjective morphism $\mathcal{F} \to \mathcal{F}_i$ induces a morphism $R^q f_*(\mathcal{F}) \to R_q f_*(\mathcal{F}_i)$. In fact, $R^q f_*(\mathcal{F}_i)$ is an $\mathcal{O}_Y/\mathcal{I}^{i+1}$-module, and we thus get an induced morphism

$$R^q f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{I}^{i+1} \to R^q f_*(\mathcal{F}_i).$$

Note that this is a morphism of inverse systems of $\mathcal{O}_Y$-modules, where $\big(R^q f_*(\mathcal{F}_i)\big)_{i\geq 0}$ form an inverse system with respect to the morphisms $R^q f_*(\mathcal{F}_j) \to R^q f_*(\mathcal{F}_i)$, for

[1] In this setting, a map is proper if the inverse image of any compact set is compact.

$j \geq i$, induced by the canonical surjections $\mathcal{F}_j \to \mathcal{F}_i$. We will be concerned with the corresponding morphism between the inverse limits:

$$\varprojlim \big(R^q f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{I}^{i+1}\big) \to \varprojlim R^q f_*(\mathcal{F}_i). \tag{14.1.1}$$

EXAMPLE 14.1.2. If Y is affine, with $\mathcal{O}(Y) = A$ and $\mathcal{I}(Y) = I$, then the morphism (14.1.1) induces by taking global sections on both sides a morphism

$$H^q(X,\mathcal{F})\widehat{\ } \to \varprojlim H^q(X,\mathcal{F}_i),$$

where on the left-hand side we have the completion of the finitely generated A-module $H^i(X,\mathcal{F})$, with respect to the ideal I.

EXAMPLE 14.1.3. If $y \in Y$ and $\mathcal{I} = \mathfrak{m}_y$, the radical ideal sheaf defining a point, then the ideal $\mathcal{I}^{i+1}\mathcal{O}_X$ corresponds to the "$i^{\rm th}$ infinitesimal neighborhood" of the fiber $f^{-1}(y)$. Given an affine open neighborhood U of y, we have isomorphisms $H^q\big(f^{-1}(U),\mathcal{F}_i\big) \simeq H^q(X,\mathcal{F}_i)$ (see Remark 10.5.19) and thus the morphism in (14.1.1) induces by taking sections over U a morphism

$$R^q f_*(\mathcal{F})\widehat{{}_y} \to \varprojlim H^q(X,\mathcal{F}_i),$$

where on the left-hand side we have the completion of the stalk $R^q f_*(\mathcal{F})_y$ with respect to the maximal ideal in $\mathcal{O}_{Y,y}$.

The following theorem, known as the *Formal Function theorem*, is due to Grothendieck:

THEOREM 14.1.4. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties, $\mathcal{F}$ is a coherent sheaf on X, and $q \geq 0$, then the morphism (14.1.1) is an isomorphism.*

Before giving the proof of the theorem, we give some applications. All these come from the following corollary, known as *Zariski's Main theorem.*

COROLLARY 14.1.5. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties such that the canonical morphism $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an isomorphism, then f has connected fibers.*

PROOF. Let $y \in Y$ and suppose that the fiber X_y over y is the union of two disjoint open subsets W_1 and W_2. We will apply the Formal Function theorem for $\mathcal{F} = \mathcal{O}_X$ and write $\mathcal{O}_{X_i}$ for $\mathcal{F}_i$. If $j\colon X_y \hookrightarrow X$ is the inclusion map, since each sheaf $\mathcal{O}_{X_i}$ has support contained in X_y, the canonical morphism $\mathcal{O}_{X_i} \to j_*\big(j^{-1}(\mathcal{O}_{X_i})\big)$ is an isomorphism. The decomposition $X_y = W_1 \sqcup W_2$ thus induces a product decomposition

$$\Gamma(X,\mathcal{O}_{X_i}) \simeq A_i \times B_i,$$

where $A_i = \Gamma\big(W_1, j^{-1}(\mathcal{O}_{X_i})\big)$ and $B_i = \Gamma\big(W_2, j^{-1}(\mathcal{O}_{X_i})\big)$ are k-algebras. Note that both A_i and B_i are non-zero, since they each contain a copy of k. Moreover, we have natural morphisms $A_{i+1} \to A_i$ and $B_{i+1} \to B_i$ compatible with the homomorphisms $\Gamma(X,\mathcal{O}_{X_{i+1}}) \to \Gamma(X,\mathcal{O}_{X_i})$. By passing to projective limit, we obtain a ring isomorphism

$$\varprojlim \Gamma((,\mathcal{O}_{X_i}) \simeq A \times B, \quad \text{where} \quad A = \varprojlim A_i, B = \varprojlim B_i.$$

Note again that A and B are both non-zero, since they each contain a copy of k.

By the theorem, we thus obtain an isomorphism

$$f_*(\mathcal{O}_X)\widehat{{}_y} \simeq A \times B,$$

and thus, using the hypothesis, a ring isomorphism $\widehat{\mathcal{O}_{Y,y}} \simeq A \times B$. However, $\widehat{\mathcal{O}_{Y,y}}$ is a local ring (see Remark G.2.8) and a local ring doesn't decompose non-trivially as a product. This gives a contradiction, completing the proof. $\square$

EXAMPLE 14.1.6. An important example which satisfies the hypothesis in Corollary 14.1.5 is that of a proper, birational morphism of irreducible varieties $f\colon X \to Y$, with Y normal. Indeed, consider the canonical morphism of sheaves $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$. In order to show that this is an isomorphism, we may and will assume that Y is affine. Let $U \subseteq X$ and $V \subseteq Y$ be affine open subsets such that f induces an isomorphism $U \simeq V$, and consider the commutative diagram

$$\begin{tikzcd} \Gamma(Y,\mathcal{O}_Y) \arrow[r] \arrow[d] & \Gamma(X,\mathcal{O}_X) \arrow[d] \\ \Gamma(V,\mathcal{O}_V) \arrow[r] & \Gamma(U,\mathcal{O}_U), \end{tikzcd}$$

in which the vertical maps are given by restriction. We thus conclude that $\Gamma(X,\mathcal{O}_X)$ is a subring of the function field $k(X) = k(Y)$ that contains $\Gamma(Y,\mathcal{O}_Y)$. Since it is finite over $\Gamma(Y,\mathcal{O}_Y)$ by Theorem 11.3.1, it is contained in the integral closure of $\Gamma(Y,\mathcal{O}_Y)$ in its fraction field. However, Y is normal, and thus $\Gamma(X,\mathcal{O}_X) = \Gamma(Y,\mathcal{O}_Y)$. We thus conclude, by the previous corollary, that in this case f has connected fibers.

We now show that every proper morphism admits a decomposition into a finite map and one that satisfies the conclusion of Corollary 14.1.5. This is known as the *Stein factorization*.

COROLLARY 14.1.7. *If $f\colon X \to Y$ is a proper morphism, then f factors as $X \xrightarrow{g} Z \xrightarrow{u} Y$, where u is finite and g is proper, with $\mathcal{O}_Z \to g_*(\mathcal{O}_X)$ an isomorphism. Moreover, if $X \xrightarrow{h} W \xrightarrow{v} Y$ is another such factorization, then there is an isomorphism $\alpha\colon W \to Z$ such that $\alpha \circ h = g$ and $u \circ \alpha = v$.*

PROOF. Consider the quasi-coherent sheaf of $\mathcal{O}_Y$-algebras $f_*(\mathcal{O}_X)$. Since f is proper, this is coherent by Theorem 11.3.1. For every affine open subset U of Y, the ring $\mathcal{O}_Y\big(f^{-1}(U)\big)$ is reduced, hence $f_*(\mathcal{O}_X)$ is a reduced $\mathcal{O}_Y$-algebra. We thus conclude that we have a finite morphism $u\colon Z = \mathcal{M}ax\mathcal{S}pec\big(f_*(\mathcal{O}_X)\big) \to Y$. Moreover, corresponding to the identity map $f_*(\mathcal{O}_X) \to f_*(\mathcal{O}_X)$, we have a morphism $g\colon X \to Z$ of varieties over Y (see Remark 8.6.7). Let us check that the canonical morphism $\mathcal{O}_Z \to g_*(\mathcal{O}_X)$ is an isomorphism. If $Y = \bigcup_{i=1}^r V_i$ is a finite affine cover of Y, then $Z = \bigcup_{i=1}^r u^{-1}(V_i)$ is a finite affine open cover of Z, and by definition, we have

$$\Gamma\big(u^{-1}(V_i),\mathcal{O}_Z\big) \simeq \Gamma\big(f^{-1}(V_i),\mathcal{O}_X\big),$$

such that the pull-back map induced by g corresponds to the identity. We thus see that the factorization satisfies the conclusion of the corollary (note that since f is proper, we also have that g is proper by Remark 5.1.8).

Suppose now that $X \xrightarrow{h} W \xrightarrow{v} Y$ is another such decomposition. Since v is a finite morphism, we have $W \simeq \mathcal{M}ax\mathcal{S}pec\big(v_*(\mathcal{O}_Z)\big)$ as varieties over Y. On the other hand, since the canonical morphism $\mathcal{O}_W \to h_*(\mathcal{O}_X)$ is an isomorphism, by applying v_*, we obtain $v_*(\mathcal{O}_Z) \simeq v_*\big(h_*(\mathcal{O}_X)\big) = f_*(\mathcal{O}_X)$. We thus have an isomorphism $\alpha\colon W \to Z$ of varieties over Y. The fact that $g = \alpha \circ h$ follows using

Remark 8.6.7 and noticing that the map $u_*(\mathcal{O}_Z) \to (u \circ g)_*(\mathcal{O}_X)$ corresponding to g and the map $u_*(\mathcal{O}_Z) \to u_*\big((\alpha \circ h)_*(\mathcal{O}_X)\big)$ corresponding to $\alpha \circ u$ are equal. $\square$

An important consequence of the results in this section is the following:

COROLLARY 14.1.8. *If $f\colon X \to Y$ is a proper morphism that has finite fibers, then f is finite.*

PROOF. Consider the Stein factorization $X \xrightarrow{g} Z \xrightarrow{u} Y$ of f. It is enough to show that g is an isomorphism. Note that since f has finite fibers, g has finite fibers, too. Since $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, it follows from Corollary 14.1.5 that g has connected fibers. We thus conclude that each fiber is either empty or contains just one point.

Since g is proper, we thus conclude that it gives a homeomorphism onto a closed subvariety Z' of Z. In fact, we have $Z' = Z$. Indeed, if $j\colon Z' \hookrightarrow Z$ is the inclusion, then the composition

$$\mathcal{O}_Z \to j_*(\mathcal{O}_{Z'}) \to g_*(\mathcal{O}_Z)$$

is an isomorphism, hence the morphism $\mathcal{O}_Z \to j_*(\mathcal{O}_{Z'})$ is injective. Its kernel is the radical ideal sheaf corresponding to Z', and thus $Z' = Z$. Since $g\colon X \to Z$ is a homeomorphism such that the induced morphism $\mathcal{O}_Z \to g_*(\mathcal{O}_X)$ is an isomorphism, it follows that g is an isomorphism, completing the proof of the corollary. $\square$

EXAMPLE 14.1.9. If X is a complete variety and $f\colon X \to \mathbf{P}^n$ is an arbitrary morphism, then $f^*\big(\mathcal{O}_{\mathbf{P}^n}\big)$ is globally generated and if f is finite, then $f^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ is also ample (see Corollary 11.6.17). We can now see that the converse holds: if $\mathcal{L}$ is an ample and globally generated line bundle on X, then there is a finite morphism $f\colon X \to \mathbf{P}^n$ such that $\mathcal{L} \simeq f^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$. Indeed, since $\mathcal{L}$ is globally generated, we have a morphism $f\colon X \to \mathbf{P}^n$ such that $f^*\big(\mathcal{O}_{\mathbf{P}^n}(1)\big) \simeq \mathcal{L}$. We claim that any such morphism f is finite. Indeed, by Corollary 14.1.8, it is enough to show that f has finite fibers. However, if Z is a fiber of f, then $\mathcal{L}|_Z \simeq \mathcal{O}_Z$, and since Z is complete, it is a finite set (see Remark 11.6.5).

Another consequence of the Formal Function theorem is the following

COROLLARY 14.1.10. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties and $\mathcal{F}$ is a coherent sheaf on X such that*

$$\dim\big(\mathrm{Supp}(\mathcal{F}) \cap f^{-1}(y)\big) \leq n \quad \textit{for all} \quad y \in Y,$$

then $R^\ell f_*(\mathcal{F}) = 0$ for all $\ell > n$.

The assertion follows easily from Theorem 14.1.4 if we know that for a complete n-dimensional variety Z, we have $H^\ell(Z, \mathcal{G}) = 0$ for every $\ell > n$ and every coherent sheaf $\mathcal{G}$ on Z. However, we haven't proved this yet (cf. Corollary 10.6.7), so that we have to prove this as part of an inductive approach. The proof will make use of Chow's lemma and the approach in the proof of Theorem 11.3.1.

PROOF OF COROLLARY 14.1.10. We first note that it is enough to prove the assertion when $\dim\big(f^{-1}(y)\big) \leq n$ for every $y \in Y$. Indeed, if $\mathcal{I}$ is the radical ideal sheaf defining $\mathrm{Supp}(\mathcal{F})$, then we have a finite filtration of $\mathcal{F}$, with successive quotients that are annihilated by $\mathcal{I}$ (see Remark 8.4.21). Each such quotient is of the form $j_*(\mathcal{G})$, where $j\colon \mathrm{Supp}(\mathcal{F}) \hookrightarrow X$ is the inclusion map and $\mathcal{G}$ is a coherent sheaf on $\mathrm{Supp}(\mathcal{F})$. Since $R^\ell f_*\big(j_*(\mathcal{G})\big) \simeq R^\ell(f \circ j)_*(\mathcal{G})$ by Example 10.5.17, if we know that $R^\ell(f \circ j)_*(\mathcal{G}) = 0$ for all such $\mathcal{G}$ and all $i > n$, then we obtain $R^\ell f_*(\mathcal{F}) = 0$ for all $i > n$ using the long exact sequence for higher direct images.

We prove by induction on n that if $\dim\big(f^{-1}(y)\big) \leq n$ for all $y \in Y$, then $R^\ell f_*(\mathcal{F}) = 0$ for all $\ell > n$. Of course, it is enough to show that $R^\ell f_*(\mathcal{F})_y = 0$ for all $y \in Y$, and since a finitely generated $\mathcal{O}_{Y,y}$ embeds in its completion (see Remark G.1.4), it is enough to show that the completion of $R^\ell f_*(\mathcal{F})_y$ is 0 for all $y \in Y$. By Theorem 14.1.4 (applied with $\mathcal{I}$ being the radical ideal sheaf defining $y \in Y$), it is enough to show that for every such y and every $i \geq 0$, we have $H^\ell(X, \mathcal{F}_i) = 0$ for all $\ell > n$. By considering again a finite filtration of $\mathcal{F}_i$ whose successive quotients are annihilated by the radical ideal sheaf corresponding to $f^{-1}(y) \hookrightarrow X$, we thus see that it is enough to show that for every n-dimensional complete variety X and every coherent sheaf $\mathcal{F}$ on X, we have $H^\ell(X, \mathcal{F}) = 0$ for $\ell > n$. This is trivially true if $n = 0$ and thus it is enough to show that if $n \geq 1$ and the assertion holds for smaller values of n, then it also holds for n.

If X is a projective variety, then the assertion holds by Corollary 10.6.7. In order to prove the general case, we argue as in the proof of Theorem 11.3.1. We apply Chow's lemma (see Theorem 5.2.1) to get a proper morphism $g\colon W \to X$, with W a projective variety, and that induces an isomorphism $g^{-1}(U) \to U$, where U is open and dense in X and $g^{-1}(U)$ is open and dense in W. We consider the canonical morphism $\phi\colon \mathcal{F} \to g_*\big(\mathcal{G}\big)$, where $\mathcal{G} = g^*(\mathcal{F})$ (note that $g_*(\mathcal{F})$ is coherent by Theorem 11.3.1). Since both $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are supported on $X \smallsetminus U$, whose dimension is $\leq n-1$, we deduce from the inductive hypothesis, and using the long exact sequences in cohomology for the short exact sequences

$$0 \to \ker(\phi) \to \mathcal{F} \to \operatorname{Im}(\phi) \to 0$$

and

$$0 \to \operatorname{Im}(\phi) \to g_*(\mathcal{G}) \to \operatorname{coker}(\phi) \to 0$$

that we have isomorphisms

$$H^\ell(X, \mathcal{F}) \simeq H^\ell\big(X, \operatorname{Im}(\phi)\big) \simeq H^\ell\big(X, g_*(\mathcal{G})\big) \quad \text{for all} \quad q > n.$$

Consider now the Leray spectral sequence for g and the coherent sheaf $\mathcal{G}$ on W:

$$E_2^{p,q} = H^p\big(X, R^q g_*(\mathcal{G})\big) \Rightarrow_p H^{p+q}(W, \mathcal{G}).$$

Given $\ell > n$, we need to show that $E_2^{\ell,0} = 0$. Note that $E_\infty^{\ell,0} = 0$, being a sub-quotient of $H^\ell(W, \mathcal{G})$, which is 0, since W is a projective variety of dimension n. Since $E_\infty^{\ell,0} = E_r^{\ell,0}$ for $r \gg 0$, in order to conclude that $E_2^{\ell,0} = 0$, it is enough to show that for every $r \geq 2$, if $E_{r+1}^{\ell,0} = 0$, then $E_r^{\ell,0} = 0$. Since $E_{r+1}^{\ell,0}$ is the cohomology of

$$E_r^{\ell-r,r-1} \to E_r^{\ell,0} \to E_r^{\ell+r,1-r} = 0,$$

it follows that it is enough to show that $E_r^{\ell-r,r-1} = 0$. This is a sub-quotient of $E_2^{\ell-r,r-1}$, hence it is enough to show that

$$H^{\ell-r}\big(X, R^{r-1}g_*(\mathcal{G})\big) = 0 \quad \text{for} \quad r \geq 2. \tag{14.1.2}$$

Since all fibers of g have dimension $\leq n-1$, it follows from the inductive assumption that $R^j g_*(\mathcal{G}) = 0$ for $j > n-1$. We may thus assume that $r \leq n$.

Let $V = \big\{x \mid \dim\big(g^{-1}(x)\big) \leq r-2\big\}$ (note that this is open in X by Theorem 5.4.1). Since $r - 2 \leq n - 2$, it follows from the inductive assumption that $R^{r-1}g_*(\mathcal{G})|_V = 0$, hence $\operatorname{Supp}\big(R^{r-1}g_*(\mathcal{G})\big) \subseteq X \smallsetminus V$. On the other hand, since f induces an isomorphism $g^{-1}(U) \to U$, with $g^{-1}(U)$ open and dense in W and U open and dense in X, it follows from the theorem on fiber dimension that

$\dim(X \smallsetminus V) \leq (n-1)-(r-1) = n-r$ and thus the vanishing in (14.1.2) follows from the inductive assumption, since $\ell - r > n - r$. This completes the proof of the corollary. $\square$

We now turn to the proof of Theorem 14.1.4, for which we follow closely [Gro61, §4]. The key ingredient is the following finiteness result:

PROPOSITION 14.1.11. *Let $f\colon X \to Y$ be a proper morphism. We consider an $\mathbf{N}$-graded $\mathcal{O}_Y$-algebra $\mathcal{T}$ and an $\mathbf{N}$-graded $\mathcal{O}_X$-algebra $\mathcal{S}$ that satisfy the usual conditions*[2]*. If we have a surjective morphism of $\mathcal{O}_X$-algebras $f^*(\mathcal{T}) \to \mathcal{S}$, then for every graded $\mathcal{S}$-module $\mathcal{M} = \bigoplus_{i\in\mathbf{Z}} \mathcal{M}$ which is quasi-coherent as an $\mathcal{O}_X$-module and locally finitely generated over $\mathcal{S}$, then for every $j \geq 0$, the $\mathcal{T}$-module $\bigoplus_{i\in\mathbf{Z}} R^j f_*(\mathcal{M}_i)$ is locally finitely generated.*

PROOF. Since we have a surjective morphism $f^*(\mathcal{T}) \to \mathcal{S}$, it follows that we have a commutative diagram

$$\begin{tikzcd}
\widetilde{X} = \mathcal{M}ax\mathcal{P}roj(\mathcal{S}) \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
\widetilde{Y} = \mathcal{M}ax\mathcal{P}roj(\mathcal{T}) \arrow[r, "v"] & Y.
\end{tikzcd}$$

The graded $\mathcal{S}$-module $\mathcal{M}$ gives a coherent sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{X}$ and Theorem 11.3.1 implies that $R^q g_*(\widetilde{\mathcal{M}})$ is a coherent sheaf on $\widetilde{Y}$ for every q.

The same result implies that for every i, the $\mathcal{O}_Y$-module $R^j f_*(\mathcal{M}_i)$ is a coherent $\mathcal{O}_Y$-module, Moreover, since $\mathcal{M}$ is locally finitely generated over $\mathcal{S}$, it follows that $\mathcal{M}_i = 0$ for $i \ll 0$. We thus conclude that it is enough to show that $\bigoplus_{i\geq 0} R^j f_*(\mathcal{M}_i)$ is locally finitely generated over $\mathcal{T}$.

We consider the two Leray spectral sequences corresponding to the compositions $v \circ g = f \circ u$ and to the quasi-coherent sheaf $\mathcal{P} := \bigoplus_{i\geq 0} \big(\mathcal{M} \otimes \mathcal{O}_{\widetilde{X}}(i)\big)$ on $\widetilde{X}$. In what follows, we freely use the fact that the cohomology commutes with direct sums of quasi-coherent sheaves (see Remark 10.6.9). The first of these two spectral sequences is

$$E_2^{p,q} = R^p v_*\big(\oplus_{i\geq 0} R^q g_*\big(\widetilde{\mathcal{M}} \otimes \mathcal{O}_{\widetilde{X}}(i)\big)\big) \Rightarrow_p R^{p+q}(v\circ g)_*(\mathcal{P}).$$

Note that since $g^*\big(\mathcal{O}_{\widetilde{Y}}(i)\big) \simeq \mathcal{O}_{\widetilde{X}}(i)$, using the projection formula we obtain

$$E_2^{p,q} \simeq R^p v_*\big(\oplus_{i\geq 0} \big(R^q g_*(\widetilde{\mathcal{M}}) \otimes \mathcal{O}_{\widetilde{Y}}(i)\big)\big).$$

Since $R^q g_*(\widetilde{\mathcal{M}})$ is a coherent $\mathcal{O}_{\widetilde{Y}}$-module, it follows from Theorem 11.2.1 that $E_2^{p,q}$ is a coherent $\mathcal{O}_Y$-module for $p > 0$, while Corollary 11.2.4 implies that $E_2^{0,q}$ is locally finitely generated over $\mathcal{T}$. Finally, note that by Corollary 10.6.6, there is a positive integer N such that $R^p v_*(\mathcal{M}) = 0$ for every $p > N$ and every quasi-coherent sheaf $\mathcal{M}$ on $\widetilde{Y}$, hence $E_2^{p,q} = 0 = E_\infty^{p,q}$, unless $0 \leq p \leq N$. The spectral sequence thus implies that for every d, $R^d(v\circ g)_*(\mathcal{P})$ has a finite filtration such that the successive quotients are locally finitely generated over $\mathcal{T}$, and therefore $R^d(v\circ g)_*(\mathcal{P})$ is locally finitely generated over $\mathcal{T}$.

[2] This means that $\mathcal{T}$ is quasi-coherent, reduced, generated over $\mathcal{T}_0$ by $\mathcal{T}_1$, and such that $\mathcal{T}_0$ and $\mathcal{T}_1$ are coherent $\mathcal{O}_Y$-modules, and $\mathcal{S}$ satisfies the analogous conditions on X.

Consider now the spectral sequence

$$\overline{E}_2^{p,q} = R^p f_*\big(R^q u_*(\oplus_{i\geq 0}\widetilde{\mathcal{M}}\otimes \mathcal{O}_{\widetilde{X}}(i))\big) \Rightarrow_p R^{p+q}(v\circ g)_*(\mathcal{P}).$$

Again, Theorem 11.2.1 implies that the $\mathcal{O}_X$-module $R^q u_*\big(\oplus_{i\geq 0}\widetilde{\mathcal{M}}\otimes\mathcal{O}_{\widetilde{X}}(i))\big)$ is coherent for every $q>0$, and thus $\overline{E}_2^{p,q}$ is a coherent $\mathcal{O}_Y$-module for $q>0$ by Theorem 11.3.1. On the other hand, Corollary 11.2.4 implies that

$$\mathcal{N} := u_*\big(\oplus_{i\geq 0}\widetilde{\mathcal{M}}\otimes\mathcal{O}_{\widetilde{X}}(i)\big)$$

is locally finitely generated over $\mathcal{S}$. The canonical morphism $\oplus_{i\geq 0}\mathcal{M}_i\to\mathcal{N}$ is an isomorphism in large enough degrees by Corollary 11.2.3 and thus it is enough to show that $\overline{E}_2^{p,0}$ is locally finitely generated over $\mathcal{T}$ for every p.

For $r\gg 0$, we have $\overline{E}_r^{p,0}=\overline{E}_\infty^{p,0}$, and this is a subquotient of $R^d(v\circ g)_*(\mathcal{P})$, hence it is locally finitely generated over $\mathcal{T}$. In order to conclude the proof, it is thus enough to show that if $r\geq 2$ and $\overline{E}_{r+1}^{p,0}$ is locally finitely generated over $\mathcal{T}$, then the same holds for $\overline{E}_r^{p,0}$. Recall that we have the maps

$$\overline{E}_r^{p-r,r-1}\xrightarrow{d_r}\overline{E}_r^{p,0}\to\overline{E}_r^{p+r,1-r}=0$$

such that $\overline{E}_{r+1}^{p,0}\simeq\operatorname{coker}(d_r)$. Since this is locally finitely generated over $\mathcal{T}$ and $\overline{E}_r^{p-r,r-1}$ has the same property (in fact, this is coherent over $\mathcal{O}_Y$), the same holds for $\overline{E}_r^{p,0}$. This concludes the proof. $\square$

We can now give the proof of the Formal Function theorem.

Proof of Theorem 14.1.4. In order to show that the morphism (14.1.1) is an isomorphism, it is enough to show that we get an isomorphism whenever we take sections over an affine open subset of Y. We thus may and will assume that Y is affine, with $R=\mathcal{O}(Y)$ and $I=\mathcal{I}(Y)$. Let $T=\bigoplus_{i\geq 0}\mathcal{I}^i$. It is clear that $\mathcal{T}:=\widetilde{T}$ and $\mathcal{S}:=\bigoplus_{i\geq 0}\mathcal{I}^i\mathcal{O}_X$ satisfy the conditions in Proposition 14.1.11. Moreover, $\mathcal{M}:=\bigoplus_{i\geq 0}\mathcal{I}^{i+1}\mathcal{F}$ is locally finitely generated over $\mathcal{S}$, hence by the proposition, the T-module $N^{(q)}=\bigoplus_{i\geq 0}H^q(X,\mathcal{I}^{i+1}\mathcal{F})$ is finitely generated for every $q\geq 0$.

For every $i\geq 0$, the short exact sequence

$$0\to\mathcal{I}^{i+1}\mathcal{F}\to\mathcal{F}\to\mathcal{F}_i\to 0$$

gives via the long exact sequence in cohomology an exact complex

$$0\to A_i\to H^q(X,\mathcal{F})\to H^q(X,\mathcal{F}_i)\to B_i\to 0,$$

where

$$A_i=\operatorname{Im}\big(H^q(X,\mathcal{I}^{i+1}\mathcal{F})\to H^q(X,\mathcal{F})\big)\quad\text{and}\quad B_i=\operatorname{Im}\big(H^q(X,\mathcal{F}_i)\to H^{q+1}(X,\mathcal{I}^{i+1}\mathcal{F})\big).$$

We thus obtain an exact sequence

$$0\to\varprojlim\big(H^q(X,\mathcal{F})/A_i\big)\to\varprojlim H^q(X,\mathcal{F}_i)\to\varprojlim B_i. \tag{14.1.3}$$

For every $m\geq 0$ and every $h\in I^m$, we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{i+1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot h} & & \downarrow{\scriptstyle \cdot h} & & \downarrow{\scriptstyle \cdot h} & & \\ 0 & \longrightarrow & \mathcal{I}^{i+m+1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{i+m} & \longrightarrow & 0, \end{array}$$

and the connecting homomorphisms in the long exact sequences in cohomology fit in a commutative diagram

$$\begin{tikzcd} H^q(X,\mathcal{F}_i) \arrow[r] \arrow[d] & H^{q+1}(X,\mathcal{I}^{i+1}\mathcal{F}) \arrow[d] \\ H^q(X,\mathcal{F}_{i+m}) \arrow[r] & H^{q+1}(X,\mathcal{I}^{i+m+1}\mathcal{F}). \end{tikzcd}$$

This implies that $B := \bigoplus_{i\geq 0} B_i$ is a T-submodule of $N^{(q+1)}$; in particular, it is finitely generated over T. Since T is generated as a T_0-algebra by T_1, it follows that there is i_1 such that $B_{m+i_1} = T_m \cdot B_{i_1}$ for all $m \geq 0$. Given $h \in T_m$ and $i \geq 0$, consider the composition

$$B_i \xrightarrow{\cdot h} B_{i+m} \to B_i,$$

where the second map is one of the structural maps of the inverse system $(B_i)_{i\geq 0}$. It is easy to see that this composition is the multiplication map by h, using the R-module structure of B_i. For every i, note that I^{i+1} annihilates $H^q(X,\mathcal{F}_i)$, hence it also annihilates its quotient B_i. We thus see that if $m \geq i_1 + 1$, then the map $B_{m+i} \to B_i$ is the zero map. Therefore $\varprojlim B_i = 0$.

Similarly, for every $m \geq 0$ and every $h \in I^m$, we have a commutative diagram

$$\begin{tikzcd} H^q(X,\mathcal{I}^{i+1}\mathcal{F}) \arrow[r] \arrow[d] & H^q(X,\mathcal{F}) \arrow[d] \\ H^q(X,\mathcal{I}^{i+m+1}\mathcal{F}) \arrow[r] & H^q(X,\mathcal{F}), \end{tikzcd}$$

where the vertical maps are induced by multiplication with h. This implies that $A := \bigoplus_{i\geq 0} A_i$ has a structure of T-module; in fact, it is a quotient of $N^{(q)}$, hence it is finitely generated over T.

Note that

$$I^{i+1} \cdot H^q(X,\mathcal{F}) \subseteq A_i \quad \text{for all} \quad i \geq 0 \tag{14.1.4}$$

(this follows from the fact that $H^q(X,\mathcal{F}_{i+1})$ is annihilated by I^{i+1}). On the other hand, since A is a finitely generated T-module, and T is generated as a T_0-algebra by T_1, it follows that there is i_2 such that $A_{m+i_2} = T_m \cdot A_{i_2}$ for all $m \geq 0$. For every $h \in T_m$ and every i, the composition

$$A_i \xrightarrow{\cdot h} A_{m+i} \hookrightarrow A_i$$

is the multiplication map by h using the R-module structure of A_i. We thus see that

$$A_{m+i_2} = I^m \cdot A_{i_2} \quad \text{for all} \quad m \geq 0. \tag{14.1.5}$$

The inclusion in (14.1.4) implies that we have a canonical morphism

$$\varprojlim \big(H^q(X,\mathcal{F})/I^{i+1} \cdot H^q(X,\mathcal{F})\big) \to \varprojlim \big(H^q(X,\mathcal{F})/A_i\big)$$

and (14.1.5) implies that this is an isomorphism. Since we have already seen that $\varprojlim B_i = 0$, the conclusion of the theorem follows from (14.1.3). $\square$

REMARK 14.1.12. In the setting of Theorem 14.1.4, if $\mathcal{J}$ is a coherent ideal in $\mathcal{O}_X$ with $V(\mathcal{J}) = V(\mathcal{I} \cdot \mathcal{O}_X)$ and if we put $\mathcal{F}'_i = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}^{i+1}$, then for every q we have a canonical isomorphism

$$\varprojlim R^q f_*(\mathcal{F}_i) \simeq \varprojlim R^q f_*(\mathcal{F}'_i).$$

Indeed, the hypothesis implies that we have positive integers m and n such that $\mathcal{I}^m \subseteq \mathcal{J}$ and $\mathcal{J}^n \subseteq \mathcal{I}$, which induces canonical morphisms $\mathcal{F}_{mi+m-1} \to \mathcal{F}'_i$ and $\mathcal{F}'_{ni+n-1} \to \mathcal{F}_i$ for every i. Using the fact that an inverse limit does not change if we pass to a final subset of the index set, we obtain morphisms

$$\varprojlim \mathcal{F}_i = \varprojlim \mathcal{F}_{mi+m-1} \to \varprojlim \mathcal{F}'_i \quad \text{and} \quad \varprojlim \mathcal{F}'_i = \varprojlim \mathcal{F}'_{ni+n-1} \to \varprojlim \mathcal{F}_i$$

which are mutual inverses.

In the context of Theorem 14.1.3, one can show that the sheaves $R^q f_*(\mathcal{F}_i)$ satisfy the so-called Mittal-Leffler condition. We explain this now, discussing also the motivation for considering this condition. However, we will not make use of this in what follows.

DEFINITION 14.1.13. Consider an inverse system $(A_n, \phi_{m,n})$ with objects in some Abelian category $\mathcal{A}$, indexed by the non-negative integers (with the usual order). This satisfies the *Mittag-Leffler condition* (ML, for short) if for every $n \geq 0$, the non-increasing sequence $\big(\mathrm{Im}(A_m \to A_n)\big)_{m \geq n}$ of sub-objects of A_n is stationary, that is

$$\mathrm{Im}(A_m \to A_n)\big) = \mathrm{Im}(A_{m+1} \to A_n)\big) \quad \text{for} \quad m \gg 0.$$

In what follows we mostly consider the case when $\mathcal{A}$ is the category of modules over a fixed commutative ring R.

EXAMPLE 14.1.14. Every inverse system $(A_n, \phi_{m,n})$ of finite-dimensional vector spaces over a field clearly satisfies that the ML condition.

REMARK 14.1.15. Suppose that $(A_n, \phi_{m,n})$ is an inverse system of R-modules that satisfies the ML condition. For every $n \geq 1$, put

$$A'_n := \mathrm{Im}(A_q \to A_n)\big) \quad \text{for} \quad q \gg 0.$$

It is clear that for every $m \geq n$, $\phi_{m,n}$ induces a morphism $\phi'_{m,n} \colon A'_m \to A'_n$ and these satisfy the required compatibilities. Note that each $\phi'_{m,n}$ is surjective: if q is large enough, then

$$A'_n = \mathrm{Im}(A_q \to A_n) \quad \text{and} \quad A'_m = \mathrm{Im}(A_q \to A_m),$$

and the assertion follows. The inclusion maps give a morphism of inverse systems

$$(A'_n, \phi'_{m,n}) \to (A_n, \phi_{m,n})$$

and the induced morphism

$$\varprojlim A'_n \to \varprojlim A_n$$

is an isomorphism. Indeed, the injectivity is clear. The surjectivity follows from the fact that if $(a_n)_{n \geq 1} \in \varprojlim A_n$, then $a_n \in A'_n$ for all $n \geq 1$.

REMARK 14.1.16. Under the assumptions in Theorem 14.1.4, the inverse system of coherent sheaves $\big((R^q f_*(\mathcal{F}_i)\big)_{i \geq 0}$ satisfies the ML condition. Since these are coherent sheaves, it is clear that it is enough to check this when Y is affine. With the notation in the proof of Theorem 14.1.3, we have seen that given any i, the map $B_m \to B_i$ is the zero map for $m \gg 0$. The commutative diagram with exact rows

$$\begin{tikzcd}
H^q(X,\mathcal{F}) \arrow[r] & H^q(X,\mathcal{F}_i) \arrow[r] & B_i \arrow[r] & 0 \\
H^q(X,\mathcal{F}) \arrow[r] \arrow[u] & H^q(X,\mathcal{F}_m) \arrow[r] \arrow[u] & B_m \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

then implies that

$$\operatorname{Im}\bigl(H^q(X,\mathcal{F}_m) \to H^q(X,\mathcal{F}_i)\bigr) = \operatorname{Im}\bigl(H^q(X,\mathcal{F}) \to H^q(X,\mathcal{F}_i)\bigr) \quad \text{for} \quad m \gg 0.$$

This gives our assertion.

The ML condition is useful, since it provides a criterion for the exactness of inverse limits, as follows.

Proposition 14.1.17. *If we have morphisms of inverse systems of R-modules*

$$(A_n)_{n\geq 0} \xrightarrow{(u_n)} (B_n)_{n\geq 0} \xrightarrow{(v_n)} (C_n)_{n\geq 0}$$

such that for every n, we have an exact sequence

$$0 \to A_n \xrightarrow{u_n} B_n \xrightarrow{v_n} C_n \to 0,$$

and if $(A_n)_{n\geq 0}$ satisfies the ML condition, then after taking the inverse limit, we have an exact sequence

$$0 \to \varprojlim A_n \xrightarrow{u} \varprojlim B_n \xrightarrow{v} \varprojlim C_n \to 0.$$

Proof. The exactness of

$$0 \to \varprojlim A_n \xrightarrow{u} \varprojlim B_n \xrightarrow{v} \varprojlim C_n$$

is easy and does not need the ML condition. The interesting statement is the surjectivity of v. Of course, we may assume that $A_n \subseteq B_n$ and $C_n = B_n/A_n$. For $m \geq n$, we denote by $\psi_{m,n}$ the structural morphism $B_m \to B_n$.

We begin by treating two special cases. Suppose first that every morphism $A_m \to A_n$, for $m \geq n$, is surjective. Given $c = (c_n)_{n\geq 0} \in \varprojlim C_n$, we construct recursively $b_n \in B_n$ such that

$$v_n(b_n) = c_n \quad \text{and} \quad \psi_{n+1,n}(b_{n+1}) = b_n \quad \text{for} \quad n \geq 0. \tag{14.1.6}$$

In this case $b = (b_n)_{n\geq 0} \in \varprojlim B_n$ and $v(b) = c$. To begin with, we choose $b_0 \in B_0$ such that $v_0(b_0) = c_0$. Suppose now that we have chosen $b_0, \ldots, b_m$ such that the conditions in (14.1.6) hold for $0 \leq n \leq m$. Let $b'_{m+1} \in B_{m+1}$ be such that $v_{m+1}(b'_{m+1}) = c_{m+1}$. In this case $b_m - \psi_{m+1,m}(b'_{m+1}) \in A_m$. By assumption, we can find an element $a \in A_{m+1}$ such that $\psi_{m+1,m}(a) = b_m - \psi_{m+1,m}(b'_{m+1})$. In this case, $b_{m+1} = b'_{m+1} + a$ satisfies

$$v_{m+1}(b_{m+1}) = c_{m+1} \quad \text{and} \quad \psi_{m+1,m}(b_{m+1}) = b_m,$$

completing the proof of the recursion step.

Suppose now that $(A_n)_{n\geq 0}$ satisfies the condition that for every n, the map $A_q \to A_n$ is 0 for $q \gg 0$. Given $(c_n)_{n\geq 0} \in \varprojlim C_n$, consider for every n some $b'_n \in B_n$ with $v_n(b'_n) = c_n$. Given n, consider $q \geq n$ such that the map $A_q \to A_n$ is 0. For every $m \geq q$, we have $b'_m - \psi_{m+1,m}(b'_{m+1}) \in A_m$, hence $\psi_{m,n}(b'_m) = \psi_{m+1,n}(b'_{m+1})$; we put $b_n = \psi_{m,n}(b'_m)$, which is thus independent of $m \geq q$. It is then clear that $b = (b_n)_{n\geq 0} \in \varprojlim B_n$ and $v(b) = c$.

We now treat the general case. As in Remark 14.1.15, for every $n \geq 0$, consider $A'_n = \mathrm{Im}(A_q \to A_n)$ for $q \gg 0$. The morphism v factors as

$$\varprojlim B_n \to \varprojlim B_n/A'_n \to \varprojlim B_n/A_n.$$

The first map is surjective by the first case treated above, while the second map is surjective by the second case. Therefore the composition is surjective. $\square$

14.2. Serre duality

In this section we discuss Serre duality for projective varieties. While similar results hold for complete varieties, we only treat the projective case (which will be enough for our applications). Serre duality is an analogues of Poincaré duality in the setting of algebraic varieties. However, it is much more general: on one hand, it applies to arbitrary coherent sheaves, and on the other hand, it also holds on certain singular varieties.

Definition 14.2.1. Let X be a complete variety, with $\dim(X) = n$. We will say that *Serre duality holds on* X if there is a coherent sheaf ω_X° on X such that for every $i \geq 0$ and every coherent sheaf $\mathcal{F}$ on X, we have a functorial isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \simeq H^{n-i}(X, \mathcal{F})^\vee.$$

Remark 14.2.2. Note that by Remark 10.7.14, for every coherent sheaf ω_X° on X, the sequence of contravariant functors $\big(\mathrm{Ext}^i_{\mathcal{O}_X}(-, \omega_X^\circ)\big)_{i\geq 0}$ form a contravariant δ-functor from the category $\mathcal{C}oh(X)$ to the category of k-vector spaces. Similarly, if X is a complete variety of dimension n, since $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and all coherent sheaves $\mathcal{F}$ on X (see Corollary 14.1.10), the sequence of contravariant functors $\big(H^{n-i}(X, -)\big)_{i\geq 0}$ form a contravariant δ-functor from $\mathcal{C}oh(X)$ to the category of k-vector spaces.

We will say that *Serre duality holds in strong form* on X if the above two contravariant δ-functors are isomorphic. We will see in Theorem 14.2.12 below that the two conditions for Serre duality are in fact equivalent, at least for projective varieties.

Remark 14.2.3. Note that for every complete variety X and every locally free sheaf $\mathcal{E}$ on X, we have canonical isomorphisms

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E}, \omega_X^\circ) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X^\circ) \simeq H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{E}^\vee)$$

(see Examples 10.7.10 and 10.7.9). It follows that if Serre duality holds on X, then for every locally free sheaf $\mathcal{E}$ on X, we have a functorial isomorphism

$$H^i(X, \omega_X^\circ \otimes_{\mathcal{O}_X} \mathcal{E}^\vee) \simeq H^{n-i}(X, \mathcal{E})^\vee.$$

Definition 14.2.4. A contravariant[3] δ-functor $F = (F_i)_{i\geq 0} \colon \mathcal{A} \to \mathcal{B}$ between two Abelian categories is *coeffacebale* if for every $A \in \mathrm{Ob}(\mathcal{A})$ and every $i > 0$, there is an epimorphism $P \to A$ such that $F_i(P) = 0$. In what follows we only need the case when $\mathcal{A}$ is the category of coherent sheaves on a variety, so the reader can restrict to this case.

The above property is useful because it implies universality:

[3]A similar definition works for covariant δ-functors.

PROPOSITION 14.2.5. *A coeffaceable contravariant δ-functor $F = (F_i)_{i\geq 0}\colon \mathcal{A} \to \mathcal{B}$ is universal, that is, for every contravariant δ-functor $G = (G_i)_{i\geq 0}$ and every natural transformation $\alpha_0\colon F_0 \to G_0$, there is a unique natural transformation of δ-functors $\alpha\colon F \to G$ that extends α_0.*

PROOF. We construct the natural transformations $\alpha_i\colon F_i \to G_i$ by induction on $i \geq 0$, such that for every short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

the diagram

$$\begin{tikzcd} F_i(A') \arrow[r] \arrow[d, "\alpha_i(A')"'] & F_{i+1}(A'') \arrow[d, "\alpha_{i+1}(A'')"] \\ G_i(A') \arrow[r] & G_{i+1}(A'') \end{tikzcd} \tag{14.2.1}$$

is commutative. With α_0 being given, suppose that we have constructed α_i, for some $i \geq 0$, and let us construct α_{i+1}.

Given any A, by hypothesis there is a short exact sequence

$$0 \to B \to P \to A \to 0$$

such that $F_{i+1}(P) = 0$. The commutative diagram with exact rows

$$\begin{tikzcd} F_i(P) \arrow[r] \arrow[d, "\alpha_i(P)"'] & F_i(B) \arrow[r] \arrow[d, "\alpha_i(B)"] & F_{i+1}(A) \arrow[r] & F_{i+1}(P) = 0 \\ G_i(P) \arrow[r] & G_i(B) \arrow[r] & G_{i+1}(A) \arrow[r] & G_{i+1}(P) \end{tikzcd}$$

induces a unique morphism $\alpha_{i+1}(A)\colon F_{i+1}(A) \to G_{i+1}(A)$ making the new square commutative.

The first thing that needs to be checked is the independence on the choice of the short exact sequence. We only sketch the argument. It is easy to see that if we consider two short exact sequences as above, that fit in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & P' \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

then the morphisms $F_{i+1}(A) \to G_{i+1}(A)$ induced by the two exact sequences agree. Given two arbitrary short exact sequences as above

$$0 \to B' \to P' \to A \to 0$$

and

$$0 \to B'' \to P'' \to A \to 0,$$

we construct a new short exact sequence

$$0 \to B \to P \to A \to 0$$

with the same property, that maps to each of the other two. For this, we consider $P_1 \times_A P_2 \to A$ and then take an epimorphism $P \to P_1 \times_A P_2$ such that $F_{i+1}(P) = 0$. This gives the independence of the choice of short exact sequence in the definition.

In order to show that α_{i+1} is a natural transformation, consider a morphism $u\colon A'\to A$. The commutativity that we need follows easily if we compute $\alpha_{i+1}(A')$ and $\alpha_{i+1}(A)$ using a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & B' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & B \arrow[r] & P \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

such that $F_{i+1}(P)=0=F_{i+1}(P')$. This is easy to construct: given an epimorphism $P\to A$ with $F_{i+1}(P)=0$, we consider $P\times_A A'\to A'$ and choose an epimorphism $P'\to P\times_A A'$ such that $F_{i+1}(P')=0$.

Finally, given an arbitrary short exact sequence, we need to check the commutativity of (14.2.1). We leave this as an exercise for the reader. $\square$

Remark 14.2.6. If X is a projective variety, then for every coherent sheaf ω_X°, the δ-functor $\big(\mathrm{Ext}^i_{\mathcal{O}_X}(-,\omega_X^\circ)\big)_{i\geq 0}$ is coeefaceable. Indeed, given an ample line bundle $\mathcal{L}$ on X, for every coherent sheaf $\mathcal{F}$, if $q\gg 0$, then we have a surjective morphism $(\mathcal{L}^{-q})^{\oplus N_q}\to\mathcal{F}$ and

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^{-q},\omega_X^\circ)\simeq H^i(X,\omega_X^\circ\otimes_{\mathcal{O}_X}\mathcal{L}^q)=0 \quad \text{for all} \quad i\geq 1$$

if $q\gg 0$, by Theorem 11.2.1.

By Proposition 14.2.5, it follows that Serre duality holds in strong form on a projective n-dimensional variety X if and only if the following two conditions hold:

i) There is a coherent sheaf ω_X° on X such that for every coherent sheaf $\mathcal{F}$, we have a functorial isomorphism
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X^\circ)\simeq H^n(X,\mathcal{F})^\vee.$$

ii) The δ-functor $\big(H^{n-i}(X,-)^\vee\big)_{i\geq 0}$ is coeffaceable.

Motivated by i) above, we make the following

Definition 14.2.7. Given an n-dimensional complete variety X, a *dualizing sheaf* on X is a coherent sheaf ω_X° that represents the functor $H^n(X,-)^\vee$; in other words, for every coherent sheaf $\mathcal{F}$ on X, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X^\circ)\simeq H^n(X,\mathcal{F})^\vee.$$

Like every object representing a functor, a dualizing sheaf is unique up to a canonical isomorphism.

The starting point in the study of Serre duality is the case of the projective space.

Proposition 14.2.8. *The projective space $\mathbf{P}^n$ satisfies Serre duality in strong form, with $\omega_{\mathbf{P}^n}^\circ=\omega_{\mathbf{P}^n}$.*

Proof. We only need to check conditions i) and ii) in Remark 14.2.6. For ii), note that for every coherent sheaf $\mathcal{F}$ and every $q\gg 0$, we have a surjective morphism $\mathcal{O}_{\mathbf{P}^n}(-q)^{\oplus N_q}\to\mathcal{F}$ and

$$H^{n-i}\big(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(-q)\big)=0 \quad \text{for} \quad i\geq 1, q>0$$

by Theorem 11.2.2. Therefore the δ-functor $\big(H^{n-i}(\mathbf{P}^n,-)^\vee\big)_{i\geq 0}$ is coeffaceable.

In order to complete the proof, it is thus enough to show that a dualizing sheaf on $\mathbf{P}^n$ is given by $\omega_{\mathbf{P}^n} \simeq \mathcal{O}_{\mathbf{P}^n}(-n-1)$. Recall that by Theorem 11.2.2 we have an isomorphism $H^n(\mathbf{P}^n, \omega_{\mathbf{P}^n}) \simeq k$. This induces a natural transformation

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{F}, \omega_{\mathbf{P}^n}) \to \mathrm{Hom}_k\big(H^n(\mathbf{P}^n, \mathcal{F}), H^n(\mathbf{P}^n, \omega_{\mathbf{P}^n})\big) \simeq H^n(\mathbf{P}^n, \mathcal{F})^\vee$$

for every coherent sheaf $\mathcal{F}$. This is an isomorphism if $\mathcal{F} = \mathcal{O}_{\mathbf{P}^n}(m)$, with $m \in \mathbf{Z}$, by the last assertion in Theorem 11.2.2. For an arbitrary coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$, there is an exact complex

$$\mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{F} \to 0,$$

where both $\mathcal{E}_1$ and $\mathcal{E}_2$ are direct sums of line bundles. We thus have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{F}, \omega_{\mathbf{P}^n}) \arrow[r] \arrow[d, "\phi_0"'] & \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{E}_1, \omega_{\mathbf{P}^n}) \arrow[r] \arrow[d, "\phi_1"'] & \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{E}_2, \omega_{\mathbf{P}^n}) \arrow[d, "\phi_2"'] \\
0 \arrow[r] & H^n(\mathbf{P}^n, \mathcal{F})^\vee \arrow[r] & H^n(\mathbf{P}^n, \mathcal{E}_1)^\vee \arrow[r] & H^n(\mathbf{P}^n, \mathcal{E}_2)^\vee.
\end{tikzcd}$$

Since ϕ_1 and ϕ_2 are isomorphisms, it follows that ϕ_0 is an isomorphism as well. This completes the proof of the proposition. $\square$

We next show that every projective variety has a dualizing sheaf. We actually prove the following more general result.

PROPOSITION 14.2.9. *Let X be an n-dimensional closed subvariety of the complete algebraic variety Y. If Y is Cohen-Macaulay, of pure dimension N, and Serre duality holds on Y, with the dualizing sheaf ω_Y° being a line bundle, then $\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y^\circ)$ is a dualizing sheaf on X, where $r = N - n$.*

PROOF. Note first that $\omega_X^\circ := \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y^\circ)$ is a coherent sheaf on Y by Proposition 10.7.16. It is easy to see that it is annihilated by the radical ideal sheaf $\mathcal{I}_X$ corresponding to X, since this is the case for $\mathcal{O}_X$ (this will follow, for example, from the computation below, using an injective resolution of ω_Y°). We may thus consider ω_X° as a coherent sheaf on X.

We next observe that

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y^\circ) = 0 \quad \text{for} \quad i < r. \tag{14.2.2}$$

Indeed, for every $x \in Y$, using the fact that ω_Y° is a line bundle, we have by Proposition 10.7.16,

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y^\circ)_x \simeq \mathrm{Ext}^i_{\mathcal{O}_{Y,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}).$$

This is clearly 0 if $x \not\in X$, while for $x \in X$, this vanishes by Theorem 12.1.5, since Y being Cohen-Macaulay, of pure dimension N, we have

$$\mathrm{depth}(\mathcal{I}_{X,x}, \mathcal{O}_{Y,x}) = \mathrm{codim}(\mathcal{I}_{X,x}) = N - \dim(\mathcal{O}_{X,x}) \geq r.$$

Given a coherent sheaf $\mathcal{F}$ on X, using Serre duality on Y, we have a functorial isomorphism

$$H^n(X, \mathcal{F})^\vee \simeq H^n(Y, \mathcal{F})^\vee \simeq \mathrm{Ext}^r_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^\circ). \tag{14.2.3}$$

If $r = 0$, then we are done: in this case, for every coherent sheaf $\mathcal{F}$ on X, we clearly have

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^\circ) \simeq \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y^\circ)\big) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ),$$

hence we are done by (14.2.3). Suppose now that $r \geq 1$, and consider a resolution $\omega_Y^\circ \to \mathcal{I}^\bullet$ by injective $\mathcal{O}_Y$-modules. Consider the complex $\mathcal{J}^\bullet = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{I}^\bullet)$. For every $\mathcal{O}_X$-module $\mathcal{G}$ and every i, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J}^i) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{I}^i),$$

hence $\mathcal{J}^i$ is an injective $\mathcal{O}_X$-module. If we put $\mathcal{Q} = \mathrm{Im}(\mathcal{J}^{r-1} \to \mathcal{J}^r)$, it follows from what (14.2.2) that we have an exact complex

$$0 \to \mathcal{J}^0 \to \mathcal{J}^1 \to \ldots \to \mathcal{J}^{r-1} \to \mathcal{Q} \to 0.$$

Since all $\mathcal{J}^i$ are injective $\mathcal{O}_X$-modules, it follows that this sequence breaks into split short exact sequences; in particular, $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module, too. This implies that if $\mathcal{P} = \ker(\mathcal{J}^r \to \mathcal{J}^{r+1})$, the short exact sequence

$$0 \to \mathcal{Q} \to \mathcal{P} \to \mathcal{P}/\mathcal{Q} \to 0 \tag{14.2.4}$$

splits. First, note that by definition, we have $\omega_X^\circ = \mathcal{P}/\mathcal{Q}$. Second, for every coherent sheaf $\mathcal{F}$ on X, we have

$$\mathrm{Ext}^r_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^\circ) = \frac{\ker\big(\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{I}^r) \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{I}^{r+1})\big)}{\mathrm{Im}\big(\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{I}^{r-1}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{I}^r)\big)}$$

$$\simeq \frac{\ker\big(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^r) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^{r+1})\big)}{\mathrm{Im}\big(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^{r-1}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^r)\big)} \simeq \frac{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{P})}{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})} \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}/\mathcal{P});$$

the second to last isomorphism follows from the left exactness of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ and the fact that the morphism $\mathcal{J}^{r-1} \to \mathcal{Q}$ is a split surjection, while the last isomorphism follows from the fact that (14.2.4) splits. Using (14.2.3), we obtain a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \simeq H^n(X, \mathcal{F})^\vee,$$

hence ω_X° is a dualizing sheaf of X. □

Remark 14.2.10. It will follow from Theorem 14.2.12 below that in the above proposition, the condition that Serre duality holds on Y is a consequence of the fact that Y is Cohen-Macaulay, of pure dimension. Also, the hypothesis that ω_Y° can be dropped, but we will not need this degree of generality.

Corollary 14.2.11. *Every projective variety X has a dualizing sheaf.*

Proof. Since X is projective, we can embed X as a closed subvariety of some $\mathbf{P}^N$. By Proposition 14.2.8, $\mathbf{P}^N$ has a dualizing sheaf, namely $\omega_{\mathbf{P}^N}$, which is a line bundle. Since $\mathbf{P}^N$ is clearly Cohen-Macaulay, of pure dimension, the proposition implies that X has a dualizing sheaf. □

We now come to the main result of this section.

Theorem 14.2.12. *Given a projective, n-dimensional variety X, and a closed embedding $X \hookrightarrow \mathbf{P}^N$, the following are equivalent:*

i) *Serre duality holds on X.*
ii) *Serre duality holds on X in strong form.*
iii) *X is Cohen-Macaulay, of pure dimension.*
iv) *For every locally free sheaf $\mathcal{E}$ on X, if $j \gg 0$, then*

$$H^i\big(X, \mathcal{E}(-j)\big) = 0 \quad \textit{for all} \quad i < n.$$

v) *For $j \gg 0$, we have*

$$H^i\big(X, \mathcal{O}_X(-j)\big) = 0 \quad \textit{for all} \quad i < n.$$

We will need the following characterization of projective dimension:

LEMMA 14.2.13. *If $x \in X$ is a smooth point on an algebraic variety, $R = \mathcal{O}_{X,x}$, and M is a finitely generated R-module, then $\mathrm{pd}_R(M) \leq r$ if and only if $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > r$.*

PROOF. The "only if" part is clear (see Proposition 10.7.6). For the "if" part, consider the minimal free resolution of M:

$$0 \to F_q \to \ldots \to F_1 \to F_0 \to M$$

(note that this is finite by Proposition 12.2.16 and Corollary 12.1.18, since R is the local ring of a smooth point). If $q > r$, then the hypothesis implies that the induced map $\mathrm{Hom}_R(F_{q-1}, R) \to \mathrm{Hom}_R(F_q, R)$ is surjective. Being a morphism of free modules, it is split surjective, contradicting the fact that modulo the maximal ideal, it induces the 0 map. $\square$

PROOF OF THEOREM 14.2.12. The implication ii)⇒i) is trivial and i)⇒iv) follows from Remark 14.2.3: if $\mathcal{E}$ is locally free, then

$$H^i\big(X, \mathcal{E}(-j)\big) \simeq H^{n-i}\big(X, \omega_X^\circ \otimes_{\mathcal{O}_X} \mathcal{E}^\vee(j)\big)$$

and this vanishes for $j \gg 0$ and $n - i > 0$ by Theorem 11.2.1.

The implication iv)⇒v) is trivial, too and v)⇒ii) follows from Remark 14.2.6. Indeed, since X has a dualizing sheaf by Corollary 14.2.11, we only need to show that the contravariant δ-functor $\big(H^{n-i}(X, -)^\vee\big)_{i \geq 0}$ is coeffaceable. Given a coherent sheaf $\mathcal{F}$ on X, if $q \gg 0$, then we have a surjection $\mathcal{O}_X(-j)^{\oplus N_j} \to \mathcal{F}$. On the other hand, it follows from v) that $H^{n-i}\big(X, \mathcal{O}_X(-j)\big) = 0$ for all $i > 0$ and $j \gg 0$, proving coeffaceability.

In order to complete the proof, it is enough to show the equivalence iii)⇔v). For every $x \in X$, we have $\mathrm{depth}(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x}) \leq n$, hence iii) holds if and only if for all $x \in X$, we have $\mathrm{depth}(\mathcal{O}_{X,x}) \geq n$. Note now that the depth of $\mathcal{O}_{X,x}$ is the same when considering it as a module over itself or as a module over $\mathcal{O}_{\mathbf{P}^N,x}$. Since $\mathcal{O}_{\mathbf{P}^N,x}$ is a regular local ring of dimension N, every finitely generated module over it has finite projective dimension (see Proposition 12.2.16), and it follows from the Auslander-Buchsbaum formula that the condition in iii) is equivalent to the fact that

$$\mathrm{pd}_{\mathcal{O}_{\mathbf{P}^N,x}}(\mathcal{O}_{X,x}) \leq N - n = r \quad \text{for all} \quad x \in X.$$

Using again the fact that $\mathbf{P}^N$ is smooth, it follows from Lemma 14.2.13 that the above condition is equivalent to

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbf{P}^N,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbf{P}^N,x}) = 0 \quad \text{for all} \quad x \in X,\, i > r.$$

By Proposition 10.7.16, the above condition is equivalent to $\mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}) = 0$ for $i > r$.

Recall now that by Proposition 10.7.13, for every $j \in \mathbf{Z}$, we have a spectral sequence

$$E_2^{p,q} = H^p\big(X, \mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j))\big) \Rightarrow_p \mathrm{Ext}_{\mathcal{O}_{\mathbf{P}^N}}\big(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j)\big).$$

Note that by Example 10.7.10, we have

$$E_2^{p,q} \simeq H^p\big(X, \mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N})(j)\big),$$

and thus Theorem 11.2.1 implies that for $j \gg 0$, we have $E_2^{p,q} = 0$ for $p \neq 0$, hence the spectral sequence gives isomorphisms

$$\Gamma\big(X, \mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j))\big) \simeq \operatorname{Ext}^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j)\big) \quad \text{for all} \quad q \geq 0. \tag{14.2.5}$$

Moreover, for $q \gg 0$, the sheaf

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j)\big) \simeq \mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}) \otimes_{\mathcal{O}_{\mathbf{P}^N}} \mathcal{O}_{\mathbf{P}^N}(j)$$

is globally generated, hence by (14.2.5), we have $\mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}) = 0$ if and only if $\operatorname{Ext}^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j)\big) = 0$. Finally, Serre duality on $\mathbf{P}^N$ implies that

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}(j)\big) = 0 \quad \text{for} \quad i > r \quad \text{and} \quad j \gg 0$$

if and only if

$$H^{N-i}(\mathbf{P}^N, \mathcal{O}_X(-j)\big) = 0 \quad \text{for} \quad N - i < n \quad \text{and} \quad j \gg 0.$$

This completes the proof of the theorem. $\square$

The same argument in the proof of the implication iii)⇒v) in the above theorem gives the following

COROLLARY 14.2.14. *If $X \hookrightarrow \mathbf{P}^N$ is a projective variety such that* $\operatorname{depth}(\mathcal{O}_{X,x}) \geq 2$ *for every $x \in X$, then*

$$H^1\big(X, \mathcal{O}_X(-j)\big) = 0 \quad \textit{for} \quad j \gg 0.$$

This in turn implies the following connectedness result:

COROLLARY 14.2.15 (Enriques-Zariski-Severi). *Let X be an irreducible normal projective variety, with* $\dim(X) \geq 2$. *If D is an effective Cartier divisor on X such that $\mathcal{O}_X(D)$ is ample, then* $\operatorname{Supp}(D)$ *is connected.*

PROOF. Let $m > 0$ be such that $\mathcal{O}_X(mD)$ is a very ample line bundle, and consider the closed immersion $X \hookrightarrow \mathbf{P}^N$ such that $\mathcal{O}_X(1) \simeq \mathcal{O}_X(mD)$. Since X is normal, irreducible, of dimension ≥ 2, it follows from Serre's criterion for normality (see Theorem 12.1.27) that $\operatorname{depth}(\mathcal{O}_{X,x}) \geq 2$ for every $x \in X$. We thus conclude from the previous corollary that $H^1\big(X, \mathcal{O}_X(-qmD)\big) = 0$ for $q \gg 0$. Given such q, the long exact sequence in cohomology for the short exact sequence

$$0 \to \mathcal{O}_X(-qmD) \to \mathcal{O}_X \to \mathcal{O}_{qmD} \to 0$$

gives an exact sequence

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{qmD}) \to H^1\big(X, \mathcal{O}_X(-qmD)\big) = 0.$$

Since X is connected, we have $H^0(X, \mathcal{O}_X) = k$, hence $H^0(X, \mathcal{O}_{qmD}) = k$ (of course, we can't have $D = 0$, since $\mathcal{O}_X(D)$ is ample and X has positive dimension). Therefore $\operatorname{Supp}(qmD) = \operatorname{Supp}(D)$ is connected. $\square$

Our next goal is to describe more explicitly, under extra assumptions, the dualizing sheaf.

PROPOSITION 14.2.16. *Suppose that Y is a Cohen-Macaulay variety of pure dimension N and X is a closed subvariety, regularly embedded in Y, of pure dimension $n = N - r$. If ω_Y° is a line bundle, then*

$$\omega_X^\circ \simeq \omega_Y^\circ|_X \otimes_{\mathcal{O}_X} \det(N_{X/Y}).$$

In particular, ω_X° is a line bundle.

PROOF. By Theorem 14.2.12, Y satisfies Serre duality, hence we may apply Theorem 14.2.9 to deduce $\omega_X^\circ \simeq \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y^\circ)$. We will compute this Ext sheaf locally. Suppose first that that there is a locally free sheaf $\mathcal{E}$ of rank r on Y and a section $s \in \Gamma(Y, \mathcal{E})$ such that the corresponding ideal $\mathcal{I}(s)$ is equal to the radical ideal sheaf $\mathcal{I}_{X/Y}$ corresponding to X. Since Y is Cohen-Macaulay, it follows that s is a regular section and thus, by Example 12.2.22, we have a locally free resolution of $\mathcal{O}_X$ given by the Koszul complex associated to s:

$$0 \to \wedge^r \mathcal{E}^\vee \to \wedge^{r-1}\mathcal{E}^\vee \to \ldots \to \mathcal{E}^\vee \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

By Proposition 10.7.15, we can use this resolution to compute ω_X° and get

$$\omega_X^\circ \simeq \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y^\circ) \simeq \operatorname{coker}\big(\omega_Y^\circ \otimes_{\mathcal{O}_Y} \wedge^{r-1}\mathcal{E} \xrightarrow{-\wedge s} \omega_Y^\circ \otimes_{\mathcal{O}_Y} \wedge^r \mathcal{E}\big).$$

Since $\mathcal{E}$ is locally free, of rank r, it is straightforward to see that

$$\operatorname{coker}\big(\wedge^{r-1}\mathcal{E} \xrightarrow{-\wedge s} \wedge^r\mathcal{E}\big) \simeq \wedge^r\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_X.$$

Using the isomorphism $N_{Y/X} \simeq \mathcal{E}|_X$ (see Example 12.2.22), we obtain

$$\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y^\circ) \simeq \omega_Y^\circ|_X \otimes_{\mathcal{O}_X} \det(\mathcal{E}|_X) \simeq \omega_Y^\circ|_X \otimes_{\mathcal{O}_X} \det(N_{Y/X}).$$

In general, there is no such vector bundle $\mathcal{E}$. However, for every $x \in X$, there is an open neighborhood U of x, a vector bundle $\mathcal{E}$ on U and $s \in \Gamma(U, \mathcal{E})$ such that $\mathcal{I}(s) = \mathcal{I}_{X/Y}|_U$. Indeed, we can choose an affine neighborhood U of x such that we have $f_1, \ldots, f_r \in \mathcal{O}_Y(U)$ that generate $\Gamma(U, \mathcal{I}_{X/Y})$. In this case $s = (f_1, \ldots, f_r) \in \Gamma(U, \mathcal{O}_U^{\oplus r})$ satisfies the required condition. The above argument gives an isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y^\circ)|_U \simeq \big(\omega_Y^\circ|_X \otimes_{\mathcal{O}_X} \det(N_{Y/X})\big)|_U$$

and it is straightforward to check that this is independent of the choice of $(\mathcal{E}, s)$. These isomorphisms thus glue to give the desired global isomorphism. $\square$

EXAMPLE 14.2.17. If X is a smooth, irreducible projective variety, then we have a closed immersion $X \hookrightarrow \mathbf{P}^N$. Note that X is regularly embedded in $\mathbf{P}^N$, since $\mathbf{P}^N$ and X are smooth (see Example 12.3.19). We can thus apply the above proposition to conclude that

$$\omega_X^\circ \simeq \omega_{\mathbf{P}^N}^\circ|_X \otimes_{\mathcal{O}_X} \det(N_{X/\mathbf{P}^N}) \simeq \omega_{\mathbf{P}^N}|_X \otimes_{\mathcal{O}_X} \det(N_{X/\mathbf{P}^N}).$$

Using Corollary 8.7.27, we conclude that $\omega_X^\circ \simeq \omega_X$.

EXAMPLE 14.2.18. Let X be a smooth, irreducible, projective variety and Z a closed subvariety of X, of pure codimension 1. In this case Z is an effective Cartier divisor on X. Using the proposition and the previous example, we obtain

$$\omega_Z^\circ \simeq \omega_X|_Z \otimes \mathcal{O}_Z(Z),$$

where $\mathcal{O}_Z(Z) := \mathcal{O}_X(Z)|_Z$. This is known as the *adjunction formula*.

EXAMPLE 14.2.19. If X is a smooth, irreducible, n-dimensional projective variety, then $h^{p,q}(X) = h^{n-p,n-q}(X)$. Indeed, it follows from Remark 8.5.25 that

$$(\Omega_X^p)^\vee \simeq \Omega_X^{n-p} \otimes_{\mathcal{O}_X} \omega_X^{-1},$$

hence Serre duality gives

$$h^{p,q}(X) = h^q(X, \Omega_X^p) = h^{n-q}\big(X, \omega_X \otimes_{\mathcal{O}_X} (\Omega_X^{n-p})^\vee\big) = h^{n-q}(X, \Omega_X^{n-p}) = h^{n-p,n-q}(X).$$

CHAPTER 15

Algebraic curves

In this chapter we give an introduction to algebraic curves. In the first section we introduce the degree of a divisor on a smooth projective curve, prove the Riemann-Roch theorem, and give some applications. In the second section we discuss morphisms between curves. We begin by describing the equivalence between the category of smooth, projective curves and that of finitely generated fields of transcendence degree 1 over the ground field, and then turn to the case of finite morphisms and prove, in particular, the Riemann-Hurwitz theorem. Finally, in the last section we discuss several other topics about the geometry of curves: hyperelliptic curves, Clifford's theorem, and the fact that every smooth projective curve can be embedded in $\mathbf{P}^3$.

15.1. The Riemann-Roch theorem

Recall that a *curve* is an algebraically variety of pure dimension 1. Moreover, in this chapter we assume that all curves are *irreducible*.

REMARK 15.1.1. We will make extensive use of Serre duality, which implies that for a vector bundle $\mathcal{E}$ on a smooth, projective curve, we have functorial isomorphisms

$$H^i(X,\mathcal{E})^\vee \simeq H^{1-i}(X,\omega_X\otimes_{\mathcal{O}_X}\mathcal{E}^\vee) \quad \text{for} \quad i=0,1. \tag{15.1.1}$$

Indeed, X satisfies Serre duality by Theorem 14.2.12, since it is is smooth, and we have $\omega_X^\circ \simeq \omega_X$ by Proposition 14.2.16. In fact, in this case we don't need Theorem 14.2.12: the isomorphism for $i=1$ follows from the definition of the dualizing sheaf, and by replacing $\mathcal{E}$ with $\omega_X\otimes_{\mathcal{O}_X}\mathcal{E}^\vee$, we obtain the isomorphism for $i=0$. In what follows, we will only need the case when $\mathcal{E}$ is a line bundle.

REMARK 15.1.2. Recall that if X is a complete curve, then the arithmetic genus of X is given by

$$p_a(X) = (-1)^{\dim(X)}\big(\chi(\mathcal{O}_X)-1\big) = h^1(X,\mathcal{O}_X),$$

where we use the fact that $h^0(X,\mathcal{O}_X)=1$ since X is irreducible. A first consequence of Serre duality is that if X is a smooth, projective curve, then $h^1(X,\mathcal{O}_X) = h^0(X,\omega_X)$, that is, the arithmetic genus and the geometric genus coincide. Hence in this case, we simply call this the *genus* of X.

15.1.1. Degree of divisors on smooth, projective curves. We now introduce the basic invariant of a divisor on a smooth, projective curve. Recall that on smooth, irreducible varieties, we do not distinguish between Weil and Cartier divisors.

DEFINITION 15.1.3. If $D = \sum_{i=1}^{r} n_i P_i$ is a divisor on the smooth projective curve X, then the *degree* of D is given by

$$\deg(D) := \sum_{i=1}^{r} n_i \in \mathbf{Z}.$$

REMARK 15.1.4. It is clear from definition that given two divisors D and E as above, we have

$$\deg(D + E) = \deg(D) + \deg(E).$$

The following result is the cornerstone of the theory of algebraic curves:

THEOREM 15.1.5 (Riemann-Roch). *If D is a divisor on a smooth, projective curve X of genus g, then*

$$\chi\big(X, \mathcal{O}_X(D)\big) = \deg(D) - g + 1. \tag{15.1.2}$$

PROOF. The theorem holds if $D = 0$, since

$$\chi(X, \mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(X, \mathcal{O}_X) = 1 - g.$$

By adding or subtracting one point at a time, we can get from D to 0, so that it is enough to show that for every divisor D and every $P \in X$, the theorem holds for D if and only if it holds for $E = D - P$. Of course, we have $\deg(E) = \deg(D) - 1$, and thus it is enough to show that $\chi\big(X, \mathcal{O}_X(E)\big) = \chi\big(X, \mathcal{O}_X(D)\big) - 1$. By tensoring the exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0$$

with the line bundle $\mathcal{O}_X(D)$, we obtain the exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_X(D) \to \mathcal{O}_P \to 0.$$

By taking Euler-Poincaré characteristics, we get

$$\chi\big(X, \mathcal{O}_X(D)\big) - \chi\big(X, \mathcal{O}_X(E)\big) = \chi(X, \mathcal{O}_P) = 1,$$

completing the proof of the theorem. □

COROLLARY 15.1.6. *If D_1 and D_2 are linearly equivalent divisors on the smooth, projective curve X, then*

$$\deg(D_1) = \deg(D_2).$$

PROOF. The assertion follows from the fact that the left-hand side of (15.1.2) only depends on the line bundle associated to D. □

DEFINITION 15.1.7. Let X be a smooth, projective curve. Recall that if $\mathcal{L}$ is a line bundle on X, then there is a divisor D on X such that $\mathcal{O}_X(D) \simeq \mathcal{L}$ (see Proposition 9.4.11). In this case, we define the *degree* $\deg(\mathcal{L})$ of $\mathcal{L}$ to be equal to $\deg(D)$ (by the above corollary, this is independent of the choice of D).

EXAMPLE 15.1.8. If $\mathcal{L}$ is a line bundle on a smooth, projective curve such that $h^0(X, \mathcal{L}) \geq 1$, then $\deg(\mathcal{L}) \geq 0$, with equality if and only if $\mathcal{L} \simeq \mathcal{O}_X$. Indeed, a divisor $D \in |\mathcal{L}|$ is effective, hence we clearly have $\deg(D) \geq 0$, with equality if and only if $D = 0$.

EXAMPLE 15.1.9. If $\mathcal{L}$ is a very ample line bundle on the smooth, projective curve X, giving a closed immersion $X \hookrightarrow \mathbf{P}^n$ such that $\mathcal{L} \simeq \mathcal{O}_X(1)$, then the degree of $\mathcal{L}$ is equal to the degree of X with respect to this embedding. Indeed, the Hilbert polynomial of X with respect to this embedding is given by

$$P_X(m) = \chi\big(X, \mathcal{O}_X(m)\big) = \chi(X, \mathcal{L}^m) = m \cdot \deg(\mathcal{L}) - g + 1,$$

where the last equality follows from the Riemann-Roch theorem. This implies that the degree of X is equal to $\deg(\mathcal{L})$.

We now give some easy consequences of the Riemann-Roch theorem.

COROLLARY 15.1.10. *If X is a smooth, projective curve of genus g, then*

$$\deg(\omega_X) = 2g - 2.$$

PROOF. The Riemann-Roch theorem gives

$$\chi(X, \omega_X) = \deg(\omega_X) - g + 1.$$

On the other hand, Serre duality gives

$$\chi(X, \omega_X) = h^0(X, \omega_X) - h^1(X, \omega_X) = h^1(X, \mathcal{O}_X) - h^0(X, \mathcal{O}_X) = g - 1,$$

and we obtain the formula in the corollary. □

EXAMPLE 15.1.11. If $\mathcal{L}$ is a line bundle on a smooth, projective curve of genus g, such that $h^1(\mathcal{L}) \geq 1$, then $\deg(\mathcal{L}) \leq 2g-2$, with equality if and only if $\mathcal{L} \simeq \omega_X$. Indeed, Serre duality gives $h^0(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \geq 1$, and we deduce from Example 15.1.8 that

$$2g - 2 - \deg(\mathcal{L}) = \deg(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \geq 0,$$

with equality if and only if $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \simeq \mathcal{O}_X$.

REMARK 15.1.12. For a line bundle $\mathcal{L}$, the Riemann-Roch formula allows us to compute one of $h^0(X, \mathcal{L})$ and $h^1(X, \mathcal{L})$ whenever we know the other one; in particular, it is useful to know when one of these is 0. Examples 15.1.8 and 15.1.11 provide such criteria.

More precisely, Example 15.1.8 says that if $\deg(\mathcal{L}) < 0$, then $h^0(X, \mathcal{L}) = 0$, hence $h^1(X, \mathcal{L}) = g - 1 - \deg(\mathcal{L})$. Similarly, Example 15.1.11 says that if $\deg(\mathcal{L}) \geq 2g - 1$, then $h^1(X, \mathcal{L}) = 0$, hence $h^0(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$.

REMARK 15.1.13. An immediate consequence of the Riemann-Roch theorem is that if X is a smooth, projective curve of genus g, and $\mathcal{L}$ is a line bundle with $\deg(\mathcal{L}) \geq g$, then

$$h^0(X, \mathcal{L}) \geq \chi(X, \mathcal{L}) \geq 1.$$

The next proposition gives a necessary and sufficient condition for when a line bundle on a smooth, projective curve is globally generated or very ample.

NOTATION 15.1.14. If $\mathcal{L}$ is a line bundle on a smooth projective curve and D is a divisor on X, we put $\mathcal{L}(D) := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

PROPOSITION 15.1.15. *Let $\mathcal{L}$ be a line bundle on the smooth, projective curve X.*

i) *The line bundle $\mathcal{L}$ is globally generated if and only if for every $P \in X$, we have*

$$h^0\big(X, \mathcal{L}(-P)\big) = h^0(X, \mathcal{L}) - 1.$$

ii) *The line bundle $\mathcal{L}$ is very ample if and only if for every $P, Q \in X$ (not necessarily distinct), we have*

$$h^0\big(X, \mathcal{L}(-P-Q)\big) = h^0(X, \mathcal{L}) - 2.$$

PROOF. For every $P \in X$, by tensoring with $\mathcal{L}$ the short exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0,$$

we obtain the exact sequence

$$0 \to \mathcal{L}(-P) \to \mathcal{L} \to \mathcal{L}|_P \to 0.$$

By taking global sections, we obtain an exact sequence

$$0 \to H^0\big(X, \mathcal{L}(-P)\big) \to H^0(X, \mathcal{L}) \xrightarrow{\phi} \mathcal{L}_{(P)} \simeq k,$$

where $\phi(s) = s(P)$. We thus conclude that we have either $h^0\big(X, \mathcal{L}(-P)\big) = h^0(X, \mathcal{L}) - 1$ or $h^0\big(X, \mathcal{L}(-P)\big) = h^0(X, \mathcal{L})$, and that the former equality holds if and only if P is not a base-point of $|\mathcal{L}|$. This gives the assertion in i).

We now see that if the condition in ii) holds, then $\mathcal{L}$ is, in particular, globally generated. We henceforth assume that this is the case. The equivalence in ii) follows from the characterization in Proposition 11.5.18 if we show the following:

(α) Given two distinct points $P, Q \in X$, the complete linear system $|\mathcal{L}|$ separates P and Q if and only if

$$h^0\big(X, \mathcal{L}(-P-Q)\big) = h^0(X, \mathcal{L}) - 2.$$

(β) Given $P \in X$, the complete linear system $|\mathcal{L}|$ separates the tangent directions at P if and only if

$$h^0\big(X, \mathcal{L}(-2P)\big) = h^0(X, \mathcal{L}) - 2.$$

We first consider (α). We have seen that for every $P \in X$, the codimension 1 linear subspace $H^0\big(X, \mathcal{L}(-P)\big) \subseteq H^0(X, \mathcal{L})$ consists of those sections $s \in H^0(X, \mathcal{L})$ such that $s(P) = 0$. We deduce that for two distinct points $P, Q \in X$, the linear system $|\mathcal{L}|$ separates P and Q if and only if $H^0\big(X, \mathcal{L}(-P)\big) \neq H^0\big(X, \mathcal{L}(-Q)\big)$, which is the case if and only if the intersection of these two subspaces has codimension 2 in $H^0(X, \mathcal{L})$. On the other hand, we have an exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(-P-Q) \to \mathcal{O}_X \to \mathcal{O}_{\{P,Q\}} \to 0,$$

which after tensoring with $\mathcal{L}$ and taking global sections gives an exact sequence

$$0 \to H^0\big(X, \mathcal{L}(-P-Q)\big) \to H^0(X, \mathcal{L}) \xrightarrow{\psi} \mathcal{L}_{(P)} \oplus \mathcal{L}_{(Q)} \simeq k^{\oplus 2},$$

where $\psi(s) = \big(s(P), s(Q)\big)$. We thus have

$$H^0\big(X, \mathcal{L}(-P-Q)\big) = H^0\big(X, \mathcal{L}(-P)\big) \cap H^0\big(X, \mathcal{L}(-Q)\big).$$

This implies that the left-hand side has codimension 2 in $H^0(X, \mathcal{L})$ if and only if $|\mathcal{L}|$ separates P and Q.

The argument for (β) is similar. For every $P \in X$, we have $\dim_k T_P X = 1$, hence $|\mathcal{L}|$ separates the tangent directions at P if and only if there is $s \in H^0(X, \mathcal{L})$ such that $s(P) = 0$, but $s_P \not\in \mathfrak{m}_P^2 \mathcal{L}_P$, where $\mathfrak{m}_P$ is the maximal ideal in $\mathcal{O}_{X,P}$. By tensoring with $\mathcal{L}$ the short exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(-2P) \to \mathcal{O}_X \to \mathcal{O}_{2P} \to 0,$$

and taking global sections, we obtain an exact sequence

$$0 \to H^0\big(X, \mathcal{L}(-2P)\big) \to H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{2P}) \simeq \mathcal{O}_{X,P}/\mathfrak{m}_P^2.$$

This identifies $H^0\big(\mathcal{L}(-2P)\big)$ with the space of sections $s \in H^0(X, \mathcal{L})$ such that $s_P \in \mathfrak{m}_P^2\mathcal{L}_P$. This immediately gives the assertion in (β). $\square$

Corollary 15.1.16. *Let X be a smooth, projective curve of genus g. If $\mathcal{L}$ is a line bundle on X, with* $\deg(\mathcal{L}) \geq 2g$*, then $\mathcal{L}$ is globally generated. If* $\deg(\mathcal{L}) \geq 2g+1$*, then $\mathcal{L}$ is very ample.*

Proof. Note that if $\deg(\mathcal{L}) \geq 2g$, then it follows from Remark 15.1.12 that $h^0(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$. Moreover, for every $P \in X$, we have $\deg\big(\mathcal{L}(-P)\big) = \deg(\mathcal{L}) - 1 \geq 2g - 1$, hence another application of the same remark gives

$$h^0\big(X, \mathcal{L}(-P)\big) = \deg\big(\mathcal{L}(-P)\big) - g + 1 = h^0(X, \mathcal{L}) - 1.$$

We deduce that $\mathcal{L}$ is globally generated using the first assertion in the proposition. If $\deg(\mathcal{L}) \geq 2g + 1$, we deduce similarly that $\mathcal{L}$ is very ample using the second assertion in the proposition. $\square$

Corollary 15.1.17. *A line bundle $\mathcal{L}$ on a smooth, projective curve X is ample if and only if* $\deg(\mathcal{L}) > 0$*.*

Proof. If $\deg(\mathcal{L}) > 0$, then it follows from the previous corollary that $\mathcal{L}^m$ is very ample for every m such that $m \cdot \deg(\mathcal{L}) \geq 2g + 1$, where g is the genus of X. Conversely, suppose that $\mathcal{L}$ is ample, and let $m > 0$ be such that $\mathcal{L}^m$ is very ample. By Example 15.1.9, we see that $\deg(\mathcal{L}^m) = m \cdot \deg(\mathcal{L})$ is equal to the degree of X with respect to the embedding given by the complete linear system $|\mathcal{L}^m|$; in particular, this is a positive integer. $\square$

When classifying curves, the basic trichotomy is the following.

Example 15.1.18 (Case $g = 0$). A smooth projective curve X has genus $g = 0$ if and only if $X \simeq \mathbf{P}^1$. Indeed, it is clear that $p_g(\mathbf{P}^1) = 0$. Conversely, if $g = 0$, and $P \in X$ is any point, then it follows from Proposition 15.1.15 that the line bundle $\mathcal{L} = \mathcal{O}_X(P)$ is very ample. Moreover, we have $h^0(X, \mathcal{L}) = 2$ by Remark 15.1.12, hence $|\mathcal{L}|$ gives a closed immersion $i \colon X \hookrightarrow \mathbf{P}^1$. Of course, this is an isomorphism.

Note that in this case, the line bundle ω_X^{-1} is ample.

Example 15.1.19 (Case $g = 1$). A smooth projective curve X of genus $g = 1$ is an *elliptic curve*. Note that by Corollary 15.1.10, in this case $\deg(\omega_X) = 0$. In fact, we have $\omega_X \simeq \mathcal{O}_X$: indeed, since $h^0(\omega_X) = 1$ by assumption, we have an effective divisor D such that $\omega_X \simeq \mathcal{O}_X(D)$. Since $\deg(D) = 0$, it follows that $D = 0$.

Note that if X is an elliptic curve, then for every line bundle $\mathcal{L}$ on X of degree 3, it follows from Proposition 15.1.15 that $\mathcal{L}$ is very ample. Moreover, by Remark 15.1.12, we have $h^0(X, \mathcal{L}) = 3$, hence the complete linear system $|\mathcal{L}|$ gives a closed immersion $X \hookrightarrow \mathbf{P}^2$, with X of degree 3 (see Remark 15.1.9). Conversely, if $X \subseteq \mathbf{P}^2$ is a smooth curve of degree 3, then it follows from Corollary 8.7.27 that

$$\omega_X \simeq \omega_{\mathbf{P}^2} \otimes_{\mathcal{O}_{\mathbf{P}^2}} \mathcal{O}_{\mathbf{P}^2}(X)|_X \simeq \mathcal{O}_X,$$

hence X is an elliptic curve.

EXAMPLE 15.1.20 (The general case). The case $g \geq 2$ is the case when we have $\deg(\omega_X) > 0$, hence ω_X is ample by Corollary 15.1.17. This is the "general case". We will see in Example 16.2.3 below that for every $g \geq 2$, there are smooth, projective curves of genus g.

We now give some examples of complete intersection curves. Suppose first that X is a smooth curve in $\mathbf{P}^2$ of degree $d \geq 4$. In this case, it follows from Corollary 8.7.27 that $\omega_X \simeq \mathcal{O}_X(d-3)$. In particular, it follows from Example 11.2.6 that the genus g of X is equal to $\dim_k \big(k[x_0, x_1, x_2]/(f)\big)_{d-3}$, where f is an equation of X. We thus obtain

$$g = \binom{d-1}{2}.$$

More generally, suppose that $X \subseteq \mathbf{P}^n$ is a smooth curve, which is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-1} \geq 1$. In this case, it follows from Example 12.3.21 that $\omega_X \simeq \mathcal{O}_X(d_1 + \ldots + d_{n-1} - n - 1)$. On the other hand, a repeated application of the formula in Example 11.4.10 gives

$$\deg\big(\mathcal{O}_X(1)\big) = d_1 \cdots d_{n-1},$$

and we deduce using Corollary 15.1.10 that if g is the genus of X, then

$$2g - 2 = d_1 \cdots d_{n-1}(d_1 + \ldots + d_{n-1} - n - 1).$$

Note that if X is non-degenerate (equivalently, $d_i \geq 2$ for all i) and $n \geq 3$, then $g \geq 1$, with equality if and only if $n = 3$ and $d_1 = d_2 = 2$.

REMARK 15.1.21. The genus is the fundamental invariant in the classification of smooth algebraic curves. An important result, going back to Riemann, says that for every $g \geq 1$, there is a *moduli space* $\mathcal{M}_g$ of smooth, projective curves of genus g. This is an irreducible, quasi-projective variety whose points are in bijection with the smooth, projective curves of genus g. If $g = 1$, then $\mathcal{M}_g$ is isomorphic to $\mathbf{A}^1$, while for $g \geq 2$, $\mathcal{M}_g$ has dimension $3g - 3$. For a thorough discussion, see [HM98].

REMARK 15.1.22. A basic result in the theory of curves is that if X is a smooth, projective curve, then there is an irreducible projective variety $\mathrm{Pic}^0(X)$ whose points are in bijection with the isomorphism classes of line bundles of degree 0. Moreover, the tensor product of line bundles makes $\mathrm{Pic}^0(X)$ an algebraic group (a connected algebraic group which is a projective variety is an *Abelian variety*). This is the *Picard variety* of X.

EXERCISE 15.1.23. Let X be a smooth projective curve.

i) Show that if for a vector bundle $\mathcal{E}$ on X, we put $\deg(\mathcal{E}) := \deg\big(\det(\mathcal{E})\big)$, then we obtain a group homomorphism $\deg\colon K^0(X) \to \mathbf{Z}$.
ii) Via the canonical isomorphism $K^0(X) \to K_0(X)$, the homomorphism in i) gives a group homomorphism $\deg\colon K_0(X) \to \mathbf{Z}$. Show that if $\mathcal{M}$ is a coherent sheaf that is supported on a finite set, then $\deg(\mathcal{M}) = h^0(X, \mathcal{M})$.
iii) Show that if $\mathcal{E}$ is a vector bundle on X, then the canonical morphism $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ has a section $s\colon X \to \mathbf{P}(\mathcal{E})$ (that is, we have $\pi \circ s = \mathrm{id}_X$). Deduce that there is a surjective morphism $\mathcal{E} \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle.
iv) Show that every vector bundle $\mathcal{E}$ on X has a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a line bundle for every i, with $1 \leq i \leq r$.

v) Show that if $\mathcal{E}$ is a vector bundle of rank r, then we have the following version of the Riemann-Roch formula:

$$\chi(X, \mathcal{E}) = \deg(\mathcal{E}) - \mathrm{rk}(\mathcal{E}) \cdot (g-1),$$

where g is the genus of X.

Exercise 15.1.24. Let X be a smooth projective curve and consider the group homomorphism

$$\alpha\colon K^0(X) \to \mathrm{Pic}(X) \oplus \mathbf{Z}, \quad \alpha\big([\mathcal{E}]\big) = \big(\det(\mathcal{E}), \mathrm{rank}(\mathcal{E})\big).$$

i) Show that we have a group homomorphism $\mathrm{Pic}(X) \to K^0(X)$ that maps the isomorphism class of $\mathcal{L}$ to $[\mathcal{L}] - [\mathcal{O}_X]$.
ii) Deduce that α is an isomorphism.

15.1.2. Arithmetic genus of singular curves. We have not discussed so far singular curves. We only prove one result, relating the arithmetic genus of a singular curve to that of its normalization.

Given an arbitrary curve X, consider the normalization morphism $\pi\colon \widetilde{X} \to X$. The canonical morphism $\mathcal{O}_X \to \pi_*(\mathcal{O}_{\widetilde{X}})$ is injective and let $\mathcal{M}$ be its cokernel, so that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_{\widetilde{X}}) \to \mathcal{M} \to 0. \tag{15.1.3}$$

Since π is an isomorphism over X_{sm}, it follows that $\mathcal{M}$ is supported on the singular locus of X. Moreover, for every $P \in X_{\mathrm{sing}}$, we have

$$\mathcal{M}_P \simeq \pi_*(\mathcal{O}_{\widetilde{P}})_P / \mathcal{O}_{X,P} \simeq \widetilde{\mathcal{O}_{X,P}} / \mathcal{O}_{X,P},$$

where $\widetilde{\mathcal{O}_{X,P}}$ is the integral closure of $\mathcal{O}_{X,P}$ in $k(X)$. We put $\delta_P = \ell_{\mathcal{O}_{X,P}}(\mathcal{M}_P)$. It is clear that $\delta_P > 0$ for every singular point P.

Proposition 15.1.25. *If X is a complete curve and $\pi\colon \widetilde{X} \to X$ is the normalization morphism, then*

$$p_a(X) = p_a(\widetilde{X}) + \sum_{P \in X_{\mathrm{sing}}} \delta_P.$$

In particular, we have $p_a(X) \geq p_a(\widetilde{X})$, with strict inequality if X is singular.

Proof. By taking the long exact sequence in cohomology corresponding to (15.1.3), we obtain

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to H^0(X, \mathcal{M})$$
$$\to H^1(X, \mathcal{O}_X) \to H^1\big(X, \pi_*(\mathcal{O}_{\widetilde{X}})\big) \to H^1(X, \mathcal{M}) \to 0.$$

Since $\mathcal{M}$ is supported on a finite set, we have $H^1(X, \mathcal{M}) = 0$. Note also that the morphism

$$H^0(X, \mathcal{O}_X) \to H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$$

is an isomorphism, both cohomology groups being canonically isomorphic to k. Finally, since π is a finite morphism, we have an isomorphism

$$H^1\big(X, \pi_*(\mathcal{O}_{\widetilde{X}})\big) \simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$$

by Example 10.5.16. We thus have a short exact sequence

$$0 \to H^0(X, \mathcal{M}) \to H^1(X, \mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to 0,$$

and the equality in the proposition follows. $\square$

Corollary 15.1.26. *If X is a complete curve with $p_a(X) = 0$, then X is smooth; if X is projective*[1]*, then it is isomorphic to $\mathbf{P}^1$.*

Proof. It follows from the proposition that if X is singular, then $0 = p_a(X) > p_a(\widetilde{X}) \geq 0$, a contradiction. Therefore X is smooth and if it is projective, then it follows from Example 15.1.18 that $X \simeq \mathbf{P}^1$. □

15.2. Morphisms between algebraic curves

15.2.1. Rational maps vs. morphisms. We begin by discussing some general facts about morphisms of curves. Note first that since we are in dimension 1, smoothness is the same as normality. Second, we recall that by Remark 9.1.7, any rational map $X \dashrightarrow Y$, where X is a smooth curve and Y is a complete variety, is a morphism.

Remark 15.2.1. Every non-constant morphism $f \colon X \to Y$, where X and Y are complete curves, is finite. Indeed, since f is non-constant, every fiber of f is finite. Since X and Y are complete, f is proper by Remark 5.1.8, and thus f is finite by Corollary 14.1.8.

Proposition 15.2.2. *Every birational map between smooth, complete curves is an isomorphism. Moreover, every smooth, complete curve is projective.*

Proof. If $\phi \colon X \dashrightarrow Y$ is a birational map between smooth, complete curves, we have seen that both ϕ and ϕ^{-1} are morphisms. Therefore ϕ is an isomorphism.

Suppose now that X is an arbitrary smooth, complete curve. If U is an affine open subset of X, by embedding U in an affine space $\mathbf{A}^n$ and taking the closure in $\mathbf{P}^n$, we see that X is birational to a projective curve Y. If $\pi \colon \widetilde{Y} \to Y$ is the normalization map, since Y is projective, it follows that $\widetilde{Y}$ is projective (see Remark 11.6.18). We conclude that X is birational to the smooth, projective curve $\widetilde{Y}$. By the first assertion in the proposition, X and $\widetilde{Y}$ are isomorphic, hence X is projective. □

Remark 15.2.3. The category of smooth, projective curves, with non-constant morphisms, is equivalent to the category of finitely generated field extensions K/k, with $\mathrm{trdeg}(K/k) = 1$; this equivalence is given by the functor which associates to X its function field. This follows from the the following two facts:

i) A dominant rational map between smooth, projective curves is the same as a non-constant morphism.
ii) If K/k is a finitely generated field extension, of transcendence degree 1, then there is a smooth, projective curve X such that $k(X) \simeq K$. For this, we argue as in the proof of the above proposition: we have an affine curve U such that $k(U) \simeq K$. By taking the closure of U in some projective space and then taking the normalization, we obtain a smooth, projecctive curve X with $k(X) \simeq K$.

Remark 15.2.4. If $f \colon X \to Y$ is a finite, birational morphism between two curves, with Y smooth, then f is an isomorphism. This follows from the fact that if A is a normal domain, then every finite extension of A contained in the fraction field of A is equal to A.

[1] In fact, we will see in Proposition 15.2.2 below that if X is a smooth, complete curve, then X is automatically projective.

Remark 15.2.5. We can show that every complete curve X is projective. Indeed, let $\pi\colon \widetilde{X} \to X$ be the normalization of X. If P is a smooth point of X, we claim that the line bundle $\mathcal{O}_X(P)$ is ample, hence X is projective. Indeed, we have seen that $\widetilde{X}$ is projective. By assumption, f is an isomorphism over a neighborhood of P, hence $f^*\big(\mathcal{O}_X(P)\big) \simeq \mathcal{O}_{\widetilde{X}}\big(f^{-1}(P)\big)$ is a line bundle of degree 1, hence ample by Corollary 15.1.17. We conclude that $\mathcal{O}_X(P)$ is ample by Exercise 11.6.20.

Proposition 15.2.6. *If X is a smooth curve, then there is an open immersion $X \hookrightarrow Y$, where Y is a smooth, projective curve. Moreover, any birational morphism between smooth curves is an open immersion.*

Proof. Choose a smooth projective curve Y such that $k(X) \simeq k(Y)$. We thus have a birational map $X \dashrightarrow Y$, hence a birational morphism $f\colon X \to Y$. We will show that this is an open immersion. We first show that if U is an affine open subset of X, then the restriction of f to U is an open immersion. Let $\overline{U}$ be the closure of U in some projective space and $\pi\colon V \to \overline{U}$ the normalization morphism. Since U is smooth, π is an isomorphism over U, hence we have an open immersion $U \hookrightarrow V$. The morphism $f|_U$ has an extension to a birational morphism $g\colon V \to Y$, which is an isomorphism. This implies that $f|_U$ is an open immersion.

In order to conclude that f is an open immersion, it is enough to show that f is injective. If this is not the case, then there are $P \neq Q$ in X such that $f(P) = f(Q)$. Let U be an affine open neigbourhood of P. Since $f|_U$ is an open immersion, it follows that f maps U isomorphically onto an open subset $f(U)$ of Y. In particular, $Q \not\in U$. If $U' = U \cup \{Q\}$, then U' is an open subset of X such that f induces a morphism $U' \to f(U)$. Since the composition $U \hookrightarrow U' \to f(U)$ is a closed immersion, it follows that $U \hookrightarrow U'$ is a closed immersion (see Remark 5.1.8), contradicting the fact that U' is connected. This completes the proof of the first assertion in the proposition.

Suppose now that $f\colon X_0 \to Y_0$ is an arbitrary birational morphism between smooth curves. If Z is a smooth, projective curve that is birational to X_0 and Y_0, we already know that we have open immersions $X \hookrightarrow Z$ and $Y \hookrightarrow Z$. The morphism f then extends to a birational morphism $Z \to Z$, which must be an isomorphism. This implies that f is an open immersion. $\square$

Proposition 15.2.7. *If X is a curve that is not projective, then X is affine.*

Proof. Since X is not projective, it follows from Remark 15.2.5 that X is not complete. Let $\pi\colon \widetilde{X} \to X$ be the normalization morphism. Since $\widetilde{X}$ is smooth, it follows from Proposition 15.2.6 that we have an open immersion $\widetilde{X} \hookrightarrow Y$, where Y is a smooth projective curve. Note that $\widetilde{X} \neq Y$, since X is not complete. The complement $Y \smallsetminus \widetilde{X}$ is a non-empty closed subset of Y, hence it is a finite set $\{P_1, \ldots, P_r\}$, with $r \geq 1$. If $D = \sum_{i=1}^r P_i$, then it follows from Corollary 15.1.17 that $\mathcal{L} = \mathcal{O}_Y(D)$ is ample. Let m be a positive integer such that $\mathcal{L}^m$ is very ample and let $i\colon Y \hookrightarrow \mathbf{P}^n$ be the closed immersion defined by the complete linear system $|\mathcal{L}^m|$. Let H be a hyperplane in $\mathbf{P}^n$ such that $H|_Y = mD$. We see that $\widetilde{X}$ is a closed subvariety of the affine variety $\mathbf{P}^n \smallsetminus H$, hence $\widetilde{X}$ is affine. Since π is finite and surjective, we conclude that X is affine (see Exercise 10.5.21). $\square$

15.2.2. Pull-back of divisors via finite morphisms. We now turn to the study of non-constant morphisms between smooth, projective curves. Let $f\colon X \to$

Y be such a morphism. As we have already mentioned, f is finite. Moreover, it is also flat: this follows, for example, from Example 12.3.10.

The *ramification index* $e_P(f)$ at a point $P \in X$ is defined as follows. We have an induced local homomorphism $\phi\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ and if $u \in \mathcal{O}_{Y,f(P)}$ and $v \in \mathcal{O}_{X,P}$ are local uniformizers (that is, generators for the respective maximal ideals), then we can write $\phi(u) = v^e w$ for an invertible element $w \in \mathcal{O}_{X,P}$ and a non-negative integer e. We put $e_P(f) = e$ (it is clear that this is independent of the choice of u and v). Note that $e_P(f)$ is the same as the multiplicity of P in the fiber $f^{-1}\big(f(P)\big)$. Proposition 13.1.1 thus gives the following

Proposition 15.2.8. *If $f\colon X \to Y$ is a finite morphism between smooth, projective curves, then for every $Q \in Y$, we have*

$$\sum_{P \in f^{-1}(Q)} e_P(f) = \deg(f).$$

If $f\colon X \to Y$ is as above, then for every $Q \in Y$, we may consider Q as a Cartier divisor on Y, and we may consider its inverse image $f^*(Q)$. It then follows from definition that

$$f^*(Q) = \sum_{P \in f^{-1}(Q)} e_P(f) \cdot P.$$

In particular, the above proposition gives the following

Corollary 15.2.9. *If $f\colon X \to Y$ is a finite morphism between smooth, projective curves, then for every divisor D on Y, we have*

$$\deg\big(f^*(D)\big) = \deg(f) \cdot \deg(D).$$

Remark 15.2.10. If $f\colon X \to Y$ is a finite morphism between smooth, projective curves, and if D is a divisor on X, then we may consider its push-forward $f_*(D)$. It follows from definition that $\deg\big(f_*(D)\big) = \deg(D)$.

Remark 15.2.11. Let $f\colon X \to Y$ be a finite morphism between smooth, projective curves. Note that by definition f is unramified at $P \in X$ if and only if $e_P(f) = 1$. By Remark 13.2.12, this is the case if and only if the induced morphism $df_P\colon T_PX \to T_{f(P)}Y$ is an isomorphism (which is the case if and only if it is non-zero).

Example 15.2.12. We can use the above framework to see that every finite étale morphism $f\colon X \to \mathbf{P}^1$, where X is a connected variety, is an isomorphism. Indeed, given such f, it follows from Proposition 13.2.10 that X is smooth, hence it is a smooth, projective curve. Since f is étale, we have an isomorphism $\omega_X \simeq f^*(\omega_Y)$ (see Remark 13.2.9) and using the formula for the degrees of the canonical line bundles, we deduce that if g is the genus of X, then

$$2g - 2 = \deg(f) \cdot (-2).$$

If f is not birational, then $\deg(f) \geq 2$, hence $2g-2 \leq -4$, a contradiction. Therefore f is birational, hence an isomorphism.

Example 15.2.13. Let us show that if X is a smooth, projective curve, which is not isomorphic to $\mathbf{P}^1$, then for every point $P \in X$, we have $h^0\big(X, \mathcal{O}_X(P)\big) = 1$. Indeed, suppose that this is not the case, hence there is an effective divisor D on X, linearly equivalent to P, and different from P. We have $\deg(D) = 1$, hence $D = Q$, for some point Q different from P. Since $P \sim Q$, it follows that there is $\phi \in k(X) \smallsetminus \{0\}$ such that $\operatorname{div}(\phi) = P - Q$. Note that ϕ gives a rational map $X \dashrightarrow \mathbf{P}^1$, and thus a morphism $f \colon X \to \mathbf{P}^1$. Note that $\operatorname{div}(\phi) = f^*(0 - \infty)$. We thus see that $f^*(0) = P$ and it follows from Corollary 15.2.9 that $\deg(f) = 1$. This means that f is birational, hence an isomorphism.

EXAMPLE 15.2.14. We can now give an example of an ample line bundle which is not globally generated; in particular, it is not very ample. Indeed, suppose that X is a curve of genus ≥ 1 and let $\mathcal{L} = \mathcal{O}_X(P)$ for some point $P \in X$. In this case, $\mathcal{L}$ is ample by Corollary 15.1.17. On the other hand, it follows from the above example that $|\mathcal{L}|$ has only one element, and thus P lies in the base-locus of $\mathcal{L}$.

15.2.3. Degree of line bundles on singular curves. So far, we have only considered the degree of a line bundle on a smooth, projective curve. We can extend this to arbitrary projective curves, as follows.

DEFINITION 15.2.15. If $\mathcal{L}$ is a line bundle on the projective curve X, and $\pi \colon \widetilde{X} \to X$ is the normalization, then we define the *degree* of $\mathcal{L}$ to be

$$\deg(\mathcal{L}) := \deg\big(\pi^*(\mathcal{L})\big).$$

We can similarly define the degree of a Cartier divisor D on X and we have $\deg(D) = \deg\big(\mathcal{O}_X(D)\big)$. It is clear that the degree gives a group homomorphism $\operatorname{Pic}(X) \to \mathbf{Z}$.

REMARK 15.2.16. The formula in Corollary 15.2.9 holds in this more general setting. Indeed, suppose that $f \colon X \to Y$ is a finite morphism between two projective curves and let $\pi_X : \widetilde{X} \to X$ and $\pi_Y : \widetilde{Y} \to Y$ be the corresponding normalization morphisms. We thus obtain a dominant rational map $\pi_Y^{-1} \circ f \circ \pi_X$, which is in fact a morphism $\widetilde{f}$ such that $\deg(f) = \deg(\widetilde{f})$. Using the definition of degree on X and Y and Corollary 15.2.9, we thus see that if $\mathcal{L}$ is a line bundle on Y, then

$$\begin{aligned}\deg\big(f^*(\mathcal{L})\big) &= \deg\big(\pi_X^*\big(f^*(\mathcal{L})\big)\big) = \deg\Big(\widetilde{f}^*\big(\pi_Y^*(\mathcal{L})\big)\Big)\\ &= \deg(\widetilde{f}) \cdot \deg\big(\pi_Y^*(\mathcal{L})\big) = \deg(f) \cdot \deg(\mathcal{L}).\end{aligned}$$

REMARK 15.2.17. If X is a projective curve embedded in $\mathbf{P}^n$, then the formula in Example 15.1.9 also holds in the possibly singular case, namely $\deg\big(\mathcal{O}_X(1)\big) = \deg(X)$. Indeed, let $\pi_X \colon \widetilde{X} \to X$ be the normalization morphism. By tensoring with $\mathcal{O}_X(m)$ the short exact sequence

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_{\widetilde{X}}) \to \mathcal{M} \to 0$$

discussed in §15.1.2, we obtain using the projection formula the exact sequence

$$0 \to \mathcal{O}_X(m) \to \pi_*\big(\pi^*(\mathcal{O}_X(m))\big) \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \to 0. \tag{15.2.1}$$

Since $\mathcal{M}$ is supported on a finite set of points, we have $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \mathcal{M}$ for every m. On the other hand, since π is a finite morphism, it follows from Example 10.5.16 that

$$\chi\big(X, \pi_*(\pi^*(\mathcal{O}_X(m)))\big) = \chi\big(\widetilde{X}, \pi^*(\mathcal{O}_X(m))\big).$$

By taking Euler-Poincaré characteristics in (15.2.1) and using the Riemann-Roch formula on $\widetilde{X}$, we thus obtain

$$\chi\big(X, \mathcal{O}_X(m)\big) = \chi\big(\widetilde{X}, \pi^*(\mathcal{O}_X(m))\big) - \chi(X, \mathcal{M})$$

$$= \deg\bigl(\pi^*(\mathcal{O}_X(1))\bigr)\cdot m + \bigl(1 - g - \chi(X, \mathcal{M})\bigr)$$

for all $m \in \mathbf{Z}$, where g is the genus of $\widetilde{X}$. We thus obtain

$$\deg(X) = \deg\bigl(\pi^*(\mathcal{O}_X(1))\bigr) = \deg\bigl(\mathcal{O}_X(1)\bigr).$$

EXAMPLE 15.2.18. Suppose that $X \subseteq \mathbf{P}^n$ is a projective curve and $Q \in \mathbf{P}^n \smallsetminus X$. The projection $\mathbf{P}^n \smallsetminus \{Q\} \to \mathbf{P}^{n-1}$ induces a finite surjective morphism $f\colon X \to Y$, with $f^*\bigl(\mathcal{O}_Y(1)\bigr) \simeq \mathcal{O}_X(1)$. We thus see that

$$\deg(X) = \deg(f)\cdot \deg(Y).$$

EXERCISE 15.2.19. Let $Q \in \mathbf{P}^n$ be a point and consider the projection map $\phi\colon \mathbf{P}^n \smallsetminus \{Q\} \to \mathbf{P}^{n-1}$.

i) Show that if $\pi\colon W \to \mathbf{P}^n$ is the blow-up of $\mathbf{P}^n$ at the point Q (that is, the blow-up of the radical ideal sheaf corresponding to Q), then the rational map $\phi \circ \pi^{-1}$ is in fact a morphism $f\colon W \to \mathbf{P}^{n-1}$ such that

$$f^*\bigl(\mathcal{O}_{\mathbf{P}^{n-1}}(1)\bigr) \simeq \pi^*\bigl(\mathcal{O}_{\mathbf{P}^n}(1)\bigr) \otimes_{\mathcal{O}_W} \mathcal{O}_W(-E),$$

where E is the exceptional divisor on the blow-up.

ii) Suppose that $X \subseteq \mathbf{P}^n$ is a curve such that $Q \in X$ is a smooth point. Show that the induced morphism $\widetilde{X} \to X$ is an isomorphism, where $\widetilde{X}$ is the strict transform of X. Deduce that the restriction of π to $X \smallsetminus \{Q\}$ extends to a morphism $f\colon X \to \mathbf{P}^{n-1}$ such that

$$f^*\bigl(\mathcal{O}_{\mathbf{P}^{n-1}}(1)\bigr) \simeq \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-Q).$$

EXAMPLE 15.2.20. Let $X \subseteq \mathbf{P}^n$ be a projective curve and let $Q \in X$ be a smooth point. It follows from the above exercise that the projection map $\mathbf{P}^n \smallsetminus \{Q\} \to \mathbf{P}^{n-1}$ induces a surjective morphism $f\colon X \to Y \subseteq \mathbf{P}^{n-1}$. If X is not a line, then Y is a curve and we have

$$\deg(X) - 1 = \deg(f)\cdot \deg(Y).$$

EXAMPLE 15.2.21. We have seen in Example 11.4.11 that if $X \subseteq \mathbf{P}^n$ is a projective, non-degenerate curve, then $\deg(X) \geq n$. We now show that we have $\deg(X) = n$ if and only if X is a rational normal curve. The "if" part is clear (see Example 11.4.7), hence we only need to prove the "only if" part. We argue by induction on n, the case $n = 1$ being trivial.

For the induction step, suppose that $n \geq 2$ and let $Q \in X$ be a smooth point. If $f\colon X \to Y$ is the finite, surjective morphism induced by the projection $\mathbf{P}^n \smallsetminus \{Q\} \to \mathbf{P}^{n-1}$, then we have seen in the previous example that

$$\deg(X) - 1 = \deg(f)\cdot \deg(Y).$$

It is clear that Y is non-degenerate, hence if $\deg(X) = n$, we have

$$n - 1 = \deg(f)\cdot \deg(Y) \geq \deg(f)\cdot (n-1).$$

This implies that $\deg(f) = 1$ and $\deg(Y) = n - 1$. By the inductive assumption, Y is a rational normal curve in $\mathbf{P}^{n-1}$. In particular, it is smooth, and since f is birational, we conclude that f is an isomorphism (see Remark 15.2.4). Since Y is isomorphic to $\mathbf{P}^1$, it follows that X is isomorphic to $\mathbf{P}^1$. Since $\deg(X) = n$, we have $\mathcal{O}_X(1) \simeq \mathcal{O}_{\mathbf{P}^1}(n)$, and since $h^0\bigl(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n)\bigr) = n + 1$, is now clear that X is a rational normal curve.

15.2.4. The Riemann-Hurwitz theorem. Suppose that $f\colon X \to Y$ is a finite morphism between smooth, projective curves. Our goal is to relate, in this case, the canonical line bundles on X and Y. Recall that we have an exact sequence

$$f^*(\omega_Y) \xrightarrow{\alpha} \omega_X \to \Omega_{X/Y} \to 0 \tag{15.2.2}$$

(see Proposition 8.7.20). Note that α is a morphism between two line bundles. This implies that we have a section $s \in \Gamma\big(X, \omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}\big)$ such that α is given by tensoring with $f^*(\omega_Y)$ the morphism $\mathcal{O}_X \to \omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}$ corresponding to s.

Suppose now that f is separable, that is, the induced finite field extension $k(X)/k(Y)$ is separable (for example, this always holds if $\mathrm{char}(k) = 0$). In this case, it follows from Lemma 13.2.19 that there is an open subset $U \subseteq X$ such that the induced morphism $U \to Y$ is smooth, of relative dimension 0 (hence étale). This implies that $\alpha|_U$ is an isomorphism of line bundles. Equivalently, $s|_U$ is everywhere non-zero, and thus $s \neq 0$. In particular, this implies that the morphism α in (15.2.2) is injective. The *ramification divisor* of f, denoted by Ram_f, is the effective divisor on X corresponding to the section s; we thus have

$$\mathcal{O}_X(\mathrm{Ram}_f) \simeq \omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}.$$

Note that we have

$$\Omega_{X/Y} \simeq f^*(\omega_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_{\mathrm{Ram}_f},$$

hence for every $P \in X$, the coefficient of P in Ram_f is equal to $\ell\big((\Omega_{X/Y})_P\big)$. In order to compute this, we choose local uniformizers $u \in \mathcal{O}_{Y,f(P)}$ and $v \in \mathcal{O}_{X,P}$ and write $\phi^*(u) = v^e w$, where $\phi\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is the local homomorphism induced by f at P and $w \in \mathcal{O}_{X,P}$ is invertible; therefore $e = e_P(f)$. If $\mathfrak{m}_P$ is the maximal ideal in $\mathcal{O}_{X,P}$, then we have an isomorphism $(\Omega_X)_{(P)} \simeq \mathfrak{m}_P/\mathfrak{m}_P^2$ and our choice of v thus implies that $d(v)$ generates the free $\mathcal{O}_{X,P}$-module $(\Omega_X)_P$. Similarly, $d(u)$ generates the free $\mathcal{O}_{Y,f(P)}$-module $(\Omega_Y)_{f(P)}$. The exact sequence (15.2.2) thus implies that $\ell\big((\Omega_{X/Y})_P\big)$ is the smallest non-negative integer m such that $\alpha\big(d(u) \otimes 1\big)$ lies in $(v^m) \cdot d(v)$. Since $\phi(u) = v^e w$, it follows that

$$\alpha\big(d(u) \otimes 1\big) = d\big(\phi(u)\big) = d(v^e w) = ev^{e-1}wd(v) + v^e d(w).$$

This shows that unless $\mathrm{char}(k) = p > 0$ and p divides $e_P(f)$, the coefficient of P in Ram_f is equal to $e_P(f) - 1$; on the other hand, if $\mathrm{char}(k) = p > 0$ and p divides $e_P(F)$, then the only thing we can conclude is that this coefficient is $\geq e_P(f)$.

We are thus led to the following terminology.

DEFINITION 15.2.22. Let $f\colon X \to Y$ be a separable, finite morphism of smooth, projective curves. We say that f is *wildly ramified* at P if $\mathrm{char}(k) = p > 0$ and p divides $e_P(f)$; otherwise, f is *tamely ramified* at P.

We collect in the following theorem, known as the Riemann-Hurwitz theorem, the conclusion of the above discussion:

THEOREM 15.2.23. *If $f\colon X \to Y$ is a finite, separable morphism of smooth, projective curves, then we have an effective divisor* Ram_f *on X such that*

$$\omega_X \simeq f^*(\omega_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mathrm{Ram}_f). \tag{15.2.3}$$

Moreover, if we write $\mathrm{Ram}_f = \sum_{P \in X} a_P P$, *then for every P, we have $a_P = e_P(f) - 1$ if f is tamely ramified at P, and $a_P \geq e_P(f)$ if f is wildly ramified at P.*

COROLLARY 15.2.24. *If $f\colon X \to Y$ is a finite, separable morphism of smooth, projective curves, with g_X and g_Y the genus of X and Y, respectively, then*

$$2g_X - 2 = \deg(f) \cdot (2g_Y - 2) + \deg(\mathrm{Ram}_f).$$

In particular, if f is tamely ramified, we have

$$2g_X - 2 = \deg(f) \cdot (2g_Y - 2) + \sum_{P \in X} \big(e_P(f) - 1\big).$$

PROOF. The assertion follows by computing the degrees on each side of (15.2.3). □

COROLLARY 15.2.25. *If $f\colon X \to Y$ is a finite, separable morphism of smooth, projective curves, with g_X and g_Y the genus of X and Y, respectively, then $g_X \geq g_Y$. Moreover, equality holds if and only if f is an isomorphism, or $X \simeq Y \simeq \mathbf{P}^1$, or both X and Y are elliptic curves.*

PROOF. Since Ram_f is an effective divisor, it follows from the previous corollary that

$$g_X - 1 \geq \deg(f) \cdot (g_Y - 1).$$

If $g_Y = 0$, then it is clear that $g_X \geq g_Y$. Moreover, if $g_X = g_Y$, then $X \simeq Y \simeq \mathbf{P}^1$.

On the other hand, if $g_Y \geq 1$, since $\deg(f) \geq 1$, we conclude that $g_X \geq g_Y$. Moreover, in this case equality implies that either $g_X = g_Y = 1$ or $\deg(f) = 1$ (that is, f is birational, hence an isomorphism). □

15.3. Hyperelliptic curves, Clifford's theorem, and embeddings in $\mathbf{P}^3$

In this section we discuss some further topics concerning algebraic curves.

15.3.1. Hyperelliptic curves. We begin by discussing a new invariant of smooth projective curves: the smallest degree of a finite morphism to $\mathbf{P}^1$. More precisely, we make the following

DEFINITION 15.3.1. Given a smooth, projective curve X, the *gonality* $\mathrm{gon}(X)$ is the smallest degree of a line bundle $\mathcal{L}$ on X such that $h^0(X, \mathcal{L}) \geq 2$.

REMARK 15.3.2. Note that if $\mathcal{L}$ is a line bundle of minimal degree having $h^0(X, \mathcal{L}) \geq 2$, then for every $P \in X$, we have $h^0\big(X, \mathcal{L}(-P)\big) < 2$. First, this implies that $h^0(X, \mathcal{L}) = 2$. Second, by Proposition 15.1.15, we see that $\mathcal{L}$ is globally generated, and thus it defines a finite morphism $f\colon X \to \mathbf{P}^1$ such that $f^*\big(\mathcal{O}_{\mathbf{P}^1}(1)\big) \simeq \mathcal{L}$. Therefore we have $\deg(f) = \deg(\mathcal{L})$.

Conversely, given a finite morphism $f\colon X \to \mathbf{P}^1$, the line bundle $\mathcal{L} = f^*\big(\mathcal{O}_{\mathbf{P}^1}(1)\big)$ clearly satisfies $h^0(X, \mathcal{L}) \geq 2$ and $\deg(\mathcal{L}) = \deg(f)$. It follows that the gonality of X is equal to the smallest degree of a finite morphism $X \to \mathbf{P}^1$.

It follows from the above remark that $\mathrm{gon}(X) = 1$ if and only if $X = \mathbf{P}^1$. The next possible case is that of gonality 2. Note that a curve of genus $g \geq 1$ has gonality 2 if and only if there is a line bundle of degree 2 on X with $h^0(X, \mathcal{L}) \geq 2$ (in which case, this is in fact an equality).

EXAMPLE 15.3.3. Every elliptic curve has gonality 2. Indeed, if $\mathcal{L}$ is a line bundle of degree 2 on an elliptic curve X, then it follows from Remark 15.1.12 that $h^0(X, \mathcal{L}) = 2$.

DEFINITION 15.3.4. A smooth, projective curve of genus $g \geq 2$ is *hyperelliptic* if $\mathrm{gon}(X) = 2$.

EXAMPLE 15.3.5. Every smooth, projective curve of genus $g = 2$ is hyperelliptic. Indeed, we have $h^0(X, \omega_X) = g = 2$ and $\deg(\omega_X) = 2g - 2 = 2$.

Let X be a smooth, projective curve of genus $g \geq 2$. We first note that the line bundle ω_X is globally generated. Indeed, by Proposition 15.1.15, it is enough to show that

$$h^0\big(X, \omega_X(-P)\big) = h^0(X, \omega_X) - 1 \quad \text{for all} \quad P \in X.$$

By Serre duality, we have $h^0\big(X, \omega_X(-P)\big) = h^1\big(X, \mathcal{O}_X(P)\big)$. Since $h^0\big(X, \mathcal{O}_X(P)\big) = 1$ by Example 15.2.13, the Riemann-Roch formula gives

$$h^1\big(X, \mathcal{O}_X(P)\big) = 1 - \deg\big(\mathcal{O}_X(P)\big) + g - 1 = g - 1 = h^0(X, \omega_X) - 1.$$

Moreover, Proposition 15.1.15 implies that ω_X is not very ample if and only if there are $P, Q \in X$ such that

$$h^0\big(X, \omega_X(-P - Q)\big) = g - 1.$$

By Serre duality, this is equivalent to $h^1\big(X, \mathcal{O}_X(P+Q)\big) = g-1$, and by Riemnann-Roch, this is further equivalent to $h^0\big(X, \mathcal{O}_X(P + Q)\big) = 2$. It is clear that we have such P and Q if and only if X is hyperelliptic. We thus proved the following

PROPOSITION 15.3.6. *For every smooth, projective curve of genus $g \geq 2$, the line bundle ω_X is globally generated, and it is very ample if and only if X is not hyperelliptic.*

DEFINITION 15.3.7. A *canonical curve* is a non-hyperelliptic, smooth, projective curve X of genus $g \geq 3$, embedded in $\mathbf{P}^{g-1}$ by the complete linear system $|\omega_X|$.

EXAMPLE 15.3.8. The canonical curves of genus 3 are precisely the smooth plane curves of degree 4.

PROPOSITION 15.3.9. *If X is a hyperelliptic curve of genus $g \geq 2$, then there is a unique line bundle $\mathcal{L}$ on X, up to isomorphism, with $h^0(X, \mathcal{L}) = 2$ and $\deg(\mathcal{L}) = 2$. We have an isomorphism $\omega_X \simeq \mathcal{L}^{g-1}$; moreover, every $E \in |\omega_X|$ can be written as $D_1 + \ldots + D_{g-1}$, where $D_1, \ldots, D_{g-1} \in |\mathcal{L}|$.*

PROOF. Let $\mathcal{L}$ be any line bundle on X with $h^0(X, \mathcal{L}) = 2$ and $\deg(\mathcal{L}) = 2$. For every $P \in X$, we have $h^0\big(X, \mathcal{L}(-P)\big) = 1$, hence there is a unique $\sigma(P) \in X$ such that $P + \sigma(P) \in |\mathcal{L}|$.

Consider the morphism $f\colon X \to \mathbf{P}\big(H^0(X, \omega_X)^\vee\big) \simeq \mathbf{P}^{g-1}$ associated to $|\omega_X|$ and let Y be its image. In this case, we have

$$2g - 2 = \deg(\omega_X) = \deg(f) \cdot \deg(Y). \tag{15.3.1}$$

By definition, Y is non-degenerate in $\mathbf{P}^{g-1}$, hence $\deg(Y) \geq g - 1$ by Example 11.4.11.

We claim that f is not birational, hence $\deg(f) \geq 2$. Indeed, for every $P \in X$, the argument in the proof of Proposition 15.3.6 shows that

$$H^0\big(X, \omega_X(-P)\big) = H^0\big(X, \omega_X(-P - \sigma(P))\big).$$

Therefore one of the following conditions hold:

i) If $\sigma(P) \neq P$, then $f(P) = f\big(\sigma(P)\big)$, hence f is not injective over U, for every open neighborhood U of $f(P)$.
ii) If $\sigma(P) = P$, then df_P is the 0 map, hence f is not an isomorphism over U, for every open neighborhood U of $f(P)$.

This holds for every $P \in X$, hence f is not birational.

Since $\deg(f) \geq 2$, we conclude from (15.3.1) that $\deg(Y) = g-1$ and $\deg(f) = 2$. In particular, it follows from Example 15.2.21 that Y is a rational normal curve in $\mathbf{P}^{g-1}$. We thus have an isomorphism $\phi\colon \mathbf{P}^1 \to Y$ such that $\phi^*\big(\mathcal{O}_Y(1)\big) \simeq \mathcal{O}_{\mathbf{P}^1}(g-1)$. Let us consider the degree 2 morphism $\psi = \phi^{-1} \circ f\colon X \to \mathbf{P}^1$. It is clear that if $\mathcal{M} = \psi^*\big(\mathcal{O}_{\mathbf{P}^1}(1)\big)$, then

$$\mathcal{M}^{g-1} \simeq f^*\big(\mathcal{O}_Y(1)\big) \simeq \omega_X.$$

Moreover, every element of $|\omega_X|$ is of the form $f^*(H|_Y)$ for some hyperplane H in $\mathbf{P}^{g-1}$. Since $\phi^*(H|_Y)$ is a divisor on $\mathbf{P}^1$ of degree $g-1$, it follows that we have $Q_1, \ldots, Q_{g-1} \in \mathbf{P}^1$ such that

$$f^*(H|_Y) = \psi^*(P_1 + \ldots + P_{g-1}) = D_1 + \ldots + D_{g-1}, \quad \text{where} \quad D_i = \psi^*(Q_i) \in |\mathcal{M}|.$$

In order to complete the proof of the proposition, it is enough to show that $\mathcal{L} \simeq \mathcal{M}$. Given any $P \in X$, consider $Q \in X$ such that $P + Q = \psi^*\big(\psi(P)\big)$. This implies that $\psi(P) = \psi(Q)$ and if $P = Q$, then $d\psi_P$ is the 0 map. Since $d\psi_P = 0$ if and only if $df_P = 0$ and similarly, $\psi(P) = \psi(P')$ if and only if $f(P) = f(P')$, it is now straightforward to see that $Q = \sigma(P)$, and thus $\mathcal{M} \simeq \mathcal{L}$. □

15.3.2. The Clifford theorem. When $\mathcal{L}$ is a line bundle on a smooth, projective curve X, of genus g, the Riemann-Roch formula gives the lower bound $h^0(X, \mathcal{L}) \geq \deg(L) - g + 1$, with equality if $h^1(X, \mathcal{L}) = 0$. Similarly, we always have the lower bound $h^1(X, \mathcal{L}) \geq g - 1 - \deg(\mathcal{L})$, with equality if $h^0(X, \mathcal{L}) = 0$. However, when both $h^0(X, \mathcal{L}) > 0$ and $h^1(X, \mathcal{L}) > 0$, we have the following interesting upper bound for $h^0(X, \mathcal{L})$, known as Clifford's theorem.

THEOREM 15.3.10. *If $\mathcal{L}$ is a line bundle on a smooth, projective curve X, of genus g, such that $h^0(X, \mathcal{L}) > 0$ and $h^1(X, \mathcal{L}) > 0$, then*

$$h^0(X, \mathcal{L}) \leq \frac{1}{2}\deg(\mathcal{L}) + 1.$$

Moreover, equality holds if and only if we are in one of the following situations:

i) *$\mathcal{L} \simeq \mathcal{O}_X$ or $\mathcal{L} \simeq \omega_X$ (with $g > 0$).*
ii) *X is a hyperelliptic curve and $\mathcal{L} \simeq \mathcal{M}^r$, with $1 \leq r \leq g-1$, where $\mathcal{M}$ is the unique line bundle of degree 2 on X, with $h^0(X, \mathcal{M}) = 2$.*

In fact, the first part of the theorem holds much more generally. In order to state the result, we introduce the following

DEFINITION 15.3.11. Let U, V, and W be finite-dimensional vector spaces over k. A linear map $\phi\colon U \otimes_k V \to W$ is *1-generic* if for every $u \in U \smallsetminus \{0\}$ and $v \in V \smallsetminus \{0\}$, we have $\phi(u, v) \neq 0$.

EXAMPLE 15.3.12. If X is an irreducible variety and $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free sheaves on X, then multiplication of sections gives a linear map

$$H^0(X, \mathcal{E}_1) \otimes_k H^0(X, \mathcal{E}_2) \to H^0(X, \mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2)$$

which is 1-generic.

PROPOSITION 15.3.13. *If $\phi\colon U\otimes_k V\to W$ is a 1-generic linear map, with U and V non-zero, finite dimensional k-vector spaces, then*

$$\dim_k(W)\geq \dim_k(U)+\dim_k(V)-1.$$

PROOF. If $\ker(\phi)=0$, then

$$\dim(W)\geq \dim(U\otimes_k V)=\dim_k(U)\cdot\dim_k(V)\geq \dim_k(U)+\dim_k(V)-1,$$

since $\dim_k(U),\dim_k(V)\geq 1$. From now on we assume that $\ker(\phi)\neq 0$. Moreover, after replacing W by the image of ϕ, we may assume that ϕ is surjective.

We here switch from our usual convention and denote by $\mathbf{P}(V)$ the projective space parametrizing the *lines* in V. Inside the projective space $\mathbf{P}(U\otimes_k V)$ we have the linear subspace $\mathbf{P}\big(\ker(\phi)\big)$. We also have the image of the Segre embedding

$$\mathbf{P}(U)\times\mathbf{P}(V)\hookrightarrow \mathbf{P}(U\otimes_k V),\quad \big([u],[v]\big)\to[u\otimes v].$$

By assumption, these two subvarieties do not intersect, hence by Corollary 4.2.12, we have

$$\dim\big(\mathbf{P}(\ker(\phi))\big)+\dim\big(\mathbf{P}(U)\times\mathbf{P}(V)\big)\leq \dim\big(\mathbf{P}(U\otimes_k V)\big)-1.$$

We thus obtain

$$\dim_k(U\otimes_k V)-\dim_k(W)-1=\dim\big(\ker(\phi)\big)-1=\dim\Big(\mathbf{P}\big(\ker(\phi)\big)\Big)$$

$$\leq \dim\big(\mathbf{P}(U\otimes_k V)\big)-\dim\big(\mathbf{P}(U)\times\mathbf{P}(V)\big)-1=\dim(U\otimes_k V)-\dim_k(U)-\dim_k(V),$$

which gives the inequality in the proposition. □

By combining this with Example 15.3.12, we obtain the following

COROLLARY 15.3.14. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector bundles on the irreducible variety X, with $H^0(X,\mathcal{E}_1)\neq 0$ and $H^0(X,\mathcal{E}_2)\neq 0$, then*

$$h^0(X,\mathcal{E}_1\otimes_{\mathcal{O}_X}\mathcal{E}_2)\geq h^0(X,\mathcal{E}_1)+h^0(X,\mathcal{E}_2)-1.$$

PROOF OF THEOREM 15.3.10. By assumption, we have $h^0(X,\mathcal{L})>0$ and $h^0(X,\omega_X\otimes_{\mathcal{O}_X}\mathcal{L}^{-1})=h^1(X,\mathcal{L})>0$, hence we may apply Corollary 15.3.14 with $\mathcal{E}_1=\mathcal{L}$ and $\mathcal{E}_2=\omega_X\otimes_{\mathcal{O}_X}\mathcal{L}^{-1}$, to deduce

$$g\geq h^0(X,\mathcal{L})+h^1(X,\mathcal{L})-1. \tag{15.3.2}$$

On the other hand, the Riemann-Roch formula gives

$$\deg(\mathcal{L})-g+1=h^0(X,\mathcal{L})-h^1(X,\mathcal{L}). \tag{15.3.3}$$

By adding (15.3.2) and (15.3.3), we obtain the inequality in the theorem.

The analysis of the equality case follows closely the proof of [**Har77**, Theorem IV.5.4], which in turn is based on an argument of Saint-Donat. It is clear that for $\mathcal{L}=\mathcal{O}_X$ or $\mathcal{L}=\omega_X$, the inequality in the theorem is an equality. Conversely, if $\mathcal{L}$ is a line bundle for which we have equality and $h^0(\mathcal{L})=1$ or $h^1(\mathcal{L})=1$, then $\mathcal{L}\simeq\mathcal{O}_X$ or, respectively, $\mathcal{L}\simeq\omega_X$ (this follows from Examples 15.1.8 and 15.1.11).

Suppose now that $h^0(X,\mathcal{L})=\frac{1}{2}\deg(\mathcal{L})+1$, with $h^0(X,\mathcal{L})\geq 2$ and $h^1(X,\mathcal{L})\geq 2$. We show by induction on $\deg(\mathcal{L})$, which is a positive even number, that X is hyperelliptic. If $\deg(\mathcal{L})=2$, then $h^0(X,\mathcal{L})=2$, hence X is clearly hyperelliptic (note that $g\geq 3$, since $g+1=h^0(X,\mathcal{L})+h^1(X,\mathcal{L})\geq 4$). Suppose now that $\deg(\mathcal{L})\geq 4$. Since $h^1(X,\mathcal{L})\geq 2$, we can find $E\in|\omega_X\otimes_{\mathcal{O}_X}\mathcal{L}^{-1}|$ and $E\neq 0$. We can thus choose $P\in\operatorname{Supp}(E)$ and $Q\in X\smallsetminus\operatorname{Supp}(E)$. Since $h^0(X,\mathcal{L})\geq 3$, it follows that we can choose $D\in|\mathcal{L}|$ such that $D-(P+Q)$ is effective. Let $D'=\min\{D,E\}$;

in other words, if $D = \sum_i a_iP_i$ and $E = \sum_i b_iP_i$, then $D' = \sum_i \min\{a_i, b_i\}P_i$. Note that D' is an effective divisor. By the assumptions on P and Q, we have $P \in \mathrm{Supp}(D')$ and $Q \not\in \mathrm{Supp}(D')$. In particular, we have $0 < \deg(D') < \deg(D)$.

Note that we have a short exact sequence

$$0 \to \mathcal{O}_X(D') \xrightarrow{\alpha} \mathcal{O}_X(D) \oplus \mathcal{O}_X(E) \xrightarrow{\beta} \mathcal{O}_X(D + E - D') \to 0,$$

where we consider all these sheaves as subsheaves of the constant sheaf $k(X)$, with $\alpha(u) = (u, u)$ and $\beta(u, v) = u - v$. For the surjectivity of β, note that every point of X has a neighborhood where $D' = D$ or $D' = E$. By taking global sections, we conclude that

$$h^0\big(X, \mathcal{O}_X(D')\big) + h^1\big(X, \mathcal{O}_X(D')\big) = h^0\big(X, \mathcal{O}_X(D')\big) + h^0\big(X, \mathcal{O}_X(D + E - D')\big)$$

$$\geq h^0\big(X, \mathcal{O}_X(D)\big) + h^0\big(X, \mathcal{O}_X(E)\big) = h^0\big(X, \mathcal{O}_X(D)\big) + h^1\big(X, \mathcal{O}_X(D)\big) = g + 1.$$

On the other hand, we have

$$h^0\big(X, \mathcal{O}_X(D')\big) + h^1\big(X, \mathcal{O}_X(D')\big) \leq g + 1$$

by Corollary 15.3.14, and using the Riemann-Roch formula, we conclude that

$$h^0\big(X, \mathcal{O}_X(D')\big) = \frac{1}{2}\deg(D') + 1.$$

Since $\deg(D') > 0$, we have $h^0\big(X, \mathcal{O}_X(D')\big) \geq 2$, while the fact that $D - D'$ is effective implies, via Serre duality, that

$$h^1\big(X, \mathcal{O}_X(D')\big) \geq h^1\big(X, \mathcal{O}_X(D)\big) \geq 2.$$

The induction hypothesis thus gives that X is hyperelliptic.

Suppose now that X is hyperelliptic and $\mathcal{M}$ is the unique line bundle of degree 2 such that $h^0(X, \mathcal{M}) = 2$. Recall that by Proposition 15.3.9, we have $\omega_X \simeq \mathcal{L}^{g-1}$. Note first that for every i, with $1 \leq i \leq g - 2$, we have $h^0(X, \mathcal{M}^i) = \frac{1}{2}\deg(\mathcal{M}^i) + 1$. Indeed, we have already seen that

$$h^0(X, \mathcal{M}^i) \leq \frac{1}{2}\deg(\mathcal{M}^i) + 1 = i + 1,\ \ h^0(X, \mathcal{M}^{g-1-i}) \leq \frac{1}{2}\deg(\mathcal{M}^{g-1-i}) + 1 = g - i,$$

while Corollary 15.3.14 implies

$$g = h^0(X, \omega_X) \geq h^0(X, \mathcal{M}^i) + h^0(X, \mathcal{M}^{g-1-i}) - 1,$$

which gives our assertion.

Suppose now that X is hyperelliptic and $\mathcal{L}$ is an arbitrary line bundle with $h^0(X, \mathcal{L}) \geq 2$, $h^1(X, \mathcal{L}) \geq 2$, and $h^0(X, \mathcal{L}) = \frac{1}{2}\deg(\mathcal{L}) + 1$. If $i = \frac{1}{2}\deg(\mathcal{L})$, consider the line bundle $\mathcal{L}' = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}^{g-1-i}$. Note that $\deg(\mathcal{L}') = 2g - 2$, while Corollary 15.3.14 gives

$$h^0(X, \mathcal{L}') \geq h^0(X, \mathcal{L}) + h^0(X, \mathcal{M}^{g-1-i}) - 1 = (i + 1) + (g - i) - 1 = g,$$

hence we deduce from Example 15.1.11 that $\mathcal{L}' \simeq \omega_X$, and thus $\mathcal{L} \simeq \mathcal{M}^i$. This completes the proof. $\square$

15.3.3. Embeddings in $\mathbf{P}^3$. We now want to show that every smooth, projective curve, admits a closed immersion to $\mathbf{P}^3$. More generally, we will prove the following

THEOREM 15.3.15. *If X is a smooth, irreducible, projective variety of dimension n, then there is a closed immersion $X \hookrightarrow \mathbf{P}^{2n+1}$.*

Since every projective variety can be embedded in a projective space, the theorem will follow from the following more precise statement:

PROPOSITION 15.3.16. *Let X be a smooth, irreducible, closed subvariety of $\mathbf{P}^N$. If $N > 2n+1$, where $n = \dim(X)$, and $p \in \mathbf{P}^N \smallsetminus X$ is a general point, then the projection from p induces a closed immersion $X \hookrightarrow \mathbf{P}^{N-1}$.*

We begin with some preparations. For an irreducible, closed subvariety $X \subseteq \mathbf{P}^N$, the *secant variety* $\mathrm{Sec}(X)$ is the closure of the union of all lines in $\mathbf{P}^N$ joining two distinct points of X, that is,

$$\mathrm{Sec}(X) = \overline{\bigcup_{(p,q)\in X\times X\smallsetminus\Delta_X} \langle p,q\rangle},$$

where $\langle p,q\rangle$ denotes the line spanned by the distinct points $p, q \in \mathbf{P}^N$.

Let $G = G(2, N+1)$ be the Grassmann variety parametrizing lines in $\mathbf{P}^N$. Note that we have a map

$$\Phi\colon \mathbf{P}^N \times \mathbf{P}^N \smallsetminus \Delta_{\mathbf{P}^N} \to G,$$

that maps (p,q) to the line in $\mathbf{P}^N$ spanned by p and q. It is straightforward to see that this is a morphism: if $p = [a_0, \ldots, a_N]$ and $q = [b_0, \ldots, b_N]$, then $\Phi(p,q)$ is the point in G corresponding to the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_N \\ b_0 & b_1 & \ldots & b_N \end{pmatrix}.$$

If $\dim(X) = n$, then it is clear that the closure $\mathcal{S}ec(X)$ of $\Phi(X \times X \smallsetminus \Delta_X)$ in G is irreducible, of dimension $\leq 2n$.

Recall that we have the incidence correspondence $\mathcal{Z} \subseteq \mathbf{P}^N \times G$ consisting of those pairs $\big(q, [L]\big)$, with $q \in L$. The projections onto the 2 components induce morphisms

$$\pi_1\colon \mathcal{Z} \to \mathbf{P}^N \quad \text{and} \quad \pi_2\colon \mathcal{Z} \to G.$$

Note that we have

$$\mathrm{Sec}(X) = \overline{\pi_1\big(\pi_2^{-1}(\Phi(X \times X \smallsetminus \Delta_X))\big)} = \pi_1\big(\pi_2^{-1}(\mathcal{S}ec(X))\big),$$

where the first equality follows from the definition of the secant variety and the second one follows using the fact that π_1 is a closed map and π_2 is locally trivial, with fiber $\mathbf{P}^1$ (see Exercise 7.1.10). Since $\mathcal{S}ec(X)$ is irreducible, of dimension $\leq 2n$, and π_2 is locally trivial with fiber $\mathbf{P}^1$, it follows that $\pi_2^{-1}\big(\mathcal{S}ec(X)\big)$ is irreducible, of dimension $\leq 2n+1$, and thus $\mathrm{Sec}(X)$ is irreducible, of dimension $\leq 2n+1$.

Suppose now that $X \subseteq \mathbf{P}^N$ is a smooth, irreducible subvariety of dimension n. The *tangent variety* $\mathrm{Tan}(X)$ is defined by

$$\mathrm{Tan}(X) = \bigcup_{q\in X} \mathbf{T}_q X \subseteq \mathbf{P}^N.$$

For now, this is just a subset of $\mathbf{P}^N$; we will see momentarily that it is a closed subset.

As above, we consider the Grassmann variety $G' = G(n+1, N+1)$ parametrizing n-dimensional linear subspaces of $\mathbf{P}^N$. The assumption on X implies that we can define a map $\Psi\colon X \to G'$ by $\Psi(q) = \mathbf{T}_q X$. It is easy to see that this is a morphism: if $f_1, \ldots, f_r$ generate the homogeneous ideal corresponding to X, then $\Psi(q)$ is given by the kernel of the linear map $k^{N+1} \to k^r$ corresponding to the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(q)\right)$. This matrix has rank $N-n$ at every point and on each open subset of $\mathbf{P}^N$ where a certain minor of the matrix in non-zero, we can write $\Psi(q)$ explicitly using Cramer's rule.

Let us consider again the incidence correspondence $\mathcal{Z}' \hookrightarrow \mathbf{P}^N \times G'$, with the two maps

$$\pi_1\colon \mathcal{Z}' \to \mathbf{P}^N \quad \text{and} \quad \pi_2\colon \mathcal{Z}' \to G'$$

induced by the two projections. It is clear that

$$\mathrm{Tan}(X) = \pi_1\left(\pi_2^{-1}\big(\Psi(X)\big)\right).$$

Since both Ψ and π_1 are closed maps, it follows that $\mathrm{Tan}(X)$ is a closed subset of $\mathbf{P}^N$. Moreover, it is clear that $\Psi(X)$ is irreducible, of dimension $\leq n$. Since π_2 is locally trivial, with fiber $\mathbf{P}^n$, it follows that $\pi_2^{-1}\big(\Psi(X)\big)$ is irreducible, of dimension $\leq 2n$, and thus $\mathrm{Tan}(X)$ is irreducible, of dimension $\leq 2n$. We collect the conclusion of the above discussion in the following

PROPOSITION 15.3.17. *Let $X \subseteq \mathbf{P}^n$ be an irreducible, closed subvariety, of dimension n.*

i) *The secant variety* $\mathrm{Sec}(X)$ *is irreducible, of dimension* $\leq 2n+1$.
ii) *If X is smooth, then the tangent variety* $\mathrm{Tan}(X)$ *is irreducible, of dimension* $\leq 2n$.

We can now prove the main result of this section.

PROOF OF PROPOSITION 15.3.16. Since $N > 2n+1$, it follows from Proposition 15.3.17 that both $\mathrm{Sec}(X)$ and $\mathrm{Tan}(X)$ are proper closed subsets of $\mathbf{P}^N$. We choose $p \in \mathbf{P}^N \smallsetminus \big(\mathrm{Sec}(X) \cup \mathrm{Tan}(X)\big)$, let H be a hyperplane in $\mathbf{P}^N$ that does not contain p, and let $\pi\colon \mathbf{P}^N \smallsetminus \{p\} \to H \simeq \mathbf{P}^{N-1}$ be the projection map. We claim that the induced morphism $f\colon X \to H$ is a closed immersion.

It follows from Proposition 11.5.18 (see also its proof) that f is a closed immersion if and only if f is injective and df_q is injective for every $q \in X$. Recall that by definition, $f(q)$ is the intersection of H with the line spanned by p and q. It follows that f is not injective if and only if there are two distinct points $q_1, q_2 \in X$ such that p lies on the line spanned by q_1 and q_2. However, in this case we have $p \in \mathrm{Sec}(X)$, a contradiction. This shows that f is injective.

We similarly argue to show that df_q is injective for every $q \in X$. Note that for every $q \in \mathbf{P}^n \smallsetminus \{p\}$, we may consider $T_q\mathbf{P}^n \subseteq \mathbf{T}_q\mathbf{P}^n = \mathbf{P}^n$ and $T_{\pi(q)}H \subseteq \mathbf{T}_{\pi(q)}H = H$. It is easy to see that the linear map $d\pi_q\colon T_q\mathbf{P}^n \to T_{\pi(q)}H$ extends to the rational map $\mathbf{T}_q\mathbf{P}^n \dashrightarrow \mathbf{T}_{\pi(q)}H$, which is precisely π. By assumption, for every $q \in X$, we have $p \not\in \mathbf{T}_q X$, hence π induces an injective map $\mathbf{T}_q X \to \mathbf{T}_{f(q)}H = H$. This implies that df_q is injective, completing the proof. $\square$

We end this section with two examples, showing that certain embeddings in $\mathbf{P}^3$ are complete intersections.

EXAMPLE 15.3.18. Let X be a smooth elliptic curve and $\mathcal{L}$ a line bundle of degree 4 on X. It follows from Remark 15.1.12 that $h^0(X, \mathcal{L}) = 4$, while Corollary 15.1.16 implies that $\mathcal{L}$ is very ample. Consider the closed immersion $X \hookrightarrow \mathbf{P}^3$ given by the complete linear series $|\mathcal{L}|$. Note that by Example 15.1.9, we have $\deg(X) = 4$. Since $\deg(X) = 4$ and $p_a(X) = 1$, it follows that the Hilbert polynomial of X is $P_X(t) = 4t$. Let $\mathcal{I}_X$ denote the ideal sheaf in $\mathcal{O}_{\mathbf{P}^3}$ corresponding to X.

By construction, X is not contained in any hyperplane in $\mathbf{P}^3$, that is, we have $H^0\big(\mathbf{P}^3, \mathcal{I}_X(1)\big) = 0$. By tensoring with $\mathcal{O}_{\mathbf{P}^3}(2)$ the short exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_X \to 0$$

and taking global sections, we obtain an exact sequence

$$0 \to H^0\big(\mathbf{P}^3, \mathcal{I}_X(2)\big) \to H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)\big) \to H^0\big(X, \mathcal{O}_X(2)\big) = H^0(X, \mathcal{L}^2).$$

Note that $h^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)\big) = \binom{5}{2} = 10$, while Remark 15.1.12 gives $h^0(X, \mathcal{L}^2) = 8$, hence using the above exact sequence we get $h^0\big(\mathbf{P}^3, \mathcal{I}_X(2)\big) \geq 2$. We thus have two linearly independent polynomials Q_1 and Q_2, homogeneous of degree 2, such that $X \subseteq V(Q_1, Q_2)$. Note that since X is not contained in any hyperplane, both Q_1 and Q_2 are irreducible. Since $V(Q_1)$ and $V(Q_2)$ have no common irreducible components, it follows that $\operatorname{codim}_{\mathbf{P}^3}\big(V(Q_1, Q_2)\big) = 1$. It follows from Remark 12.3.22 that Q_1, Q_2 form a regular sequence and they generate a saturated ideal in $S = k[x_0, x_1, x_2, x_3]$. Let $\mathcal{J}$ be the ideal sheaf corresponding to (Q_1, Q_2). Since the Hilbert polynomial of a quadric in $\mathbf{P}^3$ is given by

$$\binom{t+3}{3} - \binom{t+1}{3} = (t+1)^2$$

(see Example 11.4.6), it follows from Example 11.4.10 that

$$P_{\mathcal{O}_{\mathbf{P}^3}/\mathcal{J}}(t) = (t+1)^2 - (t-1)^2 = 4t.$$

This implies that $\mathcal{J} = \widetilde{I_X}$, and since both I_X and (Q_1, Q_2) are saturated, we conclude that $I_X = (Q_1, Q_2)$.

Conversely, if $X \subseteq \mathbf{P}^3$ is a smooth curve which is a complete intersection of 2 quadrics, then it follows from Example 15.1.20 that X has genus 1. Since $\mathcal{O}_X(1)$ is a line bundle of degree 4, we have $h^0\big(X, \mathcal{O}_X(1)\big) = 4$, and since X does not lie in any hyperplane, it follows that X is embedded in $\mathbf{P}^3$ by the complete linear system $|\mathcal{O}_X(1)|$.

EXAMPLE 15.3.19. Suppose now that X is a non-hyperelliptic smooth, projective curve of genus 4. In this case, we know that ω_X is a very ample line bundle, and the complete linear system $|\omega_X|$ gives a closed immersion $X \hookrightarrow \mathbf{P}^3$. Again, we know that X is not contained in any hyperplane, and we consider the short exact sequence

$$0 \to H^0\big(\mathbf{P}^3, \mathcal{I}_X(2)\big) \to H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)\big) \to H^0\big(X, \mathcal{O}_X(2)\big) = H^0(X, \omega_X^2).$$

Since $\deg(\omega_X^2) = 12 \geq 2 \cdot 4 - 1$, we have $h^1(\omega_X^2) = 0$ and the Riemann-Roch formula gives $h^0(X, \omega_X^2) = 9$. Using the fact that $h^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)\big) = 10$, we deduce from the above exact sequence that there is a homogeneous polynomial g of degree 2 that vanishes on X. Note that this is irreducible, since X is non-degenerate. Consider now the similar exact sequence

$$0 \to H^0\big(\mathbf{P}^3, \mathcal{I}_X(3)\big) \to H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)\big) \to H^0\big(X, \mathcal{O}_X(3)\big) = H^0(X, \omega_X^3).$$

We compute $h^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)\big) = 20$ and $h^0(X, \omega_X^3) = 15$, and we conclude that $h^0\big(\mathbf{P}^3, \mathcal{I}_X(3)\big) \geq 5$. This implies that there is a homogeneous polynomial h of degree 3 that vanishes on X and such that h is not of the form $g\ell$, for any linear form ℓ. We deduce that $\mathrm{codim}_{\mathbf{P}^3}\big(V(g,h)\big) = 1$. Arguing as in the previous example, we see that in fact $I_X = (g,h)$, hence X is a complete intersection.

Conversely, if $X \subseteq \mathbf{P}^3$ is a smooth curve which is a complete intersection of hypersurfaces of degrees 2 and 3, then it follows from Example 15.1.20 that X has genus 4. Moreover, we have $\omega_X \simeq \mathcal{O}_X(1)$ by Example 12.3.21, hence $h^0\big(X, \mathcal{O}_X(1)\big) = 4$. Since X is non-degenerate, it follows that it is embedded by the complete linear series $|\omega_X|$. Since ω_X is ample, we see that X is not hyperelliptic.

CHAPTER 16

Intersection numbers of line bundles

Our goal in this chapter is to define and study the basic properties of intersection numbers of line bundles on complete varieties. We follow here [**Kle66**]. We discuss in detail the case of surfaces, proving the Riemann-Roch formula in this context and the Hodge index theorem. Finally, we end this chapter with a proof of the Nakai-Moishezon ampleness criterion.

16.1. Intersection numbers

The starting point is the following result of Snapper, a wide generalization of the existence of Hilbert polynomials.

THEOREM 16.1.1. *If $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on a complete variety X and $\mathcal{F}$ is a coherent sheaf on X, then the function*

$$\mathbf{Z}^r \ni (m_1, \ldots, m_r) \to \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) \in \mathbf{Z}$$

is polynomial, of total degree $\leq \dim\big(\mathrm{Supp}(\mathcal{F})\big)$.

We begin with a general lemma.

LEMMA 16.1.2. *If $\mathcal{F}$ is a coherent sheaf on a variety X, then there is a filtration*

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_r = \mathcal{F},$$

such that for every i, with $1 \leq i \leq r$, the ideal sheaf $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}_i/\mathcal{F}_{i-1}) \subseteq \mathcal{O}_X$ is the radical ideal sheaf corresponding to a closed irreducible subvariety of X.

PROOF. We argue by Noetherian induction on $\mathrm{Supp}(\mathcal{F})$. We have seen in Remark 8.4.21 that if $\mathcal{I}$ is the radical ideal sheaf defining $\mathrm{Supp}(\mathcal{F})$, then we have a finite filtration of $\mathcal{F}$ such that $\mathcal{I}$ annihilates each successive quotient. If $V(\mathcal{I}) \neq X$, then by the inductive assumption, each of these successive quotients has a filtration as in the lemma, and thus $\mathcal{F}$ has one as well. Therefore we may and will assume that $V(\mathcal{I}) = X$, that is, $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}) = 0$.

Let $X_1, \ldots, X_n$ be the irreducible components of X. If $n = 1$, then we are done. Otherwise, if $\mathcal{I}_j$ is the radical ideal sheaf defining X_j in X, then we have $\mathcal{I}_1 \cap \ldots \cap \mathcal{I}_n = 0$. Since $\mathcal{I}_1\mathcal{F}$ is annihilated by $\mathcal{I}_2 \cap \ldots \cap \mathcal{I}_n \neq 0$ and $\mathcal{F}/\mathcal{I}_1\mathcal{F}$ is annihilated by $\mathcal{I}_1 \neq 0$, it follows from the inductive assumption that both $\mathcal{I}_1\mathcal{F}$ and $\mathcal{F}/\mathcal{I}_1\mathcal{F}$ have filtrations as in the lemma, hence so does $\mathcal{F}$, $\square$

PROOF OF THEOREM 16.1.1. We argue by induction on $d = \dim\big(\mathrm{Supp}(\mathcal{F})\big)$. The assertion is clear if $d = -1$ (we here make the convention that $\dim(\emptyset) = -1$ and the zero polynomial has degree -1).

We note that if we have a short exact sequence of coherent sheaves on X

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0,$$

then for every $m_1, \ldots, m_r \in \mathbf{Z}$, we have

$$\chi(X, \mathcal{G} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) = \chi(X, \mathcal{G}' \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) + \chi(X, \mathcal{G}'' \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}).$$

This implies that if two of these functions are polynomial, of degree $\leq d$, then the third one satisfies the same property, too.

In particular, it follows from the lemma that we may assume that $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the radical ideal sheaf defining an irreducible closed subvariety Y of X. After replacing X by Y, we may thus assume that X is irreducible and $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}) = 0$, hence $d = \dim(X)$.

Suppose first that X is projective. In this case, it follows from Remark 11.6.10 that we can write $\mathcal{L}_1 \simeq \mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2^{-1}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are very ample line bundles. Let $A \in |\mathcal{M}_1|$ and $B \in |\mathcal{M}_2|$ be general elements, so that, in particular, A and B do not contain any associated subvarieties of $\mathcal{F}$. First, this gives exact sequences

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-B) \otimes \mathcal{L}_1^{m_1-1} \to \mathcal{F} \otimes \mathcal{L}_1^{m_1} \to \mathcal{F} \otimes \mathcal{O}_A \otimes \mathcal{L}_1^{m_1} \to 0$$

and

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-B) \otimes \mathcal{L}_1^{m_1-1} \to \mathcal{F} \otimes \mathcal{L}_1^{m_1-1} \to \mathcal{F} \otimes \mathcal{O}_B \otimes \mathcal{L}_1^{m_1-1} \to 0.$$

By tensoring these with $\mathcal{L}_2^{m_2} \otimes \ldots \otimes \mathcal{L}_r^{m_r}$ and taking the long exact sequences in cohomology, we obtain using the additivity of the Euler-Poincaré characteristic

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{m_1-1} \otimes \ldots \otimes \mathcal{L}_r^{m_r})$$
$$= \chi(X, \mathcal{F} \otimes \mathcal{O}_A \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) - \chi(X, \mathcal{F} \otimes \mathcal{O}_B \otimes \mathcal{L}_1^{m_1-1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}).$$

Second, it is clear that $\dim\big(\mathrm{Supp}(\mathcal{F} \otimes \mathcal{O}_A)\big) \leq d-1$ and $\dim\big(\mathrm{Supp}(\mathcal{F} \otimes \mathcal{O}_B)\big) \leq d-1$. It follows from these inequalities and the inductive assumption that the function

$$\mathbf{Z}^r \ni (m_1, \ldots, m_n) \to \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) - \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1-1} \otimes \ldots \otimes \mathcal{L}_r^{m_r})$$

is polynomial of total degree $\leq (d-1)$. Since the same assertion clearly also holds with respect to the other variables, it is now easy to deduce, using Lemma 11.4.2, that the function

$$\mathbf{Z}^r \ni (m_1, \ldots, m_r) \to \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r})$$

is polynomial, of total degree $\leq d$. This completes the proof in the case when X is projective.

In the general case we use Chow's lemma to obtain a birational morphism $f \colon Y \to X$, where Y is an irreducible projective variety. Let $\mathcal{G} = f^*(\mathcal{F})$ and consider the canonical morphism $\phi \colon \mathcal{F} \to f_*(\mathcal{G})$. Note that $f_*(\mathcal{G})$ is coherent since f is proper; moreover, since f is birational, it follows that ϕ is an isomorphism on some open subset of X, hence

$$\dim\big(\mathrm{coker}(\phi)\big) \leq d-1 \quad \text{and} \quad \dim\big(\ker(\phi)\big) \leq d-1.$$

Using the exact sequences

$$0 \to \mathrm{Im}(\phi) \to f_*(\mathcal{G}) \to \mathrm{coker}(\phi) \to 0$$

and

$$0 \to \ker(\phi) \to \mathcal{F} \to \mathrm{Im}(\phi) \to 0,$$

the observation at the beginning of the proof, and the inductive assumption, we see that it is enough to prove that the function

$$\mathbf{Z}^r \ni (m_1, \ldots, m_r) \to \chi\big(X, f_*(\mathcal{G}) \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}\big)$$

is polynomial, of total degree $\leq d$.

The projective case implies that the function

$$\mathbf{Z}^r \ni (m_1, \ldots, m_r) \to \chi\big(Y, \mathcal{G} \otimes f^*(\mathcal{L}_1)^{m_1} \otimes \ldots \otimes f^*(\mathcal{L}_r)^{m_r}\big)$$

is polynomial, of total degree $\leq d$. On the other hand, the Leray spectral sequence, the additivity of the Euler-Poincaré characteristic, and the projection formula give

$$\chi\big(Y, \mathcal{G} \otimes f^*(\mathcal{L}_1)^{m_1} \otimes \ldots \otimes f^*(\mathcal{L}_r)^{m_r}\big) = \sum_{p \geq 0} (-1)^p \chi\big(X, R^p f_*(\mathcal{G}) \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}\big).$$

Note that the sum on the right-hand side is a finite sum. Moreover, for every $p > 0$, we have $\dim\big(\mathrm{Supp}(R^p f_*(\mathcal{G}))\big) \leq d-1$ (we use again the fact that f is birational), hence the inductive assumption implies that the function

$$\mathbf{Z}^r \ni (m_1, \ldots, m_r) \to \chi\big(X, R^p f_*(\mathcal{G}) \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}\big)$$

is polynomial, of total degree $\leq d-1$. We thus conclude that the corresponding function for $p = 0$ is also polynomial, of total degree $\leq d$, completing the proof of the theorem. $\square$

DEFINITION 16.1.3. Let X be a complete variety. If $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on X and $\mathcal{F}$ is a coherent sheaf on X with $\dim\big(\mathrm{Supp}(\mathcal{F})\big) \leq r$, then the *intersection number* $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F})$ is defined as the coefficient of $m_1 \cdots m_r$ in the polynomial $P(m_1, \ldots, m_r)$ such that $P(m_1, \ldots, m_r) = \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r})$ for all $(m_1, \ldots, m_r) \in \mathbf{Z}^r$.

If $\mathcal{F} = \mathcal{O}_Y$, for a closed subvariety Y of X, then we write $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r \cdot Y)$ instead of $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{O}_Y)$ and simply $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r)$ if $Y = X$. Of course, we have

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r \cdot Y) = (\mathcal{L}_1|_Y \cdot \ldots \cdot \mathcal{L}_r|_Y).$$

Furthermore, if $\mathcal{L}_1 = \ldots = \mathcal{L}_r = \mathcal{L}$, then we write $(\mathcal{L}^r; \mathcal{F})$, $(\mathcal{L}^r \cdot Y)$ and $(\mathcal{L}^r)$ for the corresponding intersection numbers. If $D_1, \ldots, D_r$ are Cartier divisors on an irreducible complete variety X and $\mathcal{F}$ is as above, then we also write $(D_1 \cdot \ldots \cdot D_r; \mathcal{F})$ for $\big(\mathcal{O}_X(D_1) \cdot \ldots \cdot \mathcal{O}_X(D_r); \mathcal{F}\big)$ and similarly for the other variants of intersection numbers.

The following elementary lemma will allow us to describe the intersection numbers as alternating sums of Euler-Poincaré characteristics.

LEMMA 16.1.4. *If $P \in R[x_1, \ldots, x_r]$ is a polynomial with coefficients in a ring R such that the total degree of P is $\leq r$, then the coefficient of $x_1 \cdots x_r$ in P is equal to*

$$\sum_{J \subseteq \{1, \ldots, r\}} (-1)^{|J|} P(\delta_{J,1}, \ldots, \delta_{J,r}),$$

where the sum is over all subsets J of $\{1, \ldots, r\}$ (including the empty subset) and where $\delta_{J,j} = -1$ if $j \in J$ and $\delta_{J,j} = 0$ if $j \notin J$.

PROOF. The assertion follows by induction on r, the case $r = 1$ being trivial. For the induction step, it is enough to show that the coefficient of $x_1 \cdots x_r$ in P is equal to the coefficient of $x_1 \cdots x_{r-1}$ in

$$Q(x_1, \ldots, x_{r-1}) = P(x_1, \ldots, x_{r-1}, 0) - P(x_1, \ldots, x_{r-1}, -1),$$

whose total degree is $\leq (r-1)$. This in turn follows by considering the effect of taking the difference on the right-hand side for each of the monomials in P. $\square$

COROLLARY 16.1.5. *If $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on a complete variety X and $\mathcal{F}$ is a coherent sheaf on X with* $\dim\big(\mathrm{Supp}(\mathcal{F})\big) \leq r$, *then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = \sum_{J \subseteq \{1,\ldots,r\}} (-1)^{|J|} \chi\big(\mathcal{F} \otimes (\otimes_{j\in J} \mathcal{L}_j^{-1})\big).$$

In the next proposition we give the basic properties of intersection numbers.

PROPOSITION 16.1.6. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be line bundles on the complete variety X and $\mathcal{F}$ a coherent sheaf on X, with* $\dim\big(\mathrm{Supp}(\mathcal{F})\big) \leq r$.

i) *If* $\dim\big(\mathrm{Supp}(\mathcal{F})\big) < r$, *then* $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = 0$.

ii) *The intersection number $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F})$ is an integer. The map*

$$\mathrm{Pic}(X)^r \ni (\mathcal{L}_1, \ldots, \mathcal{L}_r) \to (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) \in \mathbf{Z}$$

is multilinear and symmetric.

iii) *If $Y_1, \ldots, Y_s$ are the r-dimensional irreducible components of* $\mathrm{Supp}(\mathcal{F})$, *then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = \sum_{i=1}^{s} \ell_{\mathcal{O}_{X,Y_i}}(\mathcal{F}_{Y_i}) \cdot (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r \cdot Y_i). \tag{16.1.1}$$

iv) *(Projection formula) Suppose that $f\colon X \to Y$ is a surjective morphism of complete irreducible varieties, with* $\dim(X) \leq r$. *Given line bundles $\mathcal{M}_i$ on Y such that $\mathcal{L}_i \simeq f^*(\mathcal{M}_i)$ for every i, we have $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r) = d \cdot (\mathcal{M}_1 \cdot \ldots \cdot \mathcal{M}_r)$ if f is generically finite of degree d, and $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r) = 0$, otherwise.*

v) *If X is irreducible and $\mathcal{L}_r = \mathcal{O}_X(D)$ for some effective Cartier divisor D that does not contain any associated subvariety of $\mathcal{F}$, then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1}; \mathcal{F} \otimes \mathcal{O}_D),$$

with the convention that when $r = 1$, the right-hand side is equal to $h^0(X, \mathcal{F} \otimes \mathcal{O}_D)$.

PROOF. The assertion in i) follows from definition and Theorem 16.1.1. The fact that intersection numbers are integers is clear by Corollary 16.1.5. The symmetry of the application in ii) is obvious, hence in order to prove ii) we only need to show that

$$\big((\mathcal{L}_1 \otimes \mathcal{L}_1') \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}\big) - (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) - (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = 0. \tag{16.1.2}$$

An easy computation using the formula in Corollary 16.1.5 shows that the difference in (16.1.2) is equal to $-(\mathcal{L}_1 \cdot \mathcal{L}_1' \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F})$, which vanishes by i).

We note that iii) clearly holds if $\dim(\mathrm{Supp}(\mathcal{F})) < r$. It follows from definition and the additivity of the Euler-Poincaré characteristic that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of coherent sheaves on X, then

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}') + (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F}''). \tag{16.1.3}$$

Since $\ell_{\mathcal{O}_{X,Y_i}}(\mathcal{F}_{Y_i}) = \ell_{\mathcal{O}_{X,Y_i}}(\mathcal{F}'_{Y_i}) + \ell_{\mathcal{O}_{X,Y_i}}(\mathcal{F}''_{Y_i})$ for every i, we conclude that if (16.1.3) holds for two of $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$, then it also holds for the third one.

Recall that by Lemma 16.1.2, $\mathcal{F}$ has a finite filtration such that the annihilator of each of the successive quotients is the radical ideal corresponding to an irreducible closed subset of X. We conclude that in order to prove (16.1.2), we may assume that X is an irreducible variety. We also see that if $\mathcal{G}$ is another sheaf such that we have a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ that is an isomorphism over a non-empty open subset of X, then (16.1.2) holds for $\mathcal{F}$ if and only if it holds for $\mathcal{G}$ (note that in this case both $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are supported in dimension $< r$). In particular, by replacing $\mathcal{F}$ by $\mathcal{F} \otimes \mathcal{O}_X(D)$, where D is a suitable effective divisor, with $\mathcal{O}_X(D)$ ample, we may assume that $\mathcal{F}$ is generated by global sections. If d is the dimension over $k(X)$ of the stalk of $\mathcal{F}$ with respect to X, and $s_1, \ldots, s_d \in \Gamma(X, \mathcal{F})$ are general sections, then the induced morphism $\mathcal{O}_X^{\oplus d} \to \mathcal{F}$ is an isomorphism on some non-empty open subset of X. Since (16.1.2) clearly holds for $\mathcal{O}_X^{\oplus d}$, this completes the proof of iii).

In order to prove iv), note first that the additivity of the Euler-Poincaré characteristic, the Leray spectral sequence, and the projection formula imply that

$$\chi(X, \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r}) = \sum_{i \geq 0} (-1)^i \chi\big(Y, R^i f_*(\mathcal{O}_X) \otimes \mathcal{M}_1^{m_1} \otimes \ldots \otimes \mathcal{M}_r^{m_r}\big),$$

hence by definition of intersection numbers we have

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r) = \sum_{i \geq 0} (-1)^i \big(\mathcal{M}_1 \cdot \ldots \cdot \mathcal{M}_r; R^i f_*(\mathcal{O}_X)\big).$$

If f is not generically finite, then all intersection numbers on the right-hand side are zero since $\dim(Y) < r$. Suppose now that f is generically finite and $\deg(f) = d$. In this case we have an open subset $U \subseteq Y$ such that f has finite fibers over U, hence it is finite by Corollary 14.1.8. This implies that for all $i \geq 1$, the support of $R^i f_*(\mathcal{O}_X)$ is contained in $Y \smallsetminus U$, while the dimension over $k(Y)$ of the stalk of $f_*(\mathcal{O}_X)$ with respect to Y is d. The formula in iv) now follows from iii) and i).

In order to prove v), we use Corollary 16.1.5 by considering first the subsets contained in $\{1, \ldots, r-1\}$, and then the ones contaning r. We obtain

$$\begin{aligned}\big(\mathcal{L}_1 \cdot \ldots \mathcal{L}_{r-1} \cdot \mathcal{O}_X(D); \mathcal{F}\big) &= \sum_{J \subseteq \{1, \ldots, r-1\}} (-1)^{|J|} \chi\big(X, \mathcal{F} \otimes (\otimes_{i \in J} \mathcal{L}_i^{-1})\big) \\ &+ \sum_{J \subseteq \{1, \ldots, r-1\}} (-1)^{|J|+1} \chi\big(\mathcal{F} \otimes \mathcal{O}_X(-D) \otimes (\otimes_{i \in J} \mathcal{L}_i^{-1})\big) \\ &= \sum_{J \subseteq \{1, \ldots, r-1\}} (-1)^{|J|} \chi\big((\otimes_{i \in J} \mathcal{L}_i^{-1}) \otimes \mathcal{F} \otimes \mathcal{O}_D\big) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1}; \mathcal{F} \otimes \mathcal{O}_D),\end{aligned}$$

where the second equality follows by tensoring the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

with $\mathcal{F} \otimes (\otimes_{i \in J} \mathcal{L}_i^{-1})$, and using the additivity of the Euler-Poincaré characteristic. This completes the proof of the proposition. $\square$

REMARK 16.1.7. Suppose that X is an n-dimensional Cohen–Macaulay, irreducible variety, and $D_1, \ldots, D_n$ are effective Cartier divisors on X such that

$$\dim(D_1 \cap \ldots \cap D_i) = n - i \quad \text{for} \quad 1 \leq i \leq n.$$

In this case, at every point $x \in D_1 \cap \ldots \cap D_i$, the local equations of $D_1, \ldots, D_i$ form a regular sequence in $\mathcal{O}_{X,x}$. By successively applying assertion v) in the above proposition, we see that

$$(D_1 \cdot \ldots \cdot D_n) = (D_2 \cdot \ldots \cdot D_n; \mathcal{O}_{D_1}) = \ldots = h^0(X, \mathcal{O}_{D_1} \otimes \ldots \otimes \mathcal{O}_{D_n}).$$

If, in addition, the intersection points of $D_1, \ldots, D_n$ are smooth points of X and of each of the D_i and the intersection is transversal, then $(D_1 \cdot \ldots \cdot D_n)$ is equal to the number of points in $D_1 \cap \ldots \cap D_n$.

REMARK 16.1.8. It is easy to see that properties i)–v) in Proposition 16.1.6 uniquely determine the intersection numbers $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r; \mathcal{F})$. Indeed, we argue by induction on r. It follows from iii) that a general such intersection number is determined if we know the intersection numbers of the form $(\mathcal{L}_1 \cdot \ldots \mathcal{L}_r)$ when X is an r-dimensional complete variety. Moreover, by Chow's lemma we can find a birational morphism $f\colon W \to X$, with W a projective variety, and property iv) gives $(\mathcal{L}_1 \cdot \ldots \mathcal{L}_r) = \big(f^*(\mathcal{L}_1) \cdot \ldots \cdot f^*(\mathcal{L}_r)\big)$. Therefore we may assume that X is projective. By multilinearity, if we write $\mathcal{L}_r \simeq \mathcal{O}_X(A - B)$, with A and B effective Cartier divisors, then

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r) = \big(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1} \cdot \mathcal{O}_X(A)\big) - \big(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1} \cdot \mathcal{O}_X(B)\big).$$

On the other hand, property v) gives

$$\big(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1} \cdot \mathcal{O}_X(A)\big) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1}; \mathcal{O}_A) \quad \text{and}$$
$$\big(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1} \cdot \mathcal{O}_X(B)\big) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{r-1}; \mathcal{O}_B),$$

hence we are thus done by induction.

REMARK 16.1.9. If $Q(x)$ is a polynomial in one variable of degree d and we consider the polynomial in r variables $P(x_1, \ldots, x_r) = Q(x_1 + \ldots + x_r)$, then the total degree of P is d and the coefficient of $x_1 \cdots x_r$ in P is $d! \cdot a$, where a is the coefficient of x^d in Q. It follows from Theorem 16.1.1 that if $\mathcal{L}$ is a line bundle on an n-dimensional complete variety X, then we have a polynomial $Q(x)$ of degree $\leq n$ such that $Q(m) = \chi(X, \mathcal{L}^m)$ for all $m \in \mathbf{Z}$. We deduce that

$$\chi(X, \mathcal{L}^m) = Q(m) = \frac{(\mathcal{L}^n)}{n!} m^n + \text{lower order terms in } m.$$

This expression is known as the *asymptotic Riemann-Roch formula.*

REMARK 16.1.10. Suppose that $\mathcal{L}$ is a very ample line bundle on the projective variety X, with $\dim(X) = n$, and consider a closed immersion $X \hookrightarrow \mathbf{P}^N$ such that $\mathcal{L} \simeq \mathcal{O}_X(1)$. If P_X is the Hilbert polynomial of X with respect to this embedding, then $P_X(m) = \chi(X, \mathcal{L}^m)$ for every $m \in \mathbf{Z}$. It follows from Remark 16.1.9 that $(\mathcal{L}^n) = \deg(X)$. In particular, we have $(\mathcal{L}^n) > 0$.

This implies, more generally, that if $\mathcal{M}$ is an ample line bundle on X, then for every closed subvariety V of X of dimension r, we have $(\mathcal{M}^r \cdot V) > 0$. Indeed, if $\mathcal{M}^m$ is very ample and $\mathcal{L} = \mathcal{M}^m|_V$, then

$$(\mathcal{M}^r \cdot V) = \frac{1}{m^r} (\mathcal{L}^r) > 0.$$

From now on, we will often use this property without further comment.

EXAMPLE 16.1.11. Suppose that X is an irreducible, projective curve. We claim that if $\mathcal{L}$ is a line bundle on X, then $(\mathcal{L}) = \deg(\mathcal{L})$. First, we may assume that X is smooth: if $\pi\colon \widetilde{X} \to X$ is the normalization morphism, then we have

$$\deg(\mathcal{L}) = \deg\big(\pi^*(\mathcal{L})\big) \quad \text{and} \quad (\mathcal{L}) = \big(\pi^*(\mathcal{L})\big),$$

where the first equality follows from the definition of degree and the second equality follows from assertion iv) in Proposition 16.1.6. In this case, by additivity, it is enough to show that if $\mathcal{L} = \mathcal{O}_X(P)$, for some $P \in X$, then $(\mathcal{L}) = 1$. This follows from assertion v) in Proposition 16.1.6.

Applying Corollary 16.1.5, we thus obtain $\deg(\mathcal{L}) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}^{-1})$. By replacing $\mathcal{L}$ with $\mathcal{L}^{-1}$, we conclude that

$$\chi(X, \mathcal{L}) = \deg(\mathcal{L}) + \chi(X, \mathcal{O}_X).$$

This is an extension of the Riemann-Roch theorem to the singular case.

EXAMPLE 16.1.12. Suppose that X is a smooth, complete surface. If $\mathcal{L}$ and $\mathcal{M}$ are line bundles on X and we write $\mathcal{L} = \mathcal{O}_X(D)$, where $D = \sum_{i=1}^r a_i D_i$, then it follows from assertion v) in Proposition 16.1.6 and the previous example that

$$(\mathcal{L} \cdot \mathcal{M}) = \sum_{i=1}^r a_i \cdot \deg(\mathcal{M}|_{D_i}).$$

We end this section by introducing the relation of numerical equivalence.

DEFINITION 16.1.13. If X is a complete variety, then two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are *numerically equivalent* (written $\mathcal{L}_1 \equiv \mathcal{L}_2$) if

$$(\mathcal{L}_1 \cdot C) = (\mathcal{L}_2 \cdot C)$$

for every irreducible curve $C \subseteq X$. We similarly define numerical equivalence for Cartier divisors on an irreducible variety. It is clear that the set of isomorphism classes of line bundles that are *numerically trivial* (that is, they are numerically equivalent to 0) is a subgroup of $\mathrm{Pic}(X)$. The quotient group $\mathrm{N}^1(X) := \mathrm{Pic}(X)/\equiv$ is the *Néron-Severi* group of X.

PROPOSITION 16.1.14. *If $f\colon X \to Y$ is a morphism of complete algebraic varieties and $\mathcal{L} \in \mathrm{Pic}(Y)$ is such that $\mathcal{L} \equiv 0$, then $f^*(\mathcal{L}) \equiv 0$. The converse holds if f is surjective.*

PROOF. Let C be an irreducible curve in X. It follows from the projection formula that $\big(f^*(\mathcal{L}) \cdot C\big) = 0$ if $f(C)$ is a point and we have

$$\big(f^*(\mathcal{L}) \cdot C\big) = d \cdot (\mathcal{L} \cdot C')$$

if $C' = f(C)$ is a curve and d is the degree of the induced morphism $C \to C'$. This gives the first assertion in the proposition and the second one also follows if we show that if f is surjective, then for every irreducible curve C' in Y, there is an irreducible curve C in X, whose image is C'. This is a consequence of the more general lemma below. $\square$

LEMMA 16.1.15. *If $f\colon X \to Y$ is a morphism of algebraic varieties, with Y irreducible and $f(X)$ dense in Y, then there is an irreducible, closed subvariety X' of X, with $\dim(X') = \dim(Y)$, and such that $f(X')$ is dense in Y.*

PROOF. After replacing X with an irreducible component that dominates Y, we may assume that X is irreducible. We know that $\dim(X) \geq \dim(Y)$. If equality holds, then we can take $X' = X$. Otherwise, arguing by induction on $\dim(X) - \dim(Y)$, it is enough to show that there is an irreducible closed subvariety Z of X that dominates Y and such that $\dim(Z) = \dim(X) - 1$. By Theorem 3.4.2, there is a non-empty open subset $V \subseteq Y$ such that every fiber $f^{-1}(y)$, with $y \in V$, has pure dimension equal to $\dim(X) - \dim(Y)$. Let $U \subseteq f^{-1}(V)$ be a non-empty affine open subset and consider a closed immersion $U \hookrightarrow \mathbf{A}^n$. Given $x \in U$, a general affine hyperplane $H \subseteq \mathbf{A}^n$ containing x does not contain any irreducible component of $f^{-1}\big(f(x)\big)$. If W is an irreducible component of $H \cap U$ containing x, then $\dim(W) = \dim(X) - 1$ and $\dim\big(W \cap f^{-1}(f(x))\big) = \dim(X) - \dim(Y) - 1$. By Theorem 3.4.1, it follows that W dominates Y, hence we may take Z to be the closure of W in X. □

Proposition 16.1.14 implies that for every morphism of complete varieties $f\colon X \to Y$, we have an induced group homomorphism

$$f^*\colon \mathrm{N}^1(Y) \to \mathrm{N}^1(X), \quad \mathcal{L} \to f^*(\mathcal{L}).$$

This is injective if f is surjective. Moreover, if $g\colon Y \to Z$ is another morphism of complete varieties, then it is clear that $(g \circ f)^* = f^* \circ g^*$.

By definition, two line bundles are numerically equivalent if and only if their restrictions to irreducible curves have the same degree. The next proposition shows that in fact, their intersection numbers with any other line bundles on X are equal.

PROPOSITION 16.1.16. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles on the complete variety X. If $\mathcal{L}_1 \equiv \mathcal{L}_2$, then for every line bundles $\mathcal{M}_2, \ldots, \mathcal{M}_r \in \mathrm{Pic}(X)$ and every coherent sheaf $\mathcal{F}$ on X such that $\dim\big(\mathrm{Supp}(\mathcal{F})\big) \leq r$, we have*

$$(\mathcal{L}_1 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_r; \mathcal{F}) = (\mathcal{L}_2 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_r; \mathcal{F}).$$

PROOF. We argue by induction on $r \geq 1$. Note that by property iii) in Proposition 16.1.6, we may assume that X is irreducible and $\mathcal{F} = \mathcal{O}_X$. In this case, if $r = 1$, then we are done by definition. Note also that we may assume that X is projective. Indeed, by Chow's lemma, we can find a birational morphism $f\colon W \to X$, with W projective. Since $f^*(\mathcal{L}_1) \equiv f^*(\mathcal{L}_2)$ by Proposition 16.1.14, the reduction follows using property iv) in Proposition 16.1.6. If $r \geq 2$, then we can write $\mathcal{M}_r \simeq \mathcal{O}_X(A - B)$, for effective Cartier divisors A and B, and property v) in Proposition 16.1.6 gives

$$(\mathcal{L}_1 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_r) = (\mathcal{L}_1 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_{r-1}; \mathcal{O}_A) - (\mathcal{L}_1 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_{r-1}; \mathcal{O}_B)$$

and similarly

$$(\mathcal{L}_2 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_r) = (\mathcal{L}_2 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_{r-1}; \mathcal{O}_A) - (\mathcal{L}_2 \cdot \mathcal{M}_2 \cdot \ldots \cdot \mathcal{M}_{r-1}; \mathcal{O}_B).$$

We conclude using the inductive hypothesis. □

The following is a fundamental result concerning the Néron-Severi groups, known as the *theorem of the base*. It was proved by Severi over the complex numbers and by Néron in general. The proof, however, goes beyond the methods discussed in these notes.

THEOREM 16.1.17. *If X is a complete variety, then $\mathrm{N}^1(X)$ is a finitely generated Abelian group.*

We note that it is a trivial consequence of the definition that $\mathrm{N}^1(X)$ has no torsion. The above theorem says that its rank, called the *Picard rank* of X and denoted by $\rho(X)$, is finite. It is often convenient to tensor with $\mathbf{R}$ and consider the finite-dimensional real vector space

$$\mathrm{N}^1(X)_{\mathbf{R}} := \mathrm{N}^1(X) \otimes_{\mathbf{Z}} \mathbf{R}.$$

16.2. Intersection numbers on surfaces

In this section we consider the case when X is a smooth projective surface (always assumed to be irreducible) and discuss some results concerning the intersection form on line bundles that we defined in the previous section. We follow closely the presentation in [**Har77**]. In general, it is more convenient to use the divisorial notation when computing intersection numbers, so we begin by introducing the following notation.

NOTATION 16.2.1. If X is a smooth, irreducible variety, then we write K_X for any divisor on X such that $\mathcal{O}_X(K_X) \simeq \omega_X$. Of course, this is only defined up to linear equivalence.

We first give a useful consequence of adjunction.

PROPOSITION 16.2.2 (Adjunction formula). *If C is an irreducible curve on the smooth, projective surface X, then*

$$2p_a(C) - 2 = (C^2) + (C \cdot K_X).$$

PROOF. Since X is smooth, it follows from Example 14.2.18 that we have $\omega_C^\circ \simeq \mathcal{O}_X(K_X + C)|_C$. In particular, ω_C° is a line bundle. On the other hand, we have

$$\big((K_X + C) \cdot C\big) = \deg\big(\mathcal{O}_X(K_X + C)|_C\big),$$

hence it is enough to show that

$$2p_a(C) - 2 = \deg(\omega_C^\circ).$$

We know this when C is smooth and the argument extends easily to our setting, as follows. Note that by definition of the dualizing sheaf, for every line bundle $\mathcal{L}$ on C, we have $h^1(C, \mathcal{L}) = h^0(C, \omega_C^\circ \otimes \mathcal{L}^{-1})$. Applying this with $\mathcal{L} = \omega_C^\circ$ and $\mathcal{L} = \mathcal{O}_C$, we obtain

$$h^1(C, \omega_C^\circ) = h^0(C, \mathcal{O}_C) = 1 \quad \text{and} \quad h^0(C, \omega_C^\circ) = h^1(C, \mathcal{O}_C) = p_a(C).$$

By the Riemann-Roch formula in the singular case (see Example 16.1.11), we have

$$\chi(C, \omega_C^\circ) = \deg(\omega_C^\circ) + \chi(C, \mathcal{O}_C),$$

and thus

$$\deg(\omega_C^\circ) = \big(p_a(C) - 1\big) - \big(1 - p_a(C)\big) = 2p_a(C) - 2.$$

□

EXAMPLE 16.2.3. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $\pi_1 \colon X \to \mathbf{P}^1$ and $\pi_2 \colon X \to \mathbf{P}^1$ be the projections onto the first and second component, respectively. It follows from Example 9.4.29 that $\operatorname{Pic}(X)$ is freely generated by the line bundles associated to L_1 and L_2, where $L_1 = \pi_1^*(Q_1)$ and $L_2 = \pi_2^*(Q_2)$, for some points $Q_1, Q_2 \in \mathbf{P}^1$. Since any two points on $\mathbf{P}^1$ are linearly equivalent, it is clear that L_1 is independent, up to linear equivalence, of the choice of the point Q_1, and similarly for L_2. Moreover, since any two distinct fibers of π_1 do not intersect, it follows that $(L_1^2) = 0$ and similarly $(L_2^2) = 0$. Since a fiber of π_1 and a fiber of π_2 are smooth and they intersect transversally in a point, it follows that $(L_1 \cdot L_2) = 1$.

Suppose now that C is an arbitrary irreducible curve in $\mathbf{P}^1 \times \mathbf{P}^1$. We say that C is of type (a, b) if $C \sim aL_1 + bL_2$. Note that we have $a, b \geq 0$. Indeed, if $\pi_1(C)$ is not a point, then $C \cap L_1$ is a finite set, hence $b = (C \cdot L_1) > 0$. On the other hand, if $\pi_1(C)$ is a point, then we clearly have $b = (C \cdot L_1) = 0$. We similarly see that $a = (C \cdot L_2) \geq 0$, with equality if and only if $\pi_2(C)$ is a point.

Recall that $\omega_X \simeq \pi_1^*(\omega_{\mathbf{P}^1}) \otimes \pi_2^*(\omega_{\mathbf{P}^1})$, hence $K_X \sim -2L_1 - 2L_2$. We thus conclude from the adjunction formula that if C has type (a, b), then

$$2p_a(C) - 2 = \big((aL_2 + bL_2) \cdot ((a-2)L_1 + (b-2)L_2)\big) = a(b-2) + b(a-2),$$

and we get $p_a(C) = (a-1)(b-1)$.

In particular, for every $g \geq 1$, we may consider a general element of $C_g \in |\mathcal{O}_X(2, g+1)|$. Such a curve is smooth and irreducible by Theorem 6.4.1 and Corollary 14.2.15 and the above computation shows that it has genus g. Note that C_g is hyperelliptic: the morphism π_2 induces a morphism $f \colon C \to \mathbf{P}^1$ such that $\deg\big(f^*(Q_2)\big) = (C \cdot L_2) = 2$, hence $\deg(f) = 2$.

THEOREM 16.2.4 (Riemann-Roch). *If $\mathcal{L}$ is a line bundle on the smooth, projective surface X, then*

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + \frac{1}{2}\big((\mathcal{L}^2) - (\mathcal{L} \cdot \omega_X)\big).$$

PROOF. Given two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, if we apply the formula in Corollary 16.1.5 for $\mathcal{L}_1^{-1}$ and $\mathcal{L}_2^{-1}$, we obtain

$$(\mathcal{L}_1 \cdot \mathcal{L}_2) = (\mathcal{L}_1^{-1} \cdot \mathcal{L}_2^{-1}) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}_1) - \chi(X, \mathcal{L}_2) + \chi(X, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

If we take $\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{L}_2 = \omega_X \otimes \mathcal{L}^{-1}$, we obtain

$$-(\mathcal{L}^2) + (\mathcal{L} \cdot \omega_X) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}) - \chi(X, \omega_X \otimes \mathcal{L}^{-1}) + \chi(X, \omega_X). \tag{16.2.1}$$

Serre duality gives $\chi(X, \mathcal{L}) = \chi(X, \omega_X \otimes \mathcal{L}^{-1})$ and $\chi(X, \mathcal{O}_X) = \chi(X, \omega_X)$, hence (16.2.1) implies the formula in the theorem. □

The following consequence of the Riemann-Roch theorem is often useful:

COROLLARY 16.2.5. *If $\mathcal{L}$ is a line bundle on the smooth, projective surface X such that $(\mathcal{L}^2) > 0$, then there is $c > 0$ such that we have either $h^0(X, \mathcal{L}^m) > cm^2$ for $m \gg 0$ or $h^0(X, \mathcal{L}^{-m}) > cm^2$ for $m \gg 0$. Moreover, if $\mathcal{M}$ is an ample line bundle, then $(\mathcal{L} \cdot \mathcal{M}) \neq 0$ and we are in the first situation above if $(\mathcal{L} \cdot \mathcal{M}) > 0$ and in the second situation if $(\mathcal{L} \cdot \mathcal{M}) < 0$.*

PROOF. Note that since $(\mathcal{L}^2) > 0$ we have $\mathcal{L} \not\equiv 0$ by Proposition 16.1.16. If C is an irreducible curve on X such that $(\mathcal{L} \cdot C) \neq 0$, and if we write $\mathcal{O}_X(C) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, with $\mathcal{L}_1$ and $\mathcal{L}_2$ ample, we see that $(\mathcal{L} \cdot \mathcal{L}_1) \neq 0$ or $(\mathcal{L} \cdot \mathcal{L}_2) \neq 0$. We thus have an ample line bundle $\mathcal{M}'$ such that $(\mathcal{L} \cdot \mathcal{M}') \neq 0$. After possibly replacing $\mathcal{L}$ by $\mathcal{L}^{-1}$, we may assume that $(\mathcal{L} \cdot \mathcal{M}') > 0$.

The Riemann-Roch theorem implies that for every $m \in \mathbf{Z}$, we have

$$\chi\big(X, \mathcal{L}^m) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(\mathcal{L}^2) \cdot m^2 - \frac{1}{2}(\mathcal{L} \cdot \omega_X) \cdot m.$$

Since $(\mathcal{L}^2) > 0$, this implies that there is $c > 0$ such that

$$h^0(X, \mathcal{L}^m) + h^2(X, \mathcal{L}^m) \geq \chi(X, \mathcal{L}^m) > cm^2 \quad \text{for} \quad m \gg 0. \tag{16.2.2}$$

In particular, we see that for $m \gg 0$, at least one of $h^0(X, \mathcal{L}^m)$ and $h^2(X, \mathcal{L}^m)$ is > 0. Note that by Serre duality, we have $h^2(X, \mathcal{L}^m) = h^0(X, \omega_X \otimes \mathcal{L}^{-m})$, and if this is positive, then

$$(\omega_X \cdot \mathcal{M}') - m \cdot (\mathcal{L} \cdot \mathcal{M}') > 0,$$

which gives a contradiction for $m \gg 0$, since $(\mathcal{L} \cdot \mathcal{M}') > 0$. We thus conclude that for $m \gg 0$ we have $h^2(X, \mathcal{L}^m) = 0$, and thus $h^0(X, \mathcal{L}^m) > cm^2$.

Finally, given any ample line bundle $\mathcal{M}$, we note that since $h^0(X, \mathcal{L}^m) > 0$ for $m \gg 0$, we have

$$(\mathcal{L}, \mathcal{M}) = \frac{1}{m}(\mathcal{L}^m \cdot \mathcal{M}) > 0.$$

This completes the proof of the corollary. □

We end this section with the following result, known as the *Hodge index theorem*.

THEOREM 16.2.6. *Let X be a smooth projective surface. If H is an ample line bundle on X and D is a divisor on X such that $(H \cdot D) = 0$, then $(D^2) \leq 0$. Moreover, we have $(D^2) = 0$ if and only if $D \equiv 0$.*

PROOF. Note first that if $(D^2) > 0$, then $(D \cdot H) \neq 0$ by Corollary 16.2.5, a contradiction. Suppose now that $(D^2) = 0$. If $D \not\equiv 0$, then we can find a divisor A on X such that $(D \cdot A) \neq 0$. After replacing A by $(H^2)A - (A \cdot H)H$, we may assume that $(A \cdot H) = 0$ and after possibly replacing A by $-A$, we may assume that $(D \cdot A) > 0$. Since $\big((mD + A) \cdot H\big) = 0$, we have seen that

$$0 \geq \big((mD + A)^2\big) = 2m \cdot (D \cdot A) + (A^2).$$

By taking $m \gg 0$, we obtain a contradiction. □

REMARK 16.2.7. Let us explain the name of the above result. If X is a smooth, projective surface, then in follows from definition of numerical equivalence that we have the a symmetric, bilinear map

$$\mathrm{N}^1(X) \times \mathrm{N}^1(X) \to \mathbf{Z}, \quad (\mathcal{L}, \mathcal{M}) \to (\mathcal{L} \cdot \mathcal{M}).$$

This induces a corresponding symmetric bilinear form of real vector spaces, the *intersection form*

$$\mathrm{N}^1(X)_{\mathbf{R}} \times \mathrm{N}^1(X)_{\mathbf{R}} \to \mathbf{R}.$$

Again, it follows from definition of numerical equivalence that the induced linear map

$$\mathrm{N}^1(X)_{\mathbf{R}} \to \mathrm{N}^1(X)_{\mathbf{R}}^{\vee}$$

is injective. By Theorem 16.1.17, the real vector space $\mathrm{N}^1(X)_{\mathbf{R}}$ is finite-dimensional, and thus the intersection form is non-degenerate. Like every non-degenerate bilinear form on a real vector space, it has a *signature* and an *index*: we can find $\mathcal{L}_1, \ldots, \mathcal{L}_\rho \in \mathrm{Pic}(X)$ giving a basis of $\mathrm{N}^1(X)_{\mathbf{R}}$ such that $(\mathcal{L}_i \cdot \mathcal{L}_j) = 0$ for $i \neq j$, and if r is the number of those i with $(\mathcal{L}_i^2) > 0$ and $s = \rho(X) - r$ is the number of those i with $(\mathcal{L}^2) < 0$, then the signature is (r, s) and the index is s. The above theorem thus says that the signature of the intersection form on $\mathrm{N}^1(X)_{\mathbf{R}}$ is $(1, \rho - 1)$.

This in turn leads to a stronger version of the theorem: if H is any divisor on X such that $(H^2) > 0$ and D is a divisor such that $(D \cdot H) = 0$, then $(D^2) \leq 0$, with equality if and only if $D \equiv 0$.

EXAMPLE 16.2.8. The stronger version of the Hodge Index theorem in the above remark has the following useful consequence. Suppose that $f \colon X \to Y$ is a surjective morphism between projective surfaces, with X smooth. If $E_1, \ldots, E_r$ are irreducible curves on X that are contracted by f (that is, each $f(E_i)$ is a point), then the intersection matrix

$$\big((E_i \cdot E_j)\big)_{1\leq i,j\leq r}$$

is negative definite. Equivalently, for every divisor $E=\sum_{i=1}^r a_iE_i$, we have $(E^2)\leq 0$, with equality if and only if $E=0$.

Indeed, if H is an ample divisor on Y, then it follows from the projection formula that $\big(f^*(H)^2\big)=\deg(f)\cdot(H^2)>0$ and $\big(f^*(H)\cdot E_i\big)=0$ for all i. The version of the Hodge Index theorem in Remark 16.2.7 thus implies that if $E=\sum_{i=1}^r a_iE_i$, then $(E^2)\leq 0$, with equality if and only if $E\equiv 0$. In order to obtain our assertion, it is enough to show that if $E\equiv 0$, then $a_i=0$ for all i. Let us write

$$E=\sum_{i,a_i>0} a_iE_i \quad\text{and}\quad E_-=-\sum_{i,a_i<0} a_iE_i,$$

hence $E=E_+-E_-$. Arguing by contradiction, let us assume that at least one of E_+ and E_- is non-zero. After replacing E by $-E$, we may assume that E_+ is non-zero. Using what we have already proved and the fact that $E\equiv 0$, we obtain

$$0\geq (E_+^2)=(E_+\cdot E_-).$$

On the other hand, since E_+ and E_- have no common components, we obtain $(E_+\cdot E_-)\geq 0$, hence $(E_+^2)=0$ and thus $E_+\equiv 0$. On the other hand, since E_+ is a non-zero effective divisor, if M is an ample divisor on X, then $(E_+\cdot M)>0$, a contradiction. This completes the argument.

Exercise 16.2.9. Show that if X is a smooth projective surface and H is a divisor on X, with $(H^2)>0$, then for every divisor D on X we have

$$(D^2)\cdot(H^2)\leq (D\cdot H)^2,$$

with equality if and only if we have integers a and b, not both of them 0, such that $aD+bH\equiv 0$.

Exercise 16.2.10. Let C be a smooth projective curve of genus $g\geq 1$. Consider $X=C\times C$ and the two projections $\pi_1\colon X\to C$ and $\pi_2\colon X\to C$.

i) Show that if $\ell_1=\pi_1^*(Q_1)$ and $\ell_2=\pi_2^*(Q_2)$, for points $Q_1,Q_2\in C$, then the classes of ℓ_1 and ℓ_2 in $\mathrm{N}^1(X)$ are independent of the choice of Q_1 and Q_2, and we have

$$(\ell_1^2)=0,\quad (\ell_2^2)=0,\quad\text{and}\quad (\ell_1\cdot\ell_2)=1.$$

ii) Show that if $\Delta\subseteq X$ is the diagonal, then

$$(\Delta\cdot\ell_1)=1,\quad (\Delta\cdot\ell_2)=1,\quad\text{and}\quad (\Delta^2)=-(2g-2).$$

Deduce that the image of Δ in $\mathrm{N}^1(X)_{\mathbf{R}}$ does not lie in the linear span of ℓ_1 and ℓ_2, hence $\rho(X)\geq 3$.

16.3. The Nakai-Moishezon ampleness criterion

Our goal in this section is to prove the following numerical ampleness criterion.

Theorem 16.3.1 (Nakai-Moishezon). *If $\mathcal{L}$ is a line bundle on the complete variety X, then X is ample if and only if for every irreducible closed subvariety Y of X, of dimension $r>0$, we have $(\mathcal{L}^r\cdot Y)>0$.*

PROOF. The "only if" part follows from Remark 16.1.10, hence we only need to prove the "if" part. Suppose that $(\mathcal{L}^r \cdot Y) > 0$ for every $r \geq 1$ and every r-dimensional irreducible closed subset of X. Assertion iii) in Proposition 16.1.6 implies that the same inequality holds for all r-dimensional subvarieties Y of X. Arguing by Noetherian induction, we may assume that $\mathcal{L}|_Y$ is ample for every closed subvariety Y of X, different from X. If $X_1, \ldots, X_s$ are the irreducible components of X and if $\mathcal{L}|_{X_i}$ is ample for all i, then $\mathcal{L}$ is ample (see Exercise 11.6.19). We may thus assume that X is irreducible and let $n = \dim(X)$. If $n = 0$, then every line bundle on X is ample. Suppose now that $n > 0$.

Step 1. We have $\lim_{m\to\infty} \chi(X, \mathcal{L}^m) = \infty$. This follows from the asymptotic Riemann-Roch formula (see Remark 16.1.9) and the fact that by assumption $(\mathcal{L}^n) > 0$.

Step 2. We now show that $\lim_{m\to\infty} h^0(X, \mathcal{L}^m) = \infty$. Since X is irreducible, we have a Cartier divisor E on X such that $\mathcal{L} \simeq \mathcal{O}_X(E)$. This gives an embedding of $\mathcal{L}^{-1} \simeq \mathcal{O}_X(-E)$ in the constant sheaf $k(X)$ and consider $\mathcal{I} = \mathcal{L}^{-1} \cap \mathcal{O}_X$ and $\mathcal{J} = \mathcal{I} \otimes \mathcal{L} \simeq \mathcal{I} \cdot \mathcal{L} \hookrightarrow \mathcal{O}_X$. Therefore both $\mathcal{I}$ and $\mathcal{J}$ are coherent ideal sheaves on X. Consider the following short exact sequences on X:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \otimes \mathcal{L}^m \arrow[r] \arrow[d, "\cong"] & \mathcal{L}^m \arrow[r] & (\mathcal{O}_X/\mathcal{I}) \otimes \mathcal{L}^m \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J} \otimes \mathcal{L}^{m-1} \arrow[r] & \mathcal{L}^{m-1} \arrow[r] & (\mathcal{O}_X/\mathcal{J}) \otimes \mathcal{L}^{m-1} \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{I}$ and $\mathcal{J}$ are non-zero ideal sheaves, it follows from the inductive assumption that $\mathcal{L}|_{V(\mathcal{I})}$ and $\mathcal{L}|_{V(\mathcal{J})}$ are ample. We deduce that

$$H^i\big(X, (\mathcal{O}_X/\mathcal{I}) \otimes \mathcal{L}^m\big) = 0 \quad \text{and} \quad H^i\big(X, (\mathcal{O}_X/\mathcal{J}) \otimes \mathcal{L}^{m-1}\big) = 0$$

for all $i \geq 1$ and $m \gg 0$ (see Remark 11.6.15). Using the long exact sequences in cohomology for the above short exact sequences, we conclude that for $m \gg 0$, we have $h^i(X, \mathcal{L}^m) = h^i(X, \mathcal{L}^{m-1})$ for all $i \geq 2$. It follows that we have $C \in \mathbf{Z}$ such that for all $m \gg 0$, we have

$$h^0(X, \mathcal{L}^m) - h^1(X, \mathcal{L}^m) = \chi(X, \mathcal{L}^m) + C.$$

We thus deduce from Step 1 that $\lim_{m\to\infty} h^0(X, \mathcal{L}^m) = \infty$.

Step 3. Let us fix $m > 0$ such that $h^0(X, \mathcal{L}^m) > 0$. Since X is irreducible, we have an effective Cartier divisor D such that $\mathcal{O}_X(D) \simeq \mathcal{L}^m$. For every positive integer p, we get a short exact sequence

$$0 \to \mathcal{L}^{(p-1)m} \to \mathcal{L}^{pm} \to \mathcal{L}^{pm}|_D \to 0,$$

and the following piece from the corresponding long exact sequence

$$H^0(X, \mathcal{L}^{pm}) \xrightarrow{\phi} H^0(X, \mathcal{L}^{pm} \otimes \mathcal{O}_D) \to$$

$$\to H^1(X, \mathcal{L}^{(p-1)m}) \xrightarrow{\psi} H^1(X, \mathcal{L}^{pm}) \to H^1(X, \mathcal{L}^{pm} \otimes \mathcal{O}_D).$$

Since $\mathcal{L}|_{\mathrm{Supp}(D)}$ is ample by the inductive assumption, we have $H^1(X, \mathcal{L}^{pm} \otimes \mathcal{O}_D) = 0$ for $p \gg 0$ by Remark 11.6.15. This gives $h^1(X, \mathcal{L}^{pm}) \leq h^1(X, \mathcal{L}^{(p-1)m})$ for $p \gg 0$. Therefore the sequence $\big(h^1(X, \mathcal{L}^{pm})\big)_{p\geq 1}$ is eventually constant, which in turn implies that for $p \gg 0$, in the above exact sequence ψ is an isomorphism, hence ϕ is surjective. Note that the base-locus of $\mathcal{L}^{pm}$ is clearly contained in $\mathrm{Supp}(D)$. On the other hand, since $\mathcal{L}|_{\mathrm{Supp}(D)}$ is ample, the sheaf $\mathcal{L}^{pm} \otimes \mathcal{O}_D$ is globally generated for $p \gg 0$ (see Exercise 11.6.16). The surjectivity of ϕ hence implies that $\mathrm{Supp}(\mathcal{L}^{pm}) \cap \mathrm{Supp}(D) = \emptyset$, and we conclude that $\mathcal{L}^{pm}$ is globally generated for $p \gg 0$. Let $f\colon X \to \mathbf{P}^N$ be the morphism defined by $|\mathcal{L}^{pm}|$, so that $\mathcal{L}^{pm} \simeq f^*\big(\mathcal{O}_{\mathbf{P}^N}(1)\big)$. If C is a curve contracted by f, then the projection formula gives $(\mathcal{L}\cdot C) = 0$, a contradiction with our hypothesis. Corollary 14.1.8 then implies that f is a finite morphism, and since $\mathcal{L}^{pm}$ is the pull-back via f of an ample line bundle, we conclude that $\mathcal{L}$ is ample using Corollary 11.6.17. This completes the proof of the theorem. $\square$

Remark 16.3.2. It follows from the above theorem and Proposition 16.1.16 that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are numerically equivalent line bundles on the complete variety X, then $\mathcal{L}_1$ is ample if and only if $\mathcal{L}_2$ is ample.

CHAPTER 17

A glimpse of birational geometry

In this chapter we give an introduction to birational maps, proving in particular some basic results in dimension two. In the first section we discuss some general facts about birational morphisms and birational maps and do some computations for smooth blow-ups in arbitrary dimension. We also discuss resolutions of curves on smooth surfaces via successive blow-ups. In the second section we prove the main results about birational morphisms and birational maps between smooth algebraic surfaces. Finally, in the last section we prove Castelnuovo's criterion for constructing a blow-down morphism.

17.1. Generalities about birational morphisms and blow-ups

We begin with a discussion of exceptional loci of proper, birational morphisms.

17.1.1. Birational morphisms and exceptional loci. The starting point is the following easy consequence of Zariski's Main theorem.

PROPOSITION 17.1.1. *If $f\colon X \to Y$ is a proper, birational morphism of irreducible algebraic varieties, with Y normal, and U is the domain of the inverse rational map f^{-1}, then the following hold:*

i) *We have an induced isomorphism $f^{-1}(U) \to U$.*
ii) *For every $y \in Y \smallsetminus U$, the fiber $f^{-1}(y)$ is connected, of dimension ≥ 1.*
iii) *We have $\mathrm{codim}_X(Y \smallsetminus U) \geq 2$*

PROOF. The assumptions imply that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ (see Example 14.1.6), hence all fibers of f are connected by Zariski's Main theorem. Note that every fiber is non-empty, since $f(X)$ is closed and dense in Y. Let $W \subseteq Y$ be the subset of Y consisting of all points $y \in Y$ such that $f^{-1}(y)$ is finite (this is open in Y by Theorem 5.4.1). If $y \in Y \smallsetminus W$, then $f^{-1}(y)$ is a connected subvariety of X of dimension ≥ 1. On the other hand, for $y \in W$, the fiber $f^{-1}(y)$ is finite and connected, hence it consists of only one point. It follows that the induced morphism $g\colon f^{-1}(W) \to W$ is bijective, and being continuous and closed, it is a homeomorphism. Since $g_*(\mathcal{O}_{f^{-1}(W)}) = \mathcal{O}_W$, we conclude that g is an isomorphism.

It follows from this that the domain U of f^{-1} contains W. Moreover, since f and f^{-1} induce inverse morphisms between $f^{-1}(U)$ and U, we deduce that $U \subseteq W$, hence $U = W$. The assertion in iii) follows from the fact that f is proper and Y is normal by Proposition 9.1.6. ∎

With the notation in the above proposition, the closed subset $f^{-1}(X \smallsetminus U)$ is the *exceptional locus* $\mathrm{Exc}(f)$ of f. A prime divisor on X is *exceptional* (or f-exceptional, if f is not understood from the context) if it is contained in the exceptional locus of f. An arbitrary Weil divisor on X is *exceptional* if it is a linear combination of prime exceptional divisors.

Remark 17.1.2. It follows from the proposition that the complement $X \smallsetminus \mathrm{Exc}(f)$ can be characterized as the largest open subset V of X such that $f|_V : V \to Y$ is an open immersion.

Remark 17.1.3. Note that a prime divisor E on X is f-exceptional if and only if $\dim\big(f(E)\big) < \dim(E)$. Indeed, the "only if" part follows from assertion iii) in Proposition 17.1.1, while the "if" part follows from the fact that if E intersects the open subset $f^{-1}(U)$ in the proposition, then the induced morphism $E \to f(E)$ is birational.

Suppose now that $f : X \to Y$ is a proper, birational morphism between normal, irreducible varieties, and $U \subseteq Y$ is the domain of the rational map f^{-1}. If D is a prime divisor on Y, then $D \cap U \neq \emptyset$. The closure of $f^{-1}(D \cap U)$ is a prime divisor on X, the *strict transform* (or *proper transform*) $\widetilde{D}$ of D on X (note that this is consistent with the terminology we used in the case of blow-ups). More generally, if $D = \sum_{i=1}^r a_i D_i$ is an arbitrary Weil divisor on Y, then $\widetilde{D} := \sum_{i=1}^r a_i \widetilde{D_i}$ is the *strict transform* of D. Note that a prime divisor E on X is either f-exceptional or it is the strict transform of a prime divisor on Y (namely $f(E)$).

Proposition 17.1.4. *If $f : X \to Y$ is a proper, birational morphism between smooth, irreducible varieties, then* $\mathrm{Exc}(f)$ *has pure codimension* 1*. Moreover, there is a unique effective divisor $K_{X/Y}$ on X, with* $\mathrm{Supp}(K_{X/Y}) = \mathrm{Exc}(f)$*, such that*

$$\mathcal{O}_X(K_{X/Y}) \simeq \omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}. \tag{17.1.1}$$

The divisor $K_{X/Y}$ is the *relative canonical divisor* of f.

Proof of Proposition 17.1.4. Recall that by Proposition 8.7.20, we have an exact sequence

$$f^*(\Omega_Y) \xrightarrow{\phi} \Omega_X \to \Omega_{X/Y} \to 0.$$

Moreover, if W is an open subset of X such that $f|_W$ is an open immersion, then $\phi|_W$ is an isomorphism. Since $f^*(\Omega_Y)$ is locally free, this easily implies that ϕ is injective.

Consider now the morphism induced by ϕ at the level of top exterior powers:

$$\psi : f^*(\omega_Y) \to \omega_X.$$

This is a morphism of line bundles, hence it is determined by tensoring with $f^*(\omega_Y)$ the morphism $\mathcal{O}_X \to \omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}$ corresponding to a section s of the line bundle $\omega_X \otimes_{\mathcal{O}_X} f^*(\omega_Y)^{-1}$. Since $\phi|_W$ is an isomorphism, so is $\psi|_W$, hence $s \neq 0$. Let $K_{X/Y}$ be the effective divisor on X corresponding to s. In particular, (17.1.1) holds.

If U is as in Proposition 17.1.1, then $f^{-1}(U) \to U$ is an isomorphism; as we have already seen, in this case $\phi|_{f^{-1}(U)}$ is an isomorphism, hence $\mathrm{Supp}(K_{X/Y}) \cap f^{-1}(U) = \emptyset$. Let us show that conversely, if $x \not\in f^{-1}(U)$, then $x \in \mathrm{Supp}(K_{X/Y})$. If this is not the case, then ϕ_x is an isomorphism. Let V be an open neighborhood of x such that $\phi|_V$ is an isomorphism. This implies that for every $z \in V$, the linear map $df_z : T_zX \to T_{f(z)}Y$ is an isomorphism, hence $f|_V$ is étale. In this case, the fiber $f^{-1}\big(f(x)\big) \cap V$ is 0-dimensional, contradicting the fact that since $x \not\in f^{-1}(U)$, every irreducible component of $f^{-1}\big(f(x)\big)$ containing x has dimension ≥ 1. We thus conclude that $\mathrm{Supp}(K_{X/Y}) = \mathrm{Exc}(f)$; in particular, $\mathrm{Exc}(f)$ has pure dimension 1 in X. Finally, the uniqueness of $K_{X/Y}$ is a consequence of the general lemma below. $\square$

LEMMA 17.1.5. *If $f\colon X \to Y$ is a proper morphism of irreducible varieties, with X and Y normal, and D is an exceptional divisor on X such that $D \sim 0$, then $D = 0$.*

PROOF. By assumption, we have a non-zero $\phi \in k(X) = k(Y)$ such that $\operatorname{div}_X(\phi) = D$. If E is a prime divisor on Y and $\widetilde{E}$ is the strict transform of E, then $\operatorname{ord}_E(\phi) = \operatorname{ord}_{\widetilde{E}}(\phi) = 0$, since D is exceptional. We thus conclude that $\phi \in \mathcal{O}_Y^*(Y)$, hence $\phi \in \mathcal{O}_X^*(X)$, which implies $D = 0$. $\square$

REMARK 17.1.6. If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper, birational morphisms between irreducible, normal varieties, then

$$\operatorname{Exc}(g \circ f) = f^{-1}\big(\operatorname{Exc}(g)\big) \cup \operatorname{Exc}(f).$$

The inclusion "$\subseteq$" follows from the fact that each of the following two morphisms

$$X \smallsetminus \big(f^{-1}\big(\operatorname{Exc}(g)\big) \cup \operatorname{Exc}(f)\big) \to Y \smallsetminus \operatorname{Exc}(g) \to Z$$

is an open immersion, hence the composition is an open immersion. For the reverse inclusion, it is enough to show that for every $x \in f^{-1}\big(\operatorname{Exc}(g)\big) \cup \operatorname{Exc}(f)$, there is an irreducible subset W of positive dimension of the fiber of $g \circ f$ over $z = g\big(f(x)\big)$, such that $x \in W$. This is clear if $x \in \operatorname{Exc}(f)$ since we have such a subset in the fiber of f over $y = f(x)$. Suppose now that if $x \in f^{-1}\big(\operatorname{Exc}(g)\big) \smallsetminus \operatorname{Exc}(f)$. In this case there is an irreducible subset T of positive dimension in the fiber of g over z such that $y \in T$. Since f is an isomorphism over a suitable open neigborhood V of y, we can take W to be the closure of $f^{-1}(T \cap V)$.

When dealing with birational maps, instead of birational morphisms, a useful construction is that of the *graph* of the map. This can be defined, more generally, for arbitrary rational maps.

DEFINITION 17.1.7. Given a rational map $\phi\colon X \dashrightarrow Y$, with X irreducible, choose a morphism $f\colon U \to Y$ representing ϕ. The *graph* Γ_ϕ of ϕ is the closure in $X \times Y$ of the graph

$$\Gamma_f = \{(x, y) \in U \times Y \mid y = f(x)\}$$

of f. It is straightforward to check that this is independent of the choice of morphism representing ϕ.

REMARK 17.1.8. With the notation in the definition, note that the projections from $X \times Y$ onto the two components induce morphisms

$$p\colon \Gamma_\phi \to X \quad \text{and} \quad q\colon \Gamma_\phi \to Y.$$

The induced morphism $p^{-1}(U) \to U$ is an isomorphism with inverse $x \to \big(x, f(x)\big)$ (since Γ_f is closed in $U \times Y$, we have $\Gamma_\phi \cap (U \times Y) = \Gamma_f$). Therefore p is a birational morphism and we clearly have the equality of rational maps $\phi = q \circ p^{-1}$. In particular, we see that ϕ is birational if and only if q is birational. The graph construction thus gives a canonical way to write a birational map as the composition of a birational morphism with the inverse of another birational morphism.

Remark 17.1.9. If $\phi\colon X \dashrightarrow Y$ is a birational map between irreducible varieties, then there are open subsets U in X and V in Y such that ϕ induces an isomorphism $U \simeq V$. This implies that the isomorphism $X \times Y \to Y \times X$ that maps (P, Q) to (Q, P) maps Γ_ϕ onto $\Gamma_{\phi^{-1}}$.

Remark 17.1.10. Suppose that we have proper morphisms $\alpha\colon X \to S$ and $\beta\colon Y \to S$, with X irreducible, and $\phi\colon X \dashrightarrow Y$ is a rational map of varieties over S (that is, we have $\beta \circ \phi = \alpha$). If U is an an open subset of X such that ϕ is represented by a morphism $f\colon U \to Y$, then it is clear that $\Gamma_f \subseteq X \times_S Y$, hence Γ_ϕ is a closed subset of $X \times_S Y$. Since α and β are proper, we deduce that also the two morphisms $p\colon \Gamma_\phi \to X$ and $q\colon \Gamma_\phi \to Y$ are proper.

Under these assumptions, for a closed subset Z of X, we define its *image* by ϕ to be

$$\phi(Z) := q\big(p^{-1}(Z)\big).$$

Note that since q is proper, this is a closed subset of Y. Of course, if ϕ is a morphism, then p is an isomorphism, and this definition agrees with the usual one for the image of Z by ϕ.

Proposition 17.1.11. *Let X and Y be proper varieties over S, with X normal and irreducible, and $\phi\colon X \dashrightarrow Y$ a rational map of varieties over S. If $T \in X$ is a point that does not lie in the domain of ϕ, then $\phi(T)$ is a connected, closed subset of Y, of dimension ≥ 1.*

Proof. Consider the canonical morphisms $p\colon \Gamma_\phi \to X$ and $q\colon \Gamma_\phi \to Y$. Since T is not in the domain of ϕ, it follows that it is not in the domain of p^{-1}, hence Proposition 17.1.1 implies that $p^{-1}(T)$ is a connected, closed subset of Γ_ϕ, of dimension ≥ 1. The assertion in the statement now follows from the fact that the restriction of q to $p^{-1}(T) \subseteq \{T\} \times Y$ is a closed immersion. □

Proposition 17.1.12. *Let $\phi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ be rational maps, where X and Y are irreducible and ϕ is dominant.*

i) *We have a canonical isomorphism*

$$\Gamma_{\psi\circ\phi} \simeq \Gamma_\phi \times_Y \Gamma_\psi.$$

ii) *Suppose that X, Y, and Z are proper over S and the rational maps are maps of varieties over S. If W is a closed subset of X, then*

$$(\psi \circ \phi)(W) \subseteq \psi\big(\phi(W)\big), \tag{17.1.2}$$

with equality if ψ is a morphism.

Proof. The assertion in i) is straightforward to check when both ϕ and ψ are morphisms. The general case follows by choosing morphisms representing the two maps such that they can be composed, and by taking the closures of the corresponding graphs. Consider now the Cartesian diagram

$$\begin{tikzcd}
\Gamma_{\psi\circ\phi} \arrow[r, "\widetilde{q}"] \arrow[d, "\widetilde{u}"'] & \Gamma_\psi \arrow[d, "u"] \\
\Gamma_\phi \arrow[r, "q"'] & Y.
\end{tikzcd}$$

The inclusion in (17.1.2) follows from the obvious inclusion

$$\widetilde{q}\big(\widetilde{u}^{-1}(p^{-1}(W))\big) \subseteq u^{-1}\big(q(p^{-1}(W))\big).$$

Moreover, this is an equality if ψ is a morphism, since in this case u and $\widetilde{u}$ are isomorphisms. We thus obtain equality in (17.1.2). $\square$

17.1.2. General properties of smooth blow-ups. We now turn to the study of some very concrete birational transformations, the smooth blow-ups. We begin with a property of more general blow-ups. Recall that if $f\colon \widetilde{X} \to X$ is the blow-up of an irreducible variety X along a non-zero coherent ideal sheaf $\mathcal{I}$, then the ideal $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ is locally principal.

PROPOSITION 17.1.13. *Let X be an irreducible variety, $\mathcal{I}$ a non-zero coherent ideal sheaf on X, and $f\colon \widetilde{X} \to X$ the blow-up of X along $\mathcal{I}$. For every irreducible variety Y and every morphism $g\colon Y \to X$ such that the ideal $\mathcal{I} \cdot \mathcal{O}_Y$ is locally principal, there is a unique morphism $h\colon Y \to \widetilde{X}$ such that $f \circ h = g$.*

PROOF. By assumption, $\mathcal{I} \cdot \mathcal{O}_Y \neq 0$, hence $g(Y)$ is not contained in $V(\mathcal{I})$. Since f is an isomorphism over $X \smallsetminus V(\mathcal{I})$, it is clear that there is at most one morphism h as in the proposition. Because of uniqueness, the existence is a local problem. First, we consider an affine open cover $X = \bigcup_{i \in I} U_i$. It is enough to show that for every i, if $f_i\colon f^{-1}(U_i) \to U_i$ and $g_i\colon g^{-1}(U_i) \to U_i$ are the induced morphisms, there is $h_i\colon g^{-1}(U_i) \to f^{-1}(U_i)$ such that $f_i \circ h_i = g_i$. We thus may assume that X is affine and let $A = \mathcal{O}_X(X)$ and $I = \mathcal{I}(X)$. Let $a_1, \ldots, a_r$ be generators of I. By assumption, we have a finite cover $Y = \bigcup_{j \in J} V_j$ by affine open subsets such that for every $j \in J$, there is i with $1 \leq i \leq r$ such that $I \cdot \mathcal{O}_Y(V_j) = a_i \cdot \mathcal{O}_Y(V_j)$ and this is a non-zero ideal. It is enough to show that for every such j, there is $h_j\colon V_j \to X$ such that $f \circ h_j = g|_{V_j}$, hence we may assume that Y is affine, with $A' = \mathcal{O}_Y(Y)$, and we have i, with $1 \leq i \leq r$, such that if $\phi\colon A \to A'$ is the morphism induced by g, then $I \cdot A' = \big(\phi(a_i)\big)$ and $\phi(a_i) \neq 0$.

Recall that $\widetilde{X} = \mathrm{Proj}(S)$, where $S = \bigoplus_{m \geq 0} I^m$ and consider the element $a_i \in S_1$. We have the affine open subset $W = D_{\widetilde{X}}(a_i) \subseteq \widetilde{X}$, so that $\mathcal{O}_{\widetilde{X}}(W) = S_{(a_i)}$. By assumption, for every $u \in I^m$, we can write $\phi(u) = \phi(a_i)^m \cdot \psi_m(u)$ for a unique element $\psi_m(u) \in A'$ (recall that $\phi(a_i) \neq 0$ and A' is a domain). We can thus define a map

$$S_{(a_i)} \to A', \quad \frac{u}{a_i^m} \to \psi_m(u)$$

and it is straightforward to see that this is a morphism of A-algebras. This corresponds to a morphism of algebraic varieties $h\colon Y \to W \subseteq \widetilde{X}$ such that $f \circ h = g$. $\square$

From now on we assume that X is a smooth, irreducible variety, Z is a smooth, irreducible, closed subvariety of X, of codimension r, and $f\colon \widetilde{X} \to X$ is the blow-up of X along Z (that is, the blow-up of X along the radical ideal sheaf $\mathcal{I}_Z$ corresponding to Z). Recall that $\mathcal{I}_Z \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(1) = \mathcal{O}_{\widetilde{X}}(-E)$, for a Cartier divisor E on $\widetilde{X}$. Moreover, E is a smooth, irreducible subvariety of $\widetilde{X}$ (see Example 6.3.24). In fact, if $\mathcal{S} = \bigoplus_{m \geq 0} \mathcal{I}_Z^m$, then $E \simeq \mathcal{P}roj(\mathcal{S}/\mathcal{I}_Z\mathcal{S})$ (see Remark 4.3.18). We thus conclude that

$$E \simeq \mathcal{P}roj\big(\bigoplus_{m \geq 0} \mathcal{I}_Z^m/\mathcal{I}_Z^{m+1}\big) \simeq \mathcal{P}roj\left(\mathrm{Sym}^{\bullet}(\mathcal{I}_Z/\mathcal{I}_Z^2)\right),$$

where the second isomorphism follows from Proposition 6.3.21. We thus see that the induced morphism $E \to Z$ makes E a projective bundle over Z, such that $\mathcal{O}_{\widetilde{X}}(E)|_E \simeq \mathcal{O}_E(-1)$.

Example 17.1.14. If $f\colon \widetilde{X} \to X$ is the blow-up of a smooth, complete, n-dimensional variety at a point, and E is the exceptional divisor, then $E \simeq \mathbf{P}^{n-1}$ and

$$(E^n) = \big(\mathcal{O}_{\mathbf{P}^{n-1}}(-1)^{n-1}\big) = (-1)^{n-1}.$$

Proposition 17.1.15. *If $f\colon \widetilde{X} \to X$ is the blow-up of the smooth, irreducible variety X along the smooth, irreducible, closed subvariety Z, of codimension $r \geq 2$, and if E is the exceptional divisor, then the map*

$$\phi\colon \operatorname{Pic}(X) \oplus \mathbf{Z} \to \operatorname{Pic}(\widetilde{X}), \quad \big(\mathcal{L}, m\big) \to f^*(\mathcal{L}) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(mE)$$

is an isomorphism. Moreover, this induces an isomorphism $\psi\colon \mathrm{N}^1(X) \oplus \mathbf{Z} \simeq \mathrm{N}^1(\widetilde{X})$.

Proof. Recall that on smooth varieties, we can identify the Picard group and the class group. Let $U = X \smallsetminus Z$. Since $\widetilde{X} \smallsetminus f^{-1}(U) = E$, it follows from Example 9.3.5 that we have a short exact sequence

$$\mathbf{Z} \xrightarrow{\alpha} \operatorname{Pic}(\widetilde{X}) \xrightarrow{\beta} \operatorname{Pic}\big(f^{-1}(U)\big) \to 0,$$

where $\alpha(m) = \mathcal{O}_X(mE)$ and $\beta(\mathcal{L}) = \mathcal{L}|_{f^{-1}(U)}$. Similarly, the restriction map

$$\operatorname{Pic}(X) \to \operatorname{Pic}(U), \quad \mathcal{L} \to \mathcal{L}|_U$$

is surjective (in fact, it is an isomorphism, since $\operatorname{codim}_X(Z) \geq 2$). Since $f^{-1}(U) \to U$ is an isomorphism, we immediately deduce that the morphism ϕ in the proposition is surjective, hence also ψ is surjective.

We next show that if D is a divisor on X such that $f^*D + mE \equiv 0$, then $D \equiv 0$ and $m = 0$, hence ψ is injective. Since $r \geq 2$, for every $q \in Z$, the fiber $F = f^{-1}(q)$ is a positive-dimensional projective space and $\mathcal{O}_{\widetilde{X}}(E)|_F \simeq \mathcal{O}_F(-1)$. The projection formula thus implies that if C is a curve in F, then

$$0 = \big((f^*(D) + mE) \cdot C\big) = m \cdot (E \cdot C) = -m \cdot \deg(C).$$

Therefore $m = 0$ and we get $D \equiv 0$ since f is surjective (see Proposition 16.1.14).

It is now clear that also ϕ is injective: if D is a divisor on X such that $f^*(D) + mE \sim 0$, then in particular $f^*(D) + mE \equiv 0$, hence $m = 0$. The fact that $D \sim 0$ follows from the fact that $\pi_*(\mathcal{O}_{\widetilde{X}}) \simeq \mathcal{O}_X$ (one can use, for example, Example 14.1.6), hence by the projection formula, the isomorphism $f^*\big(\mathcal{O}_X(D)\big) \simeq \mathcal{O}_{\widetilde{X}}$ gives an isomorphism

$$\mathcal{O}_X \simeq f_*\big(f^*(\mathcal{O}_X(D))\big) \simeq \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} f_*(\mathcal{O}_{\widetilde{X}}) \simeq \mathcal{O}_X(D).$$

□

Proposition 17.1.16. *If Z is a smooth, irreducible, closed subvariety of the smooth, irreducible variety X, with corresponding radical ideal sheaf $\mathcal{I}_Z$, and $f\colon \widetilde{X} \to X$ is the blow-up of X along Z, with exceptional divisor E, then for every $m \geq 0$, the canonical morphism*

$$\mathcal{I}_Z^m \to f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big)$$

is an isomorphism and $R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big) = 0$ *for all* $i \geq 1$. *In particular, for every line bundle* $\mathcal{L}$ *on* X, *we have a canonical isomorphism*

$$H^i(X, \mathcal{L}) \simeq H^i\big(\widetilde{X}, f^*(\mathcal{L})\big) \quad \textit{for all} \quad i \geq 0.$$

PROOF. For every $m \geq 0$, consider the short exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}\big(-(m+1)E\big) \to \mathcal{O}_{\widetilde{X}}(-mE) \to \mathcal{O}_{\widetilde{X}}(-mE)|_E \to 0.$$

Recall that E is isomorphic over Z to the projective bundle $\mathcal{P}roj\big(\mathrm{Sym}(\mathcal{I}_Z/\mathcal{I}_Z^2)\big)$, such that $\mathcal{O}_{\widetilde{X}}(E)|_E \simeq \mathcal{O}_E(-1)$. We deduce that for every $m \geq 0$, we have $R^i\pi_*\big(\mathcal{O}_E(m)\big) = 0$ for all $i \geq 1$ and the canonical morphism

$$\beta_m \colon \mathcal{I}_Z^m/\mathcal{I}_Z^{m+1} \to f_*\big(\mathcal{O}_E(m)\big)$$

is an isomorphism (see Example 11.2.7). First, we see that for every $i \geq 1$ and $m \geq 0$, we have an exact sequence

$$R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-(m+1)E)\big) \to R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big) \to R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)|_E\big) = 0.$$

On the other hand, since $\mathcal{O}_{\widetilde{X}}(-E) = \mathcal{O}_{\widetilde{X}}(1)$, by applying Theorem 11.2.1 over the elements of a finite affine open cover of X, we obtain $R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big) = 0$ for all $i \geq 1$ and all $m \gg 0$. Using the above exact sequence, we then obtain $R^i f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big) = 0$ for all $i \geq 1$ and all $m \geq 0$.

Note now that for every $m \geq 0$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Z^{m+1} \arrow[r] \arrow[d, "\alpha_{m+1}"'] & \mathcal{I}_Z^m \arrow[r] \arrow[d, "\alpha_m"'] & \mathcal{I}_Z^m/\mathcal{I}_Z^{m+1} \arrow[r] \arrow[d, "\beta_m"'] & 0 \\
0 \arrow[r] & f_*\big(\mathcal{O}_{\widetilde{X}}(-(m+1)E)\big) \arrow[r] & f_*\big(\mathcal{O}_{\widetilde{X}}(-mE)\big) \arrow[r] & f_*\big(\mathcal{O}_E(m)\big) \arrow[r] & 0,
\end{tikzcd}$$

where we use the fact that $R^1 f_*\big(\mathcal{O}_{\widetilde{X}}(-(m+1)E)\big) = 0$. By Corollary 11.2.3, α_m is an isomorphism for $m \gg 0$. Since β_m is an isomorphism for all $m \geq 0$, we conclude that α_m is an isomorphism for all $m \geq 0$.

The last assertion in the theorem now follows easily using the Leray spectral sequence and the projection formula. $\square$

In order to do computations on smooth blow-ups, it is convenient to recall the description via equations. We first introduce a useful notion on arbitrary smooth varieties.

DEFINITION 17.1.17. If U is an open subset of a smooth variety X, the regular functions $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ form a *system of algebraic coordinates* (or simply, a *system of coordinates*) on U if the corresponding sections $dx_1, \ldots, dx_n$ of Ω_X give an isomorphism $\mathcal{O}_U^{\oplus n} \simeq \Omega_X|_U$. In other words, the morphism $f \colon U \to \mathbf{A}^n$ given by $(x_1, \ldots, x_n)$ has the property that $f^*(\Omega_{\mathbf{A}^n}) \simeq \Omega_U$; equivalently, f is an étale morphism (see Proposition 13.2.8). Algebraically, the condition says that for every $P \in U$, the images of $x_1 - x_1(P), \ldots, x_n - x_n(P)$ in $\mathcal{O}_{X,P}$ form a regular system of parameters. In fact, note that if this condition holds at some $P \in U$, then $dx_1(P), \ldots, dx_n(P) \in (\Omega_X)_{(P)}$ form a basis and there is an open neighborhood $V \subseteq U$ of P such that $x_1, \ldots, x_n$ give a system of coordinates on V.

Suppose now that X is a smooth, irreducible variety and $Z \subseteq X$ is a smooth, irreducible, closed subvariety, of codimension r. Given any point $P \in Z$, there is an affine open neighborhood U of P and a system of coordinates $x_1, \ldots, x_n$ on U such that $x_1, \ldots, x_r$ generate the ideal of $Z \cap U$ in U (see Proposition 6.3.21).

Let $f\colon \widetilde{X} \to X$ be the blow-up of X along Z. We have seen in Example 6.3.24 that we have a closed embedding

$$f^{-1}(U) \hookrightarrow U \times \mathbf{P}^{r-1}$$

over U such that if $y_1, \ldots, y_r$ are the homogeneous coordinates on $\mathbf{P}^{r-1}$, then $f^{-1}(U)$ is defined by the equations

$$x_i y_j = x_j y_i \quad \text{for} \quad 1 \leq i, j \leq r.$$

We can thus cover $f^{-1}(U)$ by the affine charts $V_1, \ldots, V_r$, with V_i defined by $y_i \neq 0$, such that on V_i we have algebraic coordinates $u_1, \ldots, u_n$, where $f^{\#}(x_j) = u_j$ if $j = i$ or $j > r$ and $f^{\#}(x_j) = u_i u_j$ if $1 \leq j \leq r$, with $j \neq i$.

Proposition 17.1.18. *If Z is a smooth, irreducible, closed subvariety of the smooth, irreducible variety X, with $\operatorname{codim}_X(Z) = r \geq 1$, and if $f\colon \widetilde{X} \to X$ is the blow-up of X along Z, with exceptional divisor E, then $K_{\widetilde{X}/X} = (r-1)E$. In particular, we have*

$$\omega_{\widetilde{X}} \simeq f^*(\omega_X) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}\big((r-1)E\big).$$

Proof. We know that $K_{\widetilde{X}/X} = aE$ for some $a \in \mathbf{Z}$ and need to show that $a = r-1$. In order to do this, choose a point $p \in E$ and let $q = f(p)$. We have seen that we can choose algebraic coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ in some affine open neighborhood U of q and algebraic coordinates $u_1, \ldots, u_n \in \mathcal{O}_{\widetilde{X}}(V)$ in some affine open neighborhood V of p, such that the ideal of $Z \cap U$ in U is equal to $(x_1, \ldots, x_r)$ and we have

$$x_1 = u_1, \quad x_i = u_1 u_i \quad \text{for} \quad 2 \leq i \leq r, \quad \text{and} \quad x_i = u_i \quad \text{for} \quad r+1 \leq i \leq n.$$

In this case, the morphism $\alpha\colon f^*(\Omega_X) \to \Omega_{\widetilde{X}}$ is given by $\alpha\big(f^*(dx_i)\big) = du_i$ for $i = 1$ or $r+1 \leq i \leq n$ and

$$\alpha\big(f^*(dx_i)\big) = u_1 du_i + u_i du_1 \quad \text{for} \quad 2 \leq i \leq r.$$

We thus see that $\det(\alpha)$ maps $dx_1 \wedge \ldots \wedge dx_n$ to $u_1^{r-1} du_1 \wedge \ldots \wedge du_n$, which shows that $a = r-1$. $\square$

Definition 17.1.19. If Z is a smooth, irreducible, closed subvariety of the smooth, irreducible variety X, then for an effective divisor D on X, we define the *multiplicity* (or *order*) $\operatorname{ord}_Z(D)$ of D along Z, as follows. If $R = \mathcal{O}_{X,Z}$, with maximal ideal $\mathfrak{m}$, and if $h \in R$ is the image of an equation defining D in some affine open subset intersecting Z, then $\operatorname{ord}_Z(D)$ is the largest r such that $h \in \mathfrak{m}^r$; note that since h is non-zero, this is a well-defined non-negative integer by Krull's Intersection theorem (see Theorem C.4.1). Note that $\operatorname{ord}_Z(D) \geq 1$ if and only if $Z \subseteq \operatorname{Supp}(D)$.

Example 17.1.20. Let Z be a smooth, irreducible, closed subvariety of the smooth, irreducible variety X, and $f\colon \widetilde{X} \to X$ the blow-up of X along Z, with exceptional divisor E. We claim that if $D = \sum_{i=1}^r a_i D_i$ is an effective divisor on X and $\widetilde{D}$ is the strict transform of D on $\widetilde{X}$, then

$$f^*(D) = \widetilde{D} + aE, \quad \text{where} \quad a = \operatorname{ord}_E(D).$$

Since f is an isomorphism over the complement of Z, it is clear that we have $f^*(D) = \widetilde{D} + aE$ for some $a \in \mathbf{Z}$, hence we only need to compute a. This is a local computation, that can be done by choosing local coordinates on X and on $\widetilde{X}$, and which we leave as an exercise for the reader.

17.1.3. Resolution of curves on a smooth surface. We now illustrate the above results about smooth blow-ups by describing the change of arithmetic genus for curves on a smooth surface under blow-ups. As we will see, this implies that every such curve can be resolved after finitely many blow-ups.

PROPOSITION 17.1.21. *Let X be a smooth, projective surface, $P \in X$ a point, and $f\colon \widetilde{X} \to X$ the blow-up of X at P. If C is a curve on X and $\widetilde{C}$ is its strict transform on $\widetilde{X}$, then*

$$p_a(\widetilde{C}) = p_a(C) - \frac{m(m-1)}{2}, \quad \textit{where} \quad m = \mathrm{ord}_P(C).$$

PROOF. We have seen in Example 17.1.20 that $f^*(C) = \widetilde{C} + mE$. Also, we deduce from Proposition 17.1.18 that we may take $K_{\widetilde{X}} = f^*(K_X) + E$. Finally, note that by general properties of intersection numbers, for any divisors D and D' on X, we have

$$\big(f^*(D) \cdot f^*(D')\big) = (D \cdot D') \quad \text{and} \quad \big(f^*(D) \cdot E\big) = \deg\big(f^*(D)|_E\big) = 0.$$

The Adjunction formula thus gives

$$2p_a(\widetilde{C}) - 2 = (\widetilde{C}^2) + (\widetilde{C} \cdot K_{\widetilde{X}})$$

$$= \big((f^*(C) - mE)^2\big) + \big((f^*(C) - mE) \cdot (f^*(K_X) + E)\big) = (C^2) - m^2 + (C \cdot K_X) + m$$

$$= 2p_a(C) - 2 - m^2 + m,$$

and we obtain the formula in the statement. $\square$

Recall that for every curve X, we have a resolution of singularities provided by the normalization. While this is a very convenient theoretical construction, in practice it is difficult to compute. The above proposition shows that for curves on smooth, projective surfaces, we can obtain the resolution of singularities via a finite sequence of blow-ups.

COROLLARY 17.1.22. *If C is a curve on the smooth, projective surface X, then there is a finite sequence*

$$C_r \to C_{r-1} \to \ldots \to C_1 = C,$$

with C_r smooth, where each morphism is the blow-up of the radical ideal defining a singular point,

PROOF. It follows from the proposition that if $P \in C$ is a singular point (equivalently, we have $\mathrm{ord}_P(C) \geq 2$), then $p_a(\widetilde{C}) \leq p_a(C) - 1$. Note also that the induced morphism $\widetilde{C} \to C$ is the blow-up of C with respect to the ideal of P. Since the arithmetic genus of a curve is a non-negative integer (recall that a curve is assumed to be irreducible), it follows that this process has to stop after finitely many steps. $\square$

The connection discussed in §15.1.2 between arithmetic genus and δ-invariants of curve singularities suggests another approach to resolving curves by blow-ups, which also applies to non-complete curves and to curves that do not lie on surfaces. Recall that for an (irreducible) curve C and for $P \in C$, we put

$$\delta_P = \ell_{\mathcal{O}_{C,P}}(\widetilde{\mathcal{O}_{C,P}}/\mathcal{O}_{C,P}) = \dim_k(\widetilde{\mathcal{O}_{C,P}}/\mathcal{O}_{C,P}),$$

where $\widetilde{\mathcal{O}_{C,P}}$ is the integral closure of $\mathcal{O}_{C,P}$ in $k(C)$. Note that $\delta_P = 0$ if and only if P is a smooth point of C. We also put

$$\delta(C) := \sum_{P \in C} \delta_P.$$

PROPOSITION 17.1.23. *Let C be a curve.*

i) *If $f\colon \widetilde{C} \to C$ is the blow-up along the radical ideal defining a point $P \in C$, then f is an isomorphism if and only if P is a smooth point.*
iii) *If $f\colon C' \to C$ is a finite, birational morphism, which is not an isomorphism, then $\delta(C') < \delta(C)$.*

PROOF. The assertion in i) is a special case of the fact that if $f\colon \widetilde{X} \to X$ is the blow-up of the irreducible variety X along $\mathcal{I}$, then f is an isomorphism if and only if $\mathcal{I}$ is a locally principal ideal. This is clear.

For the assertion in ii), note first that if $P \in C$ is a smooth point, then we have an open neighborhood $U \subseteq C$ of P which is smooth, and thus f is an isomorphism over U (see Remark 15.2.4). In order to prove the assertion in ii), for every $P \in C$ such that f is not an isomorphism around P, we can choose an affine open neighborhood V_P of P such that $V_P \cap C_{\mathrm{sing}} = \{P\}$. Since $\delta(C) = \sum_P \delta(V_P)$ and $\delta(C') = \sum_P \delta\big(f^{-1}(V_P)\big)$, it is enough to show that for every such P, we have $\delta(V_P) > \delta\big(f^{-1}(V_P)\big)$. Consider

$$A = \mathcal{O}_C(V_P) \hookrightarrow B = \mathcal{O}_{C'}\big(f^{-1}(V_P)\big) \hookrightarrow \widetilde{A},$$

where $\widetilde{A}$ is the integral closure of A and B in $k(C)$. In this case we have

$$\delta(V_P) - \delta\big(f^{-1}(V_P)\big) = \dim_k(\widetilde{A}/A) - \dim_k(\widetilde{A}/B) = \dim_k(B/A) > 0,$$

where the inequality follows from the fact that $B \neq A$, since f is not an isomorphism over V_P. This completes the proof. $\square$

We now deduce the following generalization of Corollary 17.1.22.

PROPOSITION 17.1.24. *Given a curve C, there is a finite sequence*

$$C_r \to C_{r-1} \to \ldots \to C_1 = C,$$

with C_r smooth, where each morphism is the blow-up of the radical ideal defining a singular point.

PROOF. If C is smooth, then there is nothing to prove. Otherwise, pick $P \in C$ and let $C_2 \to C$ be the blow-up along the ideal of P. Assertion i) in the proposition implies that f is not an isomorphism and assertion ii) implies that $\delta(C_2) < \delta(C)$. Since $\delta(C')$ is a non-negative integer for any curve C', it follows that after finitely many blow-ups we obtain a smooth curve. $\square$

17.2. Birational morphisms between smooth surfaces

Our goal in this section is to prove a structural result about birational morphisms in dimension 2, and then discuss some consequences concerning birational maps between smooth surfaces. The key result is the following

PROPOSITION 17.2.1. *Let $f\colon X\to Y$ be a proper, birational morphism between smooth surfaces. If $P\in Y$ is a point that does not lie in the domain of f^{-1} and if $\pi\colon\widetilde{Y}\to Y$ is the blow-up of Y at P, then f factors through π, that is, the rational map $\pi^{-1}\circ f$ is a morphism.*

PROOF. Suppose that there is a point $Q\in X$ that does not lie in the domain of $\phi=\pi^{-1}\circ f$. Note first that since π^{-1} is defined on $Y\smallsetminus\{P\}$, we have $f(Q)=P$. Second, it follows from Proposition 17.1.11 that $\phi(Q)$ is connected, of dimension 1. Moreover, by Proposition 17.1.12, we have $\pi\big(\phi(Q)\big)=f(Q)=P$, hence $\phi(Q)\subseteq E$, and thus $\phi(Q)=E$. Let $U\subseteq\widetilde{Y}$ be the domain of ϕ^{-1} and $g\colon U\to X$ the corresponding morphism. Since $\widetilde{Y}\smallsetminus U$ is a finite set, we have $U\cap E\neq\emptyset$. We have seen that $\{Q\}\times E\subseteq\Gamma_\phi$, hence $E\times\{Q\}\subseteq\Gamma_{\phi^{-1}}$ (see Remark 17.1.9), and therefore $g(E\cap U)=\{Q\}$. Since P is not in the domain of f^{-1}, it follows from Proposition 17.1.1 that $f^{-1}(P)$ is connected, of dimension 1. Let C be an irreducible component of $f^{-1}(P)$ containing Q. Choose $R\in U\cap E$ and consider the ring extensions induced by f and g:

$$\mathcal{O}_{Y,P}\hookrightarrow\mathcal{O}_{X,Q}\hookrightarrow\mathcal{O}_{\widetilde{Y},R}$$

and let $\mathfrak{m}_{Y,P}$, $\mathfrak{m}_{X,Q}$, and $\mathfrak{m}_{\widetilde{Y},R}$ be the corresponding maximal ideals. Note that the above extensions are morphisms of local rings.

We choose regular systems of parameters $y_1,y_2\in\mathfrak{m}_{Y,P}$ and $u_1,u_2\in\mathfrak{m}_{\widetilde{Y},R}$ such that $y_1=u_1$ and $y_2=u_1u_2$. Therefore the image in $\mathcal{O}_{\widetilde{Y},R}$ of an equation defining E around R is u_1. If $v\in\mathfrak{m}_{X,Q}$ is the image of an equation defining C around Q, since $f(C)=\{P\}$, it follows that we can write $y_1=v\cdot w_1$ and $y_2=v\cdot w_2$, for some $w_1,w_2\in\mathcal{O}_{X,Q}$. Since $y_1\not\in\mathfrak{m}^2_{\widetilde{Y},R}$, it follows that $w_1\not\in\mathfrak{m}_{X,Q}$, and thus $y_2/y_1\in\mathcal{O}_{X,Q}$. On the other hand, we have $y_2/y_1\in\mathfrak{m}_{\widetilde{Y},R}$, and we conclude that $y_2/y_1\in\mathfrak{m}_{X,Q}$. Finally, since $g(E\cap U)=\{Q\}$, it follows that $\mathfrak{m}_{X,Q}\subseteq u_1\cdot\mathcal{O}_{\widetilde{Y},R}$. We thus deduce that $u_2=y_2/y_1$ lies in $u_1\cdot\mathcal{O}_{\widetilde{Y},R}$, contradicting the fact that u_1 and u_2 generate $\mathfrak{m}_{\widetilde{Y},R}$. This completes the proof. $\square$

COROLLARY 17.2.2. *Every proper, birational morphism $f\colon X\to Y$, where X and Y are smooth surfaces, can be written as a composition $f_1\circ\ldots\circ f_n$, with $n\geq 0$, where each f_i is the blow-up of a smooth surface at a point.*

PROOF. Recall that the exceptional locus $\mathrm{Exc}(f)$ has pure codimension 1 (see Proposition 17.1.4). We argue by induction on the number r of irreducible components in $\mathrm{Exc}(f)$. Note that $r=0$ if and only if f is an isomorphism, and in this case the assertion in the corollary trivially holds. Suppose now that f is not an isomorphism and let $P\in Y$ be a point that is not in the domain of f^{-1}. If $\pi\colon\widetilde{Y}\to Y$ is the blow-up of Y at P, then it follows from the proposition that we have a morphism $g\colon X\to\widetilde{Y}$ such that $f=\pi\circ g$. Note that g is again proper and birational, hence we are done by induction if we show that the number of irreducible components of $\mathrm{Exc}(g)$ is $<r$. By Remark 17.1.6, we have

$$\mathrm{Exc}(f)=\mathrm{Exc}(g)\cup g^{-1}\big(\mathrm{Exc}(\pi)\big)=\mathrm{Exc}(g)\cup g^{-1}(E),$$

where E is the exceptional divisor of π. Since the strict transform $\widetilde{E}$ of E on X lies in $\mathrm{Exc}(f)$, but it clearly does not lie in $\mathrm{Exc}(g)$, it follows that the set of irreducible components of $\mathrm{Exc}(g)$ is a proper subset of the set of irreducible components of $\mathrm{Exc}(f)$. This completes the proof of the induction step. $\square$

Remark 17.2.3. Starting with dimension 3, it is not true that every birational morphism $f\colon X \to Y$, with X and Y smooth projective varieties, factors as a composition of smooth blow-ups.

Remark 17.2.4. Suppose that X and Y are proper varieties over S and $\phi\colon X \dashrightarrow Y$ is a birational map of varieties over S. Let us assume that we are in a setting where resolution of singularities is known to hold[1]. In this case, if X and Y are smooth, we can find a smooth variety Z, with proper birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, such that $\phi = g \circ f^{-1}$. Indeed, it is enough to consider a resolution of singularities $Z \to \Gamma_\phi$ of the graph of ϕ. For a proof in dimension 2 that does not rely on resolution of singularities (but which assumes that X and Y are projective), see [**Har77**, Theorem 5.5].

Remark 17.2.5. In dimension 2, by combining the assertion in the previous remark with Corollary 17.2.2, we conclude that if $\phi\colon X \dashrightarrow Y$ is a birational map of proper surfaces over S, with both X and Y smooth, then there is a smooth surface Z with morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, both of which factor as compositions of blow-ups of smooth varieties along smooth subvarieties, and such that $\phi = g \circ f^{-1}$. The same assertion in higher dimension is known as the *Strong Factorization conjecture* and it is widely open. A weaker statement, known as the *Weak Factorization theorem*, was proved by Abramovich, Karu, Matsuki, and Włodarczyk when $\mathrm{char}(k) = 0$, see [**Wło03**] and [**AKMW02**]. One version of this says that if $\phi\colon X \dashrightarrow Y$ is a birational map between smooth, irreducible, complete varieties, then ϕ factors as $f_1 \circ \ldots \circ f_n$, where each $f_i\colon Z_i \dashrightarrow Z_{i-1}$ is either the blow-up of a smooth, irreducible, complete variety along a smooth, irreducible, closed subvariety, or the inverse of such map. This result provides a powerful tool for proving that various invariants of smooth, complete varieties are birational invariants. To give one easy example: it implies that if X and Y are birational smooth, complete varieties, then $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y)$ for all i; in particular, we have $p_a(X) = p_a(Y)$. Indeed, by Weak Factorization, it is enough to prove this when we have a morphism $f\colon X \to Y$ which is the blow-up along the smooth subvariety Z of Y. In this case, the assertion follows from Proposition 17.1.16.

17.3. Castelnuovo's contractibility criterion

We have seen in Remark 16.2.8 that if X is a smooth, projective surface and $f\colon X \to Y$ is a morphism to another surface such that $f(C)$ is a point for some (irreducible) curve C on X, then $(C^2) < 0$. The main result in this section is a converse when the curve C is smooth and rational.

Theorem 17.3.1. *Let X be a smooth projective surface and C a curve on X such that $C \simeq \mathbf{P}^1$ and $(C^2) = -n$, for some $n > 0$. In this case there is a morphism*

[1]In other words, suppose that either $\mathrm{char}(k) = 0$ or that $\dim(X) = \dim(Y) \leq 3$. Note that resolution of singularities in positive characteristic was proved in dimension 2 by Abhyankar [**Abh56**] and in dimension 3 by Abhyankar [**Abh66**] for $\mathrm{char}(k) > 5$, with the remaining cases treated by Cossart and Piltant [**CP08**] and [**CP09**].

$f\colon X \to Y$, where Y is a projective surface, such that $f(C)$ is a point $P \in Y$ and f induces an isomorphism $f^{-1}(Y \smallsetminus \{P\}) = X \smallsetminus C \to Y \smallsetminus \{P\}$. Moreover, if R is the local ring at the origin of the affine cone over a rational normal curve in $\mathbf{P}^n$, then we have an isomorphism $\widehat{\mathcal{O}_{Y,P}} \simeq \widehat{R}$.

Proof. Let D be a very ample divisor on X. After replacing D by a suitable multiple, we may assume that the following conditions hold:

α) $H^1\big(X, \mathcal{O}_X(D)\big) = 0$.
β) $(D \cdot C) = k = jn$ for some integer $j \geq 1$.

Note first that for every i, with $0 \leq i \leq j$, we have $\big((D + iC) \cdot C\big) = (j - i)n$, and since $C \simeq \mathbf{P}^1$, we get

$$\mathcal{O}_X(D + iC)|_C \simeq \mathcal{O}_{\mathbf{P}^1}\big((j - i)n\big).$$

Claim 1. For every i, with $0 \leq i \leq j$, we have $H^1\big(X, \mathcal{O}_X\big(D + iC\big)\big) = 0$. In order to see this, we argue by induction in i, the case $i = 0$ being covered by α) above. For the induction step, note that for $i \geq 1$, we have a short exact sequence

$$0 \to \mathcal{O}_X\big(D + (i - 1)C\big)\big) \to \mathcal{O}_X(D + iC) \to \mathcal{O}_X(D + iC)|_C \to 0.$$

The long exact sequence in cohomology gives for $i \leq j$

$$H^1\big(X, \mathcal{O}_X(D + (i - 1)C)\big) \to H^1\big(X, \mathcal{O}_X(D + iC)\big) \to H^1\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}((j - i)n)\big) = 0,$$

so that $H^1\big(X, \mathcal{O}_X(D + (i - 1)C))\big) = 0$ implies $H^1\big(X, \mathcal{O}_X(D + iC)\big) = 0$. This completes the proof of the above claim.

Claim 2. The line bundle $\mathcal{O}_X(D + jC)$ is globally generated. First, since $\mathcal{O}_X(D)$ is globally generated, it follows that the base locus of $\mathcal{O}_X(D + jC)$ is contained in C. Second, since the restriction $\mathcal{O}_X(D + jC)|_C$ is isomorphic to $\mathcal{O}_C$, which is globally generated, and since the short exact sequence

$$0 \to \mathcal{O}_X(D + (j - 1)C)\big) \to \mathcal{O}_X(D + jC) \to \mathcal{O}_X(D + jC)|_C \to 0$$

induces by Claim 1 a surjective map

$$H^0\big(X, \mathcal{O}_X(D + (j - 1)C)\big) \to H^0\big(X, \mathcal{O}_X(D + jC)\big),$$

we deduce that C does not intersect the base locus of $\mathcal{O}_X(D + jC)$. We thus conclude that $\mathcal{O}_X(D + jC)$ is globally generated.

Consider the morphism $g\colon X \to \mathbf{P}^N$ defined by the complete linear system $|\mathcal{O}_X(D+jC)|$ and let Y_0 be the image. Note first that since $\deg\big(\mathcal{O}_X(D+jC)|_C\big) = 0$, it follows from the projection formula that $g(C)$ is a point P_0 in Y_0. Since $\mathcal{O}_X(D)$ is very ample, for every point $Q \in X \smallsetminus C$, we have a section of $\mathcal{O}_X(D + jC)$ which vanishes on Q, but does not vanish on some point of C; therefore $X \smallsetminus C = g^{-1}(Y_0 \smallsetminus \{P_0\})$. Moreover, from the very ampleness of $\mathcal{O}_X(D)$, it follows that the linear syatem $|\mathcal{O}_X(D+jC)|$ separates the points in $X \smallsetminus C$ and the tangent directions at the points in $X \smallsetminus C$. By Proposition 11.5.18 (see also Remark 11.5.19), we see that the induced morphism $X \smallsetminus C \to Y_0 \smallsetminus \{P_0\}$ is an isomorphism. In particular, $Y_0 \smallsetminus \{P_0\}$ is smooth.

Consider the normalization morphism $h\colon Y \to Y_0$. Since X is normal, we have a unique morphism $f\colon X \to Y$ such that $g = h \circ f$ (see Proposition 9.1.9). The image $f(C)$ is a point P in Y with $h(P) = P_0$ (this follows from the fact that h

has finite fibers). Moreover, since $Y_0 \smallsetminus \{P_0\}$ is normal, h is an isomorphism over $Y_0 \smallsetminus \{P_0\}$, hence f induces an isomorphism

$$X \smallsetminus C \to h^{-1}(Y_0 \smallsetminus \{P_0\}) = Y \smallsetminus \{P\}$$

(note that if $P' \in Y$ is such that $h(P') = P$, then $P' \in f(C) = \{P\}$).

Since f is birational and Y is normal, we have $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ (see Example 14.1.6). By the Formal Function theorem (see also Remark 14.1.12), we have

$$\widehat{\mathcal{O}_{Y,P}} \simeq f_*(\mathcal{O}_X)\widehat{\ }_P \simeq \varprojlim H^0(X, \mathcal{O}_X/\mathcal{J}^{i+1}),$$

where $\mathcal{J}$ is the radical ideal sheaf defining C in X. Since both X and C are smooth, we have

$$\mathcal{J}^i/\mathcal{J}^{i+1} \simeq \mathrm{Sym}_{\mathcal{O}_X/\mathcal{J}} \mathcal{J}/\mathcal{J}^2 \quad \text{for all} \quad i \geq 0$$

(see Proposition 6.3.21). Using the fact that $\mathcal{J}/\mathcal{J}^2 \simeq \mathcal{O}_X(-C)|_C \simeq \mathcal{O}_{\mathbf{P}^1}(n)$, we thus conclude that

$$\mathcal{J}^i/\mathcal{J}^{i+1} \simeq \mathcal{O}_{\mathbf{P}^1}(ni) \quad \text{for all} \quad i \geq 0.$$

The exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1}(ni) \to \mathcal{O}_X/\mathcal{J}^{i+1} \to \mathcal{O}_X/\mathcal{J}^i \to 0$$

induces an exact sequence

$$(17.3.1) \qquad 0 \to H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(ni)\big) \to H^0(X, \mathcal{O}_X/\mathcal{J}^{i+1}) \to H^0(X, \mathcal{O}_X/\mathcal{J}^i) \to 0.$$

Note that if S is the coordinate ring for the affine cone over the rational normal curve in $\mathbf{P}^n$, then we have a canonical injective map

$$\phi \colon S \to \bigoplus_{i \geq 0} H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(ni)\big),$$

which is an isomorphism in degree 1. In fact, since the canonical maps

$$\mathrm{Sym}^i H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n)\big) \to H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(ni)\big)$$

are surjective, it follows that ϕ is an isomorphism. It is now straightforward to see that if $\mathfrak{m}$ is the maximal ideal in S corresponding to the origin, then using the exact sequences (17.3.1), we can construct a k-algebra isomorphism

$$\varprojlim H^0(X, \mathcal{O}_X/\mathcal{J}^{i+1}) \simeq \varprojlim S/\mathfrak{m}^{i+1} \simeq \widehat{R}.$$

This completes the proof of the theorem. $\square$

Definition 17.3.2. Given a smooth, projective surface X, a curve C on X is an *exceptional curve of the first kind* if $C \simeq \mathbf{P}^1$ and $(C^2) = -1$.

Of course, if Y is a smooth, projective surface and $f \colon X \to Y$ is the blow-up of Y at a point, with exceptional divisor E, then E is an exceptional curve of the first kind. An important consequence of Theorem 17.3.1, known as *Castelnuovo's Contractibility criterion*, says that the converse holds: every exceptional curve of the first kind is the exceptional divisor of a blow-down map. More precisely, we have the following

Corollary 17.3.3. *If X is a smooth, projective surface and $C \subseteq X$ is an exceptional curve of the first kind, then we have a morphism $f \colon X \to Y$, where Y is a smooth, projective surface, and we have a point $P \in Y$ such that X is isomorphic over Y to the blow-up of Y at P, and C corresponds to the exceptional divisor.*

Proof. It follows from Theorem 17.3.1 that we have a morphism $f\colon X \to Y$, where Y is a projective surface, and we have a point $P \in Y$ such that $f^{-1}(P) = C$ and the induced morphism $X \smallsetminus C \to Y \smallsetminus \{P\}$ is an isomorphism. Moreover, since $(C^2) = -1$, it follows from the theorem that $\widehat{\mathcal{O}_{Y,P}} \simeq k[\![x,y]\!]$ and thus P is a smooth point of Y (see Remark 9.2.5). Therefore Y is smooth. Since f is birational and P is not in the domain of f^{-1}, it follows from Proposition 17.2.1 that if $\pi\colon \widetilde{Y} \to Y$ is the blow-up of Y at P, then the rational map $g = \pi^{-1} \circ f$ is a morphism. If E is the exceptional divisor of π, then its strict transform $\widetilde{E}$ on X maps to P, hence it is equal to C. It follows that there are no g-exceptional curves on X, hence g is an isomorphism by Proposition 17.1.4. $\square$

Remark 17.3.4. Another interesting case of Theorem 17.3.1 is that of a (-2)-curve, that is, a curve $C \simeq \mathbf{P}^1$ such that $(C^2) = -2$. In this case, it follows from the theorem that we have a morphism $f\colon X \to Y$ such that if $f(C) = P$, then $\widehat{\mathcal{O}_{Y,P}} \simeq \widehat{R}$, where R is the local ring at the origin for the affine cone over a smooth conic in $\mathbf{P}^2$ (for example, if $\mathrm{char}(k) \neq 2$, we have $\widehat{\mathcal{O}_{Y,P}} \simeq k[\![x,y,x]\!]/(x^2+y^2+z^2)$). Such a singular point is an A_1-*singularity.* It is not hard to see that in this case the blow-up of Y at P is smooth and then deduce that X is isomorphic over Y with this blow-up.

Definition 17.3.5. A smooth projective surface X is a *minimal surface* if there is no morphism $f\colon X \to Y$, where Y is a smooth surface. Note that by combining Corollaries 17.2.2 and 17.3.3, X is minimal if and only if there is no exceptional curve of the first kind in X.

Proposition 17.3.6. *If X is a smooth, projective surface, then there is a finite sequence of morphisms*

$$X = X_1 \to X_2 \to \ldots \to X_r = Y$$

such that all X_i are smooth, projective surfaces, with Y minimal, such that each X_{i-1} is isomorphic over X_i to the blow-up of X_i at a point $P_i \in X_i$, for $2 \leq i \leq r$.

Proof. If X contains no exceptional curve of the first kind, then X is minimal and we are done. If there is such a curve C, then by Corollary 17.3.3, X is isomorphic to the blow-up of a smooth variety X_2 at a point $P_2 \in X_2$. We then repeat. The only thing we need to prove is that this process stops after finitely many steps. This is a consequence of the fact that $\mathrm{rank}\big(\mathrm{N}^1(X_{i+1})\big) = \mathrm{rank}\big(\mathrm{N}^1(X_i)\big) - 1$ by Proposition 17.1.15 and the fact that these ranks are finite by the theorem of the base. $\square$

The starting point in the classification of surfaces says if X is a minimal surface, then either X is a *minimal model* (this means that $(K_X \cdot C) \geq 0$ for every curve C on X) or $X \simeq \mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle on a smooth, projective curve C. For a thorough discussion of classification of surfaces, see [**Bea96**] for a treatment over $\mathbf{C}$ and [**Bădut01**] for the case of arbitrary ground fields.

APPENDIX A

Finite and integral homomorphisms

A running assumption for all the appendices is that all rings are commutative, unital (that is, they have multiplicative identity), and all homomorphisms are of unital rings (that is, they map the identity to the identity). In this appendix we discuss the definition and basic properties of integral and finite ring homomorphisms.

A.1. Definitions

Let $\varphi\colon R \to S$ be a ring homomorphism. One says that φ is of *finite type* if S becomes, via φ, a finitely generated R-algebra. One says that φ is *finite* if S becomes, via φ, a finitely generated R-module. One says that φ is *integral* if every element $y \in S$ is integral over R, that is, there is a positive integer n, and elements $a_1, \ldots, a_n \in R$, such that

$$y^n + a_1 y^{n-1} + \ldots + a_n = 0 \quad \text{in} \quad S.$$

Remark A.1.1. It is clear that if φ is finite, then it is of finite type: if $y_1, \ldots, y_m \in S$ generate S as an R-module, then they also generate it as an R-algebra. The converse is of course false: for example, the inclusion $R \hookrightarrow R[x]$ is finitely generated, but not finite (the R-submodule of $R[x]$ generated by finitely many polynomials consists of polynomials of bounded degree).

Remark A.1.2. If φ is of finite type and integral, then it is finite. Indeed, if $y_1, \ldots, y_r$ generate S as an R-algebra, and we can write

$$y_i^{d_i} + a_{i,1} y_i^{d_i - 1} + \ldots + a_{i,d_i} = 0$$

for some positive integers d_i and some $a_{i,j} \in R$, then it is easy to see that

$$\{y_1^{a_1} \cdots y_r^{a_r} \mid 0 \le a_i \le d_i - 1\}$$

generate S as an R-module.

Proposition A.1.3. *If φ is finite, then it is integral.*

Proof. The assertion follows from the *determinantal trick*: suppose that $b_1, \ldots, b_n$ generate S as an R-module. For every $y \in S$, we can write for each $1 \le i \le n$:

$$y b_i = \sum_{j=1}^{n} a_{i,j} b_j \quad \text{for some} \quad a_{i,j} \in R.$$

If A is the matrix $(a_{i,j})_{1 \le i,j \le n}$ and I is the identity matrix, then we see that

$$(yI - A) \cdot \begin{pmatrix} b_1 \\ \ldots \\ b_n \end{pmatrix} = 0.$$

By multiplying with the classical adjoint of $yI - A$, we see that if $D = \det(yI - A)$, then $Db_i = 0$ for all i. This implies $D \cdot S = 0$, and in particular $D \cdot 1_S = 0$. However, it is clear that we can write

$$D = y^n + c_1 y^{n-1} + \ldots + c_n \quad \text{for some} \quad c_1, \ldots, c_n \in R.$$

We thus see that y is integral over R. $\square$

REMARK A.1.4. We will almost always consider homomorphisms of finite type. For such a homomorphism φ, it follows from Remark A.1.2 and Proposition A.1.3 that φ is finite if and only if it is integral.

A.2. Easy properties

The following property of integral morphisms is very useful.

PROPOSITION A.2.1. *If $\varphi\colon R \hookrightarrow S$ is an integral injective homomorphism of integral domains, then R is a field if and only if S is a field.*

PROOF. Suppose first that R is a field, and let $u \in S \smallsetminus \{0\}$. Since u is integral over R, it follows that we can write

$$u^n + a_1 u^{n-1} + \ldots + a_n = 0$$

for some positive integer n, and some $a_1, \ldots, a_n \in R$. We may assume that n is chosen to be minimal; in this case, since $u \neq 0$, we have $a_n \neq 0$. We see that we have $uv = 1$, where

$$v = (-a_n)^{-1} \cdot (u^{n-1} + \ldots + a_{n-2}u + a_{n-1}),$$

hence u is invertible. Since this holds for every nonzero u, it follows that S is a field.

Conversely, suppose that S is a field and let $a \in R \smallsetminus \{0\}$. Let $b = \frac{1}{a} \in S$. Since b is integral over R, we can write

$$b^r + \alpha_1 b^{r-1} + \ldots + \alpha_r = 0$$

for some positive integer r and some $\alpha_1, \ldots, \alpha_r \in R$. Since

$$\frac{1}{a} = -\alpha_1 - \alpha_2 a - \ldots - \alpha_r a^{r-1} \in A,$$

we conclude that a in invertible in R. Since this holds for every nonzero a, it follows that R is a field. $\square$

PROPOSITION A.2.2. *Given a ring homomorphism $\varphi\colon R \to S$, the subset*

$$S' := \{y \in S \mid y \text{ integral over } R\}$$

is a subring of S. This is the integral closure of R in S.

PROOF. Since it is clear that $1_S \in S'$, we only need to check that for every $y_1, y_2 \in S'$, we have $y_1 - y_2, y_1 y_2 \in S'$. Since y_1 and y_2 are integral over R, the subring $R[y_1, y_2]$ of S is finite over R (the argument is the same as in Remark A.1.2). In particular, it is integral over R by Proposition A.1.3. This implies that $y_1 - y_2$ and $y_1 y_2$, which lie in $R[y_1, y_2]$, are integral over R. $\square$

PROPOSITION A.2.3. *Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be two ring homomorphisms. If both φ and ψ are of finite type (respectively finite, integral), then $\psi \circ \varphi$ has the same property.*

PROOF. The assertion is straightforward for finite and finite type morphisms. Suppose now that φ and ψ are integral. Given $u \in T$, we can write

$$u^n + b_1 u^{n-1} + \ldots + b_n = 0$$

for some positive integer n and $b_1, \ldots, b_n \in S$. Since $b_1, \ldots, b_n$ are integral over R, it follows that $R' := R[b_1, \ldots, b_n]$ is finite over R (see Remark A.1.2). Since u is integral over R', it follows that $R'[u]$ is finite over R', and therefors it is finite over R. By Proposition A.1.3, we conclude that u is integral over R. $\square$

APPENDIX B

Noetherian rings and modules

In this appendix we discuss the definition and basic properties of Noetherian rings and modules. The main result is Hilbert's basis theorem.

B.1. Definitions

PROPOSITION B.1.1. *Given a ring R and an R-module M, the following are equivalent:*

i) *Every submodule N of M is finitely generated.*
ii) *There is no infinite strictly increasing chain of submodules of M:*

$$N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \dots .$$

iii) *Every nonempty family of submodules of M contains a maximal element.*

An R-module M is *Noetherian* if it satisfies the equivalent conditions in the proposition. The ring R is Noetherian if it is Noetherian as an R-module.

PROOF OF PROPOSITION B.1.1. Suppose first that i) holds. If there is an infinite strictly increasing sequence of submodules of M as in ii), consider $N := \bigcup_{i\geq 1} N_i$. This is a submodule of M, hence it is finitely generated by i). If $u_1, \dots, u_r$ generate N, then we can find m such that $u_i \in N_m$ for all m. In this case we have $N = N_m$, contradicting the fact that the sequence is strictly increasing.

The implication ii)⇒iii) is clear: if a nonempty family $\mathcal{F}$ has no maximal element, let us choose $N_1 \in \mathcal{F}$. Since this is not maximal, there is $N_2 \in \mathcal{F}$ such that $N_1 \subsetneq N_2$, and we continue in this way to construct an infinite strictly increasing sequence of submodules of M.

In order to prove the implication iii)⇒i), let N be a submodule of M and consider the family $\mathcal{F}$ of all finitely generated submodules of N. This is nonempty, since it contains the zero submodule. By iii), $\mathcal{F}$ has a maximal element N''. If $N'' \neq N$, then there is $u \in N \smallsetminus N''$ and the submodule $N'' + Ru$ is a finitely generated submodule of N strictly containing N'', a contradiction. Therefore $N'' = N$ and thus N is finitely generated. □

PROPOSITION B.1.2. *Given a short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of R-modules, M is Noetherian if and only if both M' and M'' are Noetherian.

PROOF. Suppose first that M is Noetherian. Since every submodule of M' is a submodule of M, hence finitely generated, it follows that M' is Noetherian. Since every submodule of $M'' \simeq M/M'$ is isomorphic toN/M', for a submodule N of M that contains M', and since N being finitely generated implies that N/M' is finitely generated, we conclude that M'' is Noetherian.

Conversely, suppose that both M' and $M'' \simeq M/M'$ are Noetherian, and let N be a submodule of M. Since $N \cap M'$ is a submodule of M', it is finitely generated, and since $N/(N \cap M')$ is isomorphic to a submodule of M/M', we have that $N/(N \cap M')$ is finitely generated. Finally, since both $N \cap M'$ and $N/(N \cap M')$ are finitely generated, it is clear that N is finitely generated. $\square$

COROLLARY B.1.3. *If R is a Noetherian ring, then an R-module M is Noetherian if and only if it is finitely generated.*

PROOF. We only need to show that if M is finitely generated, then it is Noetherian, since the converse follows from definition. Since M is finitely generated, we have a surjective morphism $R^{\oplus n} \to M$, and it follows from the proposition that it is enough to show that $R^{\oplus n}$ is Noetherian. This follows again from the proposition by induction on n. $\square$

REMARK B.1.4. If R is a Noetherian ring and I is an ideal in R, then R/I is a Noetherian ring. This is an immediate application of Corollary B.1.3.

REMARK B.1.5. If R is a Noetherian ring and $S \subseteq R$ is a multiplicative system, then the fraction ring $S^{-1}R$ is Noetherian. Indeed, every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R. If I is generated by $a_1, \ldots, a_r$, then $S^{-1}I$ is generated as an ideal of $S^{-1}R$ by $\frac{a_1}{1}, \ldots, \frac{a_r}{r}$.

B.2. Hilbert's basis theorem

The following theorem is one of the basic results in commutative algebra.

THEOREM B.2.1 (Hilbert). *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

PROOF. Let I be an ideal in $R[x]$. We consider the following recursive construction. If $I \neq 0$, let $f_1 \in I$ be a polynomial of minimal degree. If $I \neq (f_1)$, then let $f_2 \in I \smallsetminus (f_1)$ be a polynomial of minimal degree. Suppose now that $f_1, \ldots, f_n$ have been chosen. If $I \neq (f_1, \ldots, f_n)$, let $f_{n+1} \in I \smallsetminus (f_1, \ldots, f_n)$ be a polynomial of minimal degree.

If this process stops, then I is finitely generated. Let us assume that this is not the case, aiming for a contradiction. We write

$$f_i = a_i x^{d_i} + \text{lower degree terms}, \quad \text{with} \quad a_i \neq 0.$$

By our minimality assumption, we have

$$d_1 \leq d_2 \leq \ldots.$$

Let J be the ideal of R generated by the a_i, with $i \geq 1$. Since R is Noetherian, J is a finitely generated ideal, hence there is m such that J is generated by $a_1, \ldots, a_m$. In particular, we can find $u_1, \ldots, u_m \in R$ such that

$$a_{m+1} = \sum_{i=1}^{m} a_i u_i.$$

In this case, we have

$$h := f_{m+1} - \sum_{i=1}^{m} u_i x^{d_{m+1}-d_i} f_i \in I \smallsetminus (f_1, \ldots, f_m)$$

and $\deg(h) < d_{m+1}$, a contradiction. This completes the proof of the theorem. $\square$

By applying Theorem B.2.1 several times, we obtain

Corollary B.2.2. *If R is a Noetherian ring, then the polynomial ring $R[x_1, \dots, x_n]$ is Noetherian for every positive integer n.*

In particular, since a field is clearly Noetherian, we obtain

Corollary B.2.3. *For every field k and every positive integer n, the polynomial ring $k[x_1, \dots, x_n]$ is Noetherian.*

A similar argument to the one in the proof of Theorem B.2.1 gives the following variant for formal power series. We do not give a proof, since we will not really need it (the result will only be used in the proof of Proposition G.2.7, which in turn, will only be needed for the proof of Lemma 13.2.14). For a proof, see [**Mat89**, Theorem 3.3].

Theorem B.2.4. *If R is a Noetherian ring, then the formal power series ring $R[\![x]\!]$ is Noetherian. More generally, for every $n \geq 1$, the ring $R[\![x_1, \dots, x_n]\!]$ is Noetherian.*

APPENDIX C

Nakayama's lemma and Krull's intersection theorem

In this appendix we collect a few basic results on local rings and localization. We begin with Nakayama's lemma and an application to finitely generated projective modules over local rings. We then overview some general results concerning the behavior of certain properties of modules under localization. We prove the Artin-Rees lemma and deduce Krull's Intersection theorem. In the last section we introduce discrete valuation rings (we will return to this topic in a later appendix).

C.1. Nakayama's lemma

The following is one of the most basic results in commutative algebra, known as Nakayama's lemma.

PROPOSITION C.1.1. *If $(A, \mathfrak{m})$ is a local ring and M is a finitely generated module over A such that $M = \mathfrak{m}M$, then $M = 0$.*

PROOF. The proof is another application of the determinantal trick. Let $u_1, \ldots, u_n$ be generators of M over A. Since $M = \mathfrak{m}M$, for every i we can write

$$u_i = \sum_{j=1}^{n} a_{i,j} u_j \quad \text{for some} \quad a_{i,j} \in \mathfrak{m}.$$

If A is the matrix $(a_{i,j})_{1 \leq i,j \leq n}$ and I is the identity matrix, then we can rewrite the above relations as

$$(I - A) \cdot \begin{pmatrix} u_1 \\ \ldots \\ u_n \end{pmatrix} = 0.$$

By multiplying with the classical adjoint of $I - A$, we conclude that $\det(I - A) \cdot u_i = 0$ for all i. Since all entries of A lie in $\mathfrak{m}$, it is clear that

$$\det(I - A) \equiv 1 \pmod{\mathfrak{m}}.$$

Since A is local, it follows that $\det(I - A)$ is invertible, and therefore we conclude that $u_i = 0$ for all i, hence $M = 0$. □

This is sometimes applied in the following form.

COROLLARY C.1.2. *If $(A, \mathfrak{m})$ is a local ring, M is a finitely generated module over A, and N is a submodule of M such that $M = N + \mathfrak{m}M$, then $N = M$.*

PROOF. The assertion follows by applying the proposition to M/N. □

Remark C.1.3. The above corollary implies, in particular, that given elements $u_1, \ldots, u_r$ of M, they generate M if and only if their classes $\overline{u_1}, \ldots, \overline{u_r} \in M/\mathfrak{m}M$ generate $M/\mathfrak{m}M$ over $k = A/\mathfrak{m}$. We thus see that the cardinality of every minimal system of generators of M is equal to $\dim_k M/\mathfrak{m}M$.

C.2. Projective modules over local rings

Proposition C.2.1. *If $(A, \mathfrak{m})$ is a local Noetherian ring and M is a finitely generated A-module, then M is projective if and only if M is free.*

Proof. Consider a minimal system of generators $u_1, \ldots, u_n$ for M and the surjective morphism of A-modules

$$\phi\colon F = A^{\oplus n} \to M, \quad \phi(e_i) = u_i \quad \text{for} \quad 1 \leq i \leq n.$$

If $N = \ker(\phi)$, since A is Noetherian and F is a finitely generated A-module, it follows that N is a finitely generated A-module. Since M is projective, the exact sequence

$$0 \to N \to F \to M \to 0$$

is split, hence tensoring with $k = A/\mathfrak{m}$ gives an exact sequence

$$0 \to N/\mathfrak{m}N \to k^{\oplus n} \to M/\mathfrak{m}M \to 0.$$

However, we have seen in Remark C.1.3 that the elements $\overline{u_1}, \ldots, \overline{u_n} \in M/\mathfrak{m}M$ form a basis, so that we deduce from the above exact sequence that $N/\mathfrak{m}N = 0$. Since N is finitely generated, it follows from Nakayama's lemma that $N = 0$, hence $M \simeq F$ is free. □

Remark C.2.2. It is a result of Kaplansky (see [**Kap58**]) that if M is any projective module over a local ring, then M is free.

C.3. Modules and localization

We collect in this section some easy properties relating statements about modules to corresponding statements about certain localizations.

Proposition C.3.1. *Given an A-module M, the following are equivalent:*

i) $M = 0$.
ii) $M_{\mathfrak{p}} = 0$ *for all maximal ideals* $\mathfrak{p}$ *in* A.
iii) $M_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$ *in* A.
iv) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and $M_{f_i} = 0$ for all i.*

Proof. The implication iv)⇒iii) follows from the fact that if $f_1, \ldots, f_r$ generate the unit ideal, then for every prime ideal $\mathfrak{p}$ in A, there is i such that $f_i \not\in \mathfrak{p}$, in which case $M_{\mathfrak{p}}$ is a localization of M_{f_i}. Since the implications i)⇒iv) and iii)⇒ii) are trivial, in order to complete the proof it is enough to prove the implication ii)⇒i). Let $u \in M$ and consider $\mathrm{Ann}_A(u)$. For every maximal ideal $\mathfrak{p}$ in A, we have $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$, hence $\mathrm{Ann}_A(u) \not\subseteq \mathfrak{p}$. This implies that $\mathrm{Ann}_A(u) = A$, hence $u = 0$. □

Remark C.3.2. The same argument in the proof of the above proposition shows that if M is an A-module and $u \in M$, then the following assertions are equivalent:

i) $u = 0$.
ii) $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ in A.

iii) $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in A.
iv) There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and $\frac{u}{1} = 0$ in M_{f_i} for all i.

COROLLARY C.3.3. *If M is an A-module and M', M'' are submodules of M, then the following are equivalent:*

i) *$M' \subseteq M''$.*
ii) *$M'_{\mathfrak{p}} \subseteq M''_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ in A.*
iii) *$M'_{\mathfrak{p}} \subseteq M''_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in A.*
iv) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and $M'_{f_i} \subseteq M''_{f_i}$ for all i.*

PROOF. We can simply apply Proposition C.3.1 for the A-module $(M'+M'')/M''$. ☐

COROLLARY C.3.4. *Given two morphisms of A-modules*

$$M' \xrightarrow{\phi} M \xrightarrow{\psi} M'',$$

the following are equivalent:

i) *The above sequence is exact.*
ii) *The induced sequence*

$$M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$$

is exact for every prime (maximal) ideal $\mathfrak{p}$ in A.
iii) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and each induced sequence*

$$M'_{f_i} \to M_{f_i} \to M''_{f_i}$$

is exact

PROOF. The exactness of the sequence in the statement is equivalent to the two inclusions

$$\mathrm{Im}(\phi) \subseteq \ker(\phi) \quad \text{and} \quad \ker(\phi) \subseteq \mathrm{Im}(\phi).$$

The equivalence in the statement now follows by applying Corollary C.3.3 for the submodules $\mathrm{Im}(\phi)$ and $\ker(\psi)$ of M (note that localization is an exact functor, hence it commutes with taking the image and kernel). ☐

COROLLARY C.3.5. *Given an A-module M, the following are equivalent:*

i) *M is a finitely generated A-module.*
ii) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and each M_{f_i} is a finitely generated A_{f_i}-module.*

PROOF. For every i, we may choose finitely many $u_{i,j} \in M$ such that $\left\{\frac{u_{i,j}}{1} \mid j\right\}$ generate M_{f_i} as an A_{f_i}-module. It follows that if N is the A-submodule of M generated by all $u_{i,j}$, then N is finitely generated and $(M/N)_{f_i} = 0$ for all i. We then deduce from Proposition C.3.1 that $M = N$. ☐

C.4. Krull's Intersection theorem

THEOREM C.4.1. *If I is an ideal in a Noetherian ring A, M is a finitely generated A-module, and $N = \bigcap_{m\geq 1} I^m M$, then $IN = N$. In particular, if $(A, \mathfrak{m})$ is a local ring and $I \subseteq \mathfrak{m}$, then $N = 0$.*

We will deduce the theorem from the following result, known as the Artin-Rees lemma.

LEMMA C.4.2. *Let A be a Noetherian ring and I an ideal in A. If M is a finitely generated A-module and N is a submodule of M, then for every $n \geq 0$, there is $m \geq 0$ such that*

$$I^m M \cap N \subseteq I^n N.$$

PROOF. Consider the $\mathbf{N}$-graded ring

$$R(A, I) := \bigoplus_{j\geq 0} I^n t^n \subseteq A[t].$$

Note that if I is generated by $a_1, \ldots, a_r$, then $R(A, I)$ is generated over A by $a_1 t, \ldots, a_r t$. In particular, $R(A, I)$ is a Noetherian ring.

Consider now the $\mathbf{N}$-graded $R(A, I)$-module

$$T = \bigoplus_{j\geq 0} I^j M t^j \subseteq M[t] = M \otimes_A A[t].$$

Since M is finitely generated over A, it is clear that T is a finitely generated $R(A, I)$-module. Consider the $R(A, I)$-submodule of T given by

$$\bigoplus_{j\geq 0} (N \cap I^j M) t^j.$$

Since M is a finitely generated module over a Noetherian ring, it follows that M is Noetherian, hence N is finitely generated. Choose generators of N of the form $u_j t^{d_j}$ for some $u_j \in N \cap I^{d_j} M$, with $1 \leq j \leq r$. Given any $u \in N \cap I^m M$ we can thus write

$$ut^m = \sum_{j=1}^{r} (a_j t^{b_j}) \cdot (u_j t^{d_j})$$

for some $a_j \in I^{b_j}$, where $b_j = m - d_j$. We thus see that if $m \geq n + d_j$ for all j, then

$$N \cap I^m M \subseteq I^n N.$$

This completes the proof of the lemma. □

PROOF OF THEOREM C.4.1. Of course, we only need to show that $N \subseteq IN$. We apply the lemma for the submodule N of M to get a non-negative integer m such that $I^m M \cap N \subseteq IN$. However, since $N \subseteq I^m M$, this implies $N \subseteq IN$. The last assertion in the theorem is a consequence of Nakayama's lemma. □

C.5. Discrete Valuation Rings

Recall that a *discrete valuation* on a field K is a surjective map $v \colon K \to \mathbf{Z} \cup \{\infty\}$ that satisfies the following properties:

i) $v(a) = \infty$ if and only if $a = 0$.
ii) $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.
iii) $v(ab) = v(a) + v(b)$ for all $a, b \in K$.

PROPOSITION C.5.1. *Given an integral domain R, with fraction field K, the following are equivalent:*

i) *There is a discrete valuation v on K such that $R = \{a \in K \mid v(a) \geq 0\}$.*
ii) *R is a local PID, which is not a field.*
iii) *R is local, Noetherian, and the maximal ideal is principal and non-zero.*

A ring that satisfies the above equivalent properties is a *discrete valuation ring* (or *DVR*, for short).

PROOF. Let us show first that i)$\Rightarrow$ii). Let $\mathfrak{m} = \{a \in K \mid v(a) > 0\}$. It follows from the definition of a discrete valuation that $\mathfrak{m}$ is an ideal in R and that for every $u \in R \smallsetminus \mathfrak{m}$, we have $u^{-1} \in R$. Therefore R is local and $\mathfrak{m}$ is the maximal ideal of R. Given any non-zero ideal I in R, consider $a \in I$ such that $v(a)$ is minimal. Given any other $b \in I$, we have $v(b) \geq v(a)$, hence $v(ba^{-1}) \geq 0$, and therefore $b \in (a)$. This shows that $I = (a)$ and therefore R is a PID. Note that R is not a field, since an element $a \in K$ with $v(a) = 1$ is a non-invertible element of R.

Since the implication ii)$\Rightarrow$iii) is trivial, in order to complete the proof, it is enough to prove iii)$\Rightarrow$i). Suppose that $(R, \mathfrak{m})$ is a Noetherian local domain and $\mathfrak{m} = (\pi)$, for some $\pi \neq 0$. Given any non-zero element α, it follows from Theorem C.4.1 that there is $j \geq 0$ such that $\alpha \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. Therefore we can write $\alpha = u\pi^j$, with u invertible. Since K is the fraction ring of R, it follows that every non-zero element β in K can be written as $\beta = u\pi^j$ for some $j \in \mathbf{Z}$ and $u \in R \smallsetminus \mathfrak{m}$. If we put $v(\beta) = j$, then it is straightforward to check that v is a discrete valuation and $R = \{a \in K \mid v(a) \geq 0\}$. $\square$

REMARK C.5.2. Note that if R is a DVR, then $\dim(R) = 1$. Indeed, we have seen in the above proof that if the maximal ideal $\mathfrak{m}$ is generated by π, then every nonzero ideal of R is generated by some π^r, with $r \geq 0$. This implies that the only prime ideals of R are (0) and $\mathfrak{m}$.

APPENDIX D

The norm map for finite field extensions

In this appendix we define and prove some basic properties of the norm map for a finite field extension.

D.1. Definition and basic properties

Let K/L be a finite field extension. Given an element $u \in K$, we define $N_{K/L}(u) \in L$ as the determinant of the L-linear map

$$\varphi_u \colon K \to K, \quad v \to uv.$$

This is the *norm* of u with respect to K/L.

We collect in the first proposition some easy properties of this map.

PROPOSITION D.1.1. *Let K/L be a finite field extension.*

i) *We have $N_{K/L}(0) = 0$ and $N_{K/L}(u) \neq 0$ for every nonzero $u \in K$.*

ii) *We have*

$$N_{K/L}(u_1 u_2) = N_{K/L}(u_1) \cdot N_{K/L}(u_2) \quad \textit{for every} \quad u_1, u_2 \in K.$$

iii) *For every $u \in L$, we have*

$$N_{K/L}(u) = u^{[K:L]}.$$

PROOF. The first assertion in i) is clear and the second one follows from the fact that φ_u is invertible for every nonzero u. The assertion in ii) follows from the fact that

$$\varphi_{u_1} \circ \varphi_{u_2} = \varphi_{u_1 u_2} \quad \text{for every} \quad u_1, u_2 \in K$$

and the multiplicative behavior of determinants. Finally, iii) follows from the fact that for $u \in L$, the map φ_u is given by scalar multiplication. □

PROPOSITION D.1.2. *Let K/L be a finite field extension and $u \in K$. If $f \in L[x]$ is the minimal polynomial of u over L and* $\operatorname{char}(\varphi_u)$ *is the characteristic polynomial of φ_u:*

$$\operatorname{char}(\varphi_u) = \det(x \cdot \mathrm{Id} - \varphi_u),$$

then $\operatorname{char}(\varphi_u) = f^r$*, where $r = [K : L(u)]$. In particular, we have*

$$N_{K/L}(u) = (-1)^{[K:L]} \cdot f(0)^r.$$

PROOF. Let $\operatorname{char}'(\varphi_u)$ be the characteristic polynomial of $\varphi'_u = \varphi_u|_{L(u)}$. We write

$$f = x^m + a_1 x^{m-1} + \ldots + a_m,$$

where $m = [L(u) : L]$. By writing the linear map φ'_u in the basis $1, u, \ldots, u^{m-1}$ of $L(u)$ over L, we see that $x \cdot \mathrm{Id} - \varphi'_u$ is given by the matrix

$$A = \begin{pmatrix} x & 0 & \ldots & 0 & a_m \\ 1 & x & \ldots & 0 & a_{m-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & x & a_2 \\ 0 & 0 & \ldots & 1 & a_1 \end{pmatrix},$$

whose determinant can be easily computed to be equal to f. If $e_1, \ldots, e_r$ is a basis of K over $L(u)$ and we write $x \cdot \mathrm{Id} - \varphi_u$ with respect to the basis given by $u^i e_j$, for $0 \leq i \leq m-1$ and $1 \leq j \leq r$ (suitably ordered), this is the block diagonal matrix

$$\begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ 0 & 0 & \ldots & A \end{pmatrix}.$$

The first assertion in the proposition follows. The last assertion is a consequence of the fact that the constant term in $\mathrm{char}(\varphi_u)$ is $(-1)^{[K:L]} \cdot \det(\varphi_u)$. □

D.2. A property of the norm for integrally closed domains

Recall that an integral domain A with fraction field K is *integrally closed* if every element of K that is integral over A lies in A.

Proposition D.2.1. *Let $B \hookrightarrow A$ be an integral ring extension of integral domains such that the corresponding field extension $L \hookrightarrow K$ between the two fraction fields is finite. If B is integrally closed, then for every element $u \in A$, we have $v := N_{K/L}(u) \in B$. Moreover, if $u \in J$, where J is an ideal in A, then $v \in J \cap B$.*

Proof. Let $f = x^m + a_1 x^{m-1} + \ldots + a_m \in L[x]$ be the minimal polynomial of u over L. Since u is integral over B, there is a monic polynomial $g \in B[x]$ such that $g(u) = 0$. Note that f divides g in $L[x]$. Every other root of f (in some algebraic closure $\overline{K}$ of K) is automatically a root of g, and therefore it is again integral over B. Since the set of elements of $\overline{K}$ integral over B is a ring (see Proposition 2.2. in Review Sheet 1), and every a_i is (up to sign) a symmetric function of the roots of f, we conclude that a_i is integral over B. Finally, since B is integrally closed in L and the a_i lie in L, we conclude that the a_i lie in B. By Proposition D.1.2, we can write $N_{K/L}(u)$, up to sign, as a power of a_m, hence $N_{K/L}(u) \in B$.

Suppose now that $u \in J$, for an ideal J in A. Since

$$a_m = -u(u^{m-1} + a_1 u^{m-2} + \ldots + a_{m-1}),$$

and $a_i \in B \subseteq A$ for all i, we deduce that $a_m \in J$. Arguing as before, we conclude that $N_{K/L}(u) \in J \cap B$. □

APPENDIX E

Zero-divisors in Noetherian rings

In the first section we prove a basic result about prime ideals, the prime avoidance lemma. In the second section we give a direct proof for the fact that minimal prime ideals consist of zero-divisors. Finally, in the last section we discuss more generally zero-divisors on finitely generated modules over a Noetherian ring and primary decomposition.

E.1. The prime avoidance lemma

The following result, known as the Prime Avoidance lemma, is often useful.

LEMMA E.1.1. *Let R be a commutative ring, r a positive integer, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ ideals in R such that $\mathfrak{p}_i$ is prime for all $i \geq 3$. If I is an ideal in R such that $I \subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$, then $I \subseteq \mathfrak{p}_i$ for some $i \geq 1$.*

PROOF. The assertion is trivial for $r = 1$. We prove it by induction on $r \geq 2$. If $r = 2$ and $I \not\subseteq \mathfrak{p}_1$ and $I \not\subseteq \mathfrak{p}_2$, then we may choose $a \in I \smallsetminus \mathfrak{p}_1$ and $b \in I \smallsetminus \mathfrak{p}_2$. Note that since $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$, we have $a \in \mathfrak{p}_2$ and $b \in \mathfrak{p}_1$. Note that $a + b \in I$, hence $a + b \in \mathfrak{p}_1$ or $a + b \in \mathfrak{p}_2$. In the first case, we see that $a = (a + b) - b \in \mathfrak{p}_1$, a contradiction and in the second case, we see that $b = (a+b) - a \in \mathfrak{p}_2$, leading again to a contradiction. This settles the case $r = 2$.

Suppose now that $r \geq 3$ and that we know the assertion for $r - 1$ ideals. If $I \not\subseteq \mathfrak{p}_i$ for every i, it follows from the induction hypothesis that given any i, we have $I \not\subseteq \bigcup_{j \neq i} \mathfrak{p}_j$. Let us choose

$$a_i \in I \smallsetminus \bigcup_{j \neq i} \mathfrak{p}_j.$$

By hypothesis, we must have $a_i \in \mathfrak{p}_i$ for all i.

Since $\mathfrak{p}_r$ is a prime ideal and $a_i \not\in \mathfrak{p}_r$ for $i \neq r$, it follows that $\prod_{1 \leq j \leq r-1} a_j \not\in \mathfrak{p}_r$. Consider now the element

$$u = a_r + \prod_{1 \leq j \leq r-1} a_j \in I.$$

By assumption, we have $u \in \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$. If $u \in \mathfrak{p}_r$, since $a_r \in \mathfrak{p}_r$, we deduce that $\prod_{1 \leq j \leq r-1} a_j \in \mathfrak{p}_r$, a contradiction. On the other hand, if $u \in \mathfrak{p}_i$ for some $i \leq r - 1$, since $\prod_{1 \leq j \leq r-1} a_j \in \mathfrak{p}_i$, we conclude that $a_r \in \mathfrak{p}_i$, a contradiction. We thus conclude that $I \subseteq \mathfrak{p}_i$ for some i, completing the proof of the induction step. $\square$

E.2. Minimal primes and zero-divisors

Let R be a Noetherian ring. We refer to Exercise 3.1.4 for the definition of the topological space $\mathrm{Spec}(R)$. Since R is a Noetherian ring, $\mathrm{Spec}(R)$ is a Noetherian topological space, hence we can apply Proposition 1.3.12 to write it as the union of finitely many irreducible components. Since the irreducible closed subsets of Spec(R) are those of the form $V(\mathfrak{p})$, with $\mathfrak{p}$ a prime ideal in R, we conclude that there are finitely many minimal primes $\mathfrak{p}_1, \ldots \mathfrak{p}_r$ in Spec(R). The decomposition

$$\mathrm{Spec}(R) = V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r)$$

says that

$$\mathrm{rad}(0) = \bigcap_{i=1}^{r} \mathfrak{p}_i.$$

Proposition E.2.1. *With the above notation, every minimal prime ideal $\mathfrak{p}_i$ is contained in the set of zero-divisors of R.*

Proof. Given $a \in \mathfrak{p}_i$, we choose for every $j \neq i$ an element $b_j \in \mathfrak{p}_j \smallsetminus \mathfrak{p}_i$. If $b = \prod_{j \neq i} b_j$, then $b \notin \mathfrak{p}_i$, but $b \in \mathfrak{p}_j$ for all $j \neq i$. We thus have

$$ab \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r = \mathrm{rad}(0),$$

hence $(ab)^N = 0$ for some positive integer N. If a is a non-zero-divisor, we would get that $b^N = 0$, hence $b \in \mathfrak{p}_i$, a contradiction. □

Remark E.2.2. If R is reduced, then the set of zero-divisors of R is precisely the union of the minimal prime ideals. Indeed, in this case we have $\bigcap_{i=1}^r \mathfrak{p}_i = 0$. It follows that if $ab = 0$ and $a \notin \mathfrak{p}_i$ for all i, then $b \in \mathfrak{p}_i$ for all i, hence $b = 0$. In the next section we will discuss the set of zero-divisors for an arbitrary Noetherian ring (and, more generally, for a finitely generated module over such a ring).

E.3. Associated primes and zero-divisors

In this section we give a brief treatment of associated primes and primary decomposition. When dealing with associated primes, it is convenient to work more generally with modules, instead of just with the ring itself. Let us fix a Noetherian ring R.

Definition E.3.1. If M is a finitely generated R-module, an *associated prime* of M is a prime ideal $\mathfrak{p}$ in R such that

$$\mathfrak{p} = \mathrm{Ann}_R(u) \quad \text{for some} \quad u \in M, u \neq 0.$$

The set of associated primes of M is denoted Ass(M) (we write Ass$_R(M)$ if the ring is not understood from the context).

Recall that if M is an R-module, an element $a \in R$ is a *zero-divisor* of M if $au = 0$ for some $u \in M \smallsetminus \{0\}$; otherwise a is a *non-zero-divisor* of M. Note that for $M = R$, we recover the usual notion of zero-divisor in R. The third assertion in the next proposition is the main reason why associated primes are important:

Proposition E.3.2. *If M is a finitely generated R-module, then the following hold:*

i) *The set* Ass(M) *is finite.*
ii) *If $M \neq 0$, then* Ass(M) *is non-empty.*
iii) *The set of zero-divisors of M is equal to*

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

We begin with the following easy lemma:

LEMMA E.3.3. *Given an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

we have

$$\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

PROOF. The first inclusion is obvious, hence we only prove the second one. Suppose that $\mathfrak{p} \in \mathrm{Ass}(M)$, and let us write $\mathfrak{p} = \mathrm{Ann}_R(u)$, for some nonzero $u \in M$. If $u \in M'$, then clearly $\mathfrak{p} \in \mathrm{Ass}(M')$. Otherwise, the image $\overline{u}$ of u in M'' is non-zero and it is clear that $\mathfrak{p} \subseteq \mathrm{Ann}_R(\overline{u})$. If this is an equality, then $\mathfrak{p} \in \mathrm{Ass}(M'')$, hence let us assume that there is $a \in \mathrm{Ann}_R(\overline{u}) \smallsetminus \mathfrak{p}$. In this case $au \in M' \smallsetminus \{0\}$, and the fact that $\mathfrak{p}$ is prime implies that the obvious inclusion $\mathrm{Ann}_R(u) \subseteq \mathrm{Ann}_R(au)$ is an equality. Therefore $\mathfrak{p} \in \mathrm{Ass}(M')$. $\square$

PROOF OF PROPOSITION E.3.2. We may assume that M is nonzero, as otherwise all assertions are trivial. Consider the set $\mathcal{P}$ consisting of the ideals of R of the form $\mathrm{Ann}_R(u)$, for some $u \in M \smallsetminus \{0\}$. Since R is Noetherian, there is a maximal element $\mathfrak{p} \in \mathcal{P}$. We show that in this case $\mathfrak{p}$ is a prime ideal, so that $\mathfrak{p} \in \mathrm{Ass}(M)$.

By assumption, we can write $\mathfrak{p} = \mathrm{Ann}_R(u)$, for some $u \in M \smallsetminus \{0\}$. Since $u \neq 0$, we have $\mathfrak{p} \neq R$. If $b \in R \smallsetminus \mathfrak{p}$, then $bu \neq 0$ and we clearly have

$$\mathrm{Ann}_R(u) \subseteq \mathrm{Ann}_R(bu).$$

By the maximality of $\mathfrak{p}$, we conclude that this is an equality, hence for every $a \in R$ such that $ab \in \mathfrak{p}$, we have $a \in \mathfrak{p}$; we thus conclude that $\mathfrak{p}$ is a prime ideal.

In particular, this proves ii). We thus know that if M is non-zero, then we can find $u \in M \smallsetminus \{0\}$ such that $\mathrm{Ann}_R(u) = \mathfrak{p}_1$ is a prime ideal. The map $R \to M$, $a \to au$ induces thus an injection $R/\mathfrak{p} \hookrightarrow M$, so that we have a short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

with $M_1 \simeq R/\mathfrak{p}_1$. Note now that since $\mathfrak{p}_1$ is a prime ideal in R, then we clearly have $\mathrm{Ass}(R/\mathfrak{p}_1) = \{\mathfrak{p}_1\}$, and the lemma implies

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M/M_1) \cup \{\mathfrak{p}\}.$$

Therefore in order to prove that $\mathrm{Ass}(M)$ is finite it is enough to show that $\mathrm{Ass}(M/M_1)$ is finite. If $M/M_1 \neq 0$, we can repeat this argument and find $M_1 \subseteq M_2$ such that $M_2/M_1 \simeq R/\mathfrak{p}_2$, for some prime ideal $\mathfrak{p}_2$ in R. Since M is a Noetherian module, this process must terminate, hence after finitely many steps we conclude that $\mathrm{Ass}_R(M)$ is finite.

We now prove the assertion in iii). It is clear from definition that for every $\mathfrak{p} \in \mathrm{Ass}(M)$, the ideal $\mathfrak{p}$ is contained in the set of zero-divisors of M. On the other hand, if $a \in R$ is a zero-divisor, then $a \in I$, for some $I \in \mathcal{P}$. If we choose a maximal $\mathfrak{p}$ in $\mathcal{P}$ that contains I, then we have seen that $\mathfrak{p} \in \mathrm{Ass}_R(M)$, hence a lies in the union of the associated primes of M. This completes the proof of the proposition. $\square$

We record in the next corollary a useful assertion that we obtained in the above proof.

COROLLARY E.3.4. *If M is a finitely generated R-module, then there is a sequence of submodules*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ *for* $1 \leq i \leq r$*, where each* $\mathfrak{p}_i$ *is a prime deal in* R*.*

REMARK E.3.5. The results in Proposition E.3.2 are often applied as follows: if an ideal I in R has no non-zero-divisors on M, then it is contained in the union of the associated primes. Since there are finitely such prime ideals, the Prime Avoidance lemma implies that I is contained in one of them. Therefore there is $u \in M$ non-zero such that $I \cdot u = 0$.

REMARK E.3.6. If M is a finitely generated R-module, then for every multiplicative system S in R, if we consider the finitely generated $S^{-1}R$-module $S^{-1}M$, we have

$$\mathrm{Ass}_{\mathrm{S^{-1}R}}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M), S \cap \mathfrak{p} \neq 0\}.$$

Indeed, if $\mathfrak{p} = \mathrm{Ann}_R(u)$ and $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}\left(\frac{u}{1}\right)$. Conversely, if $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}\left(\frac{v}{s}\right)$, for some prime ideal $\mathfrak{p}$ in R, with $\mathfrak{p} \cap S = \emptyset$, then it is easy to see that $\mathfrak{p} = \mathrm{Ann}_R(v)$.

REMARK E.3.7. Let M be a finitely generated R-module and $I = \mathrm{Ann}_R(M)$. It is clear from definition that if $\mathfrak{p} \in \mathrm{Ass}(M)$, then $I \subseteq \mathfrak{p}$. Moreover, we have

$$\mathrm{Ass}_{R/I}(M) = \{\mathfrak{p}/I \mid \mathfrak{p} \in \mathrm{Ass}_R(M)\}.$$

We recall the easy fact that since M is a finitely generated R-module, for every prime ideal $\mathfrak{p}$ in R, we have $M_\mathfrak{p} \neq 0$ if and only if $I \subseteq \mathfrak{p}$ (see Proposition 8.4.11 and its proof). We note that every prime ideal in R that contains I and is minimal with this property lies in $\mathrm{Ass}_R(M)$ (in particular, we recover the assertion in Proposition E.2.1). Indeed, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is nonzero, hence $\mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ is non-empty by Proposition E.3.2. However, there is a unique prime ideal in $R_\mathfrak{p}$ that contains $\mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) = I_\mathfrak{p}$, namely $\mathfrak{p}R_\mathfrak{p}$. Using again the previous remark, we see that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. The primes in $\mathrm{Ass}_R(M)$ that are not minimal over $\mathrm{Ann}_R(M)$ are called *embedded primes*.

EXAMPLE E.3.8. If I is a radical ideal in R, then it follows from Remark E.2.2 that the set of zero-divisors in R/I is the union of the minimal prime ideals containing I. We deduce using Proposition E.3.2 and the Prime Avoidance lemma that every $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$ is a minimal prime containing I.

We end by discussing primary decomposition and its connection to associated primes. Since we will only need this for ideals, for the sake of simplicity, we stick to this case.

DEFINITION E.3.9. An ideal $\mathfrak{q}$ in R is *primary* if whenever $a, b \in R$ are such that $ab \in \mathfrak{q}$ and $a \not\in \mathfrak{q}$, then $b \in \mathrm{rad}(\mathfrak{q})$. It is straightforward to see that in this case $\mathfrak{p} := \mathrm{rad}(\mathfrak{q})$ is a prime ideal; one also says that $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal. A *primary decomposition* of an ideal I is an expression

$$I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n,$$

where all $\mathfrak{q}_i$ are primary ideals.

REMARK E.3.10. It follows from definition that if $I \subseteq \mathfrak{q}$ are ideals in R, then $\mathfrak{q}/I$ is a primary ideal in R/I if and only if $\mathfrak{q}$ is a primary ideal in R.

PROPOSITION E.3.11. *If* $\mathfrak{q}$ *is an ideal in* R*, then* $\mathfrak{q}$ *is a primary ideal if and only if* $\mathrm{Ass}_R(R/\mathfrak{q})$ *has only one element. Moreover, in this case the only associated prime of* $R/\mathfrak{q}$ *is* $\mathrm{rad}(\mathfrak{q})$*.*

Proof. Suppose first that $\mathfrak{q}$ is $\mathfrak{p}$-primary. Note that $\mathfrak{p}$ is the only minimal prime containing $\mathfrak{q}$, hence $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{q})$ by Remark E.3.7. On the other hand, since $\mathfrak{q}$ is $\mathfrak{p}$-primary, it follows that every zero-divisor of $R/\mathfrak{q}$ lies in $\mathfrak{p}$. Since the set of zero-divisors of $R/\mathfrak{q}$ is the union of the associated primes of $R/\mathfrak{q}$ by Proposition E.3.2, and each of these associated primes contains $\mathrm{Ann}_R(R/\mathfrak{q}) = \mathfrak{q}$, we conclude that $\mathfrak{p}$ is the only element of $\mathrm{Ass}_R(R/\mathfrak{q})$.

Conversely, suppose that $\mathrm{Ass}_R(R/\mathfrak{q})$ has only one element $\mathfrak{p}$. In this case, it follows from Remark E.3.7 that $\mathfrak{p}$ is the unique minimal prime containing $\mathfrak{q}$, hence $\mathfrak{p} = \mathrm{rad}(\mathfrak{p})$. Moreover, it follows from Proposition E.3.2 that the set of non-zero-divisors of $R/\mathfrak{q}$ is equal to $\mathfrak{p}$, which implies, by definition, that $\mathfrak{q}$ is a primary ideal. $\square$

Proposition E.3.12. *Every ideal I in R has a primary decomposition.*

Proof. After replacing R by R/I, we may assume that $I = 0$. We claim that for every $\mathfrak{p} \in \mathrm{Ass}(R)$, there is a primary ideal $\mathfrak{q}$ in R such that $\mathfrak{p} \not\in \mathrm{Ass}(\mathfrak{q})$. Indeed, consider the ideals J in R such that $\mathfrak{p} \not\in \mathrm{Ass}(J)$ (the set is non-empty since it contains 0) and since R is Noetherian, we may choose an ideal $\mathfrak{q}$ which is maximal with this property. Note that $\mathfrak{q} \neq R$, hence $\mathrm{Ass}(R/\mathfrak{q})$ is non-empty. By Proposition E.3.11, in order to show that $\mathfrak{q}$ is a primary ideal, it is enough to show that for every prime ideal $\mathfrak{p}' \neq \mathfrak{p}$, we have $\mathfrak{p}' \not\in \mathrm{Ass}(R/\mathfrak{q})$. If $\mathfrak{p}' \in \mathrm{Ass}(R/\mathfrak{q})$, then we obtain an ideal $\mathfrak{q}' \supseteq \mathfrak{q}$ such that $\mathfrak{q}'/\mathfrak{q} \simeq R/\mathfrak{p}'$. We assumed $\mathfrak{p}' \neq \mathfrak{p}$, while Lemma E.3.3 implies

$$\mathrm{Ass}(\mathfrak{q}') \subseteq \mathrm{Ass}(\mathfrak{q}) \cup \mathrm{Ass}(\mathfrak{q}'/\mathfrak{q}) = \mathrm{Ass}(\mathfrak{q}) \cup \{\mathfrak{p}'\},$$

hence $\mathfrak{p} \not\in \mathrm{Ass}(\mathfrak{q}')$, contradicting the maximality of $\mathfrak{q}$.

We thus conclude that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the associated primes of R, we can find primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ such that $\mathfrak{p}_i \not\in \mathrm{Ass}(\mathfrak{q}_i)$ for all i. If $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$, then $\mathrm{Ass}(\mathfrak{a}) \subseteq \mathrm{Ass}(R)$ and at the same time $\mathrm{Ass}(\mathfrak{a}) \subseteq \mathrm{Ass}(\mathfrak{q}_i)$ for all i, hence $\mathfrak{p}_i \not\in \mathrm{Ass}(\mathfrak{a})$. This implies that $\mathfrak{a}$ has no associated primes, hence $\mathfrak{a} = 0$. $\square$

Remark E.3.13. Note that if $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is a $\mathfrak{p}$-primary ideal. It is thus straightforward to see that given any ideal I and any primary decomposition $I = \mathfrak{q}_1, \ldots \cap \mathfrak{q}_r$, we can obtain a *minimal* such decomposition, in the sense that the following conditions are satisfied:

i) We have $\mathrm{rad}(\mathfrak{q}_i) \neq \mathrm{rad}(\mathfrak{q}_j)$ for all i and j, and
ii) For every i, with $1 \leq i \leq r$, we have $\bigcap_{j \neq i} \mathfrak{q}_j \neq I$.

Given such a minimal primary decomposition, if $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the distinct associated primes of R/I. Indeed, the injective morphism

$$R/I \hookrightarrow \bigoplus_{i=1}^{r} R/\mathfrak{q}_i$$

implies that $\mathrm{Ass}(R/I) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. On the other hand, for every i, there is $u \in \bigcap_{j \neq i} \mathfrak{q}_j$ such that $u \not\in \mathfrak{q}_i$. Moreover, after multiplying u by a suitable element in $\mathfrak{p}_i^m$, for some non-negative integer m, we may assume that $u \cdot \mathfrak{p}_i \subseteq \mathfrak{q}_i$. In this case, $\mathfrak{p}_i$ is the annihilator of the image of u in R/I, hence $\mathfrak{p}_i \in \mathrm{Ass}(R/I)$.

Remark E.3.14. In general, the primary ideals in a minimal primary decomposition of I are not unique. However, if $\mathfrak{p}$ is a minimal prime containing I, then the corresponding $\mathfrak{p}$-primary ideal $\mathfrak{q}$ in a primary decomposition of I is unique. Indeed, it is easy to check that $I \cdot R_{\mathfrak{p}} = \mathfrak{q} \cdot R_{\mathfrak{p}}$ and deduce, using that $\mathfrak{q}$ is $\mathfrak{p}$-primary, that $\mathfrak{q} = I \cdot R_{\mathfrak{p}} \cap R$.

E.4. An application: a characterization of DVRs

For the definition of a DVR, see § C.5. We can now prove the following characterization of DVRs:

PROPOSITION E.4.1. *A ring R is a DVR if and only if it is a local Noetherian domain, of dimension 1, which is integrally closed.*

We begin with a lemma that we will also use in the next section:

LEMMA E.4.2. *If R is a normal domain, then for every non-zero $a \in R$ and every $\mathfrak{p} \in \operatorname{Ass}\big(R/(a)\big)$, $\mathfrak{p}R_{\mathfrak{p}}$ is a principal ideal and* $\operatorname{codim}(\mathfrak{p}) = 1$.

PROOF. By assumption, there is $b \in R$ such that

$$\mathfrak{p} = \{h \in R \mid hb \in (a)\}. \tag{E.4.1}$$

In particular, we have $\frac{b}{a} \not\in R_{\mathfrak{p}}$ and $\mathfrak{p}R_{\mathfrak{p}} \cdot \frac{b}{a} \subseteq R_{\mathfrak{p}}$. If $\mathfrak{p}R_{\mathfrak{p}} \cdot \frac{b}{a} \subseteq \mathfrak{p}R_{\mathfrak{p}}$, then the determinantal trick (see, for example, the proof of Proposition A.1.3) implies that $\frac{b}{a}$ is integral over $R_{\mathfrak{p}}$; since R is integrally closed, it follows that $R_{\mathfrak{p}}$ is integrally closed (see Lemma 9.1.1), and thus $\frac{b}{a} \in R_{\mathfrak{p}}$, a contradiction. Therefore $\mathfrak{p}R_{\mathfrak{p}} \cdot \frac{b}{a} = R_{\mathfrak{p}}$, that is, $\frac{a}{b} \in \mathfrak{p}R_{\mathfrak{p}}$. Moreover, if $u \in \mathfrak{p}R_{\mathfrak{p}}$, then it follows from (E.4.1) that $u \in \frac{a}{b} \cdot R_{\mathfrak{p}}$. Therefore $\mathfrak{p}R_{\mathfrak{p}} = \frac{a}{b} \cdot R_{\mathfrak{p}}$, hence $R_{\mathfrak{p}}$ is a DVR by Proposition C.5.1 and thus $\operatorname{codim}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) = 1$ by Remark C.5.2. □

PROOF OF PROPOSITION E.4.1. It follows from Proposition C.5.1 that if R is a DVR, then it is a local PID. In particular, it is a local Noetherian domain, and it is a UFD, hence it is normal (see Example 1.7.28). Moreover, it follows from Remark C.5.2 that $\dim(R) = 1$.

We now prove the converse: by Proposition C.5.1, it is enough to show that the maximal ideal $\mathfrak{m}$ is principal. Since R is not a field, it follows from Nakayama's lemma that $\mathfrak{m} \neq \mathfrak{m}^2$. Let $a \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. By Proposition E.3.2, we have $\operatorname{Ass}_R\big(R/(a)\big) \neq \emptyset$, and since (0) and $\mathfrak{m}$ are the only prime ideals in R, it follows that $\mathfrak{m} \in \operatorname{Ass}_R\big(R/(a)\big)$. Lemma E.4.2 implies that $\mathfrak{m}$ is principal, completing the proof of the proposition. □

E.5. A characterization of normal rings

The following characterization of normal rings is a variant of a criterion due to Serre.

PROPOSITION E.5.1. *A Noetherian domain R is integrally closed if and only if the following two conditions hold:*

i) *For every prime ideal $\mathfrak{p}$ in R, with* $\operatorname{codim}(\mathfrak{p}) = 1$*, the ring $R_{\mathfrak{p}}$ is a DVR.*
ii) *We have $R = \bigcap_{\operatorname{codim}(\mathfrak{p})=1} R_{\mathfrak{p}}$, where the intersection is over all prime ideals $\mathfrak{p}$ in R, of codimension* 1.

Moreover, in general condition ii) *is equivalent to the following variant:*

ii') *For every $a \in R$ nonzero, and every $\mathfrak{p} \in \operatorname{Ass}_R\big(R/(a)\big)$, we have* $\operatorname{codim}(\mathfrak{p}) = 1$.

PROOF. Let K be the fraction field of R. We first prove the equivalence of ii) and ii'). Suppose first that ii') holds and consider $0 \neq \frac{b}{a} \in K$ that lies in $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in R of codimension 1. We consider a minimal primary decomposition

$$(a) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r.$$

It follows from Remark E.3.13 that if $\mathfrak{p}_j = \mathrm{rad}(\mathfrak{q}_j)$, then $\mathfrak{p}_j \in \mathrm{Ass}_R\big(R/(a)\big)$, hence $\mathrm{codim}(\mathfrak{p}_j) = 1$ for all j by ii'). By hypothesis, we have $\frac{b}{a} \in R_{\mathfrak{p}_j}$ for all j, hence there is $s_j \in R \smallsetminus \mathfrak{p}_j$ such that $s_j b \in (a) \subseteq \mathfrak{q}_j$. Since $\mathfrak{q}_j$ is a primary ideal, we conclude that $b \in \mathfrak{q}_j$ for all j, hence $\frac{b}{a} \in R$.

Conversely, suppose that ii) holds and consider $0 \neq a \in R$ and $\mathfrak{p} \in \mathrm{Ass}_R\big(R/(a)\big)$. It follows that there is $b \in R$ such that $\mathfrak{p} = \{u \in R \mid ub \in (a)\}$. In particular, we have $b \not\in (a)$, and thus by assumption, we can find a prime ideal $\mathfrak{q}$ with $\mathrm{codim}(\mathfrak{q}) = 1$, such that $\frac{b}{a} \not\in R_{\mathfrak{q}}$. This implies that

$$\mathfrak{p} = \{u \in R \mid ub \in (a)\} \subseteq \mathfrak{q}.$$

Since $\mathfrak{p} \neq 0$ (since $a \in \mathfrak{p}$) and $\mathfrak{q}$ has codimension 1, we conclude that $\mathfrak{p} = \mathfrak{q}$, and thus $\mathrm{codim}(\mathfrak{p}) = 1$.

Suppose now that conditions i) and ii) hold. It follows from i) that if $\mathfrak{p}$ is a codimension 1 prime ideal in R, then $R_{\mathfrak{p}}$ is integrally closed. We then deduce from ii) that R is integrally closed: if $u \in K$ is integral over R, then it is clearly integral over $R_{\mathfrak{p}}$, for every prime ideal $\mathfrak{p}$ in R of codimension 1, and thus $u \in \bigcap_{\mathrm{codim}(\mathfrak{p})=1} R_{\mathfrak{p}} = R$.

On the other hand, if R is integrally closed, then for every prime ideal $\mathfrak{p}$ in R, the ring $R_{\mathfrak{p}}$ is integrally closed (see Lemma 9.1.1). Since $\dim(R_{\mathfrak{p}}) = 1$, we deduce from Proposition E.4.1 that $R_{\mathfrak{p}}$ is a DVR. Moreover, property ii') follows from Lemma E.4.2. This completes the proof. $\square$

APPENDIX F

A characterization of UFDs

F.1. The UFD condition for Noetherian rings

Recall that an integral domain R is a UFD if every non-zero element is either invertible or can be written as a product of finitely many irreducible elements; moreover, this decomposition is unique, up to reordering and replacing an irreducible element $a \in R$ by ua, where u is an invertible element.

Remark F.1.1. If R is Noetherian, then the decomposition as a product of irreducible elements is automatic. Indeed, if there are non-zero non-invertible elements $a \in R$ that are not products of irreducible elements, then we may choose such a with the ideal (a) maximal among the ideals associated to such elements. In particular, a is not irreducible, hence we can write $a = a_1 a_2$, with both a_1 and a_2 non-invertible. Moreover, at least one of a_1 and a_2 (say, it is a_1) is not a product of irreducible elements. Since the ideal (a_1) strictly contains (a), we contradict the maximality of (a) in the choice of a.

Remark F.1.2. It is well-known (and easy to check) that if a domain R has the property that every element is a product of irreducible elements, then R is a UFD if and only if every irreducible element $a \in R$ is prime (that is, the ideal (a) is a prime ideal).

Proposition F.1.3. *If R is a UFD, then for every $f, g \in R$, the ideal*

$$(f) : (g) = \{h \in R \mid hg \in (f)\}$$

is principal. Conversely, if R is a Noetherian domain that satisfies this condition, then R is a UFD.

Proof. Suppose first that R is a UFD. The assertion is clear if $f = 0$ or $g = 0$, hence we may assume that f and g are non-zero. Let us write

$$f = u\pi_1^{m_1} \cdots \pi_r^{m_r} \quad \text{and} \quad v\pi_1^{n_1} \cdots \pi_r^{n_r},$$

with u and v invertible and $\pi_1, \ldots, \pi_r$ elements generating mutually distinct prime ideals. It is then straightforward to see that $(f) : (g)$ is generated by

$$\prod_{i=1}^{r} \pi_i^{\min\{m_i - n_i, 0\}}.$$

In order to prove the converse, note that by Remark F.1.2, we only need to show that if $\pi \in R$ is an irreducible element, then π is prime. Suppose that π divides ab, hence $b \in (\pi) : (a)$. Let $h \in R$ be such that $(\pi) : (a) = (h)$. We thus have $\pi \in (h)$ and since π is irreducible, we conclude that either h is invertible or $(\pi) = (h)$. In the former case we have π divides a, while in the latter case, π divides b. This completes the proof. $\square$

PROPOSITION F.1.4. *A Noetherian domain R is a UFD if and only if every prime ideal $\mathfrak{p}$ in A of codimension 1 is principal.*

PROOF. Suppose first that R is a UFD. If $\mathfrak{p}$ is a prime ideal of codimension 1, let us choose a non-zero $a \in \mathfrak{p}$. If we write $a = a_1 \cdots a_r$, with all a_i irreducible elements, since $\mathfrak{p}$ is prime, it follows that $a_i \in \mathfrak{p}$ for some i. Since R is a UFD, the ideal (a_i) is a prime ideal, and since $\operatorname{codim}(\mathfrak{p}) = 1$, it follows that $\mathfrak{p} = (a_i)$.

Conversely, suppose that every codimension 1 prime ideal in R is principal. By Remarks F.1.1 and F.1.2, we see that in order to show that R is a UFD, it is enough to show that if π is an irreducible element in R, then (π) is a prime ideal. Let $\mathfrak{p}$ be a minimal prime ideal containing (π). If follows from the Principal Ideal theorem[1] that $\operatorname{codim}(\mathfrak{p}) = 1$. By assumption, $\mathfrak{p}$ is a principal ideal. If we write $\mathfrak{p} = (b)$, the inclusion $(\pi) \subseteq (b)$ implies that $\pi = bc$, for some $c \in R$. Since π is irreducible, it follows that c is invertible, hence $(\pi) = (b)$ is a prime ideal. $\square$

[1] In these notes, we only apply the present proposition in the case when R is a (localization of an) algebra of finite type over an algebraically closed field k; this is the case in which we proved the Principal Ideal theorem in Chapter 3. However, for a proof in the general case, see Appendix J.

APPENDIX G

Completion

In this appendix we review the basic results about completion of rings and modules. For a more general treatment, we refer to [Mat89, §8]. By way of motivation, let us recall the construction of the ring of p-adic integers, where p is a positive prime integer. One defines a topology on $\mathbf{Z}$ such that two integers are "close" if their difference is divisible by a large power of p; in other words, a basis of neighborhoods of $m \in \mathbf{Z}$ is given by $(m + p^n\mathbf{Z})_{n\geq 1}$. The topology comes from a metric space structure, but the choice of metric is not important. The ring of p-adic integers $\mathbf{Z}_p$ is the completion of $\mathbf{Z}$ with respect to this topology. It can be described as the quotient of the set of Cauchy sequences in $\mathbf{Z}$ modulo a suitable equivalence relation; however, algebraically it is more convenient to describe it as

$$\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}.$$

In what follows we consider a similar construction for rather general rings and modules.

G.1. Completion with respect to an ideal

In what follows we fix a Noetherian ring A and let I be a fixed ideal in A. Note that for every $n \geq 1$ we have a canonical surjective homomorphism $A/I^{n+1} \to A/I^n$. By taking the inverse limit of these homomorphisms we obtain the *completion* of A with respect to I:

$$\widehat{A} := \varprojlim A/I^n.$$

This is a ring and we have a canonical ring homomorphism $\psi_A \colon A \to \widehat{A}$ that maps $a \in A$ to $(a \bmod I^n)_{n\geq 1}$.

Suppose now that M is an A-module. For every $n \geq 1$, we have a surjective morphism of A-modules $M/I^{n+1}M \to M/I^nM$. The *completion* of M with respect to I is

$$\widehat{M} := \varprojlim M/I^nM.$$

This is an A-module and we have a canonical morphism of A-modules $\psi_M \colon M \to \widehat{M}$ that maps $u \in M$ to $(u \bmod I^nM)_{n\geq 1}$. In fact, since each M/I^nM is an A/I^n-module, we have a natural $\widehat{A}$-module structure on $\widehat{M}$ that induces, by restriction of scalars via ψ_A, the original A-module structure on $\widehat{M}$.

If $\phi \colon M \to N$ is a morphism of A-modules, we obtain an induced morphism of $\widehat{A}$-modules $\widehat{M} \to \widehat{N}$. This gives a functor from A-modules to $\widehat{A}$-modules.

Example G.1.1. If $A = R[x_1, \ldots, x_n]$ for some Noetherian ring R, and $I = (x_1, \ldots, x_n)$, then $\widehat{A}$ is isomorphic, as an A-algebra, to $R[\![x_1, \ldots, x_n]\!]$.

Remark G.1.2. If we have a sequence of submodules $(M_n)_{n\geq 1}$ of M such that $M_{n+1}\subseteq M_n$ for every $n\geq 1$, then we have canonical morphisms $M/M_{n+1}\to M/M_n$ and we can consider $\varprojlim M/M_n$. If $I^nM\subseteq M_n$ for every n, then we have an induced morphism

$$\widehat{M}\to \varprojlim M/M_n. \tag{G.1.1}$$

If, in addition, for every n we can find ℓ such that $M_\ell\subseteq I^nM$, then (G.1.1) is an isomorphism (this follows easily using the fact that the inverse limit does not change if we pass to a final subset).

In particular, we see that the completion of M with respect to two ideals I and J are canonically isomorphic if $\operatorname{rad}(I)=\operatorname{rad}(J)$.

Remark G.1.3. If there is n such that $I^nM=0$, then it is clear that the morphism $M\to\widehat{M}$ is an isomorphism.

Remark G.1.4. By definition, the kernel of the morphism $\psi_M\colon M\to\widehat{M}$ is equal to $\bigcap_{n\geq 1} I^nM=0$. We thus see that if $(A,\mathfrak{m})$ is a local Noetherian ring, $I\subseteq\mathfrak{m}$, and M is a finitely generated A-module, then ψ_M is injective by Krull's Intersection theorem (see Theorem C.4.1).

Remark G.1.5. If $\phi\colon A\to B$ is a ring homomorphism and $I\subseteq A$ and $J\subseteq B$ are ideals such that $I\cdot B\subseteq J$, then we have a ring homomorphism $\widehat{\phi}\colon\widehat{A}\to\widehat{B}$ such that $\widehat{\phi}\circ\psi_A=\psi_B\circ\phi$ (where the completions of A and B are taken with respect to I and J, respectively). Indeed, for every n, we have an induced homomorphism $A/I^n\to B/J^n$, and by taking the inverse limit over n, we get the morphism $\widehat{\phi}$ that satisfies the required commutativity condition.

G.2. Basic properties of completion

We now derive some properties of $\widehat{A}$ and of the completion functor. We assume that A is a Noetherian ring and I is an ideal in A.

Proposition G.2.1. *Given a short exact sequence of finitely generated A-modules*

$$0\longrightarrow M'\xrightarrow{\alpha} M\xrightarrow{\beta} M''\longrightarrow 0,$$

the induced sequence

$$0\longrightarrow \widehat{M'}\longrightarrow\widehat{M}\longrightarrow\widehat{M''}\longrightarrow 0$$

is exact, too.

Proof. For every $n\geq 1$, we have an induced exact sequence of A/I^n-modules

$$0\to M'/(I^nM\cap M')\to M/I^nM\to M''/I^nM''\to 0.$$

A well-known (and easy to check) property of inverse limits implies that by passing to inverse limit we obtain an exact sequence

$$0\to\varprojlim M'/(I^nM\cap M')\to\widehat{M}\to\widehat{M''}.$$

Note first that we have a canonical morphism

$$\widehat{M'}=\varprojlim M'/I^nM'\to\varprojlim M'/(I^nM\cap M')$$

and we deduce from the Artin-Rees lemma (see Lemma C.4.2) and Remark G.1.2 that this is an isomorphism.

In order to complete the proof it is thus enough to show that the morphism $\widehat{M} \to \widehat{M''}$ is surjective. Consider $u \in \widehat{M''}$ given by $(u_n \bmod I^n M)_{n\geq 1}$, where the elements $u_n \in M''$ are such that $u_n - u_{n+1} \in I^n M''$. We construct recursively elements $v_n \in M$ such that the following hold for all $n \geq 1$:

i) $u_n = \beta(v_n)$ and
ii) $v_n - v_{n+1} \in I^n M$.

We begin by choosing $v_1 \in M$ such that $\beta(v_1) = u_1$ (this is possible since β is surjective). Suppose now that $v_1, \ldots, v_r$ are chosen such that i) holds for $1 \leq n \leq r$ and ii) holds for $1 \leq n \leq r-1$. Since $u_r - u_{r+1} \in I^r M''$, we can write $u_r - u_{r+1} = \sum_{j=1}^{s} a_j w_j$, with $a_j \in I^r$ and $w_j \in M''$. We choose $\widetilde{w}_j \in M$ such that $\beta(\widetilde{w}_j) = w_j$ and put $v_{r+1} = v_r - \sum_{j=1}^{s} a_j \widetilde{w}_j$. It is then clear that i) holds also for $n = r+1$ and ii) holds also for $n = r$. By ii), we can thus consider $v = (v_n \bmod I^n M)_{n\geq 1} \in \widehat{M}$ and it follows from i) that v maps to $u \in \widehat{M''}$. This completes the proof of the proposition. $\square$

Corollary G.2.2. *For every finitely generated A-module M, the canonical morphism*

$$\widehat{A} \otimes_A M \to \widehat{M}$$

induced by ψ_M is an isomorphism. In particular, $\widehat{M}$ is a finitely generated $\widehat{A}$-module.

Proof. The assertion is clear if M is a finitely generated, free A-module. For the general case, consider an exact sequence of A-modules

$$F_1 \to F_0 \to M \to 0,$$

where F_1 and F_0 are finitely generated, free modules. We then obtain a commutative diagram

$$\begin{tikzcd}
\widehat{A} \otimes_A F_1 \arrow[r] \arrow[d] & \widehat{A} \otimes_A F_0 \arrow[r] \arrow[d] & \widehat{A} \otimes_A M \arrow[r] \arrow[d] & 0 \\
\widehat{F_1} \arrow[r] & \widehat{F_0} \arrow[r] & \widehat{M} \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by right-exactness of the tensor product, while the bottom row is exact by the proposition. Since the first and the second vertical maps are isomorphisms, it follows that the third one is an isomorphism as well. $\square$

Corollary G.2.3. *The A-algebra $\widehat{A}$ is flat.*

Proof. We need to show that for every injective morphism of A-modules $\phi\colon M' \hookrightarrow M$, the induced morphism

$$1_{\widehat{A}} \otimes \phi\colon \widehat{A} \otimes_A M' \to \widehat{A} \otimes_A M$$

is injective. We can write $M = \varinjlim_{i\in I} M_i$, where the M_i are the finitely generated submodules of M and similarly $M' = \varinjlim_{i\in I} \phi^{-1}(M_i)$. Since the tensor product commutes with direct limits and a filtered direct limit of injective morphisms is injective, we see that it is enough to consider the case when M (and thus also M') is finitely generated. In this case, the assertion follows by combining the proposition and the previous corollary. $\square$

Corollary G.2.4. *For every $n \geq 1$ and every finitely generated A-module M, we have*
$$\widehat{I^n M} = I^n \widehat{M} = \widehat{I}^n \widehat{M}.$$
Moreover, the morphism $M \to \widehat{M}$ induces an isomorphism $M/I^n M \to \widehat{M}/I^n \widehat{M}$.

Proof. Since $\widehat{A}$ is flat over A, the canonical morphism $\widehat{A} \otimes_A I^n \to \widehat{A}$ is injective; its image is $I^n \widehat{A} = (I\widehat{A})^n$. Moreover, by Proposition G.2.1 and Corollary G.2.2, this is also the image of the morphism $\widehat{I^n} \to \widehat{A}$, which is injective. By taking $n = 1$, we see that $I\widehat{A} = \widehat{I}$, and thus
$$\widehat{I}^n = I^n \widehat{A} = \widehat{I^n}.$$

Given the finitely generated A-module M, by applying Proposition G.2.1 to the exact sequence
$$0 \to I^n M \to M \to M/I^n M \to 0,$$
we obtain an exact sequence
$$0 \to \widehat{I^n M} \to \widehat{M} \xrightarrow{p} \widehat{M/I^n M} \to 0.$$
Note also that we have an isomorphism
$$M/I^n M \simeq \widehat{M/I^n M}$$
such that p gets identified to the canonical projection $\widehat{M} \to M/I^n M$ that comes from the definition of the projective limit (see Remark G.1.3). On one hand, it follows from Corollary G.2.2 that
$$\widehat{I^n M} = \operatorname{Im}(\widehat{A} \otimes_A I^n M \to \widehat{A} \otimes_A M = \widehat{M}) = I^n \widehat{M}.$$
On the other hand, it follows from what we have already proved that
$$I^n \widehat{M} = (I^n \widehat{A}) \cdot \widehat{M} = \widehat{I}^n \widehat{M}.$$
This completes the proof of the proposition. $\square$

Given a ring A, an ideal I in A, and an A-module M, we say that M is *complete* (with respect to I) if the canonical morphism $M \to \widehat{M}$ is an isomorphism. This applies, in particular, in the case $M = A$.

Example G.2.5. Given a Noetherian ring A and a finitely generated A-module M, it follows from Corollary G.2.4 that $\widehat{M}$ is complete as an A-module (with respect to I) and as an $\widehat{A}$-module (with respect to $\widehat{I} = I\widehat{A}$).

Remark G.2.6. Let $\phi\colon A \to B$ be a ring homomorphism and $I \subseteq A$ and $J \subseteq B$ be ideals such that $I \cdot B \subseteq J$. If $\psi_A\colon A \to \widehat{A}$ is the morphism to the completion (with respect to I) and if B is complete (with respect to J), then there is a unique ring homomorphism $\rho\colon \widehat{A} \to B$ such that $\rho \circ \psi_A = \phi$.

Indeed, recall from Remark G.1.5 that we have a homomorphism $\widehat{\phi}\colon \widehat{A} \to \widehat{B}$ such that $\widehat{\phi} \circ \psi_A = \psi_B \circ \phi$. Since ψ_B is an isomorphism by assumption, we may take $\rho = \psi_B^{-1} \circ \widehat{\phi}$ and this clearly satisfies the required condition.

In order to prove uniqueness, note that if $\widehat{\phi}$ is as in the statement, then $\widehat{\phi}$ induces for every n a morphism $\phi_n\colon \widehat{A}/I^n \cdot \widehat{A} \to B/J^n B$, whose composition with the isomorphism $A/I^n \to \widehat{A}/I^n \widehat{A}$ is the morphism $A/I^n \to B/J^n$ induced by ϕ. Since we have $\psi_B \circ \widehat{\phi} = \varprojlim \phi_n$, we obtain the uniqueness of $\widehat{\phi}$.

Proposition G.2.7. *If A is Noetherian and I is an ideal in A, then the completion $\widehat{A}$ is again Noetherian.*

Proof. Since A is Noetherian, we can find $a_1, \ldots, a_d \in I$ that generate I as an ideal. It is enough to show that $\widehat{A} \simeq A[\![x_1, \ldots, x_d]\!]/(x_1 - a_1, \ldots, x_d - a_d)$; if we know this, then the assertion follows from Theorem B.2.4.

Let $B = A[x_1, \ldots, x_d]$. On B, we consider the following two ideals: $\mathfrak{a} = (x_1, \ldots, x_d)$ and $\mathfrak{b} = (x_1 - a_1, \ldots, x_d - a_d)$. Note that we have an isomorphism $B/\mathfrak{b} \simeq A$, and via this isomorphism, we have $\mathfrak{a}^n \cdot A = I^n$ for every $n \geq 1$, hence the completion $\widehat{A}$ is isomorphic to the completion of A with respect to the ideal $\mathfrak{a}$ in B. Since B is Noetherian, by considering the completion with respect to $\mathfrak{a}$, we thus obtain

$$\widehat{A} \simeq \widehat{B/\mathfrak{b}} \simeq \widehat{B}/\mathfrak{b}\widehat{B} \simeq A[\![x_1, \ldots, x_d]\!]/(x_1 - a_1, \ldots, x_d - a_d).$$

$\square$

Remark G.2.8. An important case is that when $(A, \mathfrak{m})$ is a local Noetherian ring and $I = \mathfrak{m}$. Note that in this case the morphism $\psi_A \colon A \to \widehat{A}$ is injective (see Remark G.1.4). Note also that $\mathfrak{m}\widehat{A}$ is a maximal ideal, with $\widehat{A}/\mathfrak{m}\widehat{A} \simeq A/\mathfrak{m}$ (see Corollary G.2.4).

In fact, $\mathfrak{m}\widehat{A}$ is the unique maximal ideal of $\widehat{A}$. In order to see this, it is enough to show that if $u \in \widehat{A} \smallsetminus \mathfrak{m}\widehat{A}$, then $1 - u$ is invertible. This follows from the fact that $\widehat{A} \simeq \varprojlim \widehat{A}/(\mathfrak{m}\widehat{A})^n$ and if we put $a_n = \sum_{j=0}^{n-1} u^j$ for every $n \geq 1$, then the element in $\widehat{A}$ corresponding to $(a_n)_{n \geq 1}$ is an inverse of $1 - u$.

Remark G.2.9. We did not mention the topology on the ring A associated to the ideal I, since we do not need it. However, for the interested reader, we mention the notions of Cauchy sequences and convergent sequences that come out of the topological considerations. Given a ring A, the ideal I in A, and an A-module M, we say that a sequence $(x_n)_{n \geq 1}$ of elements in M is a *Cauchy sequence* if for every m, there is N such that $x_n - x_{n+1} \in I^m M$ for all $n \geq N$. The sequence has a *limit* $x \in M$ if for every m, there is N such that $x_n - x \in I^m M$ for all $n \geq N$. One can show that M is complete if and only if every Cauchy sequence in M has a limit and this limit is unique. We leave the proof as an exercise for the reader.

APPENDIX H

Modules of finite length

We review the definition of modules of finite length and their characterization over Noetherian rings.

H.1. Finite length

Let R be a commutative ring. Recall that an R-module M is *simple* if $M \neq 0$ and for every submodule M' of M, we have either $M' = 0$ or $M' = M$. It is straightforward to see that a module M is simple if and only if it is isomorphic to $A/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of R.

DEFINITION H.1.1. An R-module M is of *finite length* if it has a *composition series*, that is, a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that M_i/M_{i-1} is a simple module for $1 \leq i \leq r$. It is a consequence of the Jordan-Hölder theorem that if M satisfies this property, then the quotients M_i/M_{i-1} are independent of the choice of composition series, up to reordering. In particular, the length r only depends on M; this is the *length* of M, denoted $\ell(M)$ (or $\ell_R(M)$ if the ring is not clear from the context).

EXAMPLE H.1.2. If R is a DVR with discrete valuation v, then for every $a \in R$, we have $\ell\big(R/(a)\big) = v(a)$.

We begin with some easy properties regarding finite length modules.

PROPOSITION H.1.3. *Given an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

the module M has finite length if and only if both M' and M'' have finite length, and in this case

$$\ell(M) = \ell(M') + \ell(M'').$$

PROOF. It is clear that if M' and M'' have finite length, then we obtain a composition series for M by concatenating the composition series for M' and M''. This implies that $\ell(M) = \ell(M') + \ell(M'')$. The converse follows from the fact, easy to check, that given a composition series for M, by intersecting each submodule with M' (respectively, by taking the image of each submodule in M'') we obtain after removing repeated submodules a composition series for M' (respectively, M''). $\square$

REMARK H.1.4. Every R-module of finite length is Artinian: if

$$M = M_0 \supsetneq M_1 \supsetneq \ldots$$

is a strictly decreasing sequence of submodules, then it follows from the above proposition that we have a strictly decreasing sequence of non-negative integers

$$\ell(M_0) > \ell(M_1) > \ldots,$$

a contradiction.

PROPOSITION H.1.5. *If R is a Noetherian ring, then an R-module M has finite length if and only if M is finitely generated and* $\dim\big(R/\mathrm{Ann}_R(M)\big) = 0$.

PROOF. Suppose first that M has a composition series

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M,$$

with $M_i/M_{i-1} \simeq R/\mathfrak{m}_i$ for $1 \leq i \leq r$, where each $\mathfrak{m}_i$ is a maximal ideal of R. Since each M_i/M_{i-1} is finitely generated, we conclude that M is finitely generated. Moreover, we have $\prod_{i=1}^r \mathfrak{m}_i \subseteq \mathrm{Ann}_R(M)$, hence the only primes containing $\mathrm{Ann}(R)$ are the $\mathfrak{m}_i$. This implies that $\dim\big(R/\mathrm{Ann}_R(M)\big) = 0$.

Conversely, if M is finitely generated over a Noetherian ring, then it follows from Corollary E.3.4 that we have submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M,$$

such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for $1 \leq i \leq r$, where each $\mathfrak{p}_i$ is a prime ideal in R. If we have $\dim\big(R/\mathrm{Ann}_R(M)\big) = 0$, then every prime ideal in $R/\mathrm{Ann}_R(M)$ is a maximal ideal. Since we clearly have $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}_i$ for all i, we conclude that each quotient M_i/M_{i-1} is a simple module, hence M has finite length. □

EXAMPLE H.1.6. If $(R, \mathfrak{m})$ is a Noetherian local ring, then an R-module M has finite length if and only if it is finitely generated and $\mathfrak{m}^r \cdot M = 0$ for some $r \geq 1$.

REMARK H.1.7. We note that if R is a Noetherian ring, with $\dim(R) = 0$, then R is the product of finitely many local rings. Indeed, given a minimal primary decomposition

$$(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r,$$

by the Chinese Remainder theorem we have

$$R \simeq \prod_{i=1}^r R/\mathfrak{q}_i$$

(note that the ideals $\mathrm{rad}(\mathfrak{q}_i)$ are mutually distinct maximal ideals, hence $\mathfrak{q}_i + \mathfrak{q}_j = R$ whenever $i \neq j$).

EXAMPLE H.1.8. If k is a field and A is a finite k-algebra, then A is clearly Noetherian and $\dim(A) = 0$ (if $\mathfrak{p}$ is a prime ideal in A, then $A/\mathfrak{p}$ is a domain which is a finite k-algebra, hence it is a field by Proposition A.2.1). In particular, we see that A has finite length as a module over itself. The previous remark implies that A is the product of finitely many local, finite k-algebras.

COROLLARY H.1.9. *If $\phi\colon R \to S$ is a finite homomorphism of Noetherian rings, with $(R, \mathfrak{m})$ a local ring, then an S-module M has finite length over S if and only if it has finite length over R, and in this case, if $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are the maximal ideals in S, then*

$$\ell_R(M) = \sum_{i=1}^r \ell_{S_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \cdot [S/\mathfrak{q}_i : R/\mathfrak{m}].$$

PROOF. Note first that since ϕ is finite, the maximal ideals in S are precisely those prime ideals whose inverse image in R is equal to $\mathfrak{m}$ and there are indeed only finitely many of these. Since $\mathrm{rad}(\mathfrak{m}S) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$, it folows that M is annihilated by a power of $\mathfrak{m}$ if and only if it is annihilated by some power of $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$. Since it is clear that M is finitely generated over S if and only if it is finitely generated over R, we deduce from Proposition H.1.5 the first assertion in the corollary. Suppose now that M has finite length and consider a composition series over S:

$$0 = M_0 \subsetneq \ldots \subsetneq M_n = M.$$

Since the formula in the corollary is additive in short exact sequences, by considering the short exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

we see that it is enough to prove the formula when $M = S/\mathfrak{q}_j$ for some j. In this case the formula follows from the fact that

$$\ell_R(S/\mathfrak{q}_j) = \ell_{R/\mathfrak{m}}(S/\mathfrak{q}_j) = [S/\mathfrak{q}_j : R/\mathfrak{m}].$$

□

H.2. The valuation of the norm of an element

In this section we prove a result that is needed for computing the push-forward of principal Weil divisors.

LEMMA H.2.1. *If R is a DVR, with discrete valuation v, and M is a finitely generated, free R-module, then for every injective morphism $\phi\colon M \to M$, the R-module* $\mathrm{coker}(\phi)$ *has finite length, and*

$$\ell\big(\mathrm{coker}(\phi)\big) = v\big(\det(\phi)\big).$$

PROOF. It follows from general results on submodules of free modules over a PID that we can find a basis $e_1, \ldots, e_n$ of M such that $\phi(M)$ is generated by $a_1e_1, \ldots, a_ne_n$ for $a_1, \ldots, a_n \in R \smallsetminus \{0\}$. In this case, we have an invertible matrix $B \in M_n(R)$ such that $\det(\phi) = \det(B) \cdot \prod_{i=1}^n a_i$, hence $v\big(\det(\phi)\big) = v(a_1 \cdots a_n)$. On the other hand, we have

$$\mathrm{coker}(\phi) \simeq \bigoplus_{i=1}^n R/(a_i),$$

hence

$$\ell\big(\mathrm{coker}(\phi)\big) = \sum_{i=1}^n \ell\big(R/(a_i)\big) = \sum_{i=1}^n v(a_i) = v(a_1 \cdots a_n).$$

□

PROPOSITION H.2.2. *Let $f\colon R \hookrightarrow S$ be a finite, injective homomorphism, where $(R, \mathfrak{m})$ is a DVR, with discrete valuation v, and S is a domain, and denote by $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ the maximal ideals of S. If $K = \mathrm{Frac}(R)$ and $L = \mathrm{Frac}(S)$, then for every non-zero $b \in S$, we have*

$$v\big(N_{L/K}(b)\big) = \sum_{i=1}^r \ell_{S_{\mathfrak{q}_i}}\big(S_{\mathfrak{q}_i}/(b)\big) \cdot [S/\mathfrak{q}_i : R/\mathfrak{m}].$$

For a much more general result, see [Ful98, Chapter A.2].

Proof of Proposition H.2.2. Note first that since S is finite over R, the maximal ideals of S are precisely the prime ideals in S lying over $\mathfrak{m}$. Since S has no torsion as an R-module, it follows from the structure theorem of finitely generated modules over a PID that S is a free R-module. Consider the injective morphism $\phi\colon S \to S$ given by multiplication by b. Since $L = S \otimes_R K$, we have $N_{L/K}(b) = \det(\phi)$ and applying the lemma, we obtain

$$v\big(N_{L/K}(b)\big) = \ell_R\big(\operatorname{coker}(\phi)\big).$$

On the other hand, it follows from Corollary H.1.9 that

$$\ell_R\big(\operatorname{coker}(\phi)\big) = \sum_{i=1}^{r} \ell_{S_{\mathfrak{q}_i}}\big(S_{\mathfrak{q}_i}/(b)\big) \cdot [S/\mathfrak{q}_i : R/\mathfrak{m}].$$

This completes the proof of the proposition. □

APPENDIX I

Embeddings in injective modules

Let R be a ring (not necessarily commutative). Recall that a left R-module Q is *injective* if the functor $\mathrm{Hom}_R(-, Q)$ is exact. Since this functor is always left exact, Q is injective if and only if for every injective morphism of R-modules $M' \hookrightarrow M$, the induced morphism of Abelian groups

$$\mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q)$$

is surjective. In this appendix we show the basic fact that the category of left R-modules has enough injectives.

I.1. The Baer criterion and embeddings in injective modules

Proposition I.1.1. *For every left R-module M, there is an injective morphism $M \hookrightarrow Q$, where Q is an injective left R-module.*

The proof proceeds by first treating the case when $R = \mathbf{Z}$. In this case, the key fact is the characterization of injective $\mathbf{Z}$-modules as divisible groups. This in turn follows from the following criterion for a module to be injective:

Proposition I.1.2. *(Baer) A left R-module Q is injective if and only if for every left ideal I in R, the induced morphism of Abelian groups*

$$Q = \mathrm{Hom}_R(R, Q) \to \mathrm{Hom}_R(I, Q)$$

is surjective.

Proof. Of course, we only need to prove the "if" part. Suppose that M is a left R-module and M' is a submodule. We need to show that for every morphism $\phi' \colon M' \to Q$, there is a morphism $\phi \colon M \to Q$ such that $\phi|_{M'} = \phi'$. We consider the set $\mathcal{M}$ of all pairs (M_1, ϕ_1), where M_1 is a submodule of M containing M' and $\phi_1 \colon M_1 \to Q$ is a morphism such that $\phi_1|_{M'} = \phi'$. We order this set by putting $(M_1, \phi_1) \leq (M_2, \phi_2)$ if $M_1 \subseteq M_2$ and $\phi_2|_{M_1} = \phi_1$.

Since we have $(M', \phi') \in \mathcal{M}$, we see that $\mathcal{M}$ is non-empty. Moreover, given a family $(M_i, \phi_i)_{i \in I}$ of elements of $\mathcal{M}$, any two of them comparable, we can take $M'' = \bigcup_{i \in I} M_i$ and $\phi'' \colon M'' \to Q$ such that $\phi''|_{M_i} = \phi_i$ for all i; in this case $(M'', \phi'') \in \mathcal{M}$ is the supremum of the family $(M_i, \phi_i)_{i \in I}$.

We can thus apply Zorn's lemma to choose a maximal element (M_0, ϕ_0) in $\mathcal{M}$. We claim that $M_0 = M$, which would complete the proof. Suppose that this is not the case and let $u \in M \smallsetminus M_0$. We will show that there is an extension of ϕ_0 to a morphism $\phi_1 \colon M_0 + Ru \to Q$; this would contradict the maximality of (M_0, ϕ_0).

Let $I = \{a \in R \mid au \in M_0\}$. Note that I is a left ideal of R and we can define a morphism $\psi \colon I \to Q$ by $\psi(a) = \phi_0(au)$. By assumption, there is $w \in Q$ such that $\psi(a) = aw$ for every $a \in I$. It is then straightforward to see that if we put

$$\phi_1(v + au) = \phi_0(v) + aw \quad \text{for} \quad v + au \in M_0 + Ru,$$

then ϕ_1 is well-defined and gives a morphism $M_0 + Ru \to Q$ such that $\phi_1|_{M_0} = \phi_0$. This completes the proof. $\square$

Recall that an Abelian group A is *divisible* if for every positive integer n, the multiplication map $A \xrightarrow{\cdot n} A$ is surjective.

Corollary I.1.3. *A* $\mathbf{Z}$*-module* Q *is injective if and only if it is a divisible Abelian group.*

Proof. Since every ideal of $\mathbf{Z}$ is of the form $n\mathbf{Z}$, for some non-negative integer n, it follows from the proposition that Q is injective if and only if for every such n, the induced morphism of Abelian groups

$$Q \to \operatorname{Hom}_{\mathbf{Z}}(n\mathbf{Z}, Q)$$

is surjective. This is clearly the case if $n = 0$. If $n > 0$, then this morphism gets identified to the morphism $Q \to Q$ given by multiplication by n, and we obtain the assertion in the corollary. $\square$

We can now prove the existence of embeddings in injective modules.

Proof of Proposition I.1.1. Suppose first that $R = \mathbf{Z}$. In this case, by the above corollary, we need to show that every Abelian group M can be embedded in a divisible Abelian group A. Write $M \simeq F/G$, where $F \simeq \mathbf{Z}^{(I)}$ is a free Abelian group. Since F is free, it has no torsion, and thus the canonical morphism $F \hookrightarrow F \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}^{(I)}$ is injective. We thus have an injective morphism $M \hookrightarrow A := (F \otimes_{\mathbf{Z}} \mathbf{Q})/G$. It is clear that $F \otimes_{\mathbf{Z}} \mathbf{Q}$ is divisible, and thus its image A is divisible, too.

Consider now the general case. By considering on M the underlying structure of $\mathbf{Z}$-module and applying what we have already proved, we get an injective morphism of $\mathbf{Z}$-modules $j \colon M \hookrightarrow A$, where A is an injective $\mathbf{Z}$-module. We claim that if we consider on $\operatorname{Hom}_{\mathbf{Z}}(R, A)$ the left R-module structure induced by the right R-module structure of R (that is, we have

$$(\lambda \cdot \phi)(r) = \phi(r\lambda) \quad \text{for all} \quad \lambda, r \in R, \phi \in \operatorname{Hom}_{\mathbf{Z}}(R, A)),$$

then $\operatorname{Hom}_{\mathbf{Z}}(R, A)$ is an injective R-module. In order to see this, it is enough to note that by the adjoint property of $R \otimes_R -$ and $\operatorname{Hom}_{\mathbf{Z}}(R, -)$, for every left R-module N, we have a canonical isomorphism

$$\operatorname{Hom}_R\big(N, \operatorname{Hom}_{\mathbf{Z}}(R, A)\big) \simeq \operatorname{Hom}_{\mathbf{Z}}(R \otimes_R N, A) \simeq \operatorname{Hom}_{\mathbf{Z}}(N, A).$$

Finally, we note that we have an injective morphism of left R-modules given by

$$M \to \operatorname{Hom}_{\mathbf{Z}}(R, A), \quad M \ni v \to \phi_v, \quad \text{where} \quad \phi_v(r) = j(rv).$$

This completes the proof. $\square$

APPENDIX J

The Principal Ideal theorem: a proof in the general case

In this appendix we give a proof of the Principal Ideal theorem for arbitrary Noetherian rings. While this is not necessary for the geometric results in these notes, the argument is so simple, assuming the results in Appendix H, that we give it for completeness.

J.1. The statement and the proof

THEOREM J.1.1. *If R is a Noetherian ring and $\mathfrak{p}$ is a minimal prime ideal containing a principal ideal (x), then* $\operatorname{codim}(\mathfrak{p}) \leq 1$.

PROOF. After replacing R by $R_{\mathfrak{p}}$, we may assume that R is a local ring and $\mathfrak{p}$ is the maximal ideal. It is enough to show that for every prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$ in R, we have $\operatorname{codim}(\mathfrak{q}) = 0$.

The ring $R/(x)$ is Noetherian and by hypothesis, has only one prime ideal, namely $\mathfrak{p}/(x)$. It follows from Proposition H.1.5 that $R/(x)$ is an R-module of finite length, hence Artinian (see Remark H.1.4). Note that if we put $\mathfrak{q}^{(n)} := \mathfrak{q}^n R_{\mathfrak{q}} \cap R$ for $n \geq 1$, then we have the non-increasing chain of R-modules in $R/(x)$:

$$\big(\mathfrak{q}^{(1)} + (x)\big)/(x) \supseteq \big(\mathfrak{q}^{(2)} + (x)\big)/(x) \supseteq \ldots \big(\mathfrak{q}^{(n)} + (x)\big)/(x) \ldots,$$

which thus must stabilize. We deduce that we have $n \geq 1$ such that $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$. This implies that for every $u \in \mathfrak{q}^{(n)}$, there are $a \in R$ and $v \in \mathfrak{q}^{(n+1)}$ such that $u = v + ax$. Since $ax \in \mathfrak{q}^{(n)}$ and $x \notin \mathfrak{q}$, we have $a \in \mathfrak{q}^{(n)}$. We thus conclude that $\mathfrak{q}^{(n)} = x \cdot \mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}$. Since x lies in the unique maximal ideal in R, it follows from Nakayama's lemma (see Corollary C.1.2) that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. This implies that $\mathfrak{q}^n R_{\mathfrak{q}} = \mathfrak{q}^{n+1} R_{\mathfrak{q}}$, and using Nakayama's lemma in $R_{\mathfrak{q}}$, we conclude that $\mathfrak{q}^n R_{\mathfrak{q}} = 0$. This implies that $\operatorname{codim}(\mathfrak{q}) = 0$ and thus completes the proof of the theorem. □

The above theorem is usually applied in the following more general form.

COROLLARY J.1.2. *If R is a Noetherian ring and $\mathfrak{p}$ is a minimal prime ideal containing $(x_1, \ldots, x_n)$, then* $\operatorname{codim}(\mathfrak{p}) \leq n$.

PROOF. The reduction to the theorem proceeds as in the proof of Corollary 3.3.7. We leave the task of rewriting algebraically that argument as an exercise for the reader. □

We also have the following partial converse to the corollary:

PROPOSITION J.1.3. *If $\mathfrak{p}$ is a prime ideal in a Noetherian ring R and* $\operatorname{codim}(\mathfrak{p}) =$ *n, then there are $x_1, \ldots, x_n \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime containing $(x_1, \ldots, x_n)$.*

PROOF. It is enough to translate algebraically the argument in the proof of Proposition 3.3.16. We leave this, as well, as an exercise for the reader. $\square$

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