Lecture notes for Math 631: Introduction to algebraic geometry

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Contents

CHAPTER 1

Affine and quasi-affine varieties

The main goal in this chapter is to establish a correspondence between various geometric notions and algebraic ones. Some references for this chapter are [**Har77**, Chapter I], [**Mum88**, Chapter I], and [**Sha13**, Chapter I].

1.1. Algebraic subsets and ideals

Let k be a fixed algebraically closed field. We do not make any assumption on the characteristic. Important examples are $\mathbf{C}$, $\overline{\mathbf{Q}}$, and $\overline{\mathbf{F}_p}$, for a prime integer p.

For a positive integer n we denote by $\mathbf{A}^n$ the *n-dimensional affine space*. For now, this is just a set, namely k^n. We assume that n is fixed and denote the polynomial ring $k[x_1, \ldots, x_n]$ by R. Note that if $f \in R$ and $u = (u_1, \ldots, u_n)$, we may evaluate f at u to get $f(u) \in k$. This gives a surjective ring homomorphism

$$k[x_1, \ldots, x_n] \to k, \quad f \to f(u),$$

whose kernel is the (maximal) ideal $(x_1 - u_1, \ldots, x_n - u_n)$.

Our goal in this section is to establish a correspondence between certain subsets of $\mathbf{A}^n$ (those *defined by polynomial equations*) and ideals in R (more precisely, radical ideals). A large part of this correspondence is tautological. The non-trivial input will be provided by Hilbert's Nullstellensatz, which we will be prove in the next section.

Definition 1.1.1. Given a subset $S \subseteq R$, the *zero-locus of S* (also called *the subset of $\mathbf{A}^n$ defined by S*) is the set

$$V(S) := \{u \in \mathbf{A}^n \mid f(u) = 0 \text{ for all } f \in S\}.$$

An *algebraic subset* of $\mathbf{A}^n$ is a subset of the form $V(S)$ for some subset S of R.

Example 1.1.2. Any linear subspace of k^n is an algebraic subset; in fact, it can be written as $V(S)$, where S is a finite set of linear polynomials (that is, polynomials of the form $\sum_{i=1}^n a_i x_i$). More generally, any translation of a linear subspace (that is, an *affine subspace*) of k^n is an algebraic subset.

Example 1.1.3. A union of two lines in $\mathbf{A}^2$ is an algebraic subset (see Proposition 1.1.6). For example, the union of the two coordinate axes can be written as $V(x_1 x_2)$.

Example 1.1.4. Another example of an algebraic subset of $\mathbf{A}^2$ is the hyperbola

$$\{u = (u_1, u_2) \in \mathbf{A}^2 \mid u_1 u_2 = 1\}.$$

Remark 1.1.5. Recall that if S is a subset of R and I is the ideal of R generated by S, then we can write

$$I = \{g_1 f_1 + \ldots + g_m f_m \mid m \geq 0, f_1, \ldots, f_m \in S, g_1, \ldots, g_m \in R\}.$$

It is then easy to see that $V(S) = V(I)$. In particular, every algebraic subset of $\mathbf{A}^n$ can be written as $V(I)$ for some ideal I in R.

We collect in the following proposition the basic properties of taking the zero locus.

PROPOSITION 1.1.6. *The following hold:*

1) $V(R) = \emptyset$*; in particular, the empty set is an algebraic subset.*
2) $V(0) = \mathbf{A}^n$*: in particular,* $\mathbf{A}^n$ *is an algebraic subset.*
3) *If* I *and* J *are ideals in* R *with* $I \subseteq J$*, then* $V(J) \subseteq V(I)$.
4) *If* $(I_\alpha)_\alpha$ *is a family of ideals in* R*, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right) = V\left(\sum_\alpha I_\alpha\right).$$

5) *If* I *and* J *are ideals in* R*, then*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

PROOF. The assertions in 1)–4) are trivial to check. Note also that the inclusions

$$V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(I \cdot J)$$

follow directly from 3). In order to show that $V(I \cdot J) \subseteq V(I) \cup V(J)$, we argue by contradiction: suppose that $u \in V(I \cdot J) \smallsetminus \big(V(I) \cup V(J)\big)$. We can thus find $f \in I$ such that $f(u) \neq 0$ and $g \in J$ such that $g(u) \neq 0$. In this case $fg \in I \cdot J$ and $(fg)(u) = f(u)g(u) \neq 0$, a contradiction with the fact that k is a domain. □

An important consequence of the assertions in the above proposition is that the algebraic subsets of $\mathbf{A}^n$ form the closed subsets for a topology of $\mathbf{A}^n$. This is the *Zariski topology* on $\mathbf{A}^n$.

The Zariski topology provides a convenient framework for dealing with algebraic subsets of $\mathbf{A}^n$. However, we will see that it has a lot less subsets than one is used to from the case of the usual Euclidean space (over $\mathbf{R}$ or over $\mathbf{C}$).

We now define a map in the other direction, from subsets of $\mathbf{A}^n$ to ideals in R. Given a subset W of $\mathbf{A}^n$, we put

$$I(W) := \{f \in R \mid f(u) = 0 \text{ for all } u \in W\}.$$

It is straightforward to see that this is an ideal in R. In fact, it is a radical[1] ideal: indeed, since k is a reduced ring, if $f(u)^q = 0$ for some positive integer q, then $f(u) = 0$. We collect in the next proposition some easy properties of this definition.

PROPOSITION 1.1.7. *The following hold:*

1) $I(\emptyset) = R$.
2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$*, then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$.
3) *If* $W_1 \subseteq W_2$*, then* $I(W_2) \subseteq I(W_1)$.

PROOF. All assertions follow immediately from definition. □

[1] An ideal I in a ring R is *radical* if whenever $f^q \in I$ for some $f \in R$ and some positive integer q, we have $f \in I$. A related concept is that of a *reduced* ring: this is a ring such that whenever $f^q = 0$ for some $f \in R$ and some positive integer q, we have $f = 0$. Note that an ideal I is radical if and only if R/I is a reduced ring.

We have thus set up two maps between subsets of $\mathbf{A}^n$ and ideals in R and we are interested in the two compositions. Understanding one of these compositions is tautological, as follows:

PROPOSITION 1.1.8. *For every subset Z of $\mathbf{A}^n$, the set $V\big(I(Z)\big)$ is equal to the closure $\overline{Z}$ of Z, with respect to the Zariski topology. In particular, if Z is an algebraic subset of $\mathbf{A}^n$, then $V\big(I(Z)\big) = Z$.*

PROOF. We clearly have
$$Z \subseteq V\big(I(Z)\big),$$
and since the right-hand side is closed by definition, we have
$$\overline{Z} \subseteq V\big(I(Z)\big).$$
In order to prove the reverse inclusion, recall that by definition of the closure of a subset, we have
$$\overline{Z} = \bigcap_W W,$$
where W runs over all algebraic subsets of $\mathbf{A}^n$ that contain Z. Every such W can be written as $W = V(J)$, for some ideal J in R. Note that we have $J \subseteq I(W)$, while the inclusion $Z \subseteq W$ gives $I(W) \subseteq I(Z)$. We thus have $J \subseteq I(Z)$, hence $V\big(I(Z)\big) \subseteq V(J) = W$. Since $V\big(I(Z)\big)$ is contained in every such W, we conclude that
$$V\big(I(Z)\big) \subseteq \overline{Z}.$$
□

The interesting statement here concerns the other composition. Recall that if J is an ideal in a ring R, then the set
$$\{f \in R \mid f^q \in J \text{ for some } q \geq 1\}$$
is a radical ideal; in fact, it is the smallest radical ideal containing J, denoted $\mathrm{rad}(J)$.

THEOREM 1.1.9. *(Hilbert's Nullstellensatz) For every ideal J in R, we have*
$$I\big(V(J)\big) = \mathrm{rad}(J).$$

The inclusion $J \subseteq I\big(V(J)\big)$ is trivial and since the right-hand side is a radical ideal, we obtain the inclusion
$$\mathrm{rad}(J) \subseteq I\big(V(J)\big).$$
This reverse inclusion is the subtle one and this is where we use the hypothesis that k is algebraically closed (note that this did not play any role so far). We will prove this in the next section, after some preparations. Assuming this, we obtain the following conclusion.

COROLLARY 1.1.10. *The two maps $I(-)$ and $V(-)$ between the algebraic subsets of $\mathbf{A}^n$ and the radical ideals in $k[x_1, \ldots, x_n]$ are inverse, order-reversing bijections.*

REMARK 1.1.11. It follows from Corollary 1.1.10 that via the above bijection, the minimal nonempty algebraic subsets correspond to the maximal ideals in R. It is clear that the minimal nonempty algebraic subsets are precisely the points in $\mathbf{A}^n$. On the other hand, given $a = (a_1, \ldots, a_n) \in \mathbf{A}^n$, the ideal $I(u)$ contains the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$, hence the two ideals are equal. We thus deduce that *every* maximal ideal in R is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. We will see in the next section that the general statement of Theorem 1.1.9 is proved by reduction to this special case.

Exercise 1.1.12. Show that the closed subsets of $\mathbf{A}^1$ are $\mathbf{A}^1$ and its finite subsets.

Exercise 1.1.13. Show that if W_1 and W_2 are algebraic subsets of $\mathbf{A}^n$, then

$$I(W_1 \cap W_2) = \mathrm{rad}\big(I(W_1) + I(W_2)\big).$$

Exercise 1.1.14. For m and $n \geq 1$, let us identify $\mathbf{A}^{mn}$ with the set of all matrices $B \in M_{m,n}(k)$. Show that the set

$$M^r_{m,n}(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \leq r\}$$

is a closed algebraic subset of $M_{m,n}(k)$.

Exercise 1.1.15. Show that the following subset of $\mathbf{A}^3$

$$W_1 = \{(t, t^2, t^3) \mid t \in k\}$$

is a closed algebraic subset, and describe $I(W_1)$. Can you do the same for

$$W_2 = \{(t^2, t^3, t^4) \mid t \in k\}?$$

How about

$$W_3 = \{(t^3, t^4, t^5) \mid t \in k\}?$$

Exercise 1.1.16. For an arbitrary commutative ring R, one can define the *maximal spectrum* $\mathrm{MaxSpec}(R)$ of R, as follows. As a set, this is the set of all maximal ideals in R. For every ideal J in R, we put

$$V(J) := \{\mathfrak{m} \in \mathrm{MaxSpec}(R) \mid J \subseteq \mathfrak{m}\}$$

and for every subset $S \subseteq \mathrm{MaxSpec}(R)$, we define

$$I(S) := \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

i) Show that $\mathrm{MaxSpec}(R)$ has a structure of topological space in which the closed subsets are the subsets of the form $V(I)$, for an ideal I in R.
ii) Show that for every subset S of $\mathrm{MaxSpec}(R)$, we have $V\big(I(S)\big) = \overline{S}$.
iii) Show that if R is an algebra of finite type over an algebraically closed field k, then for every ideal J in S, we have $I\big(V(J)\big) = \mathrm{rad}(J)$.
iv) Show that if $X \subseteq \mathbf{A}^n$ is a closed subset, then we have a homeomorphism $X \simeq \mathrm{MaxSpec}(R/J)$, where $R = k[x_1, \ldots, x_n]$ and $J = I(X)$.

1.2. Noether normalization and Hilbert's Nullstellensatz

The proof of Hilbert's Nullstellensatz is based on the following result, known as *Noether's normalization lemma*. As we will see, this has many other applications.

Before stating the result, we recall that a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module. It is straightforward to check that a composition of two finite homomorphisms is again finite. Moreover, if $A \to B$ is a finite homomorphism, then for every homomorphism $A \to C$, the induced homomorphism $C = A \otimes_A C \to B \otimes_A C$ is finite. For details about finite morphisms and the connection with integral morphisms, see Appendix A. One property that we will need is that if $A \hookrightarrow B$ is an injective finite homomorphism, with A and B domains, then A is a field if and only if B is a field (see Proposition A.2.1).

REMARK 1.2.1. If $A \hookrightarrow B$ is an injective, finite homomorphism between two domains, and $K = \mathrm{Frac}(A)$ and $L = \mathrm{Frac}(B)$, then the induced injective homomorphism $K \hookrightarrow L$ is finite. Indeed, by tensoring the inclusion $A \hookrightarrow B$ with K, we obtain a finite, injective homomorphism $K \hookrightarrow K \otimes_A B$ between domains. Note that $K \otimes_A B$ is a ring of fractions of B, hence the canonical homomorphism $K \otimes_A B \to L$ is injective. Since K is a field, it follows that $K \otimes_A B$ is a field, and thus $K \otimes_A B = L$. In particular, we see that $[L : K] < \infty$.

THEOREM 1.2.2. *Let k be a field and A a finitely generated k-algebra which is an integral domain, with fraction field K. If* $\mathrm{trdeg}(K/k) = n$, *then there is a k-subalgebra B of A, such that*

1) *B is isomorphic as a k-algebra to $k[x_1, \ldots, x_n]$, and*
2) *The inclusion $B \hookrightarrow A$ is finite.*

PROOF. We only give the proof when k is infinite. This will be enough for our purpose, since in all our applications the field k will always contain an algebraically closed (hence infinite) field. For a proof in the general case, see [**Mum88**].

The fact that k is infinite will be used via the following property: for every nonzero polynomial $f \in k[x_1, \ldots, x_r]$, there is $\lambda \in k^r$ such that $f(\lambda) \neq 0$. When $r = 1$, this follows from the fact that a nonzero polynomial in one variable has at most as many roots as its degree. The general case then follows by an easy induction on r.

Let $y_1, \ldots, y_m \in A$ be generators of A as a k-algebra. In particular, we have $K = k(y_1, \ldots, y_m)$, hence $m \geq n$. We will show, by induction on m, that we can find a change of variable of the form

$$y_i = \sum_{j=1}^{n} b_{i,j} z_j, \quad \text{for} \quad 1 \leq i \leq m, \quad \text{with} \quad \det(b_{i,j}) \neq 0,$$

(so that we have $A = k[z_1, \ldots, z_m]$) such that the inclusion $k[z_1, \ldots, z_n] \hookrightarrow A$ is finite. Note that this is enough: if $B = k[z_1, \ldots, z_n]$, then it follows from Remark 1.2.1 that the induced field extension $\mathrm{Frac}(B) \hookrightarrow K$ is finite. Therefore we have

$$n = \mathrm{trdeg}(K/k) = \mathrm{trdeg}\big(k(z_1, \ldots, z_n)/k\big),$$

hence $z_1, \ldots, z_n$ are algebraically independent.

If $m = n$, there is nothing to prove. Suppose now that $m > n$, hence $y_1, \ldots, y_m$ are algebraically dependent over k. Therefore there is a nonzero polynomial $f \in k[x_1, \ldots, x_m]$ such that $f(y_1, \ldots, y_m) = 0$. Suppose now that we write

$$y_i = \sum_{j=1}^{m} b_{i,j} z_j, \quad \text{with} \quad b_{i,j} \in k, \, \det(b_{i,j}) \neq 0.$$

Let $d = \deg(f)$ and let us write

$$f = f_d + f_{d-1} + \ldots + f_0, \quad \text{with} \quad \deg(f_i) = i \quad \text{or} \quad f_i = 0.$$

By assumption, we have $f_d \neq 0$. If we write

$$f = \sum_{\alpha \in \mathbf{Z}^m_{\geq 0}} c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

then we have

$$0 = f(y_1, \ldots, y_m) = \sum_{\alpha} c_\alpha (b_{1,1}z_1 + \ldots + b_{1,m}z_m)^{\alpha_1} \cdots (b_{m,1}z_1 + \ldots + b_{m,m}z_m)^{\alpha_m}$$
$$= f_d(b_{1,m}, \ldots, b_{m,m}) z_m^d + \text{lower degree terms in } z_m.$$

Since we assume that k is infinite, we may choose the $b_{i,j}$ such that

$$\det(b_{i,j}) \cdot f_d(b_{1,m}, \ldots, b_{m,m}) \neq 0.$$

In this case, we see that after this linear change of variable, the inclusion

$$k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite, since the right-hand side is generated as a module over the left-hand side by $1, y_m, \ldots, y_m^{d-1}$. Note that by Remark 1.2.1, the induced extension

$$k(y_1, \ldots, y_{m-1}) \hookrightarrow k(y_1, \ldots, y_m)$$

is finite, hence $\operatorname{trdeg}\big(k(y_1, \ldots, y_{m-1})/k\big) = n$. By induction, we can do a linear change of variable in $y_1, \ldots, y_{m-1}$, after which the inclusion

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}]$$

is finite, in which case the composition

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite. This completes the proof of the theorem. □

We will use Theorem 1.2.2 to prove Hilbert's Nullstellensatz in several steps.

COROLLARY 1.2.3. *If k is a field, A is a finitely generated k-algebra, and $K = A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in A, then K is a finite extension of k.*

PROOF. Note that K is a field which is finitely generated as a k-algebra. It follows from the theorem that if $n = \operatorname{trdeg}(K/k)$, then there is a finite injective homomorphism

$$k[x_1, \ldots, x_n] \hookrightarrow K.$$

Since K is a field, it follows that $k[x_1, \ldots, x_n]$ is a field, hence $n = 0$. Therefore K/k is finite. □

COROLLARY 1.2.4. *(Hilbert's Nullstellensatz, weak version) If k is an algebraically closed field, then every maximal ideal $\mathfrak{m}$ in $R = k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$, for some $a_1, \ldots, a_n \in k$.*

PROOF. It follows from Corollary 1.2.3 that if $K = R/\mathfrak{m}$, the field extension K/k is finite. Since k is algebraically closed, the canonical homomorphism $k \to K$ is an isomorphism. In particular, for every i there is $a_i \in R$ such that $x_i - a_i \in \mathfrak{m}$. Therefore we have $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$ and since both ideals are maximal, they must be equal. □

We can now prove Hilbert's Nullstellensatz, in its strong form.

PROOF OF THEOREM 1.1.9. It follows from Corollary 1.2.4 that given any ideal $\mathfrak{a}$ of R, different from R, the zero-locus $V(\mathfrak{a})$ of $\mathfrak{a}$ is nonempty. Indeed, since $\mathfrak{a} \neq R$, there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. By Corollary 1.2.4, we have

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n) \quad \text{for some} \quad a_1, \ldots, a_n \in k.$$

In particular, we see that $a = (a_1, \ldots, a_n) \in V(\mathfrak{m}) \subseteq V(J)$. We will use this fact in the polynomial ring $R[y] = k[x_1, \ldots, x_n, y]$; this is *Rabinovich's trick*.

It is clear that for every ideal J in R we have the inclusion

$$\mathrm{rad}(J) \subseteq I\big(V(J)\big).$$

In order to prove the reverse inclusion, suppose that $f \in I\big(V(J)\big)$. Consider now the ideal $\mathfrak{a}$ in $R[y]$ generated by J and by $1 - fy$. If $\mathfrak{a} \neq R[y]$, we have seen that there is $(a_1, \ldots, a_n, b) \in V(\mathfrak{a})$. By definition of $\mathfrak{a}$, this means that $g(a_1, \ldots, a_n) = 0$ for all $g \in J$ (that is, $(a_1, \ldots, a_n) \in V(J)$) and $1 = f(a_1, \ldots, a_n)g(b)$. In particular, we have $f(a_1, \ldots, a_n) \neq 0$, contradicting the fact that $f \in I\big(V(J)\big)$.

We thus conclude that $\mathfrak{a} = R$. Therefore we can find $f_1, \ldots, f_r \in J$ and $g_1, \ldots, g_{r+1} \in R[y]$ such that

$$\sum_{i=1}^{r} f_i(x)g_i(x, y) + \big(1 - f(x)y\big)g_{r+1}(x, y) = 1. \tag{1.2.1}$$

We now consider the R-algebra homomorphism $R[y] \to R_f$ that maps y to $\frac{1}{f}$. The relation (1.2.1) gives

$$\sum_{i=1}^{r} f_i(x)g_i\big(x, 1/f(x)\big) = 1$$

and after clearing the denominators (recall that R is a domain), we see that there is a positive integer N such that $f^N \in (f_1, \ldots, f_r)$, hence $f \in \mathrm{rad}(J)$. This completes the proof of the theorem. $\square$

1.3. The topology on the affine space

In this section we begin making use of the fact that the ring $k[x_1, \ldots, x_n]$ is Noetherian. Recall that a (commutative) ring R is Noetherian if the following equivalent conditions hold:

i) Every ideal in R is finitely generated.
ii) There is no infinite strictly increasing sequence of ideals of R.
iii) Every nonempty family of ideals of R has a maximal element

For this and other basic facts about Noetherian rings and modules, see Appendix B. A basic result in commutative algebra is Hilbert's basis theorem: if R is a Noetherian ring, then $R[x]$ is Noetherian (see Theorem B.2.1). In particular, since a field k is trivially Noetherian, a recursive application of the theorem implies that every polynomial algebra $k[x_1, \ldots, x_n]$ is Noetherian.

As in the previous sections, we fix an algebraically closed field k and a positive integer n. The fact that the ring $R = k[x_1, \ldots, x_n]$ is Noetherian has two immediate consequences. First, since every ideal is finitely generated, it follows that for every algebraic subset $W \subseteq \mathbf{A}^n$, there are finitely many polynomials $f_1, \ldots, f_r$ such that $W = V(f_1, \ldots, f_r)$. Second, we see via the correspondence in Corollary 1.1.10 that there is no infinite strictly decreasing sequence of closed subsets in $\mathbf{A}^n$.

Definition 1.3.1. A topological space X is *Noetherian* if there is no infinite strictly decreasing sequence of closed subsets in X.

We have thus seen that with the Zariski topology $\mathbf{A}^n$ is a Noetherian topological space. This implies that every subspace of $\mathbf{A}^n$ is Noetherian, by the following

LEMMA 1.3.2. *If X is a Noetherian topological space and Y is a subspace of X, then Y is Noetherian.*

PROOF. If we have a infinite strictly decreasing sequence of closed subsets of Y

$$F_1 \supsetneq F_2 \supsetneq \ldots,$$

consider the corresponding sequence of closures in X:

$$\overline{F_1} \supseteq \overline{F_2} \supseteq \ldots.$$

Since F_i is closed in Y, we have $\overline{F_i} \cap Y = F_i$ for all i, which implies that $\overline{F_i} \neq \overline{F_{i+1}}$ for every i. This contradicts the fact that X is Noetherian. $\square$

REMARK 1.3.3. Note that every Noetherian topological space is quasi-compact: this follows from the fact that there is no infinite strictly increasing sequence of open subsets.

EXAMPLE 1.3.4. The real line $\mathbf{R}$, with the usual Euclidean topology, is not Noetherian.

We now introduce an important notion.

DEFINITION 1.3.5. A topological space X is *irreducible* if it is nonempty and whenever we write $X = X_1 \cup X_2$, with both X_1 and X_2 closed, we have $X_1 = X$ or $X_2 = X$. We say that X is *reducible* when it is not irreducible.

REMARK 1.3.6. By passing to complements, we see that a topological space is irreducible if and only if it is nonempty and for every two nonempty open subsets U and V, the intersection $U \cap V$ is nonempty (equivalently, every nonempty open subset of X is dense in X).

REMARKS 1.3.7.
1) If Y is a subset of X (with the subspace topology), the closed subsets of Y are those of the form $F \cap Y$, where F is a closed subset of X. It follows that Y is irreducible if and only if it is nonempty and whenever $Y \subseteq Y_1 \cup Y_2$, with Y_1 and Y_2 closed in X, we have $Y \subseteq Y_1$ or $Y \subseteq Y_2$.
2) If Y is an irreducible subset of X and if $Y \subseteq Y_1 \cup \ldots \cup Y_r$, with all Y_i closed in X, then there is i such that $Y \subseteq Y_i$. This follows easily by induction on r.
3) If Y and F are subsets of X, with F closed, then $Y \subseteq F$ if and only if $\overline{Y} \subseteq F$. It then follows from the description in 1) that Y is irreducible if and only if $\overline{Y}$ is irreducible.
4) If X is irreducible and U is a nonempty open subset of X, then it follows from Remark 1.3.6 that U is dense in X. Since X is irreducible, it follows from 3) that U is irreducible.

In the case of closed subsets of $\mathbf{A}^n$, the following proposition describes irreducibility in terms of the corresponding ideal.

PROPOSITION 1.3.8. *If $W \subseteq \mathbf{A}^n$ is a closed subset, then W is irreducible if and only if $I(W)$ is a prime ideal in R.*

PROOF. Note first that $W \neq \emptyset$ if and only if $I(W) \neq R$. Suppose first that W is irreducible and let $f, g \in R$ be such that $fg \in I(W)$. We can then write

$$W = \big(W \cap V(f)\big) \cup \big(W \cap V(g)\big)$$

Since both subsets on the right-hand side are closed and W is irreducible, it follows that we have either $W = W \cap V(f)$ (in which case $f \in I(W)$) or $W = W \cap V(g)$ (in which case $g \in I(W)$). Therefore $I(W)$ is a prime ideal.

Conversely, suppose that $I(W)$ is prime and we write $W = W_1 \cup W_2$, with W_1 and W_2 closed. Arguing by contradiction, suppose that $W \neq W_i$ for $i = 1, 2$, in which case $I(W) \subsetneq I(W_i)$, hence we can find $f_i \in I(W_i) \smallsetminus I(W)$. On the other hand, we have $f_1 f_2 \in I(W_1) \cap I(W_2) = I(W)$, contradicting the fact that $I(W)$ is prime. □

EXAMPLE 1.3.9. Since R is a domain, it follows from the proposition that $\mathbf{A}^n$ is irreducible.

EXAMPLE 1.3.10. If $L \subseteq \mathbf{A}^n$ is a linear subspace, then L is irreducible. Indeed, after a linear change of variables, we have $R = k[y_1, \ldots, y_n]$ such that $I(L) = (y_1, \ldots, y_r)$ for some $r \geq 1$, and this is clearly a prime ideal in R.

EXAMPLE 1.3.11. The union of two lines in $\mathbf{A}^2$ is a reducible closed subset.

PROPOSITION 1.3.12. *Let X be a Noetherian topological space. Given a closed, nonempty subset Y, there are finitely many irreducible closed subsets $Y_1, \ldots, Y_r$ such that*

$$Y = Y_1 \cup \ldots \cup Y_r.$$

We may clearly assume that the decomposition is minimal*, in the sense that $Y_i \not\subseteq Y_j$ for $i \neq j$. In this case $Y_1, \ldots, Y_r$ are unique up to reordering.*

The closed subsets $Y_1, \ldots, Y_r$ in the proposition are the *irreducible components* of Y and the decomposition in the proposition is the *irreducible decomposition* of Y.

PROOF OF PROPOSITION 1.3.12. Suppose first that there are nonempty closed subsets Y of X that do not have such a decomposition. Since X is Noetherian, we may choose a minimal such Y. In particular, Y is not irreducible, hence we may write $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 closed and strictly contained in Y. Note that Y_1 and Y_2 are nonempty, hence by the minimality of Y, we may write both Y_1 and Y_2 as finite unions of irreducible subsets. In this case, Y is also a finite union of irreducible subsets, a contradiction.

Suppose now that we have two minimal decompositions

$$Y = Y_1 \cup \ldots \cup Y_r = Y'_1 \cup \ldots \cup Y'_s,$$

with the Y_i and Y'_j irreducible. For every $i \leq r$, we get an induced decomposition

$$Y_i = \bigcup_{j=1}^{s} (Y_i \cap Y'_j),$$

with the $Y_i \cap Y'_j$ closed for all j. Since Y_i is irreducible, it follows that there is $j \leq s$ such that $Y_i = Y_i \cap Y'_j \subseteq Y'_j$. Arguing in the same way, we see that there is $\ell \leq r$ such that $Y'_j \subseteq Y_\ell$. In particular, we have $Y_i \subseteq Y_\ell$, hence by the minimality assumption, we have $i = \ell$, and therefore $Y_i = Y'_j$. By iterating this argument and by reversing the roles of the Y_α and the Y'_β, we see that $r = s$ and the Y_α and the Y'_β are the same up to relabeling. □

Remark 1.3.13. It is clear that if X is a Noetherian topological space, W is a closed subset of X, and Z is a closed subset of W, then the irreducible decomposition of Z is the same whether considered in W or in X.

Recall that by a theorem due to Gauss, if R is a UFD, then the polynomial ring $R[x]$ is a UFD. A repeated application of this result gives that every polynomial ring $k[x_1, \ldots, x_n]$ is a UFD. In particular, a nonzero polynomial $f \in k[x_1, \ldots, x_n]$ is irreducible if and only if the ideal (f) is prime.

Example 1.3.14. Given a polynomial $f \in k[x_1, \ldots, x_n] \smallsetminus k$, the subset $V(f)$ is irreducible if and only if f is a power of an irreducible polynomial. In fact, if the irreducible decomposition of f is $f = cf_1^{m_1} \cdots f_r^{m_r}$, for some $c \in k^*$, then the irreducible components of $V(f)$ are $V(f_1), \ldots, V(f_r)$.

Exercise 1.3.15. Let Y be the algebraic subset of $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find the corresponding prime ideals.

Exercise 1.3.16. Show that if X and Y are topological spaces, with X irreducible, and $f\colon X \to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible.

Exercise 1.3.17. Let X be a topological space, and consider a finite open cover

$$X = U_1 \cup \ldots \cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i \cap U_j \neq \emptyset$.

Exercise 1.3.18. Let X be a Noetherian topological space and Y a subset X. Show that if $Y = Y_1 \cup \ldots \cup Y_r$ is the irreducible decomposition of Y, then $\overline{Y} = \overline{Y_1} \cup \ldots \cup \overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

Exercise 1.3.19. Let X be a Noetherian topological space and Y a nonempty closed subset of X, with irreducible decomposition

$$Y = Y_1 \cup \ldots \cup Y_r.$$

Show that if U is an open subset of X, then the irreducible decomposition of $U \cap Y$ is given by

$$U \cap Y = \bigcup_{i, U \cap Y_i \neq \emptyset} (U \cap Y_i).$$

We end these general topological considerations by discussing the notion of locally closed subsets.

Definition 1.3.20. Let X be a topological space. A subset V of X is *locally closed* if for every $x \in V$, there is an open neighborhood U_x of x in X such that $U_x \cap V$ is closed in U_x.

Remark 1.3.21. One should contrast the above definition with the local characterization of closed subsets: V is closed in X if and only if for every $x \in X$, there is an open neighborhood U_x of x in X such that $U_x \cap V$ is closed in U_x.

Proposition 1.3.22. *If V is a subset of a topological space X, then the following are equivalent:*

i) *V is a locally closed subset.*
ii) *V is open in $\overline{V}$.*
iii) *We can write $V = U \cap F$, with U open and F closed.*

PROOF. If V is locally closed, let us choose for every $x \in V$ an open neighborhood U_x of x as in the definition. In this case V is closed in U by Remark 1.3.21, hence $V = U \cap F$ for some F closed in X, proving i)⇒iii). In order to see iii)⇒ii), note that if if $V = U \cap F$, with U open and F closed, then $\overline{V} \subseteq F$, hence $V = U \cap \overline{V}$ is open in $\overline{V}$. Finally, the implication ii)⇒i) is clear: if $V = W \cap \overline{V}$ for some W open in X, then for every $x \in V$, if we take $U_x = W$, we have $U_x \cap V$ closed in U_x. □

Let $X \subseteq \mathbf{A}^n$ be a closed subset. We always consider on X the subspace topology. We now introduce a basis of open subsets on X.

DEFINITION 1.3.23. A *principal affine open subset* of X is an open subset of the form

$$D_X(f) := X \smallsetminus V(f) = \{x \in X \mid f(x) \neq 0\},$$

for some $f \in k[x_1, \ldots, x_n]$.

Note that $D_X(f)$ is nonempty if and only if $f \not\in I(X)$. It is clear that $D_X(f) \cap D_X(g) = D_X(fg)$. Every open subset of X can be written as $X \smallsetminus V(J)$ for some ideal J in R. Since J is finitely generated, we can write $J = (f_1, \ldots, f_r)$, in which case

$$X \smallsetminus V(J) = D_X(f_1) \cup \ldots \cup D_X(f_r).$$

Therefore every open subset of X is a finite union of principal affine open subsets of X. We thus see that the principal affine open subsets give a basis for the topology of X.

EXERCISE 1.3.24. Let X be a topological space and Y a locally closed subset of X. Show that a subset Z of Y is locally closed in X if and only if it is locally closed in Y.

1.4. Regular functions and morphisms

DEFINITION 1.4.1. An *affine algebraic variety* (or affine variety, for short) is a a closed subset of some affine space $\mathbf{A}^n$. A *quasi-affine variety* is a locally closed subset of some affine space $\mathbf{A}^n$, or equivalently, an open subset of an affine algebraic variety. A quasi-affine variety is always endowed with the subspace topology.

The above is only a temporary definition: a (quasi)affine variety is not just a topological space, but it comes with more information that distinguishes which maps between such objects are allowed. We will later formalize this as a ringed space. We now proceed describing the "allowable" maps.

DEFINITION 1.4.2. Let $Y \subseteq \mathbf{A}^n$ be a locally closed subset. A *regular function* on Y is a map $\phi \colon Y \to k$ that can locally be given by a quotient of polynomial functions, that is, for every $y \in Y$, there is an open neighborhood U_y of y in Y, and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that

$$g(u) \neq 0 \quad \text{and} \quad \phi(u) = \frac{f(u)}{g(u)} \quad \text{for all} \quad u \in U_y.$$

We write $\mathcal{O}(Y)$ for the set of regular functions on Y. If Y is an affine variety, then $\mathcal{O}(Y)$ is also called the *coordinate ring* of Y. By convention, we put $\mathcal{O}(Y) = 0$ if $Y = \emptyset$.

REMARK 1.4.3. It is easy to see that $\mathcal{O}(Y)$ is a subalgebra of the k-algebra of functions $Y \to k$, with respect to point-wise operations. For example, suppose that ϕ_1 and ϕ_2 are regular functions, $y \in Y$ and U_1 and U_2 are open neighborhoods of y, and $f_1, f_2, g_1, g_2 \in k[x_1, \ldots, x_n]$ are such that for all $u \in U_y$ we have

$$g_i(u) \neq 0 \quad \text{and} \quad \phi_i(u) = \frac{f_i(u)}{g_i(u)} \quad \text{for} \quad i = 1, 2.$$

If we take $U = U_1 \cap U_2$ and $f = f_1g_2 + f_2g_1$, $g = g_1g_2$, then for all $u \in U$, we have

$$g(u) \neq 0 \quad \text{and} \quad (\phi_1 + \phi_2)(u) = \frac{f(u)}{g(u)}.$$

REMARK 1.4.4. It follows from definition that if $\phi \colon Y \to k$ is a regular function such that $\phi(y) \neq 0$ for every $y \in Y$, then the function $\frac{1}{\phi}$ is a regular function, too.

EXAMPLE 1.4.5. If X is a locally closed subset of $\mathbf{A}^n$, then the projection π_i on the i^{th} component, given by

$$\pi_i(a_1, \ldots, a_n) = a_i$$

induces a regular function $X \to k$. Indeed, if $f_i = x_i \in k[x_1, \ldots, x_n]$, then $\pi_i(a) = f_i(a)$ for all $a \in X$.

When Y is closed in $\mathbf{A}^n$, one can describe more precisely $\mathcal{O}(Y)$. It follows by definition that we have a k-algebra homomorphism

$$k[x_1, \ldots, x_n] \to \mathcal{O}(Y)$$

that maps a polynomial f to the function $\big(u \to f(u)\big)$. By definition, the kernel of this map is the ideal $I(Y)$. With this notation, we have the following

PROPOSITION 1.4.6. *The induced k-algebra homomorphism*

$$k[x_1, \ldots, x_n]/I(Y) \to \mathcal{O}(Y)$$

is an isomorphism.

A similar description holds for principal affine open subsets of affine varieties. Suppose that Y is closed in $\mathbf{A}^n$ and $U = D_Y(h)$, for some $h \in k[x_1, \ldots, x_n]$. We have a k-algebra homomorphism

$$\Phi \colon k[x_1, \ldots, x_n]_h \to \mathcal{O}(U),$$

that maps $\frac{f}{h^m}$ to the map $\big(u \to f(u)/h(u)^m\big)$. With this notation, we have the following generalization of the previous proposition.

PROPOSITION 1.4.7. *The above k-algebra homomorphism induces an isomorphism*

$$k[x_1, \ldots, x_n]_h/I(Y)_h \to \mathcal{O}\big(D_Y(h)\big).$$

Of course it is enough to prove this more general version.

Proof of Proposition 1.4.7. The kernel of Φ consists of those fractions $\frac{f}{h^m}$ such that $\frac{f(u)}{h(u)}=0$ for every $u\in D_Y(h)$. It is clear that this condition is satisfied if $f\in I(Y)$. Conversely, if this condition holds, then $f(u)h(u)=0$ for every $u\in Y$. Therefore $fh\in I(Y)$, hence $\frac{f}{h^m}=\frac{fh}{h^{m+1}}\in I(Y)_h$. This shows that Φ is injective.

We now show that Φ is surjective. Consider $\phi\in\mathcal{O}\big(D_Y(h)\big)$. Using the hypothesis and the fact that $D_Y(h)$ is quasi-compact (being a Noetherian topological space), we can write

$$D_Y(h)=V_1\cup\ldots\cup V_r$$

and we have $f_i,g_i\in k[x_1,\ldots,x_n]$ for $1\leq i\leq r$ such that $g_i(u)\neq 0$ and $\phi(u)=\frac{f_i(u)}{g_i(u)}$ for all $u\in V_i$ and all i. Since the principal affine open subsets form a basis for the topology on Y, we may assume that $V_i=D_Y(h_i)$ for all i, for some $h_i\in k[x_1,\ldots,x_n]\smallsetminus I(Y)$. Since $g_i(u)\neq 0$ for all $u\in Y\smallsetminus V(h_i)$, it follows from Theorem 1.1.9 that

$$h_i\in\operatorname{rad}\big(I(Y)+(g_i)\big).$$

After possibly replacing each h_i by a suitable power, and then by a suitable element with the same class mod $I(Y)$, we may and will assume that $h_i\in(g_i)$. Finally, after multiplying both f_i and g_i by a suitable polynomial, we may assume that $g_i=h_i$ for all i.

We know that on $D_Y(g_i)\cap D_Y(g_j)=D_Y(g_ig_j)$ we have

$$\frac{f_i(u)}{g_i(u)}=\frac{f_j(u)}{g_j(u)}.$$

Applying the injectivity statement for $D_Y(g_ig_j)$, we conclude that

$$\frac{f_i}{g_i}=\frac{f_j}{g_j}\quad\text{in}\quad k[x_1,\ldots,x_n]_{g_ig_j}/I(Y)_{g_ig_j}.$$

Therefore there is a positive integer N such that

$$(g_ig_j)^N(f_ig_j-f_jg_i)\in I(Y)\quad\text{for all}\quad i,j.$$

After replacing each f_i and g_i by $f_ig_i^N$ and g_i^{N+1}, respectively, we may assume that

$$f_ig_j-f_jg_i\in I(Y)\quad\text{for all}\quad i,j.$$

On the other hand, we have

$$D_Y(h)=\bigcup_{i=1}^r D_Y(g_i),$$

hence $Y\cap V(h)=Y\cap V(g_1,\ldots,g_r)$, and by Theorem 1.1.9, we have

$$\operatorname{rad}\big(I(Y)+(h)\big)=\operatorname{rad}\big(I(Y)+(g_1,\ldots,g_r)\big).$$

In particular, we can write

$$h^m-\sum_{i=1}^r a_ig_i\in I(Y)\quad\text{for some}\quad m\geq 1\quad\text{and}\quad a_1,\ldots,a_r\in k[x_1,\ldots,x_n].$$

We claim that

$$\phi=\Phi\left(\frac{a_1f_1+\ldots+a_rf_r}{h^m}\right).$$

Indeed, for $u \in D_Y(g_j)$, we have

$$\frac{f_j(u)}{g_j(u)} = \frac{a_1(u)f_1(u) + \ldots + a_r(u)f_r(u)}{h(u)^m}$$

since

$$h(u)^m f_j(u) = \sum_{i=1}^{r} a_i(u)g_i(u)f_j(u) = \left(\sum_{i=1}^{r} a_i(u)f_i(u)\right) g_j(u).$$

This completes the proof of the claim and thus that of the proposition. $\square$

EXAMPLE 1.4.8. In general, it is not the case that a regular function admits a global description as the quotient of two polynomial functions. Consider, for example the closed subset W of $\mathbf{A}^4$ defined by $x_1x_2 = x_3x_4$. Inside W we have the plane L given by $x_2 = x_3 = 0$. We define the regular function $\phi\colon W \smallsetminus L \to k$ given by

$$\phi(u_1, u_2, u_3, u_4) = \begin{cases} \frac{u_1}{u_3}, & \text{if } u_3 \neq 0; \\ \frac{u_4}{u_2}, & \text{if } u_2 \neq 0. \end{cases}$$

It is an easy exercise to check that there are no polynomials $P, Q \in k[x_1, x_2, x_3, x_4]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad \phi(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in W \smallsetminus L.$$

We now turn to maps between quasi-affine varieties. If Y is a subset of $\mathbf{A}^m$ and $f\colon X \to Y$ is a map, then the composition $X \to Y \hookrightarrow \mathbf{A}^m$ is written as $(f_1, \ldots, f_m)$, with $f_i\colon X \to k$. We often abuse notation writing $f = (f_1, \ldots, f_m)$.

DEFINITION 1.4.9. If $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ are locally closed subsets, a map $f = (f_1, \ldots, f_m)\colon X \to Y$ is a *morphism* if $f_i \in \mathcal{O}(X)$ for all i.

REMARK 1.4.10. It follows from definition that $f\colon X \to Y$ is a morphism if and only if the composition

$$X \to Y \hookrightarrow \mathbf{A}^m$$

is a morphism

REMARK 1.4.11. If $X \subseteq \mathbf{A}^n$ is a locally closed subset, then a morphism $X \to \mathbf{A}^1$ is the same as a regular function $X \to k$.

EXAMPLE 1.4.12. If X is a locally closed of $\mathbf{A}^n$, then the inclusion map $\iota\colon X \to \mathbf{A}^n$ is a morphism (this follows from Example 1.4.5). This implies that the identity map $1_X\colon X \to X$ is a morphism.

PROPOSITION 1.4.13. *If X and Y are quasi-affine varieties, then every morphism $f\colon X \to Y$ is continuous.*

PROOF. Suppose that X and Y are locally closed in $\mathbf{A}^n$ and $\mathbf{A}^m$, respectively, and write $f = (f_1, \ldots, f_m)$. We will show that if $V \subseteq Y$ is a closed subset, then $f^{-1}(V)$ is a closed subset of X. By assumption, we can write

$$V = Y \cap V(I) \quad \text{for some ideal} \quad I \subseteq k[x_1, \ldots, x_n].$$

In order to check that $f^{-1}(V)$ is closed, it is enough to find for every $x \in X$ an open neighborhood U_x of x in X such that $U_x \cap f^{-1}(V)$ is closed in U_x (see Remark 1.3.21). Since each f_i is a regular function, after replacing X by a suitable open neighborhood of x, we may assume that there are $P_i, Q_i \in k[x_1, \ldots, x_n]$ such that

$$Q_i(u) \neq 0 \quad \text{and} \quad f_i(u) = \frac{P_i(u)}{Q_i(u)} \quad \text{for all} \quad u \in X.$$

For every $h \in I$, there are polynomials $A_h, B_h \in k[x_1, \ldots, x_n]$ such that

$$B_h(u) \neq 0 \quad \text{and} \quad h\left(\frac{P_1(u)}{Q_1(u)}, \ldots, \frac{P_m(u)}{Q_m(u)}\right) = \frac{A_h(u)}{B_h(u)} \quad \text{for all} \quad u \in X.$$

It is then clear that for $u \in X$ we have $u \in f^{-1}(V)$ if and only if $A_h(u) = 0$ for all $h \in I$. Therefore $f^{-1}(V)$ is closed. □

Proposition 1.4.14. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms between quasi-affine varieties, the composition $g \circ f$ is a morphism.*

Proof. Suppose that $X \subseteq \mathbf{A}^m$, $Y \subseteq \mathbf{A}^n$ and $Z \subseteq \mathbf{A}^q$ are locally closed subsets and let us write $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_q)$. We need to show that $g_i \circ f \in \mathcal{O}(X)$ for $1 \leq i \leq q$. Let us fix such i, a point $x \in X$, and let $y = f(x)$. Since $g_i \in \mathcal{O}(Y)$ is a morphism, there is an open neighborhood V_y of y and $P, Q \in k[x_1, \ldots, x_n]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad g_i(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in V_y.$$

Similarly, since f is a morphism, we can find an open neighborhood U_x of x and $A_j, B_j \in k[x_1, \ldots, x_m]$ for $1 \leq j \leq n$ such that

$$B_j(u) \neq 0 \quad \text{and} \quad f_j(u) = \frac{A_j(u)}{B_j(u)} \quad \text{for all} \quad u \in U_x.$$

It follows from Proposition 1.4.13 that $U_x \cap f^{-1}(V_y)$ is open and we have

$$g_i \circ f(u) = \frac{P\left(\frac{A_1(u)}{B_1(u)}, \ldots, \frac{A_n(u)}{B_n(u)}\right)}{Q\left(\frac{A_1(u)}{B_1(u)}, \ldots, \frac{A_n(u)}{B_n(u)}\right)}.$$

After clearing the denominators, we see that indeed, $g_i \circ f$ is a regular function in the neighborhood of x. □

It follows from Proposition 1.4.14 (and Example 1.4.12) that we may consider the *category of quasi-affine varieties* over k, whose objects are locally closed subsets of affine spaces over k, and whose arrows are the morphisms as defined above. Moreover, since a regular function on X is the same as a morphism $X \to \mathbf{A}^1$, we see that if $f\colon X \to Y$ is a morphism of quasi-affine varieties, we get an induced map

$$f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X), \quad f^{\#}(\phi) = \phi \circ f.$$

This is clearly a morphism of k-algebras. By mapping every quasi-affine variety X to $\mathcal{O}(X)$ and every morphism $f\colon X \to Y$ to $f^{\#}$, we obtain a contravariant functor from the category of quasi-affine varieties over k to the category of k-algebras.

Definition 1.4.15. A morphism $f\colon X \to Y$ is an *isomorphism* if it is an isomorphism in the above category. It is clear that this is the case if and only if f is bijective and f^{-1} is a morphism.

The following result shows that for affine varieties, this functor induces an anti-equivalence of categories. Let $\mathcal{A}fVar_k$ be the full subcategory of the category of quasi-affine varieties whose objects consist of the closed subsets of affine spaces over k and let $\mathcal{C}_k$ denote the category whose objects are reduced, finitely generated k-algebras and whose arrows are the morphisms of k-algebras.

THEOREM 1.4.16. *The contravariant functor*

$$\mathcal{A}fVar_k \to \mathcal{C}_k$$

that maps X to $\mathcal{O}(X)$ and $f\colon X \to Y$ to $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is an anti-equivalence of categories.

PROOF. Note first that if X is an affine variety, then $\mathcal{O}(X)$ is indeed a reduced, finitely generated k-algebra. Indeed, if X is a closed subset of $\mathbf{A}^n$, then it follows from Proposition 1.4.6 that we have an isomorphism $\mathcal{O}(X) \simeq k[x_1, \ldots, x_n]/I(X)$, which gives the assertion.

In order to show that the functor is an anti-equivalence of categories, it is enough to check two things:

i) For every affine varieties X and Y, the map

$$\mathrm{Hom}_{\mathcal{A}fVar_k}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_k}\big(\mathcal{O}(Y), \mathcal{O}(X)\big), \quad f \to f^{\#}$$

is a bijection.

ii) For every reduced, finitely generated k-algebra A, there is an affine variety X with $\mathcal{O}(X) \simeq A$.

The assertion in ii) is clear: since A is finitely generated, we can find an isomorphism $A \simeq k[x_1, \ldots, x_m]/J$, for some positive integer m and some ideal J. Moreover, since A is reduced, J is a radical ideal. If $X = V(J) \subseteq \mathbf{A}^m$, then it follows from Theorem 1.1.9 that $J = I(X)$ and therefore $\mathcal{O}(X) \simeq A$ by Proposition 1.4.6.

In order to prove the assertion in i), suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. By Proposition 1.4.6, we have canonical isomorphisms

$$\mathcal{O}(X) \simeq k[x_1, \ldots, x_m]/I(X) \quad \text{and} \quad \mathcal{O}(Y) \simeq k[y_1, \ldots, y_n]/I(Y).$$

If $f\colon X \to Y$ is a morphism and we write $f = (f_1, \ldots, f_n)$, then $f^{\#}(\overline{y_i}) = \overline{f}_i$. Since f is determined by the classes $\overline{f_1}, \ldots, \overline{f_n}$ mod $I(X)$, it is clear that the map in i) is injective.

Suppose now that $\alpha\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a morphism of k-algebras and let $f_i \in k[x_1, \ldots, x_m]$ be such that $\overline{f_i} = \alpha(\overline{y_i}) \in \mathcal{O}(X)$. It is then clear that $f = (f_1, \ldots, f_n)$ gives a morphism $X \to \mathbf{A}^n$. Its image lies inside Y since for every $g \in I(Y)$ we have $g(f_1, \ldots, f_n) \in I(X)$, hence $g\big(f(u)\big) = 0$ for all $u \in X$. Therefore f gives a morphism $X \to Y$ such that $f^{\#} = \alpha$. $\square$

DEFINITION 1.4.17. We extend somewhat the notion of affine variety by saying that a quasi-affine variety is affine if it is isomorphic (in the category of quasi-affine varieties) to a closed subset of some affine space.

An important example that does not come directly as a closed subset of an affine space is provided by the following proposition.

PROPOSITION 1.4.18. *Let X be a closed subset of $\mathbf{A}^n$ and $U = D_X(g)$, for some $g \in k[x_1, \ldots, x_n]$. If J is the ideal in $k[x_1, \ldots, x_n, y]$ generated by $I(X)$ and $1 - g(x)y$, then U is isomorphic to $V(J)$. In particular, U is an affine variety*[2].

PROOF. Define $\phi\colon U \to V(J)$ by $\phi(u) = \big(u, 1/g(u)\big)$. It is clear that $\phi(u)$ lies indeed in $V(J)$ and that ϕ is a morphism. Moreover, we also have a morphism $\psi\colon V(J) \to U$ induced by the projection onto the first n components. It is straightforward to check that ϕ and ψ are inverse to each other. $\square$

NOTATION 1.4.19. If X is a quasi-affine variety and $f \in \mathcal{O}(X)$, then we put

$$D_X(f) = \{u \in X \mid f(u) \neq 0\}.$$

If X is affine, say it is isomorphic to the closed subset Y of $\mathbf{A}^n$, then f corresponds to the restriction to Y of some $g \in k[x_1, \ldots, x_n]$. In this case, it is clear that $D_X(f)$ is isomorphic to $D_Y(g)$, hence it is an affine variety.

REMARK 1.4.20. If X is a locally closed subset of $\mathbf{A}^n$, then X is open in $\overline{X}$. Since the principal affine open subsets of $\overline{X}$ give a basis of open subsets for the topology of X, it follows from Proposition 1.4.18 that the open subsets of X that are themselves affine varieties give a basis for the topology of X.

EXERCISE 1.4.21. Suppose that $f\colon X \to Y$ is a morphism of affine algebraic varieties, and consider the induced homomorphism $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D_Y(u)) = D_X(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D_Y(u)) \to \mathcal{O}(D_X(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

EXERCISE 1.4.22. Let X be an affine algebraic variety, and let $\mathcal{O}(X)$ be the ring of regular functions on X. For every ideal J of $\mathcal{O}(X)$, let

$$V(J) := \{p \in X \mid f(p) = 0 \text{ for all } f \in J\}.$$

For $S \subseteq X$, consider the following ideal of $\mathcal{O}(X)$

$$I_X(S) := \{f \in \mathcal{O}(X) \mid f(p) = 0 \text{ for all } p \in S\}.$$

Show that for every subset S of X and every ideal J in $\mathcal{O}(X)$, we have

$$V\big(I_X(S)\big) = \overline{S} \quad \text{and} \quad I_X\big(V(J)\big) = \mathrm{rad}(J).$$

In particular, the maps $V(-)$ and $I_X(-)$ define order-reversing inverse bijections between the closed subsets of X and the radical ideals in $\mathcal{O}(X)$. Via this correspondence, the irreducible closed subsets correspond to the prime ideals in $\mathcal{O}(X)$ and the points of X correspond to the maximal ideals in $\mathcal{O}(X)$. This generalizes the case $X = \mathbf{A}^n$ that was discussed in Section 1.1.

[2] This justifies calling these subsets principal affine open subsets.

We have seen that a morphism $f\colon X \to Y$ between affine varieties is determined by the corresponding k-algebra homomorphism $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. For such a morphism, it follows from the above exercise that the closed subsets in X and Y are in bijection with the radical ideals in $\mathcal{O}(X)$ and, respectively, $\mathcal{O}(Y)$. In the next proposition we translate the operations of taking the image and inverse image as operations on ideals.

PROPOSITION 1.4.23. *Let $f\colon X \to Y$ be a morphism of affine varieties and $\phi = f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the corresponding k-algebra homomorphism. For a point x in X or Y, we denote by $\mathfrak{m}_x$ the corresponding maximal ideal.*

i) *If $x \in X$ and $y = f(x)$, then $\mathfrak{m}_y = \phi^{-1}(\mathfrak{m}_x)$.*
ii) *More generally, if $\mathfrak{a}$ is an ideal in $\mathcal{O}(X)$ and $W = V(\mathfrak{a})$, then $I_Y\big(\overline{f(W)}\big) = \phi^{-1}\big(I_X(W)\big)$.*
iii) *In particular, we have $I_Y\big(\overline{f(X)}\big) = \ker(\phi)$. Therefore $\overline{f(X)} = Y$ if and only if ϕ is injective.*
iv) *If $\mathfrak{b}$ is an ideal in $\mathcal{O}(Y)$ and $Z = V(\mathfrak{b})$, then $f^{-1}(Z) = V\big(\mathfrak{b} \cdot \mathcal{O}(X)\big)$.*

PROOF. The assertion in i) is a special case of that in ii), hence we begin by showing ii). We have

$$I_Y\big(\overline{f(W)}\big) = I_Y\big(f(W)\big) = \{g \in \mathcal{O}(Y) \mid g\big(f(x)\big) = 0 \text{ for all } x \in W\}$$
$$= \{g \in \mathcal{O}(Y) \mid \phi(g) \in I_X(W)\} = \phi^{-1}\big(I_X(W)\big).$$

By taking $W = X$, we obtain the assertion in iii)

Finally, if $\mathfrak{b}$ and Z are as in iv), we see that

$$f^{-1}(Z) = \{x \in X \mid g\big(f(x)\big) = 0 \text{ for all } g \in \mathfrak{b}\} = V\big(\mathfrak{b} \cdot \mathcal{O}(X)\big).$$

$\square$

REMARK 1.4.24. If $f\colon X \to Y$ is a morphism of affine varieties, then $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective if and only if f factors as $X \xrightarrow{g} Z \xrightarrow{\iota} Y$, where Z is a closed subset of Y, ι is the inclusion map, and g is an isomorphism.

EXERCISE 1.4.25. Let $Y \subseteq \mathbf{A}^2$ be the *cuspidal curve* defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbf{A}^1 \to Y$. Is it an isomorphism ?

EXERCISE 1.4.26. Suppose that $\operatorname{char}(k) = p > 0$, and consider the map $f\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $f(a_1, \ldots, a_n) = (a_1^p, \ldots, a_n^p)$. Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism.

EXERCISE 1.4.27. Use Exercise 1.3.16 to show that the affine variety

$$M_{m,n}^r(k) := \{B \in M_{m,n}(k) \mid \operatorname{rk}(B) \leq r\}$$

is irreducible.

EXERCISE 1.4.28. Let $n \geq 2$ be an integer.

i) Show that the set

$$B_n = \left\{(a_0, a_1, \ldots, a_n) \in \mathbf{A}^{n+1} \mid \operatorname{rank}\begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \leq 1\right\}$$

is a closed subset of $\mathbf{A}^{n+1}$.

ii) Show that

$$B_n = \{(s^n, s^{n-1}t, \ldots, t^n) \mid s, t \in k\}.$$

Deduce that B_n is irreducible.

EXERCISE 1.4.29. In order to get an example of a quasi-affine variety which is not affine, consider $U = \mathbf{A}^2 \smallsetminus \{0\}$. Show that the canonical homomorphism $\mathcal{O}(\mathbf{A}^2) \to \mathcal{O}(U)$ is an isomorphism and deduce that U is not affine.

EXERCISE 1.4.30. Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself.

EXERCISE 1.4.31. Show that if X is a quasi-affine variety such that $\mathcal{O}(X) = k$, then X consists of only one point.

1.5. Local rings and rational functions

Let X be a quasi-affine variety and W an irreducible closed subset of X.

DEFINITION 1.5.1. The *local ring* of X at W is the k-algebra

$$\mathcal{O}_{X,W} := \varinjlim_{U \cap W \neq \emptyset} \mathcal{O}(U).$$

Here the direct limit is over the open subsets of X with $U \cap W \neq \emptyset$, ordered by reverse inclusion, and where for $U_1 \subseteq U_2$, the map $\mathcal{O}(U_2) \to \mathcal{O}(U_1)$ is given by restriction of functions.

REMARK 1.5.2. Note that the poset indexing the above direct limit is filtering: given any two open subsets U_1 and U_2 that intersect W nontrivially, we have $U_1 \cap U_2 \cap W \neq \emptyset$ (we use here the fact that W is irreducible). Because of this, the elements of $\mathcal{O}_{X,W}$ can be described as pairs (U, ϕ), where U is open with $W \cap U \neq \emptyset$ and $\phi \in \mathcal{O}(U)$, modulo the following equivalence relation:

$$(U_1, \phi_1) \sim (U_2, \phi_2)$$

if there is an open subset $U \subseteq U_1 \cap U_2$, with $U \cap W \neq \emptyset$, such that $\phi_1|_U = \phi_2|_U$. Operations are defined by restricting to the intersection: for example, we have

$$(U_1, \phi_1) + (U_2, \phi_2) = (U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2}).$$

In order to describe $\mathcal{O}_{X,W}$, we begin with the following lemma.

LEMMA 1.5.3. *If W is an irreducible closed subset of X and V is an open subset of X with $V \cap W \neq \emptyset$, we have a canonical k-algebra isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}_{V,W \cap V}.$$

PROOF. The assertion follows from the fact that the following subset

$$\{U \subseteq V \mid U \text{ open}, U \cap W \neq \emptyset\} \subseteq \{U \subseteq X \mid U \text{ open}, U \cap W \neq \emptyset\}$$

is final. Explicitly, we have the morphism

$$\mathcal{O}_{V,W \cap V} \to \mathcal{O}_{X,W}, \quad (U, \phi) \to (U, \phi),$$

with inverse

$$\mathcal{O}_{X,W} \to \mathcal{O}_{V,W \cap V}, \quad (U, \phi) \to (U \cap V, \phi|_{U \cap V}).$$

$\square$

Given a quasi-affine variety X, the open subsets of X that are affine varieties give a basis for the topology of X (see Remark 1.4.20). By Lemma 1.5.3, we see that it is enough to compute $\mathcal{O}_{X,W}$ when X is an affine variety. This is the content of the next result.

PROPOSITION 1.5.4. *Let X be an affine variety and W an irreducible closed subset of X. If $\mathfrak{p} \subseteq \mathcal{O}(X)$ is the prime ideal corresponding to W, then we have a canonical isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

In particular, $\mathcal{O}_{X,W}$ is a local ring, with maximal ideal consisting of classes of pairs (U, ϕ), with $\phi_{U\cap W} = 0$.

PROOF. Since the principal affine open subsets of X form a basis for the topology of X, we obtain using Proposition 1.4.7 a canonical isomorphism

$$\mathcal{O}_{X,W} \simeq \varinjlim_{f} \mathcal{O}(X)_f,$$

where the direct limit on the right-hand side is over those $f \in \mathcal{O}(X)$ such that $D_X(f) \cap W \neq \emptyset$. This condition is equivalent to $f \not\in \mathfrak{p}$ and it is straightforward to check that the maps $\mathcal{O}(X)_f \to \mathcal{O}(X)_{\mathfrak{p}}$ induce an isomorphism

$$\varinjlim_{f} \mathcal{O}(X)_f \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

The last assertion in the proposition follows easily from the fact that $\mathcal{O}(X)_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}\mathcal{O}(X)_{\mathfrak{p}}$ □

There are two particularly interesting cases of this definition. First, if we take $W = \{x\}$, for a point $x \in X$, we obtain the local ring $\mathcal{O}_{X,x}$ of X *at* x. Its elements are *germs of regular functions* at x. This is a local ring, whose maximal ideal consists of germs of functions vanishing at x. As we will see, this local ring is responsible for the properties of X in a neighborhood of x. If X is an affine variety and $\mathfrak{m}$ is the maximal ideal corresponding to x, then Proposition 1.5.4 gives an isomorphism

$$\mathcal{O}_{X,x} \simeq \mathcal{O}(X)_{\mathfrak{m}}.$$

EXERCISE 1.5.5. Let $f\colon X \to Y$ be a morphism of quasi-affine varieties, and let $Z \subseteq X$ be a closed irreducible subset. Recall that by Exercise 1.3.16, we know that $W := \overline{f(Z)}$ is irreducible. Show that we have an induced morphism of k-algebras

$$g\colon \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z}$$

and that g is a local homomorphism of local rings (that is, it maps the maximal ideal of $\mathcal{O}_{Y,W}$ inside the maximal ideal of $\mathcal{O}_{X,Z}$). If X and Y are affine varieties, and

$$\mathfrak{p} = I_X(Z) \quad \text{and} \quad \mathfrak{q} = I_Y(W) = (f^{\#})^{-1}(\mathfrak{p}),$$

then via the isomorphisms given by Proposition 1.5.4, g gets identified to the homomorphism

$$\mathcal{O}(Y)_{\mathfrak{q}} \to \mathcal{O}(X)_{\mathfrak{p}}$$

induced by $f^{\#}$ via localization.

EXERCISE 1.5.6. Let X and Y be quasi-affine varieties. By the previous exercise, if $f\colon X \to Y$ is a morphism, $p \in X$ is a point, and $f(p) = q$, then f induces a local ring homomorphism $\phi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$.

i) Show that if $f'\colon X \to Y$ is another morphism with $f'(p) = q$, and induced homomorphism $\phi'\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then $\phi = \phi'$ if and only if there is an open neighborhood U of p such that $f|_U = g|_U$.
ii) Show that given any local morphism of local k-algebras $\psi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, there is an open neighborhood W of p, and a morphism $g\colon W \to Y$ with $g(p) = q$, and inducing ψ.
iii) Deduce that $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as k-algebras if and only if there are open neighborhoods W of p and V of q, and an isomorphism $h\colon W \to V$, with $h(p) = q$.

Another important example of local ring of X occurs when X is an irreducible variety and we take $W = X$. The resulting local ring is, in fact, a field, the *field of rational functions* $k(X)$ of X. Indeed, if $U \subseteq X$ is an affine open subset, then it follows from Lemma 1.5.3 and Proposition 1.5.4 that $k(X)$ is isomorphic to the field of fractions of the domain $\mathcal{O}(X)$. The elements of $k(X)$ are *rational functions* on X, that is, pairs (U, ϕ), where U is a nonempty open subset of X and $\phi\colon U \to k$ is a regular function, where we identify two such pairs if the two functions agree on some nonempty open subset of their domains (in fact, as we will see shortly, in this case they agree on the intersection of their domains). We now discuss in more detail rational functions and, more generally, rational maps.

LEMMA 1.5.7. *If X and Y are quasi-affine varieties and f_1 and f_2 are two morphisms $X \to Y$, then the subset*

$$\{a \in X \mid f_1(a) = f_2(a)\} \subseteq X$$

is closed.

PROOF. If Y is a locally closed subset in $\mathbf{A}^n$, then we write $f_i = (f_{i,1}, \ldots, f_{i,n})$ for $i = 1, 2$. With this notation, we have

$$\{a \in X \mid f_1(a) = f_2(a)\} = \bigcap_{j=1}^{n} \{a \in X \mid (f_{1,j} - f_{2,j})(a) = 0\},$$

hence this set is closed in X, since each function $f_{1,j} - f_{2,j}$ is regular, hence continuous. $\square$

DEFINITION 1.5.8. Let X and Y be quasi-affine varieties. A *rational map* $f\colon X \dashrightarrow Y$ is given by a pair (U, ϕ), where U is a dense, open subset of X and $\phi\colon U \to Y$ is a morphism, and where we identify (U_1, ϕ_1) with (U_2, ϕ_2) if there is an open dense subset $V \subseteq U_1 \cap U_2$ such that $\phi_1|_V = \phi_2|_V$. In fact, in this case we have $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$ by Lemma 1.5.7. We also note that since U_1 and U_2 are dense open subsets of X, then also $U_1 \cap U_2$ is a dense subset of X.

REMARK 1.5.9. If $f\colon X \dashrightarrow Y$ is a rational map and (U_i, ϕ_i) are the representatives of f, then we can define a map $\phi\colon U = \bigcup_i U_i \to Y$ by $\phi(u) = \phi_i(u)$ if $u \in U_i$. This is well-defined and it is a morphism, since its restriction to each of the U_i is a morphism. Moreover, (U, ϕ) is a representative of f. The open subset U, the largest one on which a representative of f is defined, is the *domain of definition* of f.

DEFINITION 1.5.10. Given a quasi-affine variety X, the set of rational functions $X \dashrightarrow k$ is denoted by $k(X)$. Since the intersection of two dense open sets is again open and dense, we may define the sum and product of two rational functions. For example, given two rational functions with representatives (U_1, ϕ_1) and (U_2, ϕ_2), we define their sum by the representative

$$(U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2}),$$

and similarly for the product. It is straightforward to see that using also scalar multiplication, $k(X)$ is a k-algebra. Note that when X is irreducible, we recover our previous definition.

EXERCISE 1.5.11. Let X be a quasi-affine variety, and let $X_1, \ldots, X_r$ be its irreducible components. Show that there is a canonical isomorphism

$$k(X) \simeq k(X_1) \times \cdots \times k(X_r).$$

EXERCISE 1.5.12. Let W be the closed subset in $\mathbf{A}^2$, defined by $x^2 + y^2 = 1$. What is the domain of definition of the rational function on W given by $\frac{1-y}{x}$?

Our next goal is to define a category in which the arrows are given by rational function. For simplicity, we only consider irreducible varieties.

DEFINITION 1.5.13. A morphism $f \colon X \to Y$ is *dominant* if $Y = \overline{f(X)}$. Equivalently, for every nonempty open subset $V \subseteq Y$, we have $f^{-1}(V) \neq \emptyset$. Note that if U is open and dense in X, then f is dominant if and only if the composition $U \hookrightarrow X \xrightarrow{f} Y$ is dominant. We can thus define the same notion for rational maps: if $f \colon X \dashrightarrow Y$ is a rational map with representative (U, ϕ), we say that f is *dominant* if $\phi \colon U \to Y$ is dominant.

Suppose that X, Y, and Z are irreducible quasi-affine varieties and $f \colon X \dashrightarrow Y$ and $g \colon Y \dashrightarrow Z$ are rational maps, with f dominant. In this case we may define the composition $g \circ f$, which is a rational map; moreover, if g is dominant, too, then $g \circ f$ is dominant. Indeed, choose a representative (U, ϕ) for f and a representative (V, ψ) for g. Since the morphism $\phi \colon U \to Y$ is dominant, it follows that $W := \phi^{-1}(V)$ is nonempty. We then take $g \circ f$ to be the rational function defined by the composition

$$W \xrightarrow{f|_W} V \longrightarrow Z.$$

It is straightforward to see that this independent of the representatives for f and g. Moreover, if g is dominant, then $g \circ f$ is dominant: if Z' is a nonempty open subset of Z, then $\psi^{-1}(Z')$ is nonempty and open since g is dominant and therefore $\phi^{-1}\big(\psi^{-1}(Z')\big)$ is nonempty, since f is dominant.

It is clear that the identity map is dominant. Moreover, composition of dominant rational map is associative. We thus obtain a category in which the objects are the irreducible quasi-affine varieties over k and the set $\mathrm{Hom}_{\mathrm{rat}}(X, Y)$ of arrows from X to Y consists of the dominant rational maps $X \dashrightarrow Y$, with the composition defined above. We are then led to the following important concept.

DEFINITION 1.5.14. A rational dominant map $f \colon X \dashrightarrow Y$ between irreducible quasi-affine varieties is *birational* if it is an isomorphism in the above category. More precisely, this is the case if there is a dominant rational map $g \colon Y \dashrightarrow X$ such that

$$g \circ f = 1_X \quad \text{and} \quad f \circ g = 1_Y.$$

A *birational morphism* is a morphism which is birational as a rational map. Two irreducible quasi-affine varieties X and Y are *birational* if there is a birational map $X \dashrightarrow Y$.

This notion plays a fundamental role in the classification of algebraic varieties. On one hand, birational varieties share interesting geometric properties. On the other hand, classifying algebraic varieties up to birational equivalence turns out to be a more reasonable endeavor than classifying varieties up to isomorphism.

EXAMPLE 1.5.15. If U is an open subset of the irreducible quias-affine variety X, then the inclusion map $i\colon U \hookrightarrow X$ is a birational morphism. Its inverse is given by the rational map represented by the identity morphism of U.

EXAMPLE 1.5.16. An interesting example, which we will come back to later, is given by the morphism

$$f\colon \mathbf{A}^n \to \mathbf{A}^n, \quad f(x_1, \ldots, x_n) = (x_1, x_1x_2, \ldots, x_1x_n).$$

Note that the linear subspace given $L = (x_1 = 0)$ is mapped to 0, but f induces an isomorphism

$$\mathbf{A}^n \smallsetminus L = f^{-1}(\mathbf{A}^n \smallsetminus L) \to \mathbf{A}^n \smallsetminus L,$$

with inverse given by $g(y_1, \ldots, y_n) = (y_1, y_2/y_1, \ldots, y_n/y_1)$.

EXAMPLE 1.5.17. Let X be the closed subset of $\mathbf{A}^2$ (on which we denote the coordinates by x and y), defined by $x^2 - y^3 = 0$. Let $f\colon \mathbf{A}^1 \to X$ be the morphism given by $f(t) = (t^3, t^2)$. Note that f is birational: if $g\colon X \smallsetminus \{(0,0)\} \to \mathbf{A}^1$ is the morphism given by $g(u, v) = \frac{u}{v}$, then g gives a rational map $X \dashrightarrow \mathbf{A}^1$ that is an inverse of f. Note that since $f^{-1}(0,0) = \{0\}$, it follows that the morphism f is bijective, However, f is not an isomorphism: otherwise, by Theorem 1.4.16 the induced homomorphism

$$f^{\#}\colon \mathcal{O}(X) = k[x,y]/(x^2 - y^3) \to k[t], \quad f^{\#}(x) = t^3, f^{\#}(y) = t^2$$

would be an isomorphism. However, it is clear that t is not in the image.

If $f\colon X \dashrightarrow Y$ is a rational, dominant map, then by taking $Z = \mathbf{A}^1$, we see that by precomposing with f we obtain a map

$$f^{\#}\colon k(Y) \to k(X).$$

It is straightforward to see that this is a field homomorphism.

THEOREM 1.5.18. *We have an anti-equivalence of categories between the category of irreducible quasi-affine varieties and dominant rational maps and the category of finite type field extensions of k and k-algebra homomorphisms, that maps a variety X to $k(X)$ and a rational dominant map $f\colon X \dashrightarrow Y$ to $f^{\#}\colon k(Y) \to k(X)$.*

PROOF. It is clear that we have a contravariant functor as described in the theorem. Note that if X is an irreducible quasi-affine variety, then $k(X)$ is a finite type extension of k: indeed, if U is an affine open subset of X, then we have $k(X) \simeq k(U) \simeq \mathrm{Frac}\big(\mathcal{O}(U)\big)$.

In order to show that this functor is an anti-equivalence, it is enough to prove the following two statements:

i) Given any two quasi-affine varieties X and Y, the map

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big)$$

is bijective.

ii) Given any finite type field extension K/k, there is an irreducible quasi-affine variety X such that $k(X) \simeq K$.

The assertion in ii) is easy to see: if $K = k(a_1, \ldots, a_n)$, let $A = k[a_1, \ldots, a_n]$. We can thus write $A \simeq k[x_1, \ldots, x_n]/P$ for some (prime) ideal P and if $X = V(P) \subseteq \mathbf{A}^n$, then X is irreducible and $k(X) \simeq K$.

In order to prove i), suppose that X and Y are locally closed in $\mathbf{A}^m$ and, respectively, $\mathbf{A}^n$. Since X and Y are open in $\overline{X}$ and $\overline{Y}$, respectively, by Proposition 1.3.22, and since inclusions of open subsets are birational, it follows that the inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ induce an isomorphism

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{rat}}(\overline{X}, \overline{Y}),$$

and also isomorphisms

$$k(X) \simeq k(\overline{X}) \quad \text{and} \quad k(Y) \simeq k(\overline{Y}).$$

We may thus replace X and Y by $\overline{X}$ and $\overline{Y}$, respectively, in order to assume that X and Y are closed subsets of the respective affine spaces.

It is clear that

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big),$$

where each set on the right-hand side consists of the dominant morphisms $D_X(g) \to Y$. Moreover, since $\mathcal{O}(Y)$ is a finitely generated k-algebra, we have

$$\mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big),$$

where each set on the right-hand side consists of the injective k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)_g$. Since the map $f \to f^{\#}$ gives a bijection

$$\mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big) \simeq \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big)$$

by Theorem 1.4.16 and Proposition 1.4.23, this completes the proof. $\square$

Corollary 1.5.19. *A dominant rational map $f\colon X \to Y$ between irreducible quasi-affine varieties X and Y is birational if and only if the induced homomorphism $f^{\#}\colon k(Y) \to k(X)$ is an isomorphism.*

Remark 1.5.20. A rational map $f\colon X \dashrightarrow Y$ between the irreducible quasi-affine varieties X and Y is birational if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \simeq V$. The "if" assertion is clear, so we only need to prove the converse. Suppose that f is defined by the morphism $\phi\colon X' \to Y$ and its inverse g is defined by the morphism $\psi\colon Y' \to X$, where $X' \subseteq X$ and $Y' \subseteq Y$ are open subsets. The equality $f \circ g = 1_Y$ as rational functions implies by Lemma 1.5.7 that the composition

$$\psi^{-1}(X') \xrightarrow{\psi} X' \xrightarrow{\phi} Y$$

is the inclusion. In particular, we deduce that

$$\psi\big(\psi^{-1}(X')\big) \subseteq \phi^{-1}\big(\psi^{-1}(X')\big) \subseteq \phi^{-1}(Y').$$

Similarly, the equality of rational functions $g \circ f = 1_X$ shows that the composition

$$\phi^{-1}(Y') \xrightarrow{\phi} Y' \xrightarrow{\psi} X$$

is the inclusion; in particular, we obtain

$$\phi\big(\phi^{-1}(Y')\big) \subseteq \psi^{-1}(X').$$

It is now clear that ϕ and ψ induce inverse morphisms between $\phi^{-1}(Y')$ and $\psi^{-1}(X')$.

EXERCISE 1.5.21. Let $X \subset \mathbf{A}^n$ be a hypersurface defined by an equation $f(x_1, \ldots, x_n) = 0$, where $f = f_{d-1} + f_d$, with f_{d-1} and f_d nonzero, homogeneous of degrees $d-1$ and d, respectively. Show that if X is irreducible, then X is birational to $\mathbf{A}^{n-1}$.

1.6. Products of (quasi-)affine varieties

We begin by showing that for positive integers m and n, the Zariski topology on $\mathbf{A}^m \times \mathbf{A}^n = \mathbf{A}^{m+n}$ is finer than the product topology.

PROPOSITION 1.6.1. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets, then $X \times Y$ is a closed subset of $\mathbf{A}^{m+n}$.*

PROOF. The assertion follows from the fact that if $X = V(I)$ and $Y = V(J)$, for ideals $I \subseteq k[x_1, \ldots, x_m]$ and $J \subseteq k[y_1, \ldots, y_n]$, then

$$X \times Y = V(I \cdot R + J \cdot R),$$

where $R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. $\square$

COROLLARY 1.6.2. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are open (respectively, locally closed) subsets, then $X \times Y$ is an open (respectively, locally closed) subset of $\mathbf{A}^{m+n} = \mathbf{A}^m \times \mathbf{A}^n$. In particular, the topology on $\mathbf{A}^m \times \mathbf{A}^n$ is finer than the product topology.*

PROOF. The assertion for open subsets follows from Proposition 1.6.1 and the fact that

$$\mathbf{A}^{m+n} \smallsetminus X \times Y = \big(\mathbf{A}^m \times (\mathbf{A}^n \smallsetminus Y)\big) \cup \big((\mathbf{A}^m \smallsetminus X) \times \mathbf{A}^n\big).$$

The assertion for locally closed subsets follows immediately from the assertions for open and closed subsets. $\square$

COROLLARY 1.6.3. *Given any quasi-affine varieties X and Y, the topology on $X \times Y$ is finer than the product topology.*

PROOF. If X and Y are locally closed subsets of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, then $X \times Y$ is a locally closed subset of $\mathbf{A}^{m+n}$. Since the topology on $\mathbf{A}^{m+n}$ is finer than the product topology by the previous corollary, we are done. $\square$

EXAMPLE 1.6.4. The topology on $\mathbf{A}^m \times \mathbf{A}^n$ is strictly finer than the product topology. For example, the diagonal in $\mathbf{A}^1 \times \mathbf{A}^1$ is closed (defined by $x - y \in k[x, y]$), but it is not closed in the product topology.

REMARK 1.6.5. If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then $X \times Y \subseteq \mathbf{A}^{m+n}$ is a locally closed subset, and the two projections induce morphisms $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$. These make $X \times Y$ the product of X and Y in the category of quasi-affine varieties over k. Indeed, given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, it is clear that there is a unique morphism $\phi\colon Z \to X \times Y$ such that $p \circ \phi = f$ and $q \circ \phi = g$, namely $\phi = (f, g)$.

This implies, in particular, that if $f\colon X \to X'$ and $g\colon Y \to Y'$ are isomorphisms, then the induced map $X \times Y \to X' \times Y'$ is an isomorphism.

PROPOSITION 1.6.6. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then the two projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are open*[3].

PROOF. We show that p is open, the argument for q being entirely similar. Note first that by Remark 1.6.5, we may replace X and Y by isomorphic quasi-affine varieties. Moreover, if we write $X = \bigcup_i X_i$ and $Y = \bigcup_j Y_j$, then for any open subset W of $X \times Y$, we have

$$p(W) = p\left(\bigcup_{i,j} W \cap (X_i \times Y_j)\right),$$

hence if order to show that p is open, it is enough to show that each projection $X_i \times Y_j \to X_i$ is open. By Remark 1.4.20, both X and Y can be covered by open subsets that are affine varieties. We may thus assume that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. Let $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$ be the rings corresponding to $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively. Using again the fact that every open subset of $X \times Y$ is a union of principal affine open subsets, we see that it is enough to show that $p(W)$ is open in $\mathbf{A}^m$ for a nonempty subest $W = D_{X \times Y}(h)$, where $h \in k[x, y]$.

Let us write

$$h = \sum_{i=1}^{r} f_i(x) g_i(y). \tag{1.6.1}$$

We may and will assume that for the given set W, h and the expression (1.6.1) are chosen such that r is minimal. Note that in this case, the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. Indeed, if this is not the case and $\sum_{i=1}^{r} \lambda_i g_i = P(y) \in I(Y)$, such that $\lambda_j \neq 0$ for some j, then we may take $h' = h - \lambda_j^{-1} f_j(x) P(y)$; we then have $D_{X \times Y}(h') = D_{X \times Y}(h)$ and we can write

$$h' = \sum_{i, i \neq j} \big(f_i(x) - \lambda_i \lambda_j^{-1} f_j(x)\big) g_i(y),$$

contradicting the minimality of r.

Suppose now that $u \in p(W)$. This implies that $u \in X$ such that there is $v \in Y$, with $h(u, v) \neq 0$. In particular, there is j such that $f_j(u) \neq 0$. It is enough to show that in this case $D_X(f_j)$, which contains u, is contained in $p(W)$. Suppose, arguing by contradiction, that there is $u' \in D_X(f_j) \smallsetminus p(W)$. This implies that for every $v \in Y$, we have $\sum_{i=1}^{r} f_i(u') g_i(v) = 0$, hence $\sum_{i=1}^{r} f_i(u') g_i \in I(Y)$. Since $f_j(u') \neq 0$, this contradicts the fact that the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. ☐

COROLLARY 1.6.7. *If X and Y are irreducible quasi-affine varieties, then $X \times Y$ is irreducible.*

PROOF. We need to show that if U and V are nonempty, open subsets of $X \times Y$, then $U \cap V$ is nonempty. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projections. By the proposition, the nonempty subsets $p(U)$ and $p(V)$ of X are open. Since X is irreducible, we can find $a \in p(U) \cap p(V)$. In this case, the subsets $\{b \in Y \mid (a, b) \in U\}$ and $\{b \in Y \mid (a, b) \in V\}$ of Y are nonempty. They are also open: this follows from the fact that the map $Y \to X \times Y$, $y \to (a, y)$ is

[3]Recall that a continuous map $\phi\colon Z_1 \to Z_2$ is *open* if for every open subset U of Z_1, its image $\phi(U)$ is open in Z_2.

a morphism, hence it is continuous. Since Y is irreducible, these two subsets must intersect, hence there is a point $(a, b) \in U \cap V$. $\square$

Our next goal is to describe the ideal defining the product of two affine varieties. Suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. We have seen in the proof of Proposition 1.6.1 that if $I(X) \subseteq \mathcal{O}(\mathbf{A}^m)$ and $I(Y) \subseteq \mathcal{O}(\mathbf{A}^n)$ are the ideals defining X and Y, respectively, then $X \times Y$ is the algebraic subset of $\mathbf{A}^{m+n}$ defined by

$$J := I(X) \cdot \mathcal{O}(\mathbf{A}^{m+n}) + I(Y) \cdot \mathcal{O}(\mathbf{A}^{m+n}).$$

We claim that, in fact, J is the ideal defining $X \times Y$, that is, J is a radical ideal. Note that $\mathcal{O}(\mathbf{A}^{m+n})$ is canonically isomorphic to $\mathcal{O}(\mathbf{A}^m) \otimes_k \mathcal{O}(\mathbf{A}^n)$ and by the right-exactness of the tensor product, we have

$$\mathcal{O}(\mathbf{A}^{m+n})/J \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

The assertion that J is a radical ideal (or equivalently, that $\mathcal{O}(\mathbf{A}^{m+n})/J$ is a reduced ring is the content of the following

PROPOSITION 1.6.8. *If X and Y are affine varieties, then the ring $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced.*

Before giving the proof of the proposition, we need some algebraic preparations concerning separable extensions.

LEMMA 1.6.9. *If k is any field and K/k is a finite, separable field extension, then for every field extension k'/k, the ring $K \otimes_k k'$ is reduced.*

PROOF. Since K/k is finite and separable, it follows from the Primitive Element theorem that there is an element $u \in K$ such that $K = k(u)$. Moreover, separability implies that if $f \in k[x]$ is the minimal polynomial of u, then all roots of f in some algebraic closure of k are distinct. The isomorphism $K \simeq k[x]/(f)$ induces an isomorphism

$$K \otimes_k k' \simeq k'[x]/(f).$$

If $g_1, \ldots, g_r$ are the irreducible factors of f in $k'[x]$, any two of them are relatively prime (otherwise f would have multiple roots in some algebraic closure of k). It then follows from the Chinese Remainder theorem that we have an isomorphism

$$K \otimes_k k' \simeq \prod_{i=1}^{r} k'[x]/(g_i).$$

Since each factor on the right-hand side is a field (the polynomial g_i being irreducible), the product is a reduced ring. $\square$

LEMMA 1.6.10. *If k is a perfect[4] field and K/k is a finitely generated field extension, then there is a transcendence basis $x_1, \ldots, x_n$ of K over k such that K is separable over $k(x_1, \ldots, x_n)$.*

PROOF. Of course, the assertion is trivial if $\mathrm{char}(k) = 0$, hence we may assume that $\mathrm{char}(k) = p > 0$. Let us write $K = k(x_1, \ldots, x_m)$. We may assume that $x_1, \ldots, x_n$ give a transcendence basis of K/k, and suppose that $x_{n+1}, \ldots, x_{n+r}$ are not separable over $K' := k(x_1, \ldots, x_n)$, while $x_{n+r+1}, \ldots, x_m$ are separable over K'. If $r = 0$, then we are done. Otherwise, since x_{n+1} is not separable over K',

[4] Recall that a field k is perfect if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and $k = k^p$. Equivalently, a field is perfect if every finite extension K/k is separable.

it follows that there is an irreducible polynomial $f \in K'[T]$ such that $f \in K'[T^p]$ and such that $f(x_{n+1}) = 0$. We can find a nonzero $u \in k[x_1, \ldots, x_n]$ such that $g = uf \in k[x_1, \ldots, x_n, T^p]$.

We claim that there is $i \leq n$ such that $\frac{\partial g}{\partial x_i} \neq 0$. Indeed, otherwise we have $g \in k[x_1^p, \ldots, x_n^p, T^p]$, and since k is perfect, we have $k = k^p$, hence $g = h^p$ for some $h \in k[x_1, \ldots, x_n, T]$; this contradicts the fact that f is irreducible.

After relabeling the variables, we may assume that $i = n$. The assumption on i says that x_n is (algebraic and) separable over $K'' := k(x_1, \ldots, x_{n-1}, x_{n+1})$. Note that since x_n is algebraic over K'' and K is algebraic over $k(x_1, \ldots, x_{n-1}, x_n)$, it follows that K is algebraic over K'', and since all transcendence bases of K over k have the same number of elements, we conclude that $x_1, \ldots, x_{n-1}, x_{n+1}$ is a transcendence basis of K over k. We may thus switch x_n and x_{n+1} to lower r. After finitely many steps, we obtain the conclusion of the lemma. $\square$

Proposition 1.6.11. *If k is a perfect field, then for every field extensions K/k and k'/k, the ring $K \otimes_k k'$ is reduced.*

Proof. We may assume that K is finitely generated over k. Indeed, we can write

$$K = \varinjlim_i K_i,$$

where the direct limit is over all $k \subseteq K_i \subseteq K$, with K_i/k finitely generated. Since we have an induced isomorphism

$$K \otimes_k k' \simeq \varinjlim_i K_i \otimes_k k',$$

and a direct limit of reduced rings is reduced, we see that it is enough to prove the proposition when K/k is finitely generated.

In this case we apply Lemma 1.6.10 to find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $K_1 := k(x_1, \ldots, x_n)$. We have

$$K \otimes_k k' = K \otimes_{K_1} K_1 \otimes_k k'.$$

Since $K_1 \otimes_k k'$ is a ring of fractions of $k'[x_1, \ldots, x_n]$, we have an injective homomorphism

$$K_1 \otimes_k k' \hookrightarrow K_2 := k'(x_1, \ldots, x_n).$$

By tensoring with K, we get an injective homomorphism

$$K \otimes_k k' \hookrightarrow K \otimes_{K_1} K_2.$$

Since K/K_1 is a finite separable extension, we deduce from Lemma 1.6.9 that $K \otimes_{K_1} K_2$ is reduced, hence $K \otimes_k k'$ is reduced. $\square$

We can now prove our result about the coordinate ring of the product of two affine varieties.

Proof of Proposition 1.6.8. We will keep using the fact that the tensor product over k is an exact functor. Note first that we may assume that X and Y are irreducible. Indeed, let $X_1, \ldots, X_r$ be the irreducible components of X and $Y_1, \ldots, Y_s$ the irreducible components of Y. Since $X = X_1 \cup \ldots \cup X_r$, it is clear that the canonical homomorphism

$$\mathcal{O}(X) \to \prod_{i=1}^{r} \mathcal{O}(X_i)$$

is injective. Similarly, we have an injective homomorphism

$$\mathcal{O}(Y) \to \prod_{j=1}^{s} \mathcal{O}(Y_i)$$

and we thus obtain an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow \prod_{i,j} \mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j).$$

The right-hand side is a reduced ring if each $\mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j)$ is reduced, in which case $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. We thus may and will assume that both X and Y are irreducible.

We know that in this case $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains and let $k(X)$ and $k(Y)$ be the respective fraction fields. Since k is algebraically closed, it is perfect, hence $k(X) \otimes_k k(Y)$ is a reduced ring by Proposition 1.6.11. The inclusions

$$\mathcal{O}(X) \hookrightarrow k(X) \quad \text{and} \quad \mathcal{O}(Y) \hookrightarrow k(Y)$$

induce an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow k(X) \otimes_k k(Y),$$

which implies that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. □

We now give another application of Lemma 1.6.10. We first make a definition.

Definition 1.6.12. A hypersurface in $\mathbf{A}^n$ is a closed subset of the form

$$\{u \in \mathbf{A}^n \mid f(u) = 0\} \quad \text{for some} \quad f \in k[x_1, \ldots, x_n] \smallsetminus k.$$

Proposition 1.6.13. *Every irreducible variety is birational to an (irreducible) hypersurface in an affine space* $\mathbf{A}^n$.

Proof. Let X be an irreducible variety, with function field $K = k(X)$. By Lemma 1.6.10, we can find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $k(x_1, \ldots, x_n)$. In this case, it follows from the Primitive Element theorem that there is $u \in K$ such that $K = k(x_1, \ldots, x_n, u)$. If $f \in k(x_1, \ldots, x_n)[t]$ is the minimal polynomial of u, then

$$K \simeq k(x_1, \ldots, x_n)[t]/(f).$$

It is easy to see that after multiplying u by a suitable nonzero element of $k[x_1, \ldots, x_n]$, we may assume that $f \in k[x_1, \ldots, x_n, t]$ and f is irreducible. In this case, we see by Theorem 1.5.18 that X is birational to the affine variety $V(f) \subseteq \mathbf{A}^{n+1}$. □

We end this section with some exercises about linear algebraic groups. We begin with a definition.

Definition 1.6.14. A *linear algebraic group* over k is an affine variety G over k that is also a group, and such that the multiplication $\mu\colon G \times G \to G$, $\mu(g, h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. If G_1 and G_2 are linear algebraic groups, a *morphism of algebraic groups* is a morphism of affine varieties $f\colon G_1 \to G_2$ that is also a group homomorphism.

Linear algebraic groups over k form a category. In particular, we have a notion of isomorphism between linear algebraic groups: this is an isomorphism of affine algebraic varieties that is also a group isomorphism.

EXERCISE 1.6.15. i) Show that $(k, +)$ and $(k^*, \cdot)$ are linear algebraic groups.

ii) Show that the set $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k has a structure of linear algebraic group.

iii) Show that the set $\mathrm{SL}_n(k)$ of $n \times n$ matrices with coefficients in k and with determinant 1 has a structure of linear algebraic group.

iv) Show that if G and H are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) *torus* $(k^*)^n$ is a linear algebraic group with respect to componentwise multiplication.

DEFINITION 1.6.16. Let G be a linear algebraic group and X a quasi-affine variety. An *algebraic group action* of G on X is a (say, left) action of G on X such that the map $G \times X \to X$ giving the action is a morphism of algebraic varieties.

EXERCISE 1.6.17. Show that $\mathrm{GL}_n(k)$ has an algebraic action on $\mathbf{A}^n$.

EXERCISE 1.6.18. Let G be a linear algebraic group acting algebraically on an affine variety X. Show that in this case G has an induced linear action on $\mathcal{O}(X)$ given by

$$(g \cdot \phi)(u) = \phi\big(g^{-1}(u)\big).$$

While $\mathcal{O}(X)$ has in general infinite dimension over k, show that the action of G on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace V of $\mathcal{O}(X)$ that is preserved by the G-action (Hint: consider the image of f by the corresponding k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

EXERCISE 1.6.19. Let G and X be as in the previous problem. Consider a system of k-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \mathrm{GL}_N(k)$, and an isomorphism of X with a closed subset of $\mathbf{A}^N$, such that the action of G on X is induced by the standard action of $\mathrm{GL}_N(k)$ on $\mathbf{A}^N$. Use a similar argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_N(k)$.

EXERCISE 1.6.20. Show that the linear algebraic group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbf{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action ? Note that the orbits are locally closed subsets of $M_{m,n}(k)$ (as we will see later, this is a general fact about orbits of algebraic group actions).

1.7. Affine toric varieties

In this section we discuss a class of examples of affine varieties that are associated to semigroups.

DEFINITION 1.7.1. A *semigroup* is a set S endowed with an operation $+$ (we will use in general the additive notation) which is commutative, associative and has a unit element 0. If S is a semigroup, a *subsemigroup* of S is a subset $S' \subseteq S$ closed under the operation in S and such that $0_S \in S'$ (in which case, S' becomes a semigroup with the induced operation). A map $\phi\colon S \to S'$ between two semigroups is a *semigroup morphism* if $\phi(u_1 + u_2) = \phi(u_1) + \phi(u_2)$ for all u_1 and u_2, and if $\phi(0) = 0$.

EXAMPLE 1.7.2. i) Every Abelian group is a semigroup.
ii) The field k, endowed with the multiplication, is a semigroup.
iii) The set $\mathbf{N}$ of non-negative integers, with the addition, is a semigroup.
iv) The set $\{m \in \mathbf{N} \mid m \neq 1\}$ is a subsemigroup of $\mathbf{N}$.
v) If S_1 and S_2 are semigroups, then $S_1 \times S_2$ is a semigroup, with componentwise addition.

Given a semigroup S, we consider the semigroup algebra $k[S]$. This has a basis over k indexed by the elements of S. We denote the elements of this basis by χ^u, for $u \in S$. The multiplication is defined by $\chi^{u_1} \cdot \chi^{u_2} = \chi^{u_1+u_2}$ (hence $1 = \chi^0$). This is a k-algebra. Note that if $\phi\colon S_1 \to S_2$ is a morphism of semigroups, then we get a morphism of k-algebras $k[S_1] \to k[S_2]$ that maps χ^u to $\chi^{\phi(u)}$.

EXAMPLE 1.7.3. We have an isomorphism

$$k[\mathbf{N}^r] \simeq k[x_1, \ldots, x_r], \quad \chi^{e_i} \to x_i,$$

where e_i is the tuple that has 1 on the i^{th} component and 0 on all the others. We similarly have an isomorphism

$$k[\mathbf{Z}^r] \simeq k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}].$$

EXAMPLE 1.7.4. In general, if S_1 and S_2 are semigroups, we have a canonical isomorphism

$$k[S_1 \times S_2] \simeq k[S_1] \otimes_k k[S_2].$$

We will assume that our semigroups satisfy two extra conditions. First, we will assume that they are *finitely generated*: a semigroup S satisfies this property if it has finitely many generators $u_1, \ldots, u_r \in S$ (this means that every element in S can be written as $\sum_{i=1}^r a_i u_i$, for some $a_1, \ldots, a_r \in \mathbf{N}$). In other words, the unique morphism of semigroups $\mathbf{N}^r \to S$ that maps e_i to u_i for all i is surjective. Note that in this case, the induced k-algebra homomorphism

$$k[x_1, \ldots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is onto, hence $k[S]$ is finitely generated.

We will also assume that S is *integral*, that is, it is isomorphic to a subsemigroup of a finitely generated, free Abelian group. Since we have an injective morphism of semigroups $S \hookrightarrow \mathbf{Z}^r$, we obtain an injective k-algebra homomorphism $k[S] \hookrightarrow k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$. In particular, $k[S]$ is a domain.

EXERCISE 1.7.5. Suppose that S is the image of a morphism of semigroups $\phi\colon \mathbf{N}^r \to \mathbf{Z}^m$ (this is how semigroups are usually described). Show that the kernel of the induced surjective k-algebra homomorphism

$$k[x_1, \ldots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is the ideal

$$\big(x^a - x^b \mid a, b \in \mathbf{N}^r, \phi(a) = \phi(b)\big).$$

We have seen that if S is an integral, finitely generated semigroup, then $k[S]$ is a finitely generated k-algebra, which is also a domain. Therefore it corresponds to an irreducible affine variety over k, uniquely defined up to canonical isomorphism. We will denote this variety[5] by $\mathrm{TV}(S)$. Its points are in bijection with the maximal ideals in $k[S]$, or equivalently, with the k-algebra homomorphisms $k[S] \to k$. Such

[5]This is not standard notation in the literature.

homomorphisms in turn are in bijection with the semigroup morphisms $S \to (k, \cdot)$. Via this bijection, if we consider $\phi\colon S \to (k, \cdot)$ as a point in $\mathrm{TV}(S)$ and $\chi^u \in k[S]$, then

$$\chi^u(\phi) = \phi(u) \in k.$$

Given a morphism of finitely generated, integral semigroups $S \to S'$, the k-algebra homomorphism $k[S] \to k[S']$ corresponds to a morphism $\mathrm{TV}(S') \to \mathrm{TV}(S)$.

The affine variety $TV(S)$ carries more structure, induced by the semigroup S, which we now describe. First, we have a morphism

$$TV(S) \times TV(S) \to TV(S)$$

corresponding to the k-algebra homomorphism

$$k[S] \to k[S] \otimes_k k[S], \quad \chi^u \to \chi^u \otimes \chi^u.$$

At the level of points (identified, as above, to semigroup morphisms to k), this is given by

$$(\phi, \psi) \to \phi \cdot \psi, \quad \text{where} \quad (\phi \cdot \psi)(u) = \phi(u) \cdot \psi(u).$$

It is clear that the operation is commutative, associative, and has an identity element, given by the morphism $S \to k$ that takes constant value 1.

Remark 1.7.6. If $S \to S'$ is a morphism between integral, finitely generated semigroups, it is clear that the induced morphism of affine varieties $\mathrm{TV}(S') \to \mathrm{TV}(S)$ is compatible with the operation defined above.

Example 1.7.7. If $S = \mathbf{N}^r$, then the operation that we get on $\mathrm{TV}(S) = \mathbf{A}^r$ is given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n).$$

In particular, note that $\mathrm{TV}(S)$ is *not* a group.

Example 1.7.8. With the operation defined above, $\mathrm{TV}(\mathbf{Z})$ is a linear algebraic group isomorphic to $(k^*, \cdot)$. In general, if M is a finitely generated, free Abelian group, then the above operation makes $\mathrm{TV}(M)$ a linear algebraic group. In fact, we have $M \simeq \mathbf{Z}^r$, for some r, and therefore $\mathrm{TV}(M)$ is isomorphic, as an algebraic group, to the torus $(k^*)^r$ (see Exercise 1.6.15 for the definition of the algebraic tori). It follows from the lemma below that we can recover M from $\mathrm{TV}(M)$, together with the group structure, as

$$M \simeq \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M), k^*\big).$$

Lemma 1.7.9. *For every finitely generated, free Abelian groups M and M', the canonical map*

$$\mathrm{Hom}_{\mathbf{Z}}(M, M') \to \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M'), \mathrm{TV}(M)\big)$$

is a bijection.

Proof. A morphism of algebraic groups $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is given by a k-algebra homomorphism $f\colon k[M] \to k[M']$ such that the induced diagram

$$\begin{tikzcd} k[M] \arrow[r, "f"] \arrow[d, "\alpha_M"] & k[M'] \arrow[d, "\alpha_{M'}"] \\ k[M] \otimes k[M] \arrow[r, "f \otimes f"] & k[M'] \otimes k[M'], \end{tikzcd}$$

is commutative, where α_M and $\alpha_{M'}$ are the k-algebra homomorphisms inducing the group structure. Given $u \in M$, we see that if $f(\chi^u) = \sum_{u' \in M'} a_{u,u'} \chi^{u'}$, then

$$\sum_{u' \in M'} a_{u,u'} \chi^{u'} \otimes \chi^{u'} = \sum_{u', v' \in M'} a_{u,u'} a_{u,v'} \chi^{u'} \otimes \chi^{v'}.$$

First, this implies that if u', $v' \in M'$ are distinct, then $a_{u,u'} \cdot a_{u,v'} = 0$. Therefore there is a unique $u' \in M'$ such that $a_{u,u'} \neq 0$ (note that $\chi^u \in k[M]$ is invertible, hence $f(\chi^u) \neq 0$). Moreover, for this u' we have $a_{u,u'}^2 = a_{u,u'}$, hence $a_{u,u'} = 1$. This implies that we have a (unique) map $\phi\colon M \to M'$ such that f is given by $f(\chi^u) = \chi^{\phi(u)}$. Since f is a ring homomorphism, we see that ϕ is a semigroup morphism. This shows that the map in the lemma is bijective. $\square$

EXERCISE 1.7.10. Given an integral semigroup S, show that there is an injective semigroup morphism $\iota\colon S \hookrightarrow S^{\mathrm{gp}}$, where S^{gp} is a finitely generated Abelian group, that satisfies the following universal property: given any semigroup morphism $h\colon S \to A$, where A is an Abelian group, there is a unique group morphism $g\colon S^{\mathrm{gp}} \to A$ such that $g \circ \iota = h$. Hint: if $S \hookrightarrow M$ is an injective semigroup morphism, where M is a finitely generated, free Abelian group, then show that one can take S^{gp} to be the subgroup of M generated by S. Note that it follows from this description that S^{gp} is finitely generated (since M is) and S^{gp} is generated as a group by S.

Suppose now that S is an arbitrary integral, finitely generated semigroup. The semigroup morphism $\iota\colon S \to S^{\mathrm{gp}}$ induces a k-algebra homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ and correspondingly a morphism of affine algebraic varieties $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$.

LEMMA 1.7.11. *With the above notation, the morphism* $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ *is an isomorphism onto a principal affine open subset of* $\mathrm{TV}(S)$.

PROOF. Suppose that $u_1, \ldots, u_r$ is a finite system of generators of S. In this case S^{gp} is generated as a semigroup by $u_1, \ldots, u_r$, and $-(u_1 + \ldots + u_r)$. This shows that we can identify the homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ with the localization homomorphism of $k[S]$ at $\chi^{u_1 + \ldots + u_s}$. $\square$

Since the morphism $\mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is compatible with the operations on the two varieties, we conclude that in particular, the action of the torus $\mathrm{TV}(S^{\mathrm{gp}})$, considered as an open subset of $\mathrm{TV}(S)$, extends to an action of $\mathrm{TV}(S^{\mathrm{gp}})$ on $\mathrm{TV}(S)$. We are thus led to the following

DEFINITION 1.7.12. An affine toric variety is an irreducible affine variety X, together with an open subset T that is (isomorphic to) a torus, such that the action of the torus on itself extends to an action of T of X.

We note that in the literature, it is common to require an affine toric variety to be normal, but we do not follow this convention. For the definition of normality and for the description in the context of toric varieties, see Definition 1.7.26 and Proposition 1.7.30 below.

We have seen that for every (integral, finitely generated) semigroup S, we obtain a toric variety $\mathrm{TV}(S)$. The following proposition shows that, in fact, every affine toric variety arises in this way.

PROPOSITION 1.7.13. *Let X be an irreducible affine variety, $T \subseteq X$ an open subset which is a torus such that the action of T on itself extends to an action on*

X. Then there is a finitely generated, integral semigroup S and an isomorphism $X \simeq \mathrm{TV}(S)$ which induces an isomorphism of algebraic groups $T \simeq \mathrm{TV}(S^{\mathrm{gp}})$, and which is compatible with the action.

Proof. Let $M = \mathrm{Hom}_{\mathrm{alg-gp}}(T, k^*)$, so that we have a canonical isomorphism $T \simeq \mathrm{TV}(M)$. The dominant inclusion morphism $T \to X$ induces an injective k-algebra homomorphism $f\colon \mathcal{O}(X) \to \mathcal{O}(T) = k[M]$, hence we may assume that $\mathcal{O}(X)$ is a subalgebra of $k[M]$. The fact that the action of T on itself extends to an action of T on X is equivalent to the fact that the k-algebra homomorphism

$$k[M] \to k[M] \otimes_k k[M], \quad \chi^u \to \chi^u \otimes \chi^u$$

induces a homomorphism $\mathcal{O}(X) \to k[M] \otimes_k \mathcal{O}(X)$. In other words, if $f = \sum_{u \in M} a_u \chi^u$ lies in $\mathcal{O}(X)$, then $\sum_{u \in M} a_u \chi^u \otimes \chi^u$ lies in $k[M] \otimes_k \mathcal{O}(X)$. This implies that for every $u \in M$ such that $a_u \neq 0$, we have $\chi^u \in \mathcal{O}(X)$. It follows that if $S = \{u \in M \mid \chi^u \in \mathcal{O}(X)\}$, then $\mathcal{O}(X) = k[S]$. It is clear that S is integral and since $k[S]$ is a finitely generated k-algebra, it follows easily that S is a finitely generated semigroup. In order to complete the proof of the proposition, it is enough to show that $M = S^{\mathrm{gp}}$.

It follows from Exercise 1.7.10 that we may take S^{gp} to be the subgroup of M generated by S. By hypothesis, the composition

$$TV(M) \xrightarrow{g} \mathrm{TV}(S^{\mathrm{gp}}) \xrightarrow{h} X = \mathrm{TV}(S)$$

is an isomorphism onto an open subset of X. Since we also know that h is an isomorphism onto an open subset of X, it follows that g gives is an isomorphism onto an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$. In particular, this implies that g is injective. We now show that $M = S^{\mathrm{gp}}$.

Since M is a finitely generated, free Abelian group, we can find a basis $e_1, \ldots, e_n$ of M such that S^{gp} has a basis given by $a_1 e_1, \ldots, a_r e_r$, for some $r \leq n$ and some positive integers $a_1, \ldots, a_r$. In this case g gets identified to the morphism

$$(k^*)^n \to (k^*)^r, \quad (t_1, \ldots, t_n) \to (t_1^{a_1}, \ldots, t_r^{a_r}).$$

Since g is injective, we see that $r = n$. Moreover, if $a_j > 1$ for some j, then $\mathrm{char}(k) = p > 0$ and for every i we have $a_i = p^{e_i}$ for some nonnegative integer e_i. It is easy to see that in this case g is surjective (cf. Exercise 1.4.26). Since we know that it gives an isomorphism of $\mathrm{TV}(M)$ with an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$, it follows that g is an isomorphism. However, this implies $a_i = 1$ for all i. Therefore we have $S^{\mathrm{gp}} = M$. $\square$

We now turn to the description of toric morphisms. Suppose that X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$.

Definition 1.7.14. With the above notation, a *toric morphism* $X \to Y$ is a morphism of algebraic varieties $f\colon X \to Y$ that induces a morphism of algebraic groups $g\colon T_X \to T_Y$.

Remark 1.7.15. Note that if $f\colon X \to Y$ is a toric morphism as above, then f is a morphism of varieties with torus action, in the sense that

$$f(t \cdot x) = g(t) \cdot f(x) \quad \text{for every } t \in T_X, x \in X.$$

Indeed, this follows by Lemma 1.5.7 from the fact that we have this equality for $(t, x) \in T_X \times T_X$.

If $\phi\colon S_1 \to S_2$ is a semigroup morphism between two integral, finitely generated semigroups, we get an induced group morphism $S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}}$. We then obtain an induced morphism $f\colon \mathrm{TV}(S_2) \to \mathrm{TV}(S_1)$ that restricts to a morphism of algebraic groups $\mathrm{TV}(S_2^{\mathrm{gp}}) \to \mathrm{TV}(S_1^{\mathrm{gp}})$; therefore f is a toric morphism. The next proposition shows that all toric morphisms arise in this way, from a unique semigroup homomorphism.

Proposition 1.7.16. *If S_1 and S_2 are finitely generated, integral semigroups, then the canonical map*

$$\mathrm{Hom}_{\mathrm{semigp}}(S_1, S_2) \to \mathrm{Hom}_{\mathrm{toric}}\big(\mathrm{TV}(S_2), \mathrm{TV}(S_1)\big)$$

is a bijection.

Proof. By definition, a toric morphism $\mathrm{TV}(S_2) \to \mathrm{TV}(S_1)$ is given by a k-algebra homomorphism $k[S_1] \to k[S_2]$ such that the induced homomorphism $f\colon k[S_1^{\mathrm{gp}}] \to k[S_2^{\mathrm{gp}}]$ gives a morphism of algebraic groups $\mathrm{TV}(S_2^{\mathrm{gp}}) \to \mathrm{TV}(S_1^{\mathrm{gp}})$. It follows from Lemma 1.7.9 that we have a group morphism $\phi\colon S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}}$ such that $f(\chi^u) = \chi^{\phi(u)}$ for every $u \in S_1^{\mathrm{gp}}$. Since f induces a homomorphism $k[S_1] \to k[S_2]$, we have $\phi(S_1) \subseteq S_2$, hence ϕ is induces a semigroup morphism $S_1 \to S_2$. This shows that the map in the proposition is surjective and the injectivity is straightforward. $\square$

Remark 1.7.17. We can combine the assertions in Proposition 1.7.13 and 1.7.16 as saying that the functor from the category of integral, finitely generated semigroups to the category of affine toric varieties, that maps S to $\mathrm{TV}(S)$, is an anti-equivalence of categories.

Example 1.7.18. If $S = \mathbf{N}^r$, then $\mathrm{TV}(S) = \mathbf{A}^r$, with the torus $(k^*)^r \subseteq \mathbf{A}^r$ acting by component-wise multiplication.

Example 1.7.19. If $S = \{m \in \mathbb{N} \mid m \neq 1\}$, then $S^{\mathrm{gp}} = \mathbb{Z}$. If we embed X in $\mathbb{A}^2$ as the curve with equation $u^3 - v^2 = 0$, then the embedding $T \simeq k^* \hookrightarrow X$ is given by $\lambda \to (\lambda^2, \lambda^3)$. The action of T on X is described by $\lambda \cdot (u, v) = (\lambda^2 u, \lambda^3 v)$.

Exercise 1.7.20. Show that if X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$, then $X \times Y$ has a natural structure of toric variety, with torus $T_X \times T_Y$. Describe the semigroup corresponding to $X \times Y$ in terms of the semigroups of X and Y.

Exercise 1.7.21. Let S be the sub-semigroup of $\mathbb{Z}^3$ generated by e_1, e_2, e_3 and $e_1 + e_2 - e_3$. These generators induce a surjective morphism $f\colon k[\mathbb{N}^4] = k[t_1, \ldots, t_4] \to k[S]$. Show that the kernel of f is generated by $t_1t_2 - t_3t_4$. We have $S^{\mathrm{gp}} = \mathbb{Z}^3$, the embedding of $T = (k^*)^3 \hookrightarrow X$ is given by $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2/\lambda_3)$, and the action of T on X is induced via this embedding by component-wise multiplication.

The following lemma provides a useful tool for dealing with torus-invariant objects. Consider $X = \mathrm{TV}(S)$ and let $T = \mathrm{TV}(S^{\mathrm{gp}})$ be the corresponding torus. As in the case of any algebraic group action, the action of T on X induces an action of T on $\mathcal{O}(X)$ (see Exercise 1.6.18). Explicitly, in our setting this is given by

$$\phi \cdot \chi^u = \phi(u)^{-1}\chi^u \quad \text{for all} \quad u \in S, \phi \in \mathrm{Hom}_{\mathrm{gp}}(S^{\mathrm{gp}}, k^*).$$

LEMMA 1.7.22. *With the above notation, a subspace $V \subseteq k[S]$ is T-invariant (that is, $t \cdot g \in V$ for every $g \in V$) if and only if it is S-homogenous, in the sense that for every $g = \sum_{u \in S} a_u \chi^u \in V$, we have $\chi^u \in V$ whenever $a_u \neq 0$.*

PROOF. We only need to prove the "only if" part, the other direction being straightforward. By definition, V is T-invariant if and only if for every group morphism $\phi\colon S^{\mathrm{gp}} \to k^*$ and every $g = \sum_{u \in S} a_u \chi^u \in V$, we have

$$\sum_{u \in S} a_u \phi(u)^{-1} \chi^u \in V.$$

Iterating, we obtain

$$\sum_{u \in S} a_u \phi(u)^{-m} \chi^u \in V \quad \text{for all} \quad m \geq 1. \tag{1.7.1}$$

Claim. Given pairwise distinct $u_1, \ldots, u_d \in S$, we can find $\phi \in T$ such that $\phi(u_i) \neq \phi(u_{i'})$ for $i \neq i'$. Indeed, let us choose an isomorphism $S^{\mathrm{gp}} \simeq \mathbf{Z}^n$, so that each u_i corresponds to $(a_{i,1}, \ldots, a_{i,n})$. After adding to each $(a_{i,1}, \ldots, a_{i,n})$ the element $(m, \ldots, m)$ for $m \gg 0$, we may assume that $a_{i,j} \geq 0$ for all i and j. Since each polynomial

$$Q_{i,i'} = \prod_{j=1}^n x_j^{a_{i,j}} - \prod_{j=1}^n x_j^{a_{i',j}}, \quad \text{for} \quad i \neq i'$$

is nonzero, it follows that the open subset $U_{i,i'}$ defined by $Q_{i,i'} \neq 0$ is a nonempty subset of $\mathbf{A}^n$. Since $\mathbf{A}^n$ is irreducible, it follows that the intersection

$$(k^*)^n \cap \bigcap_{i \neq i'} U_{i,i'}$$

is nonempty, giving the claim.

By applying the claim to those $u \in S$ such that $a_u \neq 0$, we deduce from (1.7.1) and from the formula for the Vandermonde determinant that $\chi^u \in V$ for all u such that $a_u \neq 0$. $\square$

In the next two exercises we describe the torus-invariant subvarieties of $\mathrm{TV}(S)$ and the orbits of the torus action. We begin by defining the corresponding concept at the level of the semigroup.

DEFINITION 1.7.23. A face F of a semigroup S is a subsemigroup such that whenever $u_1, u_2 \in S$ have $u_1 + u_2 \in F$, we have $u_1 \in F$ and $u_2 \in F$.

Note that if F is a face of S, then $S \smallsetminus F$ is a subsemigroup of S. Moreover, if S is generated by $u_1, \ldots, u_n$. then a face F of S is generated by those u_i that lie in F. In particular, if S is an integral, finitely generated semigroup, then S has only finitely many faces, and each of these is an integral, finitely generated semigroup.

EXERCISE 1.7.24. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T \subset X$. A subset Y of X is *torus-invariant* if $t \cdot Y \subseteq Y$ for every $t \in T$.

i) Show that a closed subset Y of X is torus-invariant if and only if each irreducible component of Y is torus-invariant.
ii) Show that the torus-invariant irreducible closed subsets of X are precisely the closed subsets defined by ideals of the form

$$\bigoplus_{u \in S \smallsetminus F} k\chi^u,$$

where F is a face of S.

iii) Show that if Y is the closed subset defined by the ideal in ii), then we have $\mathcal{O}(Y) \simeq k[F]$, hence Y has a natural structure of affine toric variety.

EXERCISE 1.7.25. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T_X \subseteq X$.

i) Show that if $M \hookrightarrow M'$ is an injective morphism of finitely generated, free Abelian groups, then the induced morphism of tori $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is surjective.

ii) Show that if F is a face of S with corresponding closed invariant subset Y, then the inclusion of semigroups $F \subseteq S$ induces a morphism of toric varieties $f_Y : X \to Y$, which is a retract of the inclusion $Y \hookrightarrow X$. Show that the torus O_F in Y is an orbit for the action of T_X on X.

iii) Show that the map $F \to O_F$ gives a bijection between the faces of S and the orbits for the T_X-action on X.

We now discuss normality for the varieties we defined. Recall that if $R \to S$ is a ring homomorphism, then the set of elements of S that are integral over R form a subring of S, the *integral closure* of R in S (see Proposition A.2.2).

DEFINITION 1.7.26. An integral domain A is *integrally closed* if it is equal to its integral closure in its field of fractions. It is *normal* if, in addition, it is Noetherian. An irreducible, affine variety X is *normal* if $\mathcal{O}(X)$ is a normal ring.

REMARK 1.7.27. If A is an integral domain and B is the integral closure of A in its fraction field, then B is integrally closed. Indeed, the integral closure of B in K is integral over A (see Proposition A.2.3), hence it is contained in B.

EXAMPLE 1.7.28. Every UFD is integrally closed. Indeed, suppose that A is a UFD and $u = \frac{a}{b}$ lies in the fraction field of A and it is integral over A. We may assume that a and b are relatively prime. Consider a monic polynomial $f = x^m + c_1x^{m-1} + \ldots c_m \in A[x]$ such that $f(u) = 0$. Since

$$a^m = -b \cdot (c_1a^{m-1} + \ldots c_mb^{m-1}),$$

it follows that b divides a^m. Since b and a are relatively prime, it follows that b is invertible, hence $u \in A$.

In particular, we see that every polynomial ring $k[x_1, \ldots, x_n]$ is integrally closed.

DEFINITION 1.7.29. An integral, finitely generated semigroup S is *saturated* if whenever $mu \in S$ for some $u \in S^{\mathrm{gp}}$ and some positive integer m, we have $u \in S$.

PROPOSITION 1.7.30. *If S is an integral, finitely generated semigroup, the variety* $\mathrm{TV}(S)$ *is normal if and only if S is saturated.*

PROOF. The rings $k[S] \subseteq k[S^{\mathrm{gp}}]$ have the same fraction field, and $k[S^{\mathrm{gp}}] \simeq k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ for some n, so $k[S^{\mathrm{gp}}]$ is normal, being a UFD. Therefore $k[S]$ is normal if and only if it is integrally closed in $k[S^{\mathrm{gp}}]$.

Suppose first that $k[S]$ is normal. If $u \in S^{\mathrm{gp}}$ and if $mu \in S$, then $(\chi^u)^m \in k[S]$ and $\chi^u \in k[S^{\mathrm{gp}}]$. As $k[S]$ is integrally closed in $k[S^{\mathrm{gp}}]$, it follows that $\chi^u \in k[S]$, so $u \in S$.

Conversely, let us assume that S is saturated, and let R be the integral closure of $k[S]$ in $k[S^{\mathrm{gp}}]$. It is clear that R is a torus-invariant linear subspace of $k[S^{\mathrm{gp}}]$, hence it follows from Lemma 1.7.22 that it is S^{gp}-homogeneous. In order to show that $R = k[S]$ it is thus enough to check that for every $\chi^u \in R$, we have $u \in S$. By assumption, χ^u satisfies an equation of the form

$$(\chi^u)^m + a_1(\chi^u)^{m-1} + \ldots + a_m\chi^{v_m} = 0,$$

for a positive integer m and $a_1, \ldots, a_m \in k[S]$. By only considering the scalar multiples of χ^{mu}, we may assume that in fact $a_i = c_i\chi^{v_i}$ for some $c_i \in k$ and $v_i \in S$. It follows that $v_i + (m-i)u = mu$ if $a_i \neq 0$, hence $iu = v_i$. Since some a_i must be nonzero, we have $iu \in S$ for some $i \geq 1$, and because S is saturated we deduce $u \in S$. $\square$

Exercise 1.7.31. We have seen in Exercise 1.7.24 that if X is an affine toric variety and Y is a torus-invariant irreducible subset, then Y has a natural structure of toric variety. Show that if X is normal, then every such Y is normal.

CHAPTER 2

General algebraic varieties

In this chapter we introduce general algebraic varieties. Roughly speaking, these are objects obtained by gluing finitely many affine algebraic varieties and by imposing an analogue of the Hausdorff condition. The gluing could be expressed in terms of atlases (as in differential geometry), but the usual language for handling this is that of ringed spaces and we take this approach, following [**Mum88**]. We thus begin with a brief discussion of sheaves that is needed for the definition of algebraic varieties. A more detailed treatment of sheaves will be given in Chapter 8.

2.1. Presheaves and sheaves

Let X be a topological space. Recall that associated to X we have a category $\mathcal{C}at(X)$, whose objects consist of the open subsets of X and such that for every open subsets U and V of X, the set of arrows $U \to V$ contains precisely one element if $U \subseteq V$ and it is empty, otherwise.

Definition 2.1.1. Given a topological space X and a category $\mathcal{C}$, a *presheaf* on X of objects in $\mathcal{C}$ is a contravariant functor $\mathcal{F}\colon \mathcal{C}at(X) \to \mathcal{C}$. Explicitly, this means that for every open subset U of X, we have an object $\mathcal{F}(U)$ in $\mathcal{C}$, and for every inclusion of open sets $U \subseteq V$, we have a *restriction map*

$$\rho_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$$

that satisfies:

i) $\rho_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$ for every open subset $U \subseteq X$, and
ii) $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$ for every open subsets $U \subseteq V \subseteq W$ of X.
It is common to denote $\rho_{V,U}(s)$ by $s|_U$. The elements of $\mathcal{F}(U)$ are *the sections of* $\mathcal{F}$ *over* U. A common notation for $\mathcal{F}(U)$ is $\Gamma(U, \mathcal{F})$.

The important examples for us are when $\mathcal{C}$ is the category of R-modules or the category of commutative R-algebras (where R is a fixed commutative ring). In particular, when $R = \mathbf{Z}$, we have the category of Abelian groups and the category of rings.

We now introduce *sheaves*: these are presheaves in which the sections can be described locally. For the sake of concreteness, whenever dealing with sheaves, we assume that $\mathcal{C}$ is a subcategory of the category of sets and that a morphism in $\mathcal{C}$ is an isomorphism if and only if it is bijective (note that this is the case for the categories mentioned above).

Definition 2.1.2. Let X be a topological space. A presheaf $\mathcal{F}$ on X of objects in $\mathcal{C}$ is a *sheaf* if for every family of open subsets $(U_i)_{i\in I}$ of X, with $U = \bigcup_{i\in I} U_i$, given $s_i \in \mathcal{F}(U_i)$ for every i such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \text{for every} \quad i, j \in I,$$

there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

REMARK 2.1.3. A special case of the condition in the definition of a sheaf is that when $I = \emptyset$: in this case it says that $\mathcal{F}(\emptyset)$ has exactly one element.

EXAMPLE 2.1.4. If X is a topological space, then we have a presheaf $\mathcal{C}_{X,\mathbf{R}}$ of $\mathbf{R}$-algebras on X, where $\mathcal{C}_{X,\mathbf{R}}(U)$ is the $\mathbf{R}$-algebra of continuous functions $U \to \mathbf{R}$, with the restriction maps given by restriction of functions. It is clear that this is a sheaf, the *sheaf of continuous functions* on X.

EXAMPLE 2.1.5. If X is a $\mathcal{C}^\infty$-manifold, then we have a sheaf of $\mathbf{R}$-algebras $\mathcal{C}^\infty_{X,\mathbf{R}}$ on X, where $\mathcal{C}^\infty_{X,\mathbf{R}}(U)$ is the $\mathbf{R}$-algebra of $\mathcal{C}^\infty$ functions $U \to \mathbf{R}$, with the restriction maps being given by restriction of functions.

EXAMPLE 2.1.6. If X is a quasi-affine variety over an algebraically closed field k, then we have a sheaf $\mathcal{O}_X$ of k-algebras, such that $\mathcal{O}_X(U)$ is the k-algebra of regular functions $U \to k$, with the restriction maps given by restriction of functions. This is the *sheaf of regular functions* on X.

EXAMPLE 2.1.7. Given a continuous map $f\colon X \to Y$ of topological spaces, we have a sheaf of sets $\mathcal{F}$ on Y such that $\mathcal{F}(U)$ is the set of *sections of f over U*, that is, of continuous maps $s\colon U \to X$ such that $f\big(s(y)\big) = y$ for all $y \in U$; the restriction maps given by restriction of functions.

REMARK 2.1.8. If $\mathcal{C}$ is the category of R-modules, for a ring R, it is sometimes convenient to rewrite the sheaf condition for $\mathcal{F}$ as follows: given an open cover $U = \bigcup_i U_i$, we have an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \mathcal{F}(U_i \cap U_j),$$

where

$$\alpha(s) = (s|_{U_i})_i \quad \text{and} \quad \beta((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j\in I}.$$

DEFINITION 2.1.9. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X of objects in $\mathcal{C}$, a morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is given by a functorial transformation between the two contravariant functors. Explicitly, for every open subset $U \subseteq X$, we have a morphism $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ such that if $U \subseteq V$ are open subsets of X, then

$$\phi_U(s|_U) = \phi_V(s)|_U \quad \text{for every} \quad s \in \mathcal{F}(V).$$

The same definition applies for sheaves to give the notion of *morphism of sheaves*.

It is clear that morphisms of presheaves can be composed and in this way the presheaves on X of objects in $\mathcal{C}$ form a category. We also have the category of sheaves on X of objects in $\mathcal{C}$, that forms a full subcategory of the category of presheaves. In particular, we may consider isomorphisms of presheaves or of sheaves.

DEFINITION 2.1.10. Given a presheaf $\mathcal{F}$ on X (of objects in some category $\mathcal{C}$) and an open subset W of X, we obtain a presheaf $\mathcal{F}|_W$ on W such that for every open subset U of W, we take $\mathcal{F}|_W(U) = \mathcal{F}(U)$, with the restriction maps given by those for $\mathcal{F}$. This presheaf is the *restriction* of $\mathcal{F}$ to W. It is clear that if $\mathcal{F}$ is a sheaf, then $\mathcal{F}|_W$ is a sheaf. If $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, then we obtain in the obvious way an induced morphism $\phi|_W\colon \mathcal{F}|_W \to \mathcal{G}|_W$. We thus get a functor from the category of presheaves on X of objects in $\mathcal{C}$ to the category of presheaves on U of objects in $\mathcal{C}$ and a similar functor between the corresponding categories of sheaves.

From now on we assume, for simplicity, that the category $\mathcal{C}$ is either the category of R-modules or the category of R-algebras, where R is a commutative ring.

DEFINITION 2.1.11. If $\mathcal{F}$ is a presheaf on X (of R-modules or of R-algebras), then the *stalk* of $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the direct limit is over all open neighborhoods of x, ordered by reverse inclusion. Note that both categories we consider have direct limits. More generally, if W is an irreducible, closed subset of X, then the *stalk* of $\mathcal{F}$ at W is

$$\mathcal{F}_W := \varinjlim_{U \cap W \neq \emptyset} \mathcal{F}(U),$$

where the direct limit is over all open subsets U of X, with $U \cap W \neq \emptyset$.

EXAMPLE 2.1.12. If $\mathcal{O}_X$ is the sheaf of regular functions on a quasi-affine variety X and W is an irreducible closed subset of X, then the stalk of $\mathcal{O}_X$ at W is the local ring $\mathcal{O}_{X,W}$ of X at W. On general topological spaces, we typically only consider the stalks at the points of X, but in the case of algebraic varieties, it is sometimes natural to also consider the more general stalks.

REMARK 2.1.13. As in the case of a quasi-affine variety, we see that in general, the poset in the definition of $\mathcal{F}_W$ is filtering: given two open subsets U and V with $U \cap W \neq \emptyset$ and $V \cap W \neq \emptyset$, we have $(U \cap V) \cap W \neq \emptyset$, by the irreducibility of W. As a result, we may describe $\mathcal{F}_W$ as the set of all pairs (U, s), for some open subset U with $U \cap W \neq \emptyset$ and some $s \in \mathcal{F}(U)$, modulo the equivalence relation given by $(U, s) \sim (U', s')$ if there is an open subset $V \subseteq U \cap U'$, with $V \cap W \neq \emptyset$ and such that $s|_V = s'|_V$. If $s \in \mathcal{F}(U)$, for some open subset U with $U \cap W \neq \emptyset$, we write s_W for the image of s in $\mathcal{F}_W$.

REMARK 2.1.14. Note that if $\phi \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, then for every irreducible closed subset $W \subseteq X$, we have an induced morphism $\phi_W \colon \mathcal{F}_W \to \mathcal{G}_W$, that maps (U, s) to $\big(U, \phi(s)\big)$. We thus obtain a functor from the category of sheaves on X with values in $\mathcal{C}$ to $\mathcal{C}$.

REMARK 2.1.15. If $\mathcal{F}$ is a sheaf on X and $s, t \in \mathcal{F}(U)$ are such that $s_x = t_x$ for every $x \in U$, then $s = t$.

DEFINITION 2.1.16. Let $\mathcal{F}$ be a presheaf of R-modules or R-algebras on a topological space X. A *subpresheaf* of $\mathcal{F}$ is a presheaf $\mathcal{G}$ such that for every open subset U of X, $\mathcal{G}(U)$ is a submodule (respectively, an R-subalgebra) of $\mathcal{F}(U)$ and such that the restriction maps for $\mathcal{G}$ are induced by those for $\mathcal{F}$. In this case we write $\mathcal{F} \subseteq \mathcal{G}$. It is clear that in this case the inclusion maps define a morphism of presheaves $\mathcal{G} \to \mathcal{F}$. If both $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we say that $\mathcal{G}$ is a *subsheaf* of $\mathcal{F}$.

EXAMPLE 2.1.17. If X is a $\mathcal{C}^\infty$-manifold, then $\mathcal{C}^\infty_{X,\mathbf{R}}$ is a subsheaf of $\mathcal{C}_{X,\mathbf{R}}$.

DEFINITION 2.1.18. Let $\mathcal{C}$ be a category. If $f \colon X \to Y$ is a continuous map between two topological spaces and $\mathcal{F}$ is a presheaf on X of objects in $\mathcal{C}$, then we define the presheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}\big(f^{-1}(U)\big),$$

with the restriction maps being induced by those of $\mathcal{F}$. Moreover, if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, we clearly get a morphism $f_*\mathcal{F} \to f_*\mathcal{G}$ of presheaves on Y, so that we have the *push-forward* functor from the category of presheaves on X to the category of presheaves on Y. It is easy to see that if $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is a sheaf on Y.

EXAMPLE 2.1.19. If $f\colon X \to Y$ is a continuous map between topological spaces, then we have a morphism of sheaves

$$\mathcal{C}_{Y,\mathbf{R}} \to f_*\mathcal{C}_{X,\mathbf{R}}, \quad \mathcal{C}_{Y,\mathbf{R}}(U) \ni \phi \to \phi \circ f \in \mathcal{C}_{X,\mathbf{R}}\big(f^{-1}(U)\big).$$

The following exercises illustrate the advantages of working with sheaves, as opposed to presheaves.

EXERCISE 2.1.20. Show that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent:

i) The morphism ϕ is an isomorphism.
ii) There is an open cover $X = \bigcup_i U_i$ such that $\phi|_{U_i}$ is an isomorphism for all i.
iii) For every $x \in X$, the induced morphism ϕ_x is an isomorphism.

EXERCISE 2.1.21. Let $\mathcal{F}$ be a sheaf and $\mathcal{F}_1$ and $\mathcal{F}_2$ be subsheaves of $\mathcal{F}$.

i) Show that if there is an open cover $X = \bigcup_{i\in I} U_i$ such that $\mathcal{F}_1|_{U_i} \subseteq \mathcal{F}_2|_{U_i}$ for every i, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.
ii) Show that if $\mathcal{F}_{1,x} \subseteq \mathcal{F}_{2,x}$ for every $x \in X$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

EXERCISE 2.1.22. (Gluing morphisms of sheaves) Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X (of objects in some subcategory $\mathcal{C}$ that satisfies our usual requirements). If we have an open cover $X = \bigcup_{i\in I} U_i$ and for every $i \in I$ we have a morphism of sheaves $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ such that for every $i, j \in I$ we have $\phi_i|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j}$, then there is a unique morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$ for all $i \in I$.

EXERCISE 2.1.23. (Gluing sheaves). Let X be a topological space and suppose that $X = \bigcup_{i\in I} U_i$ is an open cover. Suppose that for every $i \in I$ we have a sheaf $\mathcal{F}_i$ on U_i (of objects in some subcategory $\mathcal{C}$ of the category of sets) and for every $i, j \in I$ we have isomorphisms

$$\phi_{j,i}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}, \quad \text{where} \quad U_{ij} = U_i \cap U_j$$

that satisfy the following compatibility conditions:

i) We have $\phi_{i,i} = \mathrm{Id}_{\mathcal{F}|_{U_i}}$ for every $i \in I$, and
ii) We have

$$\phi_{k,j}|_{U_{ijk}} \circ \phi_{j,i}|_{U_{ijk}} = \phi_{k,i}|_{U_{ijk}} \quad \text{for all} \quad i, j, k \in I,$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. In this case there is a sheaf $\mathcal{F}$ on X with isomorphisms $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ for all $i \in I$, such that

$$\phi_{j,i} \circ \phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \quad \text{for all} \quad i, j \in I. \tag{2.1.1}$$

Moreover, if $\mathcal{G}$ is another such sheaf, with isomorphisms $\psi_i\colon \mathcal{G} \to \mathcal{F}|_{U_i}$ for every $i \in I$ that satisfy the compatibility conditions (2.1.1), then there is a unique morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ such that $\psi_i \circ \alpha|_{U_i} = \phi_i$ for all $i \in I$.

2.2. Prevarieties

Let k be a fixed algebraically closed field. Given a topological space X and an open subset U of X, we consider the k-algebra $\mathcal{F}un_X(U)$ of functions $U \to k$, with point-wise operations. It is clear that this gives a sheaf $\mathcal{F}un_X$ of k-algebras on X, with the restriction maps being induced by restriction of functions. Note that if $f\colon X \to Y$ is a continuous map of topological spaces, then we have a canonical morphism of sheaves

$$\mathcal{F}un_Y \to f_*\mathcal{F}un_X, \quad \text{mapping} \quad \mathcal{F}un_Y(U) \ni \phi \to \phi \circ f \in \mathcal{F}un_X\big(f^{-1}(U)\big).$$

We begin by defining a category $\mathcal{T}op_k$ of topological spaces endowed with a sheaf of k-algebras, whose sections are functions on the given topological space. More precisely, the objects of this category are pairs $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ a sheaf of k-algebras on X which is a subsheaf of $\mathcal{F}un_X$. The sheaf $\mathcal{O}_X$ is the *structure sheaf*. A morphism in this category $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f\colon X \to Y$ such that the morphism of sheaves $\mathcal{F}un_Y \to f_*\mathcal{F}un_X$ induces a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$; in other words, for every open subset U of Y and every $\phi \in \mathcal{O}_Y(U)$, we have $\phi \circ f \in \mathcal{O}_X\big(\phi^{-1}(U)\big)$. It is clear that composition of continuous maps induces a composition of morphisms that makes $\mathcal{T}op_k$ a category.

EXAMPLE 2.2.1. Let $(X, \mathcal{O}_X)$ be an object in $\mathcal{T}op_k$. If U is an open subset of X, then we obtain another object $(U, \mathcal{O}_U)$ in $\mathcal{T}op_k$, where $\mathcal{O}_U = \mathcal{O}_X|_U$. Note that the inclusion map induces a morphism $(U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ in $\mathcal{T}op_k$.

REMARK 2.2.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two objects in $\mathcal{T}op_k$. If $X = \bigcup_{i \in I} U_i$ is an open cover and $\alpha_i\colon U_i \to X$ is the inclusion map, then a map $f\colon X \to Y$ is a morphism if and only if each $f \circ \alpha_i$ is a morphism. Indeed, this follows from the fact that continuity is a local property and the fact that $\mathcal{O}_X$ is a sheaf.

EXAMPLE 2.2.3. An isomorphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathcal{T}op_k$ is a homeomorphism $f\colon X \to Y$ such that for every open subset U of Y and every $\phi\colon U \to k$, we have $\phi \in \mathcal{O}_Y(U)$ if and only if $\phi \circ f \in \mathcal{O}_X\big(f^{-1}(U)\big)$.

EXAMPLE 2.2.4. If X is a locally closed subset of some $\mathbf{A}^n$, then $(X, \mathcal{O}_X)$ is an object in $\mathcal{T}op_k$. Note that if U is an open subset of X, then $\mathcal{O}_U = \mathcal{O}_X|_U$.

EXAMPLE 2.2.5. If X and Y are locally closed subsets of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, then a morphism $f\colon X \to Y$ as defined in Chapter 1 is the same as a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathcal{T}op_k$. Indeed, we know that if $f\colon X \to Y$ is a morphism, then f is continuous and for every open subset U in Y and every regular function $\phi\colon U \to k$, the composition $\phi \circ f$ is regular (see Propositions 1.4.13 and 1.4.14). Conversely, if $f\colon X \to Y$ gives a morphism in $\mathcal{T}op_k$ and if $p_i\colon Y \to k$ is induced by the i^{th} projection $\mathbf{A}^n \to k$, then it follows from definition that $p_i \circ f$ is a regular function on X for every i; therefore f is a morphism as defined in Chapter 1.

We enlarge one more time our notion of affine variety, as follows.

DEFINITION 2.2.6. We say that an object $(X, \mathcal{O}_X)$ in $\mathcal{T}op_k$ is an *affine variety* if it is isomorphic to $(V, \mathcal{O}_V)$ for some closed subset of an affine space $\mathbf{A}^n$. We say that $(X, \mathcal{O}_X)$ is a *quasi-affine* variety if it is isomorphic to $(V, \mathcal{O}_V)$ for some locally closed subspace of an affine space $\mathbf{A}^n$.

DEFINITION 2.2.7. An *algebraic prevariety* over k (or simply *prevariety*) is a pair $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ of subsheaf of k-algebras of $\mathcal{F}un_X$, such that there is a finite open covering $X = \bigcup_{i=1}^r U_i$, with each $(U_i, \mathcal{O}_{U_i})$ being an affine variety.

EXAMPLE 2.2.8. A quasi-affine variety $(V, \mathcal{O}_V)$ is a prevariety. Indeed, we may assume that V is a locally closed subset of some $\mathbf{A}^n$ and we know that there is a finite cover by open subsets $V = V_1, \ldots, V_r$ such that each $(V_i, \mathcal{O}_{V_i})$ is isomorphic to an affine variety (see Remark 1.4.20).

NOTATION 2.2.9. By an abuse of notation, we often denote a prevariety $(X, \mathcal{O}_X)$ simply by X.

DEFINITION 2.2.10. The category of algebraic prevarieties over k is a full subcategory of $\mathcal{T}op_k$. In other words, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are prevarieties, then a morphism of prevarieties $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ such that for every open subset U of Y and every $\phi \in \mathcal{O}_Y(U)$, we have $\phi \circ f \in \mathcal{O}_X\big(f^{-1}(U)\big)$.

REMARK 2.2.11. While strictly speaking we have enlarged our notion of quasi-affine varieties, in fact our old category of quasi-affine varieties and the new one are equivalent.

PROPOSITION 2.2.12. *Every prevariety X is a Noetherian topological space. In particular, it is quasi-compact.*

PROOF. By assumption, we have a finite open cover $X = U_1 \cup \ldots \cup U_r$, such that each U_i is Noetherian. Given a sequence

$$F_1 \supseteq F_2 \supseteq \ldots$$

of closed subsets of X, for every i, we can find n_i such that $F_n \cap U_i = F_{n+1} \cap U_i$ for all $n \geq n_i$. Therefore we have $F_n = F_{n+1}$ for every $n \geq \max_i n_i$, and we thus see that X is Noetherian. □

REMARK 2.2.13. For every prevariety $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ is a subsheaf of $\mathcal{C}_X$, where $\mathcal{C}_X(U)$ is the k-algebra of continuous functions $U \to k$. Indeed, this assertion can be checked locally, and thus follows from the fact that it holds on affine varieties.

REMARK 2.2.14. For every prevariety X, the affine open subsets of X give a basis for the topology of X. Indeed, this follows from the definition of a prevariety and the fact that the assertions holds if X is affine.

REMARK 2.2.15. If $(X, \mathcal{O}_X)$ is a prevariety and $\phi \in \mathcal{O}_X(U)$, for some open subset U of X, then the set

$$V := \{x \in U \mid \phi(x) \neq 0\}$$

is an open subset of X and the function $\frac{1}{\phi}$ lies in $\mathcal{O}_X(V)$. Indeed, this follows from the fact that the assertion holds on affine varieties.

REMARK 2.2.16. If X is a prevariety and W is an irreducible closed subset of X, then we can define $\mathcal{O}_{X,W}$ as in Chapter 1. This is, in fact, equal to the stalk of $\mathcal{O}_X$ at W. If U is an affine open subset with $U \cap W \neq \emptyset$ and $\mathfrak{p} \subseteq \mathcal{O}_X(U)$ is the corresponding ideal, then we have canonical isomorphisms

$$\mathcal{O}_{X,W} \simeq \mathcal{O}_{U,U\cap W} \simeq \mathcal{O}_X(U)_{\mathfrak{p}}.$$

We know that the functor mapping X to $\mathcal{O}(X)$ gives an equivalence of categories between the category of affine varieties over k and the category of reduced, finite type k-algebras. The following exercise gives an explicit construction of the inverse functor. This point of view is useful in several instances, for example when discussing the Proj construction.

EXERCISE 2.2.17. Recall that if R is any commutative ring, then we have the maximal spectrum $\mathrm{MaxSpec}(R)$, a topological space with the underlying space consisting of all maximal ideals in R (see Exercise 1.1.16). Suppose now that R is an algebra of finite type over an algebraically closed field k. Recall that in this case, for every $\mathfrak{m} \in \mathrm{MaxSpec}(R)$, the canonical homomorphism $k \to R/\mathfrak{m}$ is an isomorphism. For every open subset U of $\mathrm{MaxSpec}(R)$, let $\mathcal{O}(U)$ be the set of functions $s\colon U \to k$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and $a, b \in R$ such that for every $\mathfrak{m} \in U_x$, we have

$$b \not\in \mathfrak{m} \quad \text{and} \quad s(\mathfrak{m}) = \overline{a} \cdot \overline{b}^{-1},$$

where we denote by $\overline{u} \in k \simeq R/\mathfrak{m}$ the class of $u \in R$.

1) Show that $\mathcal{O}$ is a sheaf such that the pair $\big(\mathrm{MaxSpec}(R), \mathcal{O}\big)$ defines an element in $\mathcal{T}op_k$ that, by an abuse of notation, we denote by $\mathrm{MaxSpec}(R)$, too.
2) Show that given a homomorphism of reduced, finite type k-algebras $R \to S$, we have an induced morphism $\mathrm{MaxSpec}(S) \to \mathrm{MaxSpec}(R)$ in $\mathcal{T}op_k$, so that we get a functor from the category of reduced, finite type k-algebras to $\mathcal{T}op_k$.
3) Show that for every R as above, $\mathrm{MaxSpec}(R)$ is an affine variety. Moreover, the functor MaxSpec is an inverse of the functor from the category of affine varieties to the category of reduced, finite type k-algebras, that maps X to $\mathcal{O}(X)$.

2.3. Open and closed immersions

DEFINITION 2.3.1. Let $(X, \mathcal{O}_X)$ be an object in $\mathcal{T}op_k$. If Z is a locally closed subset of X, then we define a subsheaf $\mathcal{O}_Z$ of $\mathcal{C}_Z$, as follows. Given an open subset U of Z, a function $\phi\colon U \to k$ lies in $\mathcal{O}_Z(U)$ if for every $x \in U$, there is an open neighborhood V of x in X and $\psi \in \mathcal{O}_X(V)$ such that $\phi(u) = \psi(u)$ for $u \in V \cap X \subseteq U$. It is clear that restriction of functions makes $\mathcal{O}_Z$ a presheaf of k-algebras. Moreover, since the condition in the definition is local, $\mathcal{O}_Z$ is a sheaf, hence $(Z, \mathcal{O}_Z)$ is an object in $\mathcal{T}op_k$.

REMARK 2.3.2. If X and Z are as in the above definition and Y is a locally closed subset of Z, then it follows from the definition that the sheaves on Y defined from $(X, \mathcal{O}_X)$ and from $(Z, \mathcal{O}_Z)$ are equal.

EXAMPLE 2.3.3. If Z is open in X, then the sheaf $\mathcal{O}_Z$ defined above is just $\mathcal{O}_X|_Z$.

EXAMPLE 2.3.4. If X is a locally closed subset in $\mathbf{A}^n$, then the sheaf $\mathcal{O}_X$ on X defined from $(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n})$ is the sheaf of regular functions on X. This is an immediate consequence of the definition of regular functions on locally closed subsets of $\mathbf{A}^n$.

PROPOSITION 2.3.5. *For every prevariety $(X, \mathcal{O}_X)$ and every locally closed subset Z of X, the pair $(Z, \mathcal{O}_Z)$ is a prevariety.*

Proof. Note that by assumption, we have an open cover $X = V_1 \cup \ldots \cup V_r$ such that each $(V_i, \mathcal{O}_{V_i})$ is an affine variety. Since it is enough to show that each $(V_i \cap Z, \mathcal{O}_Z|_{V_i \cap Z})$ is a prevariety and $\mathcal{O}_Z|_{V_i\cap Z}$ is the sheaf defined on $Z \cap V_i$ as a locally closed subset of V_i (see Remark 2.3.2), it follows that we may and will assume that X is a closed subset of $\mathbf{A}^n$ and $\mathcal{O}_X$ is the sheaf of regular functions on X. In this case, it follows from Example 2.3.4 that Z is a quasi-affine variety, hence a prevariety by Example 2.2.8. $\square$

Definition 2.3.6. A locally closed *subvariety* of a prevariety $(X, \mathcal{O}_X)$ is a prevariety $(Z, \mathcal{O}_Z)$, where Z is a locally closed subset of X and $\mathcal{O}_Z$ is the sheaf defined in Definition 2.3.1. By the above proposition, this is indeed a prevariety. If Z is in fact open or closed in X, we say that we have an *open subvariety*, respectively, *closed subvariety* of X.

Definition 2.3.7. Note that if Z is a locally closed subvariety of X, then the inclusion map $i\colon Z \to X$ is a morphism of prevarieties. A morphism of prevarieties $f\colon X \to Y$ is a *locally closed* (*open*, *closed*) *immersion* (or *embedding*) if it factors as

$$X \xrightarrow{g} Z \xrightarrow{i} Y,$$

where g is an isomorphism and i is the inclusion of a locally closed (respectively, open, closed) subvariety.

Proposition 2.3.8. *If $f\colon X \to Y$ is a locally closed immersion, then for every map $g\colon W \to Y$, there is a morphism $h\colon W \to X$ such that $g = f \circ h$ if and only if $g(W) \subseteq f(X)$. Moreover, in this case h is unique.*

Proof. It is clear that if we have such h, then $g(W) = f\big(h(W)\big) \subseteq f(X)$, hence it is enough to prove the converse. Moreover, since we may replace X by an isomorphic variety, we may assume that f is the inclusion of a locally closed subvariety. Since f is injective, it is clear that if $g(W) \subseteq f(X)$, then there is a unique map $h\colon W \to X$ such that $f \circ h = g$. We need to prove that h is a morphism. Note first that since X is a subspace of Y, the map h is continuous. Furthermore, if $Y = V_1 \cup \ldots \cup V_r$ is an open cover such that each V_i is affine, in order to show that h is a morphism it is enough to show that each induced map $h^{-1}\big(f^{-1}(V_i)\big) \to f^{-1}(V)$ is a morphism (see Remark 2.2.2). Therefore we may assume that Y is an affine variety, in which case the assertion is clear. $\square$

Proposition 2.3.9. *If $f\colon X \to Y$ is a morphism of prevarieties, then the following are equivalent:*

i) *The morphism f is a closed immersion.*
ii) *For every affine open subset U of Y, its inverse image $f^{-1}(U)$ is affine, and the induced k-algebra homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X\big(f^{-1}(U)\big)$ is surjective.*
iii) *There is a finite cover $Y = U_1 \cup \ldots \cup U_r$ by affine open subsets such that for every i, the inverse image $f^{-1}(U_i)$ is affine, and the induced k-algebra homomorphism $\mathcal{O}_Y(U_i) \to \mathcal{O}_X\big(f^{-1}(U_i)\big)$ is surjective.*

Proof. We first prove the implication i)$\Rightarrow$ii). Suppose that f factors as

$$X \xrightarrow{g} Z \xrightarrow{i} Y,$$

with g an isomorphism and i the inclusion map of a closed subvariety. If $U \subseteq Y$ is an affine open subset, then $U \cap Z$ is a closed subvariety of an affine variety, hence it is affine, and the restriction map induces a surjection $\mathcal{O}(U) \to \mathcal{O}(U \cap Z)$. Since the induced morphism $f^{-1}(U) \to U \cap Z$ is an isomorphism, we obtain the assertion in i).

Since the implication ii)⇒iii) is trivial, in order to complete the proof it is enough to show iii)⇒i). With the notation in iii), we see that each induced morphism $f^{-1}(U_i) \to U_i$ is a closed immersion. In particular, it is a homeomorphism onto its image, which is a closed subset of Y. This easily implies that f is a homeomorphism onto its image, which is a closed subset of Y. Let Z be the closed subvariety of Y with underlying set $f(X)$. We need to show that the inverse map $\phi\colon Z \to X$ is a morphism. Since $X = \bigcup_i f^{-1}(U_i)$, it follows from Remark 2.2.2 that it is enough to check that each $\phi^{-1}\big(f^{-1}(U_i)\big) \to f^{-1}(U_i)$ is a morphism. This is clear, since $f^{-1}(U_i) \to U_i$ is a closed immersion. □

Remark 2.3.10. A morphism $f\colon X \to Y$ is a locally closed immersion if and only if there is an open subset U of Y such that f factors as $X \xrightarrow{g} U \xrightarrow{j} Y$, with g a closed immersion and j the inclusion morphism.

One way to construct algebraic prevarieties is by glueing. This is the content of the next exercise.

Exercise 2.3.11. Let $X_1, \ldots, X_r$ be prevarieties and for every i and j, suppose that we have open subvarieties $U_{i,j} \subseteq X_i$ and isomorphisms $\phi_{i,j}\colon U_{i,j} \to U_{j,i}$ such that

i) We have $U_{i,i} = X_i$ and $\phi_{i,i} = \mathrm{Id}_{X_i}$ for every i, and
ii) $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ on $U_{i,j} \cap \phi_{i,j}^{-1}(U_{j,k}) \subseteq U_{i,k}$.

In this case, there is a prevariety X and an open cover $X = U_1 \cup \ldots \cup U_r$ and isomorphisms $f_i\colon U_i \to X_i$ such that for every i and j, we have

$$U_i \cap U_j = f_i^{-1}(U_{i,j}) \quad \text{and} \quad \phi_{i,j} \circ f_i = f_j \quad \text{on} \quad U_i \cap U_j.$$

Moreover, if Y is another such prevariety with an open cover $Y = V_1 \cup \ldots \cup V_r$ and isomorphisms $g_i\colon V_i \to X_i$ that satisfy the same compatibility condition, then there is a unique isomorphism $h\colon X \to Y$ such that $h(U_i) = V_i$ and $g_i \circ h = f_i$ for $1 \leq i \leq r$.

Example 2.3.12. Let X and Y be two copies of $\mathbf{A}^1$ and let $U \subseteq X$ and $V \subseteq Y$ be the complement of the origin. We can apply the previous exercise to construct a prevariety W_1 by glueing X and Y along the isomorphism $U \to V$ given by the identity. This prevariety is *the affine line with the origin doubled*. On the other hand, we can glue X and Y along the isomorphism $U \to V$ corresponding to the k-algebra isomorphism

$$k[x, x^{-1}] \to k[x, x^{-1}], \quad x \to x^{-1}.$$

As we will see in Chapter 4, the resulting prevariety is the projective line $\mathbf{P}^1$.

Exercise 2.3.13. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are locally closed (respectively open, closed) immersions, then $g \circ f$ is a locally closed (respectively open, closed) immersion.

We end this section by extending to arbitrary prevarieties some properties that we proved for affine varieties. We then apply these properties to prove a sufficient criterion for a variety to be affine.

Proposition 2.3.14. *For every prevarieties X and Y, with X affine, the map*

$$\mathrm{Hom}(Y,X) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(\mathcal{O}_X(X), \mathcal{O}_Y(Y)\big)$$

that maps f to the homomorphism taking ϕ to $\phi \circ f$ is a bijection.

Proof. Recall that we know this result if Y is affine, too (see Theorem 1.4.16). We denote the map in the proposition by α_Y. We first show that α_Y is injective for all Y. Suppose that $f, g\colon Y \to X$ are morphisms such that $\alpha_Y(f) = \alpha_Y(g)$. Consider an affine open cover $Y = \bigcup_{i=1}^r U_i$. For every i, the composition

$$\mathcal{O}_X(X) \xrightarrow{\alpha_Y(f)} \mathcal{O}_Y(Y) \xrightarrow{\beta_i} \mathcal{O}_Y(U_i),$$

where β_i is given by restriction of functions, is equal to $\alpha_{U_i}(f|_{U_i})$. A similar assertion holds for g. Our assumption of f and g thus gives

$$\alpha_{U_i}(f|_{U_i}) = \alpha_{U_i}(g|_{U_i})$$

for all i, and since the U_i are affine, we conclude that $f|_{U_i} = g|_{U_i}$. This implies that $f = g$, completing the proof of injectivity.

We now prove the surjectivity of α_Y for every Y. Let $\phi\colon \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ be a k-algebra homomorphism. We consider again the affine open cover $Y = \bigcup_{i=1}^r U_i$ and consider $\phi_i = \beta_i \circ \phi$. Since each U_i is affine, there are morphisms $f_i\colon U_i \to X$ such that $\alpha_{U_i}(f_i) = \phi_i$ for all i.

Claim. For every i and j, we have $f_i|_{U_{i,j}} = f_j|_{U_{i,j}}$, where $U_{i,j} = U_i \cap U_j$. Indeed, $\alpha_{U_{i,j}}(f_i|_{U_{i,j}})$ is equal to the composition

$$\mathcal{O}_X(X) \xrightarrow{\phi} \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_Y(U_{i,j}),$$

where the second map is given by restriction of functions, and the same holds for $\alpha_{U_{i,j}}(f_j|_{U_{i,j}})$. Since we already know that $\alpha_{U_{i,j}}$ is injective, we obtain the assertion in the claim.

We deduce from the claim that we have a morphism $f\colon Y \to X$ such that $f|_{U_i} = f_i$ for all i. This implies that $\alpha_Y(f) = \phi$: indeed, since the morphism

$$\mathcal{O}_Y(Y) \to \prod_{i=1}^r \mathcal{O}_Y(U_i)$$

is injective, it is enough to note that

$$\beta_i \circ \phi = \phi_i = \alpha_Y(f_i) = \beta_i \circ \alpha_Y(f)$$

for all i. This completes the proof of the proposition. $\square$

Proposition 2.3.15. *Let X be a prevariety and $f \in \Gamma(X, \mathcal{O}_X)$. If*

$$D_X(f) = \{x \in X \mid f(x) \neq 0\},$$

then the restriction map

$$\Gamma(X, \mathcal{O}_X) \to \Gamma\big(D_X(f), \mathcal{O}_X\big)$$

induces a k-algebra isomorphism

$$\Gamma(X, \mathcal{O}_X)_f \simeq \Gamma(D_X(f), \mathcal{O}_X).$$

PROOF. Since $f(x) \neq 0$ for every $x \in D_X(f)$, it follows that $f|_{D_X(f)}$ is invertible (see Remark 2.2.15). By the universal property of localization, we see that the restriction map induces a k-algebra homomorphism

$$\tau_{X,f} \colon \Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma\big(D_X(f), \mathcal{O}_X\big).$$

We will show that this is an isomorphism. Recall that we know this when X is affine (see Proposition 1.4.7).

Consider an affine open cover $X = U_1 \cup \ldots \cup U_r$. Since $\mathcal{O}_X$ is a sheaf, we have exact sequences of $\Gamma(X, \mathcal{O}_X)$-modules

$$0 \to \Gamma(X, \mathcal{O}_X) \to \bigoplus_i \Gamma(U_i, \mathcal{O}_X) \to \bigoplus_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X)$$

and

$$0 \to \Gamma\big(D_X(f), \mathcal{O}_X\big) \to \bigoplus_i \Gamma\big(U_i \cap D_X(f), \mathcal{O}_X\big) \to \bigoplus_{i,j} \Gamma\big(U_i \cap U_j \cap D_X(f), \mathcal{O}_X\big).$$

By localizing the first sequence at f, we obtain again an exact sequence, and we thus get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{O}_X)_f \arrow[r] \arrow[d, "\tau_{X,f}"] & \prod_i \Gamma(U_i, \mathcal{O}_X)_f \arrow[r] \arrow[d, "\gamma"] & \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X)_f \arrow[d, "\delta"] \\
0 \arrow[r] & \Gamma\big(D_X(f), \mathcal{O}_X\big) \arrow[r] & \prod_i \Gamma\big(D_X(f) \cap U_i, \mathcal{O}_X\big) \arrow[r] & \prod_{i,j} \Gamma\big(D_X(f) \cap U_i \cap U_j, \mathcal{O}_X\big)
\end{tikzcd}$$

with exact rows, where

$$\gamma = (\tau_{U_i, f|_{U_i}})_i \quad \text{and} \quad \delta = (\tau_{U_i \cap U_j}, f|_{U_i \cap U_j})_{i,j}.$$

Note that since each U_i is affine, we know that γ is an isomorphism. This implies that $\tau_{X,f}$ is injective. Since this holds for all (X, f), applying the assertion for $(U_i \cap U_j, f|_{U_i \cap U_j})$, we conclude that δ is injective. An easy diagram chase then implies that $\tau_{X,f}$ is surjective. This completes the proof of the proposition. □

PROPOSITION 2.3.16. *Let X be a prevariety and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that the ideal they generate is $\Gamma(X, \mathcal{O}_X)$. If $D_X(f_i)$ is an affine variety for every i, then X is an affine variety.*

PROOF. We put $R = \Gamma(X, \mathcal{O}_X)$. This is clearly a reduced k-algebra. By assumption, we can write

$$\sum_{i=1}^{r} f_i g_i = 1 \quad \text{for some} \quad g_1, \ldots, g_r \in R.$$

We begin by showing that R is a finitely generated k-algebra. Since each $D_X(f_i)$ is affine, we know that $\Gamma\big(D_X(f_i), \mathcal{O}_X\big)$ is a finitely generated k-algebra. By Proposition 2.3.15, we have a canonical isomorphism

$$R_{f_i} \simeq \Gamma\big(D_X(f_i), \mathcal{O}_X\big),$$

hence each R_{f_i} is a finitely generated k-algebra. For each R_{f_i}, we choose finitely many generators of the form $\frac{a_{i,j}}{f_i^{m_{i,j}}}$, for suitable $a_{i,j} \in R$ and $m_{i,j} \in \mathbf{Z}_{\geq 0}$. Let $S \subseteq R$ be the k-algebra generated by the $a_{i,j}$, by the f_i, and by the g_i. It follows that S is a finitely generated k-algebra, with $f_1, \ldots, f_r \in S$, such that they generate the unit ideal in S. Moreover, we have $S_{f_i} = R_{f_i}$ for all i. This implies that if

M is the S-module R/S, we have $M_{f_i} = 0$ for all i, and therefore $M = 0$ (see Proposition C.3.1). Therefore $R = S$, hence R is a finitely generated k-algebra.

Recall that we have the functor MaxSpec on the category of reduced, finitely generated k-algebras, with values in the category of affine varieties that is the inverse of the functor that maps Y to $\Gamma(Y, \mathcal{O}_Y)$ (for what follows, the choice of an inverse functor does not actually play a role). Since R is finitely generated, it follows from Proposition 2.3.14 that we have a canonical morphism $p_X \colon X \to \mathrm{MaxSpec}(R)$ such that the induced k-algebra homomorphism

$$R \simeq \Gamma\big(\mathrm{MaxSpec}(R), \mathcal{O}_{\mathrm{MaxSpec}(R)}\big) \to \Gamma(X, \mathcal{O}_X)$$

is the identity. We show that p_X is an isomorphism.

In fact, it is easy to see explicitly what the map p_X: for every $x \in X$, we have

$$p_X(x) = \{\phi \in R \mid \phi(x) = 0\}.$$

This follows from the fact that the bijection in Proposition 2.3.14 is functorial, applied to the inclusion $\{x\} \hookrightarrow X$. The elements $f_i \in R$ define open subsets

$$U_i = \{\mathfrak{m} \in \mathrm{MaxSpec}(R) \mid f_i \not\in \mathfrak{m}\}$$

and since $f_1, \ldots, f_r$ generate the unit ideal in R, it follows that $\mathrm{MaxSpec}(R) = \bigcup_i U_i$. On the other hand, it follows from the description of p_X that $p_X^{-1}(U_i) = D_X(f_i)$ and via the isomorphism $R_{f_i} \simeq \Gamma\big(D_X(f_i), \mathcal{O}_X\big)$ provided by Proposition 2.3.15, the induced map $p_X^{-1}(U_i) \to U_i$ gets identified to

$$p_{D_X(f_i)} \colon D_X(f_i) \to \mathrm{MaxSpec}\big(\Gamma(D_X(f_i), \mathcal{O}_{D_X(f_i)})\big),$$

which is an isomorphism since $D_X(f_i)$ is affine. Since each induced morphism $p_X^{-1}(U_i) \to U_i$ is an isomorphism, it follows that p_X is an isomorphism, hence X is affine. $\square$

2.4. Products of prevarieties

We now show that the category of prevarieties has fibered products. We begin with the case of direct products.

Proposition 2.4.1. *The category of prevarieties over k has direct products.*

Proof. We show that given two prevarieties X and Y, there is a topology on the set $X \times Y$ and a subsheaf of k-algebras $\mathcal{O}_{X\times Y} \subseteq \mathcal{F}un_{X\times Y}$ that make $X \times Y$, together with the two projections, the direct product in the category of prevarieties. Let us consider open covers $X = U_1 \cup \ldots \cup U_r$ and $Y = V_1 \cup \ldots \cup V_s$, with all U_i and V_j affine varieties. We can thus write

$$X \times Y = \bigcup_{i,j} U_i \times V_j.$$

Note that each $U_i \times V_j$ has the structure of an affine variety; in particular, it is a topological space, with a topology that is finer than the product topology (see Corollary 1.6.2). Note that for every two pairs (i_1, j_1) and (i_2, j_2), we have a priori two structures of algebraic prevariety on

$$(U_{i_1} \times V_{j_1}) \cap (U_{i_2} \times V_{j_2}), \tag{2.4.1}$$

one coming from $U_{i_1} \times V_{j_1}$ and the other one from $U_{i_2} \times V_{j_2}$. However, they are the same, both being equal to the structure of prevariety on the quasi-affine variety $(U_{i_1} \cap U_{i_2}) \times (V_{j_1} \cap V_{j_2})$. This follows from the fact that if A and B are affine (or, more generally, quasi-affine) varieties and if $U_A \subseteq A$ and $U_B \subseteq B$ are open subsets, then the open subvariety $U_A \times U_B$ of $A \times B$ is the product of U_A and U_B in the category of quasi-affine varieties, which characterizes it uniquely, up to a canonical isomorphism.

It is then easy to see that if we declare that a subset W of $X \times Y$ is open if and only if $W \cap (U_i \times V_j)$ is open for all i and j, then this gives a topology on $X \times Y$ such that the topology on each $U_i \times V_j$ is the subspace topology. Note that the topology on $X \times Y$ is finer than the product topology. Moreover, if given an open subset $W \subseteq X \times Y$ and a function $\phi\colon W \to k$, we put $\phi \in \mathcal{O}_{X\times Y}(W)$ when

$$\phi|_{W\cap(U_i\times V_j)} \in \mathcal{O}_{U_i\times V_j}\big(W \cap (U_i \times V_j)\big) \quad \text{for all} \quad i, j,$$

then $\mathcal{O}_{X\times Y}$ is a subsheaf of $\mathcal{F}un_{X\times Y}$ such that $\mathcal{O}_{X\times Y}|_{U_i\times V_j} = \mathcal{O}_{U_i\times V_j}$ for all i and j.

We now show that with this structure, the two projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ make $X \times Y$ the direct product of X and Y in the category of prevarieties. Note first that since $X \times Y$ is covered by the affine open subsets $U_i \times V_j$, it follows that $X \times Y$ is a prevariety. Second, both projections p and q are morphisms: for example, for p this follows from the fact that each projection $U_i \times V_j \to U_i$ is a morphism (see Remark 2.2.2). Given a prevariety Z and morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, there is a unique map $h\colon Z \to X \times Y$ such that $p \circ h = f$ and $q \circ h = g$, namely $h(z) = \big(f(z), g(z)\big)$ for every $z \in Z$. In order to check that this is a morphism, note first that for every i and j, the subset

$$h^{-1}(U_i \times V_j) = f^{-1}(U_i) \cap g^{-1}(V_j)$$

is open in Z. Moreover, the restriction of h to this subset is a morphism: by Remark 2.2.2, in order to check this, it is enough to show that the restriction of h to the subsets in an affine open cover of $h^{-1}(U_i \times V_j)$ is a morphism; this follows from the fact that $U_i \times V_j$ is the direct product of U_i and V_j in the category of affine varieties. This completes the proof of the proposition. □

Remark 2.4.2. It follows from the proof of the proposition that the product of two prevarieties X and Y has as underlying set the Cartesian product $X \times Y$ and the topology is finer than the product topology.

Exercise 2.4.3. Show that if $f\colon Z \to X$ and $g\colon W \to Y$ are locally closed (open, closed) immersions, then we have an induced locally closed (respectively, open, closed) immersion

$$Z \times W \to X \times Y, \quad (z, w) \to \big(f(z), g(w)\big).$$

Remark 2.4.4. If X and Y are irreducible prevarieties, then $X \times Y$ is irreducible.

Proof. Consider affine open covers

$$X = U_1 \cup \ldots \cup U_r \quad \text{and} \quad Y = V_1 \cup \ldots \cup V_s.$$

Since each $U_i \times V_j$ is irreducible by Corollary 1.6.7, it is enough to note that each intersection

$$(U_i \times V_j) \cap (U_{i'} \cap V_{j'}) = (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

is nonempty (see Exercise 1.3.17). □

Definition 2.4.5. Given a morphism of prevarieties $f\colon X \to Y$, the *graph morphism* of f is the morphism $j_f\colon X \to X \times Y$ given by $j_f(x) = \big(x, f(x)\big)$. Note that this is indeed a morphism by the universal property of the product. The *graph* of f is the image Γ_f of j_f. When $f = \mathrm{id}_X$, the graph of f is the *diagonal* Δ_X of $X \times X$.

Proposition 2.4.6. *For every morphism $f\colon X \to Y$, the graph morphism $j_f\colon X \to X \times Y$ is a locally closed embedding.*

Proof. For every $x \in X$, let $V_x \subseteq Y$ be an affine open neighborhood of $f(x)$ and $U_x \subseteq f^{-1}(V_x)$ an affine open neighborhood of x. If $U = \bigcup_{x\in X} U_x \times V_x$, then it is clear that the image of j_f is contained in U. Therefore it is enough to show that the induced morphism $j'_f\colon X \to U$ is a closed immersion. We also note that since U is quasi-compact, the union in the definition of U can be taken over a finite subset of X. Since $(j'_f)^{-1}(U_x \times V_x) = U_x$ is affine, in order to complete the proof of the proposition, it is enough to show that when X and Y are affine, the morphism $j_f^{\#}\colon \mathcal{O}(X \times Y) \to \mathcal{O}(X)$ is surjective. We may assume that X is a closed subset of $\mathbf{A}^m$ and Y is a closed subset of $\mathbf{A}^n$. We denote by $x_1, \ldots, x_m$ the coordinates on $\mathbf{A}^m$ and by $y_1, \ldots, y_n$ the coordinates on $\mathbf{A}^n$. Let us write $f = (f_1, \ldots, f_n)$, with $f_i \in \mathcal{O}(X)$ for $1 \leq i \leq n$. In this case, $j_f^{\#}$ is given by

$$j_f^{\#}(x_i) = x_i \quad \text{and} \quad j_f^{\#}(y_j) = f_j \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

and it is clear that this is surjective. $\square$

We now prove the existence of fibered products in the category of prevarieties.

Proposition 2.4.7. *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be morphisms of prevarieties. If*

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

then W is a locally closed subset of $X \times Y$ and $(W, \mathcal{O}_W)$, with the restrictions of the two projections is the fiber product $X \times_Z Y$ in the category of prevarieties.

Proof. Consider the morphism $h\colon X \times Y \to Z \times Z$ given by $h(x, y) = \big(f(x), g(y)\big)$. It follows from Proposition 2.4.6 that the diagonal $\Delta_Z \subseteq Z \times Z$ is locally closed in $Z \times Z$, hence $W = h^{-1}(\Delta_Z)$ is locally closed in $X \times Y$. We now consider on W the structure of locally closed subvariety of $X \times Y$. Let $p\colon W \to X$ and $q\colon W \to Y$ be the restrictions of the two projections to W. We need to show that given a prevariety T and morphisms $\alpha\colon T \to X$ and $\beta\colon T \to Y$ such that $f\circ\alpha = g\circ\beta$, there is a unique morphism $\gamma\colon T \to W$ such that $p\circ\gamma = \alpha$ and $q\circ\gamma = \beta$. Uniqueness of γ as a map is clear: in fact, we need to have $\gamma(t) = \big(\alpha(t), \beta(t)\big)$ for all $t \in T$. In order to check that this is a morphism, note that the composition $T \to W \hookrightarrow X \times Y$ is a morphism since $X \times Y$ is the direct product of X and Y, and thus γ is a morphism by Proposition 2.3.8. $\square$

EXAMPLE 2.4.8. If $f\colon X \to Y$ is a morphism of prevarieties and Z is a locally closed subset of Y, then we have a Cartesian diagram[1]

$$\begin{tikzcd} f^{-1}(W) \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ W \arrow[r, "i"] & Y, \end{tikzcd}$$

in which i and j are the inclusion morphisms. Indeed, the assertion is an immediate application of Proposition 2.3.8.

REMARK 2.4.9. Given a Cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y, \end{tikzcd}$$

with X, Y, and Z are affine varieties, it follows that $X \times_Y Z$ is affine too: this follows from the fact that it is a closed subvariety of $X \times Y$. Moreover, the canonical homomorphism

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \to \mathcal{O}(X \times_Y Z)$$

is surjective, with the kernel being the nil-radical of $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z)$. This follows from the anti-equivalence of categories between affine varieties over k and reduced, finitely generated k-algebras, by noting that the tensor product gives the push-out in the category of k-algebras, hence the reduced tensor product gives the push-out in the category of reduced k-algebras.

2.5. Algebraic varieties

Algebraic varieties are prevarieties that satisfy an analogue of the Hausdorff condition. Note that the Zariski topology is almost never Hausdorff: if X is an irreducible prevariety, then any two nonempty open subsets intersect nontrivially. The right condition is suggested by the following observation: if X is an arbitrary topological space and if we consider on $X \times X$ the product topology, then X is Hausdorff if and only if the diagonal Δ_X is closed in $X \times X$.

DEFINITION 2.5.1. An algebraic prevariety X is *separated* if the diagonal Δ_X is a closed subset of the prevariety $X \times X$. An *algebraic variety* over k (or simply, a *variety*) is a separated algebraic prevariety.

REMARK 2.5.2. It follows from Proposition 2.3.8 that the diagonal map $X \to X \times X$, given by $x \to (x, x)$ is always a locally closed immersion for every prevariety X. Hence X is separated if and only if this map is a closed immersion.

REMARK 2.5.3. If $f, g\colon X \to Y$ are morphisms of prevarieties and Y is separated, then the set

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in X. Indeed, this is just the inverse image of the diagonal $\Delta_Y \subseteq Y \times Y$ by the morphism $X \to Y \times Y$, that maps x to $\big(f(x), g(x)\big)$. Because of this, the

[1] Recall that this means that it is a commutative diagram such that the induced morphism $f^{-1}(W) \to X \times_Y W$ given by the universal property of the fiber product is an isomorphism.

considerations in Section 1.5 about the domain of rational maps extend to the case of arbitrary algebraic varieties.

PROPOSITION 2.5.4. *The following hold:*

i) *If Z is a subvariety of the prevariety X and X is separated, then Z is separated. In particular, quasi-affine varieties are separated.*
ii) *If $f\colon X \to Y$ is a morphism of prevarieties and Y is separated, then the graph morphism $j_f\colon X \to X \times Y$, given by $j_f(x) = \big(x, f(x)\big)$ is a closed immersion.*
iii) *If X and Y are algebraic varieties, so is $X \times Y$. More generally, if $f\colon X \to Z$ and $g\colon Y \to Z$ are morphisms of varieties, then $X \times_Z Y$ is a closed subvariety of $X \times Y$, and therefore it is a variety.*

PROOF. If Z is a locally closed subvariety of X, then $Z \times Z$ is a locally closed subvariety of $X \times X$ and $\Delta_Z = (Z \times Z) \cap \Delta_X$. It follows that if Δ_X is closed in $X \times X$, then Δ_Z is closed in $Z \times Z$. Note now that if $X = \mathbf{A}^n$, with coordinates $x_1, \ldots, x_n$, then Δ_X is the closed subset of $\mathbf{A}^n \times \mathbf{A}^n$ defined by $x_1 - y_1, \ldots, x_n - y_n$. We thus conclude that every quasi-affine variety is separated.

Under the assumptions in ii), we know that j_f is a locally closed embedding by Proposition 2.4.6. Its image is the inverse image of Δ_Y by the morphism $h\colon X \times Y \to Y \times Y$ given by $h(x, y) = \big(f(x), y\big)$, hence it is closed in $X \times Y$. Therefore i_f is a closed immersion.

Suppose now that X and Y are varieties. If

$$p_{1,3}\colon (X \times Y) \times (X \times Y) \to X \times X \quad \text{and} \quad p_{2,4}\colon (X \times Y) \times (X \times Y) \to Y \times Y$$

are the projections given by

$$p_{1,3}(x_1, y_1, x_2, y_2) = (x_1, x_2) \quad \text{and} \quad p_{2,4}(x_1, y_1, x_2, y_2) = (y_1, y_2),$$

then $\Delta_{X \times Y} = p_{1,3}^{-1}(\Delta_X) \cap p_{2,4}^{-1}(\Delta_Y)$ and it is thus a closed subset of $(X \times Y) \times (X \times Y)$. This shows that $X \times Y$ is a variety. Moreover, it follows from Proposition 2.4.7 that the fiber product $X \times_Z Y$ is a locally closed subvariety of $X \times Y$, hence it is a variety by i). In fact it is a closed subvariety, since its underlying subset is the inverse image of Δ_Z via the morphism

$$X \times Y \to Z \times Z, \quad (x, y) \to \big(f(x), g(y)\big).$$

□

The following property is sometimes useful:

PROPOSITION 2.5.5. *If X is an algebraic variety and U, V are affine open subvarieties of X, then $U \cap V$ is affine, too.*

PROOF. Consider the closed immersion $i\colon X \to X \times X$ given by the diagonal map. If U and V are affine variety, then $U \times V$ is affine. Since $U \cap V = i^{-1}(U \times V)$, we see that $U \cap V$ is affine by Proposition 2.3.9. □

PROPOSITION 2.5.6. *Let X be a prevariety and suppose that we have an open cover $X = U_1 \cup \ldots \cup U_r$ by affine open subsets. Then X is separated if and only if for every i and j, the intersection $U_i \cap U_j$ is an affine variety and the homomorphism $\mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j)$ induced by the restriction homomorphisms is surjective.*

PROOF. We know that X is separated if and only if the diagonal morphism $i\colon X \to X \times X$ is a closed immersion. The assertion in the proposition is now an immediate consequence of the description of closed immersions in Proposition 2.3.9, using the fact that the canonical homomorphism

$$\mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \times_k U_j)$$

is an isomorphism (see Section 1.6). $\square$

EXAMPLE 2.5.7. Let us consider the two examples in Example 2.3.12. If X is obtained by glueing two copies of $\mathbf{A}^1$ along the identity automorphism of $\mathbf{A}^1 \smallsetminus \{0\}$, then X is covered by two affine open subsets U and V such that $U \simeq \mathbf{A}^1 \simeq V$, $U \cap V \simeq \mathbf{A}^1 \smallsetminus \{0\}$, and the morphism

$$k[x,y] = \mathcal{O}(U \times V) \to \mathcal{O}(U \cap V) = k[t, t^{-1}]$$

maps both x and y to t. This is clearly not surjective, hence X is not separated. On the other hand, if Y is obtained by glueing two copies of $\mathbf{A}^1$ along the automorphism of $\mathbf{A}^1 \smallsetminus \{0\}$ given by $t \to t^{-1}$, then Y is also covered by two affine open subsets U and V such that $U \simeq \mathbf{A}^1 \simeq V$, $U \cap V \simeq \mathbf{A}^1 \smallsetminus \{0\}$, but now the morphism

$$k[x,y] = \mathcal{O}(U \times V) \to \mathcal{O}(U \cap V) = k[t, t^{-1}]$$

maps x to t and y to t^{-1}. This is surjective, hence Y is separated.

EXERCISE 2.5.8.
i) Show that if $X_1, \ldots, X_n$ are algebraic varieties, then on the disjoint union $X = \bigsqcup_{i=1}^n X_i$ there is a unique structure of algebraic variety such that each inclusion map $X_i \hookrightarrow X$ is an open immersion .
ii) Show that every variety X is a disjoint union of connected open subvarieties; each of these is a union of irreducible components of X.
iii) Show that if X is an affine variety and $R = \mathcal{O}(X)$, then X is disconnected if and only if there is an isomorphism $R \simeq R_1 \times R_2$ for suitable nonzero k-algebras R_1 and R_2.

EXERCISE 2.5.9. Let $f\colon X \dashrightarrow Y$ be a rational map between the irreducible varieties X and Y. The *graph* Γ_f of f is defined as follows. If U is an open subset of X such that f is defined on U, then the graph of $f|_U$ is well-defined, and it is a closed subset of $U \times Y$. By definition, Γ_f is the closure of the graph of $f|_U$ in $X \times Y$.

i) Show that the definition is independent of the choice of U.
ii) Let $p\colon \Gamma_f \to X$ and $q\colon \Gamma_f \to Y$ be the morphisms induced by the two projections. Show that p is a birational morphism, and that q is birational if and only if f is.
iii) Show that if the fiber $p^{-1}(x)$ does not consist of only one point, then f is not defined at $x \in X$.

CHAPTER 3

Dimension theory

In this chapter we prove the main results concerning the dimension of algebraic varieties. We begin with some general considerations about Krull dimension in topological spaces. We then discuss finite morphisms between affine varieties and show that they are closed maps and preserve the dimension of closed subsets. We then give a proof of the Principal Ideal theorem that relies on Noether normalization and use this to deduce the main properties of dimension for algebraic varieties. The last two sections are devoted to the behavior of the dimension of the fibers of morphisms and to the Chevalley constructibility theorem.

3.1. The dimension of a topological space

Definition 3.1.1. Let X be a nonempty topological space. The *dimension* (also called *Krull dimension*) of X, denoted $\dim(X)$, is the supremum over the non-negative integers r such that there is a sequence

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r,$$

with all Z_i closed, irreducible subsets of X. We make the convention that if X is empty, then $\dim(X) = -1$.

In particular, we may consider the dimension of quasi-affine varieties, endowed with the Zariski topology. Note that in general we could have $\dim(X) = \infty$, even when X is Noetherian, but this will not happen in our setting.

Definition 3.1.2. Let $R \neq 0$ be a commutative ring. The *dimension* (also called *Krull dimension*) of R, denoted $\dim(R)$, is the supremum over the non-negative integers r such that there is a sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r,$$

with all $\mathfrak{p}_i$ prime ideals in R. We make the convention that $\dim(R) = -1$ when $R = 0$.

Remark 3.1.3. It follows from Corollary 1.1.10 and Proposition 1.3.8 that if X is an affine variety, we have $\dim(X) = \dim\big(\mathcal{O}(X)\big)$. More generally, for every commutative ring R one can interpret the dimension of R as the dimension of a topological space, as shown in the following exercise.

Exercise 3.1.4. Let R be a commutative ring and consider the *spectrum* of R:

$$\mathrm{Spec}(R) := \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal in } R\}.$$

For every ideal J in R, consider

$$V(J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid J \subseteq \mathfrak{p}\}.$$

Show that the following hold:

i) For every ideals J_1, J_2 in R, we have
$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2) = V(J_1 \cdot J_2).$$
ii) For every family $(J_\alpha)_\alpha$ of ideals in R, we have
$$\bigcap_\alpha V(J_\alpha) = V\left(\sum_\alpha J_\alpha\right).$$
iii) We have
$$V(0) = \mathrm{Spec}(R) \quad \text{and} \quad V(R) = \emptyset.$$
iv) Deduce that $\mathrm{Spec}(R)$ has a topology (the *Zariski topology*) whose closed subsets are the $V(J)$, with J an ideal in R.

v) Show that $V(J) \subseteq V(J')$ if and only if $\mathrm{rad}(J') \subseteq \mathrm{rad}(J)$. In particular, $V(J') = V(J)$ if and only if $\mathrm{rad}(J') = \mathrm{rad}(J)$.

vi) Show that the closed irreducible subsets in $\mathrm{Spec}(R)$ are those of the form $V(P)$, where P is a prime ideal in R. Deduce that
$$\dim(R) = \dim\big(\mathrm{Spec}(R)\big).$$

The following easy two lemmas show that the notion of dimension behaves as expected when it comes to some basic operations.

Lemma 3.1.5. *If Y is a subspace of X, then*
$$\dim(Y) \leq \dim(X).$$

Proof. Given a sequence of irreducible closed subsets in Y
$$Z_0 \supsetneq Z_2 \supsetneq \ldots \supsetneq Z_r,$$
by taking closures we obtain a sequence of closed subsets in X
$$\overline{Z_0} \supsetneq \overline{Z_2} \supsetneq \ldots \supsetneq \overline{Z_r}$$
(the fact that the inclusions are strict follows from $Z_i = \overline{Z_i} \cap Y$ for all i). This gives the inequality in the lemma. □

Lemma 3.1.6. *If X is a topological space, $Y_1, \ldots, Y_r$ are closed subsets of X, and $Y = Y_1 \cup \ldots \cup Y_r$, then*
$$\dim(Y) = \max_{i=1}^{r} \dim(Y_i).$$
This applies, in particular, if X is Noetherian, and $Y_1, \ldots, Y_r$ are the irreducible components of Y.

Proof. After replacing X by Y, we may assume that $X = Y$. The inequality "$\geq$" follows from Lemma 3.1.5. The opposite inequality follows from the fact that given any sequence
$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$
of irreducible, closed subsets of X, there is i such that $Z_0 \subseteq Y_i$, in which case $\dim(Y_i) \geq r$. □

The next lemma will allow us to reduce understanding the dimension of quasi-affine varieties to the case of affine varieties.

LEMMA 3.1.7. *If X is a topological space and $X = U_1 \cup \ldots \cup U_r$, with U_i open subsets of X, then*

$$\dim(X) = \max_{i=1}^{r} \dim(U_i).$$

PROOF. Again, the inequality "$\geq$" follows from Lemma 3.1.5. In order to prove the opposite inequality, consider a sequence

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$

of irreducible, closed subsets of X. Let i be such that $Z_r \cap U_i \neq \emptyset$. Since each $Z_j \cap U_i$ is irreducible and dense in Z_j (see Remarks 1.3.7), we obtain the following sequence of irreducible closed subsets of U_i:

$$Z_0 \cap U_i \supsetneq Z_1 \cap U_i \ldots \supsetneq Z_r \cap U_i,$$

hence $\dim(U_i) \geq r$. This completes the proof of the lemma. □

DEFINITION 3.1.8. If X is a topological space and Y is a closed, irreducible subset of X, then the *codimension* of Y in X, denoted $\operatorname{codim}_X(Y)$, is the supremum over the non-negative integers r for which there is a sequence

$$Z_0 \supsetneq Z_1 \supsetneq \ldots \supsetneq Z_r = Y,$$

with all Z_i closed and irreducible in X.

DEFINITION 3.1.9. Given a prime $\mathfrak{p}$ in a commutative ring R, the *codimension* (also called *height*) of $\mathfrak{p}$, denoted $\operatorname{codim}(\mathfrak{p})$, is the supremum over the non-negative integers r such that there is a sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r = \mathfrak{p},$$

with all $\mathfrak{p}_i$ prime ideals in R.

REMARK 3.1.10. It follows from Exercise 1.4.22 that if X is an affine variety and Y is an irreducible closed subset, defined by the prime ideal $\mathfrak{p} \subset \mathcal{O}(X)$, we have

$$\operatorname{codim}(\mathfrak{p}) = \operatorname{codim}_X(Y).$$

Note also that if $\mathfrak{q}$ is a prime ideal in the commutative ring R and $Z = V(\mathfrak{q}) \subseteq W = \operatorname{Spec}(R)$ is the corresponding irreducible closed subset, then

$$\operatorname{codim}(\mathfrak{q}) = \operatorname{codim}_W(Z).$$

REMARK 3.1.11. Using arguments similar to the ones in the proofs of Lemma 3.1.5 and Proposition 3.1.7, we see that if Y is an irreducible closed subset of a topological space X and U is an open subset of X such that $U \cap Y \neq \emptyset$, then

$$\operatorname{codim}_U(U \cap Y) = \operatorname{codim}_X(Y).$$

REMARK 3.1.12. If X is a Noetherian topological space, with irreducible components $X_1, \ldots, X_r$, and Y is an irreducible, closed subset of X, then

$$\operatorname{codim}_X(Y) = \max\{\operatorname{codim}_{X_i}(Y) \mid Y \subseteq X_i\}.$$

Indeed, given any chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r \subseteq X$$

of irreducible, closed subsets of X, by irreducibility of Y_r, there is i such that $Y_r \subseteq X_i$. This gives the inequality "$\leq$" and the opposite inequality is obvious.

3.2. Properties of finite morphisms

In order to prove the basic results concerning the dimension of affine algebraic varieties, we will make use of Noether's Normalization lemma. In order to exploit this, we will need some basic properties of finite morphisms. In this chapter we only discuss such morphisms between affine varieties; we will consider the general notion in Chapter 5.

DEFINITION 3.2.1. A morphism of affine varieties $f\colon X \to Y$ is *finite* if the corresponding ring homomorphism $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is finite.

EXAMPLE 3.2.2. Let Y be an affine variety and $a_1, \ldots, a_n \in \mathcal{O}(Y)$. If

$$X = \{(u,t) \in Y \times \mathbf{A}^1 \mid t^n + a_1(u)t^{n-1} + \ldots + a_n(u) = 0\},$$

then X is a closed subset of $Y \times \mathbf{A}^1$, and the composition

$$X \overset{i}{\hookrightarrow} Y \times \mathbf{A}^1 \xrightarrow{p} Y,$$

where i is the inclusion and p is the projection onto the first component, is finite. In fact, $\mathcal{O}(X)$ is free over $\mathcal{O}(Y)$, with a basis given by the classes of $1, t, \ldots, t^{n-1}$.

EXAMPLE 3.2.3. Given an irreducible closed subset $X \subseteq \mathbf{A}^N$, with $\operatorname{trdeg}\big(k(X)/k\big) = n$, it follows from Theorem 1.2.2 (and its proof) that after a linear change of coordinates $y_i = \sum_{j=1}^n a_{i,j}x_j$, with $\det(a_{i,j}) \neq 0$, the inclusion homomorphism $k[y_1, \ldots, y_n] \hookrightarrow \mathcal{O}(X)$ is finite. In other words, there is a linear automorphism $\phi\colon \mathbf{A}^N \to \mathbf{A}^N$, such that if $i\colon X \hookrightarrow \mathbf{A}^N$ is the inclusion, and $p\colon \mathbf{A}^N \to \mathbf{A}^n$ is the projection $p(u_1, \ldots, u_N) = (u_1, \ldots, u_n)$, the composition

$$X \overset{i}{\hookrightarrow} \mathbf{A}^N \xrightarrow{\phi} \mathbf{A}^N \xrightarrow{p} \mathbf{A}^n$$

is a finite morphism.

EXAMPLE 3.2.4. If X is an affine variety and Y is a closed subset of X, then Y is an affine variety and the inclusion map $Y \hookrightarrow X$ is finite. Indeed, the morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ is surjective, hence finite.

REMARK 3.2.5. It is straightforward to see that if $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms between affine varieties, then the composition $g \circ f$ is finite.

EXAMPLE 3.2.6. If X is an affine variety and Y consists of one point, then the unique morphism $f\colon X \to Y$ is finite if and only if X is a finite set. Indeed, note first that if X consists of r points, then $\mathcal{O}(X) = k^{\times r}$, hence $\mathcal{O}(X)$ is clearly a finitely generated k-vector space. For the converse, if $X_1, \ldots, X_n$ are the irreducible components of X, then for every i, the composition $X_i \hookrightarrow X \to Y$ is finite by Remark 3.2.5 and Example 3.2.4. Since it is enough to show that each X_i consists of one point, we may assume that X is irreducible. In this case, the canonical injective homomorphism $k \to \mathcal{O}(X)$ is finite, and since k is a field and $\mathcal{O}(X)$ is an integral domain, we conclude that $\mathcal{O}(X)$ is a field. The finite field extension $k \to \mathcal{O}(X)$ must be an isomorphism, since k is algebraically closed.

REMARK 3.2.7. If $f\colon X \to Y$ is a finite morphism of affine varieties and $Z \subseteq X$ and $W \subseteq Y$ are closed subsets such that $f(Z) \subseteq W$, then the induced morphism

$g\colon Z \to W$ is finite. Indeed, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}(Y) \arrow[r, "f^{\#}"] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ \mathcal{O}(W) \arrow[r, "g^{\#}"] & \mathcal{O}(Z). \end{tikzcd}$$

Since $f^{\#}$ is a finite homomorphism and the vertical homomorphisms in the diagram are surjective, it follows that $g^{\#}$ is finite as well.

In particular, using also Example 3.2.6, we see that if $f\colon X \to Y$ is finite, then for every $y \in Y$, the fiber $f^{-1}(y)$ is finite.

We collect in the next proposition some basic properties of finite ring homomorphisms (in fact, the same properties hold for integral homomorphisms).

Proposition 3.2.8. *Let $\phi\colon A \to B$ be a finite ring homomorphism.*

i) *If $\mathfrak{q}$ is a prime ideal in B and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$, then $\mathfrak{q}$ is a maximal ideal if and only if $\mathfrak{p}$ is a maximal ideal.*
ii) *If $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$ are prime ideals in B, then $\phi^{-1}(\mathfrak{q}_1) \neq \phi^{-1}(\mathfrak{q}_2)$.*
iii) *If ϕ is injective, then for every prime ideal $\mathfrak{p}$ in A, there is a prime ideal $\mathfrak{q}$ in B such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*
iv) *Given prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A and a prime ideal $\mathfrak{q}_1$ in B such that $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$, there is a prime ideal $\mathfrak{q}_2$ in B such that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$.*

Proof. Under the assumption in i), note that we have a finite, injective homomorphism of integral domains

$$A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}.$$

In this case, $A/\mathfrak{p}$ is a field if and only if $B/\mathfrak{q}$ is a field (see Proposition A.2.1). This gives i).

In order to prove ii), we first recall that the map $\mathfrak{q} \to \mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ gives a bijection between the primes $\mathfrak{q}$ in B with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and the primes in the ring $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. Since ϕ is finite, the induced homomorphism

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$$

is again finite. Given $\mathfrak{q}_1$ and $\mathfrak{q}_2$ as in ii), suppose that $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p} = \phi^{-1}(\mathfrak{q}_2)$. In this case, it follows from i) that both $\mathfrak{q}_1B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ and $\mathfrak{q}_2B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ are maximal ideals. Since the first one is strictly contained in the second one, we obtain a contradiction.

We now prove iii). Since B is a finitely generated A-module, we see that $B_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module, and it is nonzero since it contains $A_{\mathfrak{p}}$. It thus follows from Nakayama's lemma (see Proposition C.1.1) that $B_{\mathfrak{p}} \neq \mathfrak{p}B_{\mathfrak{p}}$. Since the ring $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is nonzero, it contains a prime ideal and every such prime ideal is of the form $\mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, for some prime ideal $\mathfrak{q}$ in B, with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

Finally, suppose that $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{q}_1$ are as in iv). The induced homomorphism

$$\overline{\phi}\colon A/\mathfrak{p}_1 \longrightarrow B/\mathfrak{q}_1$$

is finite and injective. We may thus apply iii) to find a prime ideal in $B/\mathfrak{q}_1$ whose inverse image via $\overline{\phi}$ is $\mathfrak{p}_2/\mathfrak{p}_1$. This ideal is of the form $\mathfrak{q}_2/\mathfrak{q}_1$, for some prime ideal $\mathfrak{q}_2$ containing $\mathfrak{q}_1$ and it is clear that $\phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$. $\square$

We now reformulate geometrically the properties of finite homomorphisms in the above proposition.

COROLLARY 3.2.9. *Let $f\colon X \to Y$ be a finite morphism of affine varieties and $\phi = f^{\#}$ the corresponding homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$.*

1) *The map f is closed, that is, $f(Z)$ is closed in Y for every closed subset Z of X. In particular, the map f is surjective if and only if ϕ is injective.*
2) *If $Z_1 \subsetneq Z_2$ are irreducible closed subsets of X, then $f(Z_1) \subsetneq f(Z_2)$ are irreducible closed subsets of Y.*
3) *If f is surjective, then given any irreducible, closed subset W of Y, there is an irreducible, closed subset Z in X such that $f(Z) = W$.*
4) *If Z_1 is an irreducible, closed subset of X and $W_1 \supseteq W_2$ are irreducible, closed subsets of Y, with $W_1 = f(Z_1)$, then there is $Z_2 \subseteq Z_1$ irreducible and closed such that $f(Z_2) = W_2$.*

PROOF. Let Z be a closed subset in X. In order to show that $f(Z)$ is closed, after writing Z as the union of its irreducible components, we see that it is enough to prove the assertion when Z is irreducible. Let $\mathfrak{q} \subseteq \mathcal{O}(X)$ be the prime ideal corresponding to Z. Recall that by Proposition 1.4.23, we have

$$\overline{f(Z)} = V\big(\phi^{-1}(\mathfrak{q})\big).$$

If $\mathfrak{m}$ is a maximal ideal in $\mathcal{O}(Y)$ containing $\phi^{-1}(\mathfrak{q})$, we deduce from assertions iv) and i) in the proposition that there is a maximal ideal $\mathfrak{n}$ in $\mathcal{O}(X)$ such that $\mathfrak{q} \subseteq \mathfrak{n}$ and $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. Therefore

$$V\big(\phi^{-1}(\mathfrak{q})\big) = f(Z)$$

and therefore $f(Z)$ is closed. In order to prove the second assertion in 1), recall that by Proposition 1.4.23, we know that ϕ is injective if and only if $\overline{f(X)} = Y$. Since $f(X)$ is closed, we obtain the assertion.

The assertions in 2), 3), and 4) now follow from assertions ii), iii), and respectively iv) in the proposition using the above description of the images of closed subsets of X. $\square$

COROLLARY 3.2.10. *If $f\colon X \to Y$ is a finite, surjective morphism of affine varieties, then*

$$\dim(X) = \dim(Y).$$

Moreover, if Z is a closed, irreducible subset of X, then

$$\operatorname{codim}_X(Z) = \operatorname{codim}_Y\big(f(Z)\big).$$

PROOF. If

$$Z_0 \supsetneq Z_1 \ldots \supsetneq Z_r$$

is a sequence of irreducible closed subsets in X, then it follows from assertions 1) and 2) in Corollary 3.2.9 that we have the following sequence of irreducible closed subsets in Y:

$$f(Z_0) \supsetneq f(Z_1) \supseteq \ldots \supsetneq f(Z_r).$$

This gives $\dim(Y) \geq \dim(X)$.

Suppose now that

$$W_0 \supsetneq W_1 \supsetneq \ldots \supsetneq W_s$$

is a sequence of irreducible closed subsets in Y. Assertion 3) in Corollary 3.2.9 gives an irreducible closed subset $T_0 \subseteq X$ such that $f(T_0) = W_0$. Using repeatedly assertion 4) in Corollary 3.2.9, we obtain a sequence of irreducible closed subsets in X

$$T_0 \supsetneq T_1 \supsetneq \ldots \supsetneq T_s$$

such that $f(T_i) = W_i$ for all i. We thus have $\dim(X) \geq \dim(Y)$ and by combining the two inequalities we get $\dim(X) = \dim(Y)$. The proof of the second assertion is entirely analogous, so we leave it as an exercise. $\square$

3.3. Main results of dimension theory

The following result, the *Principal Ideal* theorem, is the starting point of dimension theory. A similar statement holds for prime ideals in an arbitrary Noetherian ring, but we will only be concerned with our geometric setting. The proof we give follows closely [**Mum88**].

THEOREM 3.3.1. *(Krull) If X is an algebraic variety, $f \in \mathcal{O}(X)$, and Y is an irreducible component of*

$$V(f) = \{u \in X \mid f(u) = 0\},$$

then $\mathrm{codim}_X(Y) \leq 1$.

We begin with some comments about the statement.

REMARK 3.3.2. If $X_1, \ldots, X_r$ are the irreducible components of X and $f|_{X_i} \neq 0$ for all i, then $\mathrm{codim}_X(Y) \geq 1$. Indeed, since Y is irreducible, there is i such that $Y \subseteq X_i$, and our assumption on f implies that this inclusion is strict.

REMARK 3.3.3. With notation as in the theorem, if U is an open subset of X with $U \cap Y \neq \emptyset$, it is enough to prove the assertion in the theorem for U, $f|_U$, and $Y \cap U$. Indeed, it follows from Remark 3.1.11 that

$$\mathrm{codim}_X(Y) = \mathrm{codim}_U(U \cap Y),$$

while Exercise 1.3.19 implies that $U \cap Y$ is an irreducible component of $V(f|_U) = V(f) \cap U \subseteq U$.

REMARK 3.3.4. It is enough to prove the theorem when X is affine and irreducible and $Y = V(f)$. First, note that if we have a sequence

$$Z_0 \supsetneq Z_1 \supsetneq Z_2 = Y,$$

with all Z_i irreducible closed subsets of X, then $\mathrm{codim}_{Z_0}(Y) \geq 2$ and Y is an irreducible component also for $V(f|_{Z_0}) = V(f) \cap Z_0 \subseteq Z_0$. This shows that we may assume that X is irreducible. Second, let us choose an affine open subset $U \subseteq X$ that meets Y, but does not meet the other irreducible components of $V(f)$. By the previous remark, it is enough to prove the theorem for U, $f|_U$, and $Y \cap U$, and by our choice of U, we have $U \cap Y = V(f|_U)$.

REMARK 3.3.5. The theorem is easy to prove when X is affine and $\mathcal{O}(X)$ is a UFD. Indeed, the assertion is clearly true when $f = 0$ (in which case $Y = X$ and $\mathrm{codim}_X(Y) = 0$). Suppose now that $f \neq 0$. In this case, it follows from Example 1.3.14 that if the prime decomposition of f is $f = u f_1^{m_1} \cdots f_r^{m_r}$, with u invertible, then there is i such that $Y = V(f_i)$. If there is an irreducible closed subset Z with $Y \subsetneq Z \subsetneq X$ and $I_X(Z) = \mathfrak{p}$, then $\mathfrak{p} \subsetneq (f_i)$. Let $h \in \mathfrak{p}$ be any nonzero element and let m be the exponent of f_i in the prime decomposition of h is minimal. If we write $h = f_i^m h'$, since $\mathfrak{p}$ is prime and $f_i \notin \mathfrak{p}$, we have $h' \in (f_i)$, contradicting the definition of m.

The proof of the theorem makes use of Noether's Normalization lemma to reduce the general case to that treated in Remark 3.3.5. We will also need some basic facts about norm maps for finite field extensions, for which we refer to Appendix D.

Proof of Theorem 3.3.1. As we have seen in Remark 3.3.4, we may assume that X is affine and irreducible and $Y = V(f)$. Let $A = \mathcal{O}(X)$ and put $K = k(X)$. By Noether's Normalization lemma, if $n = \mathrm{trdeg}_k(K)$, we can find a k-subalgebra $B \simeq k[x_1, \ldots, x_n]$ of A such that the inclusion map $B \hookrightarrow A$ is finite (hence integral, see Proposition A.1.3). We denote by L the fraction field of B, so that the field extension K/L is finite (see Remark 1.2.1). We denote by $\mathfrak{p} \subseteq A$ the prime ideal corresponding to Y and let $\mathfrak{q} = \mathfrak{p} \cap B$.

Let $h = N_{K/L}(f)$. Note that $h \neq 0$. Moreover, since A is an integral extension of B, $f \in A$, and B is integrally closed (see Example 1.7.28), we have $h \in \mathfrak{q}$ by Proposition D.2.1.

In fact, we have $\mathfrak{q} = \mathrm{rad}(h)$. Indeed, suppose that $u \in \mathfrak{q}$. Since $\mathfrak{p} = \mathrm{rad}(f)$, it follows that we can find a positive integer m and $w \in A$ such that $u^m = fw$. By the multiplicative property of the norm and the behavior of $N_{K/L}$ on elements in L (for both properties, see Proposition D.1.1), we deduce

$$u^{m \cdot [K:L]} = N_{K/L}(u)^m = h \cdot N_{K/L}(w) \in (h).$$

Since B is a UFD, we deduce from Remark 3.3.5 that $\mathrm{codim}(\mathfrak{q}) \leq 1$. On the other hand, since the morphism $B \hookrightarrow A$ is finite and injective, it follows from Proposition 3.2.10 that $\mathrm{codim}(\mathfrak{p}) = \mathrm{codim}(\mathfrak{q})$. This completes the proof of the theorem. ☐

Remark 3.3.6. If X is an affine variety with irreducible components $X_1, \ldots, X_r$ and $f \in \mathcal{O}(X)$ is a non-zero-divisor, then $f|_{X_i} \neq 0$ for every i. Indeed, let $\mathfrak{p}_i = I_X(X_i)$ and suppose that we have $f \in \mathfrak{p}_1$. Let us choose $g_j \in \mathfrak{p}_j \smallsetminus \mathfrak{p}_1$ for $j \geq 2$. Since $\mathfrak{p}_1$ is prime, if $g = \prod_{j \geq 2} g_j$, then $g \not\in \mathfrak{p}_1$. In particular, $g \neq 0$. However, $fg \in \bigcap_{j \geq 1} \mathfrak{p}_j$, hence $fg = 0$, contradicting the fact that f is a non-zero-divisor. For a more general assertion, valid in arbitrary Noetherian rings, see Proposition E.2.1.

We thus see, by combining Theorem 3.3.1 and Remark 3.3.2, that if f is a non-zero-divisor in $\mathcal{O}(X)$, for an affine variety X, then every irreducible component of $V(f)$ has codimension 1 in X.

We now deduce from Theorem 3.3.1 the basic properties of dimension of algebraic varieties. We begin with a generalization of the theorem to the case of several functions.

Corollary 3.3.7. *If X is an algebraic variety and $f_1, \ldots, f_r$ are regular functions on X, then for every irreducible component Y of*

$$V(f_1, \ldots, f_r) = \{u \in X \mid f_1(u) = \ldots = f_r(u) = 0\},$$

we have $\mathrm{codim}_X(Y) \leq r$.

Proof. We do induction on r, the case $r = 1$ being a consequence of the theorem. Arguing as in Remarks 3.3.3 and 3.3.4, we see that we may assume that X is affine and $Y = V(f_1, \ldots, f_r)$. We need to show that for every sequence

$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_m$$

of irreducible closed subsets of X, we have $m \leq r$. By Noetherianity, we may assume that there is no irreducible closed subset Z, with $Y \subsetneq Z \subsetneq Y_1$.

By assumption, there is i (say, $i = 1$) such that $Y_1 \not\subseteq V(f_i)$. Since there are no irreducible closed subsets strictly between Y and Y_1, it follows that Y is an irreducible component of $Y_1 \cap V(f_1)$. After replacing X by an affine open subset meeting Y, but disjoint from the other components of $Y_1 \cap V(f_1)$, we may assume that in fact $Y = Y_1 \cap V(f_1)$, hence $I_X(Y) = \mathrm{rad}\big(I_X(Y_1) + (f_1)\big)$. It follows that for $2 \leq i \leq r$, we can find positive integers q_i and $g_i \in I_X(Y_1)$ such that

$$f_i^{q_i} - g_i \in (f_1). \tag{3.3.1}$$

We will show that Y_1 is an irreducible component of $V(g_2, \ldots, g_r)$. If this is the case, then we conclude by induction that $m - 1 \leq r - 1$, hence we are done. Note first that (3.3.1) gives

$$Y = V(f_1, \ldots, f_r) = V(f_1, g_2, \ldots, g_r).$$

If there is an irreducible closed subset Z such that

$$Y_1 \subsetneq Z \subseteq V(g_2, \ldots, g_r),$$

then $Y = Z \cap V(f_1)$, and the theorem implies $\mathrm{codim}_Z(Y) \leq 1$, contradicting the fact that we have $Y \subsetneq Y_1 \subsetneq Z$. Therefore Y_1 is an irreducible component of $V(g_2, \ldots, g_r)$, completing the proof of the corollary. $\square$

Corollary 3.3.8. *For every positive integer n, we have* $\dim(\mathbf{A}^n) = n$.

Proof. It is clear that $\dim(\mathbf{A}^n) \geq n$, since we have the following sequence of irreducible closed subsets in $\mathbf{A}^n$:

$$V(x_1, \ldots, x_n) \subsetneq V(x_1, \ldots, x_{n-1}) \subsetneq \ldots \subsetneq V(x_1) \subsetneq \mathbf{A}^n.$$

In order to prove the reverse inequality, it is enough to show that for every point $p = (a_1, \ldots, a_n) \in \mathbf{A}^n$, we have $\mathrm{codim}_X(\{p\}) \leq n$. This follows from Corollary 3.3.7, since $Y = V(x_1 - a_1, \ldots, x_n - a_n)$. $\square$

Corollary 3.3.9. *If X is an irreducible variety, then*

$$\dim(X) = \mathrm{trdeg}_k k(X).$$

In particular, we have $\dim(X) < \infty$.

Proof. By taking a finite cover by affine open subsets and using Lemma 3.1.7, we see that it is enough to prove the assertion when X is affine. It follows from Noether's Normalization lemma that if $n = \mathrm{trdeg}_k k(X)$, then there is a finite, surjective morphism $f \colon X \to \mathbf{A}^n$. The assertion then follows from the previous corollary via Corollary 3.2.10. $\square$

Remark 3.3.10. It follows from the previous corollary and Lemma 3.1.6 that for every algebraic variety X, we have $\dim(X) < \infty$.

Remark 3.3.11. Another consequence of Corollary 3.3.9 is that if X is an irreducible quasi-affine variety, then for every nonempty open subset U of X, we have $\dim(U) = \dim(X)$.

Definition 3.3.12. If X is a Noetherian topological space, we say that X has *pure dimension* if all its irreducible components have the same dimension.

Corollary 3.3.13. *If X is an algebraic variety, then the following hold:*

i) *If $Y \subseteq Z$ are closed irreducible subsets, then every saturated*[1] *chain*
$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r = Z$$
of irreducible closed subsets has length $r = \operatorname{codim}_Z(Y)$.

ii) *If X has pure dimension, then for every irreducible closed subset $Y \subseteq X$, we have*
$$\dim(Y) + \operatorname{codim}_X(Y) = \dim(X).$$

Proof. We begin by showing the following statement: given irreducible, closed subsets $Y \subsetneq Z$, with $\operatorname{codim}_Z(Y) = 1$, we have $\dim(Y) = \dim(Z) - 1$. For this, we may of course assume that $X = Z$. Note also that in light of Remark 3.3.11, we may replace Z by any open subset U with $U \cap Y \neq \emptyset$, since $\dim(U) = \dim(Z)$ and $\dim(U \cap Y) = \dim(Y)$. In particular, after replacing Z by an affine open subset U with $U \cap Y \neq \emptyset$, we may assume that Z is affine.

Let $f \in I_Z(Y) \smallsetminus \{0\}$. Since $\operatorname{codim}_Z(Y) = 1$, we see that Y is an irreducible component of $V(f)$. After replacing Z by an affine open subset that intersects Y, but does not intersect the other components of $V(f)$, we may assume that $Y = V(f)$. We now make use of the argument in the proof of Theorem 3.3.1. Noether's Normalization lemma gives a finite, surjective morphism $p\colon Z \to \mathbf{A}^n$ and we have seen that $p\big(V(f)\big) = V(h)$, for some nonzero $h \in \mathcal{O}(\mathbf{A}^n)$, hence the ideal $I\big(p(Y)\big) \subseteq k[x_1, \ldots, x_n]$ is principal, say generated by a polynomial g. This implies that $\dim\big(p(Y)\big) = n-1$: indeed, arguing as in the proof of Noether's Normalization lemma, we see that after a suitable linear change of coordinates, we may assume that g is a monic polynomial in x_n, with coefficients in $k[x_1, \ldots, x_{n-1}]$, in which case the morphism
$$k[x_1, \ldots, x_{n-1}] \hookrightarrow k[x_1, \ldots, x_n]/(g)$$
is finite and injective, hence we get the assertion via Corollaries 3.2.10 and 3.3.8. Since Corollary 3.2.10 gives $\dim(Z) = n$ and $\dim(Y) = \dim\big(p(Y)\big) = n - 1$, this completes the proof of our initial statement.

This assertion implies that given any saturated chain
$$Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_r = Z$$
of irreducible, closed subsets, we have $\dim(Y_i) = \dim(Y_{i-1}) + 1$ for $1 \leq i \leq r$, hence $\dim(Z) = \dim(Y) + r$. In particular, all such chains have the same length. Since there is such a chain of length $\operatorname{codim}_Z(Y)$, we obtain the assertion in i), as well as the assertion in ii) when X is irreducible.

Suppose now that we are in the setting of ii). Using Remark 3.1.12, the assertion when X is irreducible, and the fact that X is pure dimensional, we obtain
$$\operatorname{codim}_X(Y) = \max\{\operatorname{codim}_{X_i}(Y) \mid Y \subseteq X_i\}$$
$$= \max\{\dim(X_i) - \dim(Y) \mid Y \subseteq X_i\} = \dim(X) - \dim(Y),$$
completing the proof of the proposition. □

Remark 3.3.14. if X is an algebraic variety, and p is a point on X, then $\dim_p(X) := \dim(\mathcal{O}_{X,p})$ is equal to the largest dimension of an irreducible component of X that contains p. Indeed, it follows from definition that $\dim_p(X) =$

[1]This means that for every i, with $1 \leq i \leq r$, there is no closed, irreducible subset Z, with $Y_{i-1} \subsetneq Z \subsetneq Y_i$; equivalently, we have $\operatorname{codim}_{Y_i}(Y_{i-1}) = 1$.

$\mathrm{codim}_X(\{p\})$ and we deduce from Corollary 3.3.13 that if $X_1, \ldots, X_r$ are the irreducible components of X that contain p, then

$$\dim_p(X) = \max_{i=1}^{r} \mathrm{codim}_{X_i}(\{p\}) = \max_{i=1}^{r} \dim(X_i).$$

REMARK 3.3.15. Suppose that X is an algebraic variety, $f \in \mathcal{O}(X)$ is a non-zero-divisor, and

$$Y = \{x \in X \mid f(x) = 0\}.$$

In this case, for every $x \in Y$, we have

$$\dim(\mathcal{O}_{Y,x}) = \dim(\mathcal{O}_{X,x}) - 1. \tag{3.3.2}$$

In order to see this, we use the interpretation of the two dimensions given by the previous remark. Note first that it follows from Remark 3.3.6 that f does not vanish on any irreducible component of X. If Y' is an irreducible component of Y that contains x and if X' is an irreducible component of X that contains Y', then it follows from Theorem 3.3.1 that $\mathrm{codim}_{X'}(Y') = 1$ and Corollary 3.3.13 implies $\dim(Y') = \dim(X') - 1$. This gives the inequality "$\leq$" in (3.3.2). On the other hand, given any irreducible component Z of X that contains x, then every irreducible component W of $Y \cap Z$ that contains x satisfies $\mathrm{codim}_W(Z) = 1$ by Theorem 3.3.1. Using again Corollary 3.3.13, we obtain

$$\dim(\mathcal{O}_{Y,x}) \geq \dim W = \dim(Z) - 1,$$

hence we get the inequality "$\geq$" in (3.3.2).

We end this section with the following partial converse to Corollary 3.3.7.

PROPOSITION 3.3.16. *Let X be an algebraic variety. If Y is an irreducible closed subset with $\mathrm{codim}_X(Y) = r \geq 1$, then there are $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that Y is an irreducible component of $V(f_1, \ldots, f_r)$.*

PROOF. Let $X_1, \ldots, X_N$ be the irreducible components of X. Note that there is $f_1 \in I_X(Y)$ such that $X_i \not\subseteq V(f_1)$ for all i. Indeed, otherwise

$$I_X(Y) \subseteq \bigcup_{i=1}^{N} I_X(X_i).$$

Since all $I_X(X_i)$ are prime ideals and $I_X(Y) \not\subseteq I_X(X_i)$ (recall that $r \geq 1$), this contradicts the Prime Avoidance lemma (see Lemma E.1.1).

For such f_1, we have $\mathrm{codim}_{V(f_1)} Y \leq r - 1$. Iterating, we find $f_1, \ldots, f_r \in I_X(Y)$ such that $\mathrm{codim}_{V(f_1,\ldots,f_r)}(Y) = 0$, that is, Y is an irreducible component of $V(f_1, \ldots, f_r)$. $\square$

REMARK 3.3.17. In general, if X and Y are as in the proposition, it might not be possible to find $f_1, \ldots, f_r$ such that $Y = V(f_1, \ldots, f_r)$ (not even if we are willing to restrict to affine open neighborhoods of a given point). Consider, for example

$$X = V(x_1x_2 - x_3x_4) \subseteq \mathbf{A}^4 \quad \text{and} \quad Y = V(x_1, x_3).$$

In this case we have $\dim(X) = 3$ and $\dim(Y) = 2$, hence $\mathrm{codim}_X(Y) = 1$ by Corollary 3.3.13. However, for every affine open neighborhood U of the origin, there is no $f \in \mathcal{O}(U)$ such that $V(f) = Y$. Can you prove this?

EXERCISE 3.3.18. Show that if X and Y are algebraic varieties, then

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

EXERCISE 3.3.19. Show that if X is an algebraic variety and Z is a locally closed subset of X, then

$$\dim(Z) = \dim(\overline{Z}) > \dim(\overline{Z} \smallsetminus Z).$$

EXERCISE 3.3.20. Show that if X is an affine variety such that $\mathcal{O}(X)$ is a UFD, then for every closed subset $Y \subseteq X$, having all components of codimension 1, the ideal $I_X(Y)$ defining Y is principal.

EXERCISE 3.3.21. Show that if X and Y are irreducible closed subsets of $\mathbf{A}^n$, then every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$ (Hint: describe $X \cap Y$ as the intersection of $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^n$ with the diagonal $\Delta = \{(x, x) \mid x \in \mathbf{A}^n\}$).

3.4. Dimension of fibers of morphisms

We now discuss the main results concerning the dimensions of fibers of a morphism between algebraic varieties. More generally, we will be interested in the dimension of $f^{-1}(Z)$, where Z is a closed subset of Y.

We fix a dominant morphism $f \colon X \to Y$ between irreducible algebraic varieties and let $k(Y) \hookrightarrow k(X)$ be the induced extension of function fields. We put

$$r = \mathrm{trdeg}_{k(Y)} k(X) = \dim(X) - \dim(Y).$$

THEOREM 3.4.1. *With the above notation, if W is an irreducible component of $f^{-1}(Z)$ that dominates Z, then*

$$\mathrm{codim}_X(W) \leq \mathrm{codim}_Y(Z), \quad \textit{or equivalently}, \quad \dim(W) \geq \dim(Z) + r.$$

In particular, for every point y in the image of f, all irreducible components of $f^{-1}(y)$ have dimension $\geq r$.

PROOF. Note that if U is an open subset such that $Z \cap U \neq \emptyset$, since $\overline{f(W)} = Z$, we have $W \cap f^{-1}(U) \neq \emptyset$. By Corollary 3.3.11, we may thus replace f by $f^{-1}(U) \to U$. In particular, we may and will assume that Y is affine. In this case, if $s = \mathrm{codim}_Y(Z)$, it follows from Proposition 3.3.16 that there are $g_1, \ldots, g_s \in \mathcal{O}(Y)$ such that Z is an irreducible component of $V(g_1, \ldots, g_s)$. Since $W \subseteq f^{-1}(Z)$, we have $W \subseteq W' = V\big(f^{\#}(g_1), \ldots, f^{\#}(g_s)\big)$.

In fact, W is an irreducible component of W': if $W \subseteq W'' \subseteq W'$, with W'' closed and irreducible, we have

$$Z = \overline{f(W)} \subseteq \overline{f(W'')} \subseteq V(g_1, \ldots, g_s).$$

Since Z is an irreducible component of $V(g_1, \ldots, g_s)$, we deduce that $Z = \overline{f(W'')}$. In particular, we have $W'' \subseteq f^{-1}(Z)$, and since W is an irreducible component of $f^{-1}(Z)$, we conclude that $W = W''$. Therefore W is an irreducible component of W'. Corollary 3.3.7 then implies that $\mathrm{codim}_X(W) \leq s$. $\square$

THEOREM 3.4.2. *With the above notation, there is a nonempty open subset V of Y such that $V \subseteq f(X)$ and for every irreducible, closed subset $Z \subseteq Y$ with $Z \cap V \neq \emptyset$, and every irreducible component W of $f^{-1}(Z)$ that dominates Z, we have*

$$\mathrm{codim}_X(W) = \mathrm{codim}_Y(Z), \quad \textit{or equivalently}, \quad \dim(W) = \dim(Z) + r.$$

In particular, for every $y \in V$, every irreducible component of $f^{-1}(y)$ has dimension r.

PROOF. It is clear that we may replace f by $f^{-1}(U) \to U$ for any nonempty open subset, hence we may and will assume that Y is affine. We show that we may further assume that X is affine, too. Indeed, if we know the theorem in this case, we consider an open cover by affine open subsets

$$X = U_1 \cup \ldots \cup U_m$$

and let $V_i \subseteq Y$ be the nonempty open subset constructed for the morphism $U_i \to Y$. In this case it is straightforward to check that $V = \bigcap_i V_i$ satisfies the conditions in the theorem.

Suppose now that X and Y are irreducible affine varieties and let $f^{\#} \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be the induced homomorphism. This is injective, since f is dominant. We consider the $k(Y)$-algebra $S = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} k(Y)$. This is a domain with fraction field $k(X)$. By Noether's Normalization lemma, we can find $y_1, \ldots, y_r \in S$ that are algebraically independent over $k(Y)$ and such that the inclusion

$$\alpha \colon k(Y)[y_1, \ldots, y_r] \hookrightarrow S$$

is finite. After replacing each y_i by some $a_i y_i$, for a suitable nonzero $a_i \in \mathcal{O}(Y)$, we may assume that $y_i \in \mathcal{O}(X)$ for all i.

Claim. There is a nonzero $s \in \mathcal{O}(Y)$ such that the inclusion

$$\mathcal{O}(Y)_s[y_1, \ldots, y_r] \hookrightarrow \mathcal{O}(X)_s$$

is finite. In order to see this, let us choose generators $x_1, \ldots, x_N$ of $\mathcal{O}(X)$ as a k-algebra. Since α is finite, it follows that each x_i satisfies a monic equation of the form:

$$x_i^{m_i} + a_{i,1}x_i^{m_i-1} + \ldots + a_{i,m_i} = 0 \quad \text{for some} \quad a_{i,j} \in k(Y)[y_1, \ldots, y_r].$$

If $s \in \mathcal{O}(Y) \smallsetminus \{0\}$ is such that $sa_{i,j} \in \mathcal{O}(Y)[y_1, \ldots, y_r]$ for all i and j, then it follows that each x_i is integral over $\mathcal{O}(Y)_s[y_1, \ldots, y_r]$, hence $\mathcal{O}(X)_s$ is finite over $\mathcal{O}(Y)_s[y_1, \ldots, y_r]$, proving the claim.

After replacing f by $D_X\big(f^{\#}(s)\big) = f^{-1}\big(D_Y(s)\big) \to D_Y(s)$, we may thus assume that f factors as

$$X \xrightarrow{g} Y \times \mathbf{A}^r \xrightarrow{p} Y,$$

where p is the first projection and g is finite and surjective. It is clear that in this case f is surjective. Moreover, if Z and W are as in the statement of the theorem, then $g(W) \subseteq Z \times \mathbf{A}^r$, and using Corollary 3.2.9, as well as Exercise 3.3.18, we have

$$\dim(W) = \dim\big(g(W)\big) \leq \dim(Z \times \mathbf{A}^r) = \dim(Z) + r.$$

Since the opposite inequality follows by Theorem 3.4.1, we have in fact equality. $\square$

COROLLARY 3.4.3. *If $f \colon X \to Y$ is a morphism of algebraic varieties such that all fibers of f have dimension r (in particular, f is surjective), then* $\dim(X) = \dim(Y) + r$.

PROOF. If $Y_1, \ldots, Y_m$ are the irreducible components of Y, each morphism $f^{-1}(Y_i) \to Y_i$ has all fibers of dimension r. Since

$$\dim(X) = \max_{i=1}^{m} \dim\big(f^{-1}(Y_i)\big) \quad \text{and} \quad \dim(Y) = \max_{i=1}^{m} \dim(Y_i),$$

we see that it is enough to prove the assertion in the corollary when Y is irreducible.

Suppose now that $X_1, \ldots, X_s$ are the irreducible components of X. It follows from Theorem 3.4.2 that for every i, if we put

$$d_i := \dim(X_i) - \dim\left(\overline{f(X_i)}\right),$$

then there is an open subset U_i of $\overline{f(X_i)}$ such that every fiber of $X_i \to \overline{f(X_i)}$ over a point in U_i has dimension d_i. The hypothesis implies that $d_i \leq r$ for every i; moreover, there is i_0 such that $d_{i_0} = r$ and $\overline{f(X_i)} = Y$. The former fact implies that for every i, we have

$$\dim(X_i) \leq \dim\left(\overline{f(X_i)}\right) + r \leq \dim(Y) + r,$$

hence $\dim(X) \leq \dim(Y) + r$. On the other hand, the latter fact implies that $\dim(X_{i_0}) = \dim(Y) + r$, hence $\dim(X) \geq \dim(Y) + r$, completing the proof of the corollary. □

Example 3.4.4. Let a, b, and c be positive integers and let

$$f\colon \mathbf{A}^3 \to \mathbf{A}^3, \quad \text{given by} \quad f(u, v, w) = (u^a v^b w, u^c v, u).$$

This is birational, with inverse

$$g\colon V = \{(x, y, z) \in \mathbf{A}^3 \mid yz \neq 0\} \to \mathbf{A}^3$$

given by

$$g(x, y, z) = (z, yz^{-c}, xy^{-b}z^{-a+bc}).$$

Therefore f induces an isomorphism $f^{-1}(V) \to V$. In particular, for $P \in V$, the fiber $f^{-1}(P)$ is a point.

On the other hand, if $P = (x_0, y_0, 0)$, then

$$f^{-1}(P) = \begin{cases} V(u) \simeq \mathbf{A}^2, & \text{if} \quad x_0 = y_0 = 0; \\ \emptyset, & \text{otherwise} \end{cases}$$

If $P = (x_0, 0, z_0)$, with $z_0 \neq 0$, then

$$f^{-1}(P) = \begin{cases} V(v, u - z_0) \simeq \mathbf{A}^1, & \text{if} \quad x_0 = 0; \\ \emptyset, & \text{otherwise} \end{cases}$$

3.5. Constructible subsets and Chevalley's theorem

Definition 3.5.1. A subset of a topological space X is *constructible* if it is a finite union of locally closed subsets.

Proposition 3.5.2. *If X is a topological space, the set of constructible subsets of X is the smallest set that contains the open subsets of X and is closed under finite unions, finite intersections, and complements.*

Proof. The fact that a finite union of constructible sets is constructible is clear. Suppose now that A and B are constructible and let us show that $A \cap B$ is constructible. We can write

$$A = A_1 \cup \ldots \cup A_r \quad \text{and} \quad B = B_1 \cup \ldots \cup B_s,$$

with the A_i and B_j locally closed. In this case we have

$$A \cap B = \bigcup_{i,j} (A_i \cap B_j).$$

Since the intersection of two locally closed sets is locally closed, we see that $A \cap B$ is constructible.

If A is constructible and we write $A = A_1 \cup \ldots \cup A_r$, with the A_i locally closed, we have

$$X \smallsetminus A = \bigcap_{i=1}^{r} (X \smallsetminus A_i).$$

Since each A_i is locally closed, we can write it as $U_i \cap F_i$, with F_i closed and U_i open. In this case

$$X \smallsetminus A_i = (X \smallsetminus U_i) \cup (X \smallsetminus F_i)$$

is the union of a closed set with an open set, hence it is constructible. Since we have already seen that a finite intersection of constructible sets is constructible, we conclude that $X \smallsetminus A$ is constructible.

The minimality statement in the proposition is straightforward: given a set $\mathcal{C}$ of subsets of X as in the statement, this contains the open subsets by assumption, hence it also contains the closed sets, since we assume that $\mathcal{C}$ is closed under complements. Therefore $\mathcal{C}$ also contains the locally closed subsets (since it is closed under finite intersections) and therefore contains all constructible subsets (since it is closed under finite unions). $\square$

This notion is important because of the following result, due to Chevalley.

THEOREM 3.5.3. *If $f\colon X \to Y$ is a morphism between algebraic varieties, the image $f(X)$ is constructible. More generally, for every constructible subset A of X, its image $f(A)$ is constructible.*

PROOF. If A is constructible in X, we write $A = A_1 \cup \ldots \cup A_r$, with all A_i locally closed in X. Since $f(A) = f(A_1) \cup \ldots \cup f(A_r)$, it is enough to show that the image of each composition $A_i \hookrightarrow X \to Y$ is constructible. Therefore it is enough to consider the case $A = X$.

We prove that $f(X)$ is constructible by induction on $\dim(X)$. If $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X in irreducible components, we have

$$f(X) = f(X_1) \cup \ldots \cup f(X_r),$$

hence it is enough to show that each $f(X_i)$ is irreducible. We may thus assume that X is irreducible and after replacing Y by $\overline{f(X)}$, we may assume that Y is irreducible, too, and f is dominant (note that a constructible subset of $\overline{f(X)}$ is constructible also as a subset of Y). By Theorem 3.4.2, there is an open subset V of Y such that $V \subseteq f(X)$. We can thus write

$$f(X) = V \cup g(X'), \tag{3.5.1}$$

where $X' = X \smallsetminus g^{-1}(V)$ is a closed subset of X, with $\dim(X') < \dim(X)$. By induction, we know that $g(X')$ is constructible, and we deduce from (3.5.1) that $f(X)$ is constructible. $\square$

EXERCISE 3.5.4. i) Show that if Y is a topological space and A is a constructible subset of Y, then there is a subset V of A that is open and dense in $\overline{A}$ (in particular, V is locally closed in Y).

ii) Use part i) and Chevalley's theorem to show that if G is an algebraic group[2] having an algebraic action on the algebraic variety X, then every orbit is a locally closed subset of X. Deduce that X contains closed orbits.

[2]An algebraic group is defined like a *linear* algebraic group, but the variety is not necessarily affine.

CHAPTER 4

Projective varieties

In this chapter we introduce a very important class of algebraic varieties, the *projective varieties.*

4.1. The Zariski topology on the projective space

In this section we discuss the Zariski topology on the projective space, by building an analogue of the correspondence between closed subsets in affine space and radical ideals in the polynomial ring. As usual, we work over a fixed algebraically closed field k.

DEFINITION 4.1.1. For a non-negative integer n, the projective space $\mathbf{P}^n = \mathbf{P}^n_k$ is the set of all 1-dimensional linear subspaces in k^{n+1}.

For now, this is just a set. We proceed to endow it with a topology and in the next section we will put on it a structure of algebraic variety. Note that a 1-dimensional linear subspace in k^{n+1} is described by a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$, with two points $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ giving the same subspace if and only if there is $\lambda \in k^*$ such that $\lambda a_i = b_i$ for all i. In this way, we identify $\mathbf{P}^n$ with the quotient of the set $\mathbf{A}^{n+1} \smallsetminus \{0\}$ by the action of k^* given by

$$\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n).$$

Let $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ be the quotient map. We denote the image in $\mathbf{P}^n$ of a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$ by $[a_0, \ldots, a_n]$.

Let $S = k[x_0, \ldots, x_n]$. The relevant structure on S, for the study of $\mathbf{P}^n$, is that of a graded k-algebra. Recall that a graded (commutative) ring R is a commutative ring that has a decomposition as an Abelian group

$$R = \bigoplus_{m \in \mathbf{Z}} R_m$$

such that $R_i \cdot R_j \subseteq R_{i+j}$ for all i and j. We say that R is $\mathbf{N}$-graded if $R_m = 0$ for $m < 0$.

Note that the definition implies that if R is a graded ring, then R_0 is a subring of R and each R_m is an R_0-module, making R an R_0-algebra. We say that R is a graded A-algebra, for a commutative ring A, if R is a graded ring such that R_0 is an A-algebra (in which case R becomes an A-algebra, too). If R and S are graded rings, a *graded homomorphism* $\phi\colon R \to S$ is a ring homomorphism such that $\phi(R_m) \subseteq S_m$ for all $m \in \mathbf{Z}$.

The polynomial ring S is an $\mathbf{N}$-graded k-algebra, with S_m being the set of homogeneous polynomials of degree m. In general, if R is a graded ring, a nonzero element of R_m is *homogeneous of degree m*. By convention, 0 is homogeneous of degree m for every m. Given an arbitrary element $f \in R$, if we write

$$f = \sum_i f_i, \quad \text{with} \quad f_i \in R_i,$$

then the f_i are the *homogeneous componenets* of f.

REMARK 4.1.2. Note that the action of k^* on $\mathbf{A}^{n+1} \smallsetminus \{0\}$ is an algebraic action: in fact, it is induced by the algebraic action of k^* on $\mathbf{A}^{n+1}$ corresponding to the homomorphism

$$S \to k[t, t^{-1}] \otimes_k S, \quad f \to f(tx_1, \ldots, tx_n).$$

EXERCISE 4.1.3. For an ideal I in a graded ring R, the following are equivalent:

i) The ideal I can be generated by homogeneous elements of R.
ii) For every $f \in I$, all homogeneous components of f lie in I.
iii) The decomposition of R induces a decomposition $I = \bigoplus_{m \in \mathbf{Z}} (I \cap R_m)$.

An ideal that satisfies the equivalent conditions in the above exercise is a *homogeneous* (or *graded*) ideal. Note that if I is a homogeneous ideal in a graded ring R, then the quotient ring R/I becomes a graded ring in a natural way:

$$R/I = \bigoplus_{m \in \mathbf{Z}} R_m/(I \cap R_m).$$

We now return to the study of $\mathbf{P}^n$. The starting observation is that while it does not make sense to evaluate a polynomial in S at a point $p \in \mathbf{P}^n$, it makes sense to say that a *homogeneous* polynomial vanishes at p: indeed, if f is homogeneous of degree d and $\lambda \in k^*$, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d \cdot f(a_0, \ldots, a_n),$$

hence $f(\lambda a_0, \ldots, \lambda a_n) = 0$ if and only if $f(a_0, \ldots, a_n) = 0$. More generally, given any $f \in S$, we say that f vanishes at p if every homogeneous component of f vanishes at p.

Given any homogeneous ideal I of S, we define the *zero-locus* $V(I)$ of I to be the subset of $\mathbf{P}^n$ consisting of all points $p \in \mathbf{P}^n$ such that every polynomial f in I vanishes at p. Like the corresponding notion in the affine space, this notion satisfies the following basic properties. The proof is straightforward, hence we leave it as an exercise.

PROPOSITION 4.1.4. *The following hold:*

1) $V(S) = \emptyset$.
2) $V(0) = \mathbf{P}^n$.
3) *If I and J are ideals in S with $I \subseteq J$, then $V(J) \subseteq V(I)$.*
4) *If $(I_\alpha)_\alpha$ is a family of ideals in S, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

5) *If I and J are ideals in S, then*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

It follows from the proposition that we can put a topology on $\mathbf{P}^n$ (the *Zariski topology*) in which the closed subsets of $\mathbf{P}^n$ are the subsets of the form $V(I)$, where I is a homogeneous ideal in S.

REMARK 4.1.5. A closed subset $Y \subseteq \mathbf{A}^{n+1}$ is invariant by the k^*-action (that is, $\lambda \cdot Y = Y$ for every $\lambda \in k^*$) if and only if the ideal $I_{\mathbf{A}^n}(Y) \subseteq S$ is homogeneous (cf. Lemma 1.7.22). Indeed, if f is homogeneous, then for every $\lambda \in k^*$ and every $u \in \mathbf{A}^{n+1}$, we have $f(\lambda u) = 0$ if and only if $f(u) = 0$. We thus see that if I is a homogeneous ideal, then its zero-locus in $\mathbf{A}^{n+1}$ is k^*-invariant. In particular, if $I_{\mathbf{A}^n}(Y)$ is homogeneous, then Y is k^*-invariant. Conversely, if Y is k^*-invariant and $f \in I_{\mathbf{A}^n}(Y)$, let us write $f = \sum_i f_i$, with $f_i \in S_i$. For every $u \in Y$ and every $\lambda \in k^*$, we have $\lambda u \in Y$, hence

$$0 = f(\lambda u) = \sum_{i \geq 0} \lambda^i \cdot f_i(u).$$

It is easy to see that since this property holds for infinitely many λ, we have $f_i(u) = 0$ for all i, hence $I_{\mathbf{A}^n}(Y)$ is homogeneous.

REMARK 4.1.6. The topology on $\mathbf{P}^n$ is the quotient topology with respect to the k^*-action on $\mathbf{A}^{n+1} \smallsetminus \{0\}$. In other words, if $\pi \colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ is the quotient map, then a subset Z of $\mathbf{P}^n$ is closed if and only if its inverse image $\pi^{-1}(Z)$ is closed. For this, we may assume that Z is nonempty. If $\pi^{-1}(Z)$ is closed, then it is clear that $\pi^{-1}(Z) \cup \{0\}$ is closed, hence by the previous remark, there is a homogeneous ideal $I \subseteq S$ such that $\pi^{-1}(Z) \cup \{0\}$ is the zero-locus of I. In this case, it is clear that Z is the zero-locus of I in $\mathbf{P}^n$. The converse is clear.

We now construct a map in the opposite direction. Given any subset $S \subseteq \mathbf{P}^n$, let $I(S)$ be the set of polynomials in S that vanish at all points in S. Note that $I(S)$ is a homogeneous radical ideal of S (the fact that it is homogeneous follows from the fact that if $f \in I(S)$, then all homogeneous components of f lie in $I(S)$). This definition satisfies the following properties, that are straightforward to check.

PROPOSITION 4.1.7. *The following hold:*

1) $I(\emptyset) = S$.
2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$*, then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$.
3) *If* $W_1 \subseteq W_2$*, then* $I(W_2) \subseteq I(W_1)$.

We now turn to the compositions of the two maps. The first property is tautological.

PROPOSITION 4.1.8. *For every subset* S *of* $\mathbf{P}^n$*, we have* $V\big(I(S)\big) = \overline{S}$.

PROOF. The proof follows verbatim the proof in the case of affine space (see Proposition 1.1.8). $\square$

The more interesting statement concerns the other composition. This is the content of the next proposition, a graded version of the Nullstellensatz.

PROPOSITION 4.1.9. *If* $J \subseteq S$ *is a radical ideal different from* $S_+ = (x_0, \ldots, x_n)$*, then* $I\big(V(J)\big) = J$.

Note that $V(S_+) = \emptyset$, hence $I\big(V(S_+)\big) = S$. The ideal S_+, which behaves differently in this correspondence, is the *irrelevant ideal*.

PROOF OF PROPOSITION 4.1.9. The inclusion "$\supseteq$" is trivial, hence we only need to prove the reverse inclusion. It is enough to show that every homogeneous polynomial $f \in I\big(V(J)\big)$ lies in J.

We make use of the map $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$. Let Z be the closed subset of $\mathbf{A}^{n+1}$ defined by J, so that $Z \smallsetminus \{0\} = \pi^{-1}\big(V(J)\big)$. Our assumption on f says that f vanishes on $Z \smallsetminus \{0\}$. If $\deg(f) > 0$, then $f(0) = 0$, and we conclude by Hilbert's Nullstellensatz that $f \in J$. On the other hand, if $\deg(f) = 0$ and $f \neq 0$, then it follows that $V(J) = \emptyset$. This implies that $Z \subseteq \{0\}$ and another application of Hilbert's Nullstellensatz gives $S_+ \subseteq J$. Since $J \neq S_+$ by assumption, we have $J = S$, in which case $f \in J$. $\square$

Corollary 4.1.10. *The two maps $V(-)$ and $I(-)$ give inclusion reversing inverse bijections between the set of homogeneous radical ideals in S different from S_+ and the closed subsets of $\mathbf{P}^n$.*

Proof. Note that for every closed subset Z of $\mathbf{P}^n$, we have $I(Z) \neq S_+$. Indeed, if $I(Z) = S_+$, then it follows from Proposition 4.1.8 that

$$Z = V\big(I(Z)\big) = V(S_+) = \emptyset.$$

However, in this case $I(Z) = I(\emptyset) = S$. The assertion in the corollary follows directly from Propositions 4.1.8 and 4.1.9. $\square$

Exercise 4.1.11. Show that if I is a homogeneous ideal in a graded ring S, then the following hold:

i) The ideal I is radical if and only if for every *homogeneous* element $f \in S$, with $f^m \in I$ for some $m \geq 1$, we have $f \in I$.
ii) The radical $\mathrm{rad}(I)$ of I is a homogeneous ideal.

Exercise 4.1.12. Show that if I is a homogeneous ideal in a graded ring S, then I is a prime ideal if and only if for every homogeneous elements $f, g \in S$ with $fg \in I$, we have $f \in I$ or $g \in I$. Deduce that a closed subset Z of $\mathbf{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal. In particular, $\mathbf{P}^n$ is irreducible.

Definition 4.1.13. If X is a closed subset of $\mathbf{P}^n$ and I_X is the corresponding homogeneous radical ideal, then $S_X := S/I_X$ is the *homogeneous coordinate ring* of X. Note that this is an $\mathbf{N}$-graded k-algebra. In particular, S is the *homogeneous coordinate ring* of $\mathbf{P}^n$.

Suppose that X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X. For every homogeneous $g \in S_X$ of positive degree, we consider the following open subset of X:

$$D_X^+(g) = X \smallsetminus V(\widetilde{g}),$$

where $\widetilde{g} \in S$ is any homogeneous polynomial which maps to $g \in S_X$. Note that if h is another homogeneous polynomial of positive degree, we have

$$D_X^+(gh) = D_X^+(g) \cap D_X^+(h).$$

Remark 4.1.14. Every open subset of X is of the form $X \smallsetminus V(J)$, where J is a homogeneous ideal in S. By choosing a system of homogeneous generators for J, we see that this is the union of finitely many open subsets of the form $D_X^+(g)$. Therefore the open subsets $D_X^+(g)$ give a basis of open subsets for the topology of X.

Definition 4.1.15. For every closed subset X of $\mathbf{P}^n$, we define the *affine cone* $C(X)$ over X to be the union in $\mathbf{A}^{n+1}$ of the corresponding lines in X. Note that if X is nonempty, then

$$C(X) = \pi^{-1}(Z) \cup \{0\}.$$

If $X = V(I)$ is nonempty, for a homogeneous ideal $I \subseteq S$, it is clear that $C(X)$ is the zero-locus of I in $\mathbf{A}^{n+1}$. Therefore $C(X)$ is a closed subset of $\mathbf{A}^n$ for every X. Moreover, we see that $\mathcal{O}\big(C(X)\big) = S_X$.

EXERCISE 4.1.16. Show that if G is an irreducible algebraic group acting on a variety X, then every irreducible component of X is invariant under the G-action.

REMARK 4.1.17. Let X be a closed subset of $\mathbf{P}^n$, with corresponding homogeneous radical ideal $I_X \subseteq S$, and let $C(X)$ be the affine cone over X. Since $C(X)$ is k^*-invariant, it follows from the previous exercise that the irreducible components of $C(X)$ are k^*-invariant, as well. By Remark 4.1.5, this means that the minimal prime ideals containing I_X are homogeneous. They correspond to the irreducible components $X_1, \ldots, X_r$ of X, so that the irreducible components of $C(X)$ are $C(X_1), \ldots, C(X_r)$.

4.2. Regular functions on quasi-projective varieties

Our goal in this section is to define a structure sheaf on $\mathbf{P}^n$. The main observation is that if F and G are homogeneous polynomials of the same degree, then we may define a function $\frac{F}{G}$ on the open subset $\mathbf{P}^n \smallsetminus V(G)$ by

$$[a_0, \ldots, a_n] \to \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Indeed, if $\deg(F) = d = \deg(G)$, then

$$\frac{F(\lambda a_0, \ldots, \lambda a_n)}{G(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d \cdot F(a_0, \ldots, a_n)}{\lambda^d \cdot G(a_0, \ldots, a_n)} = \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Let W be a locally closed subset in $\mathbf{P}^n$. A *regular function* on W is a function $f\colon W \to k$ such that for every $p \in W$, there is an open neighborhood $U \subseteq W$ of p and homogeneous polynomials of the same degree F and G such that $G(q) \neq 0$ for every $q \in U$ and

$$f(q) = \frac{F(q)}{G(q)} \quad \text{for all} \quad q \in U.$$

The set of regular functions on W is denoted by $\mathcal{O}(W)$. Note that $\mathcal{O}(W)$ is a k-algebra with respect to the usual operations on functions. For example, if $f_1(q) = \frac{F_1(q)}{G_1(q)}$ for $q \in U_1$ and $f_2(q) = \frac{F_2(q)}{G_2(q)}$ for $q \in U_2$, where U_1 and U_2 are open neighborhoods of p, then $F_1G_2 + F_2G_1$ and G_1G_2 are homogeneous polynomials of the same degree and

$$f_1(q) + f_2(q) = \frac{(F_1G_2 + F_2G_1)(q)}{(G_1G_2)(q)} \quad \text{for} \quad q \in U_1 \cap U_2.$$

Moreover, it is clear that if V is an open subset of W, the restriction to V of a regular function on W is a regular function of V. We thus obtain in this way a subpresheaf $\mathcal{O}_W$ of k-algebras of $\mathcal{F}un_W$. In fact, this is a sheaf, as follows immediately from the fact that regular functions are defined in terms of a local property.

REMARK 4.2.1. Note that if W is a locally closed subset of $\mathbf{P}^n$, then the sheaf $\mathcal{O}_W$ we defined is the one induced from $\mathcal{O}_{\mathbf{P}^n}$ as in Section 2.3.

Our first goal is to show that all spaces defined in this way are algebraic varieties. Let U_i be the open subset defined by $x_i\neq 0$. Note that we have

$$\mathbf{P}^n=\bigcup_{i=0}^n U_i.$$

The key fact is the following assertion:

PROPOSITION 4.2.2. *For every i, with $0\leq i\leq n$, the map*

$$\psi_i\colon \mathbf{A}^n\to U_i,\quad \psi(v_1,\ldots,v_n)=[v_1,\ldots,v_i,1,v_{i+1},\ldots,v_n]$$

is an isomorphism in $\mathcal{T}op_k$.

PROOF. It is clear that ψ_i is a bijection, with inverse

$$\phi_i\colon U_i\to \mathbf{A}^n,\quad [u_0,\ldots,u_n]\to (u_0/u_i,\ldots,u_{i-1}/u_i,u_{i+1}/u_i,\ldots,u_n/u_i).$$

In order to simplify the notation, we give the argument for $i=0$, the other cases being analogous. Consider first a principal affine open subset of $\mathbf{A}^n$, of the form $D(f)$, for some $f\in k[x_1,\ldots,x_n]$. Note that if $\deg(f)=d$, then we can write

$$f(x_1/x_0,\ldots,x_n/x_0)=\frac{g(x_0,\ldots,x_n)}{x_0^d}$$

for a homogeneous polynomial $g\in S$ of degree d. It is then clear that $\phi_0^{-1}\big(D(f)\big)=D^+_{\mathbf{P}^n}(x_0g)$, hence this is open in U_0. Since the principal affine open subsets in $\mathbf{A}^n$ give a basis for the topology of $\mathbf{A}^n$, we see that ϕ_0 is continuous.

Consider now an open subset of U_0 of the form $D^+_{\mathbf{P}^n}(h)$, for some homogeneous $h\in S$, of positive degree. If we put $h_0=h(1,x_1,\ldots,x_n)$, we see that $\phi_0\big(D^+_{\mathbf{P}^n}(h)\big)=D(h_0)$ is open in $\mathbf{A}^n$. Since the open subsets of the form $D^+_{\mathbf{P}^n}(h)$ give a basis for the topology of $\mathbf{P}^n$, we conclude that ϕ_0 is a homeomorphism.

We now need to show that if U is open in $\mathbf{A}^n$ and $\alpha\colon U\to k$, then $\alpha\in\mathcal{O}_{\mathbf{A}^n}(U)$ if and only if $\alpha\circ\phi_0\in\mathcal{O}_{\mathbf{P}^n}\big(\phi_0^{-1}(U)\big)$. If $\alpha\in\mathcal{O}_{\mathbf{A}^n}(U)$, then for every point $p\in U$, we have an open neighborhood $U_p\subseteq U$ of p and $f_1,f_2\in k[x_1,\ldots,x_n]$ such that

$$f_2(u)\neq 0\quad\text{and}\quad \alpha(u)=\frac{f_1(u)}{f_2(u)}\quad\text{for all}\quad u\in U_p.$$

As above, we can write

$$f_1(x_1/x_0,\ldots,x_n/x_0)=\frac{g_1(x_0,\ldots,x_n)}{x_0^d}\quad\text{and}\quad f_2(x_1/x_0,\ldots,x_n/x_0)=\frac{g_2(x_0,\ldots,x_n)}{x_0^d}$$

for some homogeneous polynomials $g_1,g_2\in S$ of the same degree, in which case we see that

$$g_2(v)\neq 0\quad\text{and}\quad \alpha\big(\phi_0(v)\big)=\frac{g_1(v)}{g_2(v)}\quad\text{for all}\quad v\in\phi_0^{-1}(U_p).$$

Since this holds for every $p\in U$, we see that $\alpha\circ\phi_0$ is a regular function on $\phi_0^{-1}(U)$.

Conversely, suppose that $\alpha\circ\phi_0$ is a regular function on $\phi_0^{-1}(U)$. This means that for every $q\in\phi_0^{-1}(U)$, there is an open neighborhood $V_q\subseteq\phi_0^{-1}(U)$ of q and homogeneous polynomials $h_1,h_2\in S$ of the same degree such that

$$h_2(v)\neq 0\quad\text{and}\quad \alpha\big(\phi_0(v)\big)=\frac{h_1(v)}{h_2(v)}\quad\text{for all}\quad v\in V_q.$$

In this case, we have

$$h_2(1,u_1,\ldots,u_n)\neq 0 \quad\text{and}\quad \alpha(u_1,\ldots,u_n)=\frac{h_1(1,u_1,\ldots,u_n)}{h_2(1,u_1,\ldots,u_n)}$$

for all $u=(u_1,\ldots,u_n)\in\phi_0(V_q)$. Since this holds for every $q\in\phi_0^{-1}(U)$, it follows that α is a regular function on U. This completes the proof of the fact that ϕ_0 is an isomorphism. $\square$

COROLLARY 4.2.3. *For every locally closed subset W of $\mathbf{P}^n$, the space $(W,\mathcal{O}_W)$ is an algebraic variety.*

PROOF. It is enough to show the assertion for $W=\mathbf{P}^n$: the general case is then a consequence of Propositions 2.3.5 and 2.5.4. We have already seen that $\mathbf{P}^n$ is a prevariety. In order to show that it is separated, using Proposition 2.5.6, it is enough to show that each $U_i\cap U_j$ is affine and that the canonical morphism

$$\tau_{i,j}\colon \mathcal{O}(U_i)\otimes_k\mathcal{O}(U_j)\to\mathcal{O}(U_i\cap U_j) \tag{4.2.1}$$

is surjective. Suppose that $i<j$ and let us denote by $x_1,\ldots,x_n$ the coordinates on the image of ϕ_i and by $y_1,\ldots,y_n$ the coordinates on the image of ϕ_j. Note that via the isomorphism ϕ_i, the open subvariety $U_i\cap U_j$ is mapped to the open subset

$$\{(u_1,\ldots,u_n)\in\mathbf{A}^n\mid u_j\neq 0\},$$

which is affine, being a principal affine open subset of $\mathbf{A}^n$. Similarly, ϕ_j maps $U_i\cap U_j$ to the open subset

$$\{(u_1,\ldots,u_n)\in\mathbf{A}^n\mid u_{i+1}\neq 0\}.$$

Furthermore, since we have

$$\phi_j\circ\phi_i^{-1}(u_1,\ldots,u_n)=\left(\frac{u_1}{u_j},\ldots,\frac{u_i}{u_j},\frac{1}{u_j},\frac{u_{i+1}}{u_j},\ldots,\frac{u_{j-1}}{u_j},\frac{u_{j+1}}{u_j},\ldots,\frac{u_n}{u_j}\right)$$

for all $(u_1,\ldots,u_n)\in\phi_i(U_i\cap U_j)$, we see that the morphism

$$\tau_{i,j}\colon k[x_1,\ldots,x_n]\otimes_k k[y_1,\ldots,y_n]\to k[x_1,\ldots,x_n,x_j^{-1}]$$

satisfies $\tau(x_\ell)=x_\ell$ for all ℓ and $\tau(y_{i+1})=x_j^{-1}$. Therefore $\tau_{i,j}$ is surjective for all i and j, proving that $\mathbf{P}^n$ is separated. $\square$

EXAMPLE 4.2.4. The map

$$\pi\colon\mathbf{A}^{n+1}\smallsetminus\{0\}\to\mathbf{P}^n,\quad \pi(x_0,\ldots,x_n)=[x_0,\ldots,x_n]$$

is a morphism. Indeed, with the notation in the proof of Proposition 4.2.2, it is enough to show that for every i, the induced map $\pi^{-1}(U_i)\to U_i$ is a morphism. However, via the isomorphism $U_i\simeq\mathbf{A}^n$, this map becomes

$$\mathbf{A}^{n+1}\smallsetminus V(x_i)\to\mathbf{A}^n,\quad (x_0,\ldots,x_n)\to(x_0/x_i,\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots,x_n/x_i),$$

which is clearly a morphism.

DEFINITION 4.2.5. A *projective variety* is an algebraic variety that is isomorphic to a closed subvariety of some $\mathbf{P}^n$. A *quasi-projective* variety is an algebraic variety that is isomorphic to a locally closed subvariety ofsome $\mathbf{P}^n$.

Remark 4.2.6. It follows from definition that if X is a projective variety and Y is a closed subvariety of X, then Y is a projective variety as well. Similarly, if X is a quasi-projective variety and Z is a locally closed subvariety of X, then Z is a quasi-projective variety.

Remark 4.2.7. Every quasi-affine variety is quasi-projective: this follows from the fact that $\mathbf{A}^n$ is isomorphic to an open subvariety of $\mathbf{P}^n$.

Remark 4.2.8. Note that unlike the coordinate ring of an affine variety, the homogeneous coordinate ring of a projective variety $X \subseteq \mathbf{P}^n$ is not an intrinsic invariant: it depends on the embedding in the projective space.

We next show that the distinguished open subsets $D_X^+(g)$ are all affine varieties[1].

Proposition 4.2.9. *For every closed subvariety X of $\mathbf{P}^n$ and every homogeneous element $g \in S_X$ of positive degree, the variety $D_X^+(g)$ is affine.*

Proof. Since X is a closed subvariety of $\mathbf{P}^n$ and $D_X^+(g) = D_{\mathbf{P}^n}^+(\widetilde{g}) \cap X$, where $\widetilde{g} \in S$ is any lift of g, it is enough to prove the assertion when $X = \mathbf{P}^n$. Let $U = D_{\mathbf{P}^n}^+(g)$ and put $d = \deg(g)$.

Consider the regular functions $f_0, \ldots, f_n$ on U given by $f_i(u_0, \ldots, u_n) = \frac{u_i^d}{g(u)}$. Note that they generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. Indeed, since $g \in S_+ = \mathrm{rad}(x_0^d, \ldots, x_n^d)$, it follows that there is m such that $g^m \in (x_0^d, \ldots, x_n^d)$. If we write $g^m = \sum_{i=1}^n h_i x_i^d$ and if we consider the regular functions $\alpha_i \colon U \to k$ given by

$$\alpha_i(u_1, \ldots, u_n) = \frac{h_i(u)}{g(u)^{m-1}},$$

then $\sum_{i=0}^n f_i \cdot \alpha_i = 1$, hence $f_0, \ldots, f_n$ generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. By Proposition 2.3.16, we see that it is enough to show that each subset $U \cap U_i$ is affine, where U_i is the open subset of $\mathbf{P}^n$ defined by $x_i \neq 0$. However, by the isomorphism $U_i \simeq \mathbf{A}^n$ given in Proposition 4.2.2, the open subset $U \cap U_i$ becomes isomorphic to the subset

$$\{u = (u_1, \ldots, u_n) \in \mathbf{A}^n \mid g(u_1, \ldots, u_i, 1, u_{i+1}, \ldots, u_n) \neq 0\},$$

which is affine by Proposition 1.4.18. This completes the proof. $\square$

Since the open subsets $D_X^+(g)$ are affine, they are determined by their rings of regular functions. Our next goal is to describe these rings.

We begin with some general considerations regarding localization in graded rings. If S is a graded ring and $T \subseteq S$ is a multiplicative system consisting of homogeneous elements of S, then the ring of fractions $T^{-1}S$ has an induced grading, in which

$$(T^{-1}S)_m = \left\{ \frac{f}{t} \mid t \in T, f \in S_{\deg(t)+m} \right\}.$$

Note that even if S is $\mathbf{N}$-graded, $T^{-1}S$ is not, in general, $\mathbf{N}$-graded. We will use two special cases. If $g \in S$ is a homogeneous element, then S_g is a graded ring, and we denote by $S_{(g)}$ its degree 0 part. Similarly, if $\mathfrak{p}$ is a homogeneous prime ideal in S and if we take T to be the set of homogeneous elements in $S \smallsetminus \mathfrak{p}$, then $T^{-1}S$ is a graded ring and we denote its degree 0 part by $S_{(\mathfrak{p})}$. Therefore $S_{(g)}$ is the subring

[1] For another proof of this proposition, making use of the Veronese embedding, see Exercise 4.2.23 below.

of S_g consisting of fractions $\frac{h}{g^m}$, where h is a homogeneous element of S, of degree $m \cdot \deg(g)$. Similarly, $S_{(\mathfrak{p})}$ is the subring of $S_{(\mathfrak{p})}$ consisting of all fractions of the form $\frac{f}{h}$, where $f, g \in S$ are homogeneous, of the same degree, with $g \not\in \mathfrak{p}$. Note that $S_{(\mathfrak{p})}$ is a local ring, with maximal ideal

$$\{f/h \in S_{(\mathfrak{p})} \mid f \in \mathfrak{p}\}.$$

Let X be a closed subset of $\mathbf{P}^n$, with corresponding radical ideal I_X and homogeneous coordinate ring S_X. Note that if $h \in S_X$ is homogeneous, of positive degree, we have a morphism of k-algebras

$$\Phi\colon (S_X)_{(h)} \to \mathcal{O}\big(D_X^+(h)\big),$$

such that $\Phi(f/h^m)$ is given by the function $p \to \frac{\widetilde{f}(p)}{\widetilde{h}^m(p)}$, where $\widetilde{f}, \widetilde{h} \in S$ are elements mapping to $f, h \in S_X$, respectively (it is clear that $\Phi(f/h^m)$ is independent of the choice of $\widetilde{f}$ and $\widetilde{h}$).

Proposition 4.2.10. *For every X and h as above, the morphism Φ is an isomorphism.*

Proof. We will prove a more general version in Proposition 4.3.17 below. □

We end this section with the description of the dimension of a closed subset of $\mathbf{P}^n$ in terms of the homogeneous coordinate ring.

Proposition 4.2.11. *If $X \subseteq \mathbf{P}^n$ is a nonempty closed subset, with homogeneous coordinate ring S_X, then $\dim(X) = \dim(S_X) - 1$.*

Proof. Note that the morphism $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ induces a surjective morphism $f\colon C(X) \smallsetminus \{0\} \to X$ whose fibers are 1-dimensional (in fact, they are all isomorphic to $\mathbf{A}^1 \smallsetminus \{0\}$). It follows from Corollary 3.4.3 that

$$\dim\big(C(X)\big) = 1 + \dim(X).$$

Since S_X is the coordinate ring of the affine variety $C(X)$, we obtain the assertion in the proposition. □

Corollary 4.2.12. *If X and Y are nonempty closed subsets of $\mathbf{P}^n$, with $\dim(X) + \dim(Y) \geq n$, then $X \cap Y$ is nonempty and every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$.*

Proof. Note that $\big(C(X) \cap C(Y)\big) \smallsetminus \{0\} = C(X \cap Y) \smallsetminus \{0\}$. It is clear $C(X) \cap C(Y)$ is nonempty, since it contains 0. In this case, it follows from Exercise 3.3.21 that every irreducible component of $C(X) \cap C(Y)$ has dimension

$$\geq \dim\big(C(X)\big) + \dim\big(C(Y)\big) - (n+1) = \dim(X) + \dim(Y) - n + 1.$$

This implies that $C(X) \cap C(Y)$ is not contained in $\{0\}$, hence $X \cap Y$ is non-empty. Moreover, the irreducible components of $C(X) \cap C(Y)$ are of the form $C(Z)$, where Z is an irreducible component of $X \cap Y$, hence

$$\dim(Z) = \dim\big(C(Z)\big) - 1 \geq \dim(X) + \dim(Y) - n.$$

□

EXERCISE 4.2.13. A *hypersurface* in $\mathbf{P}^n$ is a closed subset defined by

$$\{[x_0,\ldots,x_n]\in\mathbf{P}^n \mid F(x_0,\ldots,x_n)=0\},$$

for some homogeneous polynomial F, of positive degree. Given a closed subset $X\subseteq\mathbf{P}^n$, show that the following are equivalent:

i) X is a hypersurface.
ii) The ideal $I(X)$ is a principal ideal.
iii) All irreducible component of X have codimension 1 in $\mathbf{P}^n$.

Note that if X is any irreducible variety and U is a nonempty open subset of X, then the map taking $Z\subseteq U$ to $\overline{Z}$ and the map taking $W\subseteq X$ to $W\cap U$ give inverse bijections (preserving the irreducible decompositions) between the nonempty closed subsets of U and the nonempty closed subsets of X that have no irreducible component contained in the $X\smallsetminus U$. This applies, in particular, to the open immersion

$$\mathbf{A}^n\hookrightarrow\mathbf{P}^n,\quad (x_1,\ldots,x_n)\to[1,x_1,\ldots,x_n].$$

The next exercise describes this correspondence at the level of ideals.

EXERCISE 4.2.14. Let $S=k[x_0,\ldots,x_n]$ and $R=k[x_1,\ldots,x_n]$. For an ideal J in R, we put

$$J^{\rm hom}:=\left(f^{\rm hom}\mid 0\neq f\in J\right),$$

where $f^{\rm hom}=x_0^{\deg(f)}\cdot f(x_1/x_0,\ldots,x_n/x_0)\in S$. On the other hand, if $\mathfrak{a}$ is a homogeneous ideal in S, then we put $\overline{\mathfrak{a}}:=\{h(1,x_1,\ldots,x_n)\mid h\in\mathfrak{a}\}\subseteq R$.

An ideal $\mathfrak{a}$ in S is called x_0-saturated if $(\mathfrak{a}\colon x_0)=\mathfrak{a}$ (recall that $(\mathfrak{a}\colon x_0):=\{u\in S\mid x_0u\in\mathfrak{a}\}$).

i) Show that the above maps give inverse bijections between the ideals in R and the x_0-saturated homogeneous ideals in S.
ii) Show that we get induced bijections between the radical ideals in R and the homogeneous x_0-saturated radical ideals in S. Moreover, a homogeneous radical ideal $\mathfrak{a}$ is x_0-saturated if and only if either no irreducible component of $V(\mathfrak{a})$ is contained in the hyperplane $(x_0=0)$, or if $\mathfrak{a}=S$.
iii) The above correspondence induces a bijection between the prime ideals in R and the prime ideals in S that do not contain x_0.
iv) Consider the open immersion

$$\mathbf{A}^n\hookrightarrow\mathbf{P}^n,\ (u_1,\ldots,u_n)\to(1\colon u_1\colon\ \ldots\colon u_n),$$

which allows us to identify $\mathbf{A}^n$ with the complement of the hyperplane $(x_0=0)$ in $\mathbf{P}^n$. Show that for every ideal J in R we have $\overline{V_{\mathbf{A}^n}(J)}=V_{\mathbf{P}^n}(J^{\rm hom})$.
v) Show that for every homogeneous ideal $\mathfrak{a}$ in S, we have $V_{\mathbf{P}^n}(\mathfrak{a})\cap\mathbf{A}^n=V_{\mathbf{A}^n}(\overline{\mathfrak{a}})$.

EXERCISE 4.2.15. Recall that $GL_{n+1}(k)$ denotes the set of invertible $(n+1)\times(n+1)$ matrices with entries in k. Let $PGL_{n+1}(k)$ denote the quotient $GL_{n+1}(k)/k^*$, where k^* acts on $GL_{n+1}(k)$ by

$$\lambda\cdot(a_{i,j})_{i,j}=(\lambda a_{i,j})_{i,j}.$$

i) Show that $PGL_{n+1}(k)$ has a natural structure of linear algebraic group, and that it is irreducible.

ii) Prove that $PGL_{n+1}(k)$ acts algebraically on $\mathbf{P}^n$.

DEFINITION 4.2.16. Two subsets of $\mathbf{P}^n$ are *projectively equivalent* if they differ by an automorphism in $PGL_{n+1}(k)$ (we will see later that these are, indeed, all automorphisms of $\mathbf{P}^n$).

DEFINITION 4.2.17. A *linear subspace* of $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ defined by an ideal generated by homogeneous polynomials of degree one. A *hyperplane* is a linear subspace of codimension one.

EXERCISE 4.2.18. Consider the projective space $\mathbf{P}^n$.

i) Show that a closed subset Y of $\mathbf{P}^n$ is a linear subspace if and only if the affine cone $C(Y) \subseteq \mathbf{A}^{n+1}$ is a linear subspace.
ii) Show that if L is a linear subspace in $\mathbf{P}^n$ of dimension r, then there is an isomorphism $L \simeq \mathbf{P}^r$.
iii) Show that the hyperplanes in $\mathbf{P}^n$ are in bijection with the points of "another" projective space $\mathbf{P}^n$, called the *dual* of $\mathbf{P}^n$, and usually denoted by $(\mathbf{P}^n)^*$. We denote the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H by $[H]$.
iv) Show that the subset
$$\big\{\big(p,[H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H\big\}$$
is closed in $\mathbf{P}^n \times (\mathbf{P}^n)^*$.
v) Show that given two sets of points in $\mathbf{P}^n$
$$\Gamma = \{P_0, \ldots, P_{n+1}\} \text{ and } \Gamma' = \{Q_0, \ldots, Q_{n+1}\},$$
such that no $(n+1)$ points in the same set lie in a hyperplane, there is a unique $A \in PGL_{n+1}(k)$ such that $A \cdot P_i = Q_i$ for every i.

EXERCISE 4.2.19. Let $X \subseteq \mathbf{P}^n$ be an irreducible closed subset of codimension r. Show that if $H \subseteq \mathbf{P}^n$ is a hypersurface such that X is not contained in H, then every irreducible component of $X \cap H$ has codimension $r+1$ in $\mathbf{P}^n$.

EXERCISE 4.2.20. Let $X \subseteq \mathbf{P}^n$ be a closed subset of dimension r. Show that there is a linear space $L \subseteq \mathbf{P}^n$ of dimension $(n-r-1)$ such that $L \cap X = \emptyset$.

EXERCISE 4.2.21. (The Segre embedding). Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1)-1$, and let us denote the coordinates on $\mathbf{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\phi_{m,n} \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism
$$f_{m,n} \colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n], \quad f_{m,n}(z_{i,j}) = x_i y_j.$$
Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if X and Y are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.

Exercise 4.2.22. (The Veronese embedding). Let n and d be positive integers, and let $M_0, \ldots, M_N$ be all monomials in $k[x_0, \ldots, x_n]$ of degree d (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism $\nu_{n,d}\colon \mathbf{P}^n \to \mathbf{P}^N$ that takes the point $[a_0, \ldots, a_n]$ to the point $\big[M_0(a), \ldots, M_N(a)\big]$.
2) Consider the ring homomorphism $f_d\colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_d)$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\nu_{n,d}$ (in particular, this image is closed).
3) Show that $\nu_{n,d}$ is a closed immersion.
4) Show that if Z is a hypersurface of degree d in $\mathbf{P}^n$ (this means that $I(Z) = (F)$, where F is a homogeneous polyomial of degree d), then there is a hyperplane H in $\mathbf{P}^N$ such that for every projective variety $X \subseteq \mathbf{P}^n$, the morphism $\nu_{n,d}$ induces an isomorphism between $X \cap Z$ and $\nu_{n,d}(X) \cap H$. This shows that the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane.
5) The *rational normal curve* in $\mathbf{P}^n$ is the image of the Veronese embedding $\nu_{1,d}\colon \mathbf{P}^1 \to \mathbf{P}^d$, mapping $[a, b]$ to $[a^d, a^{d-1}b, \ldots, b^d]$. Show that the rational normal curve is the zero-locus of the 2×2-minors of the matrix
$$\begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$

Exercise 4.2.23. Use the Veronese embedding to deduce the assertion in Proposition 4.2.9 from the case when h is a linear form (which follows from Proposition 4.2.2).

Exercise 4.2.24. A *plane Cremona transformation* is a birational map of $\mathbf{P}^2$ into itself. Consider the following example of *quadratic* Cremona transformation: $\phi\colon \mathbf{P}^2 \to \mathbf{P}^2$, given by $\phi(x\colon y\colon z) = (yz\colon xz\colon xy)$, when no two of x, y, or z are zero.

1) Show that ϕ is birational, and its own inverse.
2) Find open subsets $U, V \subset \mathbf{P}^2$ such that ϕ induces an isomorphism $U \simeq V$.
3) Describe the open sets on which ϕ and ϕ^{-1} are defined.

4.3. A generalization: the MaxProj construction

We now describe a generalization of the constructions in the previous two sections. A key idea introduced by Grothendieck in algebraic geometry is that it is often better to study morphisms $f\colon X \to Y$, instead of varieties X (the case of a variety being recovered as the special case when Y is a point). More precisely, instead of studying varieties with a certain property, one should extend this property to morphisms and study it in this context. We begin with one piece of terminology.

Definition 4.3.1. Given a variety Y, a *variety over* Y is a morphism $f\colon X \to Y$, where X is another variety. A morphism between varieties $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ is a morphism of varieties $g\colon X_1 \to X_2$ such that $f_2 \circ g = f_1$. It is clear that we can compose morphisms of varieties over Y and we get, in this way, a category.

Following the above philosophy, we introduce in this section the Proj construction, that allows us to study projective varieties over Y, when Y is affine (as we will see, these are simply closed subvarieties of a product $Y \times \mathbf{P}^n$). We will return later to the case when Y is an arbitrary variety, after discussing quasi-coherent sheaves.

The setting is the following: we fix an $\mathbf{N}$-graded, reduced, finitely generated k-algebra $S = \bigoplus_{m \in \mathbf{N}} S_m$. This implies that S_0 is a finitely generated k-algebra and it is also easy to see that each S_m is a finitely generated S_0-module. We put $S_+ = \bigoplus_{m>0} S_m$.

EXERCISE 4.3.2. Given homogeneous elements $t_0, \ldots, t_n \in S_+$, show that they generate S as an S_0-algebra if and only if they generate S_+ as an ideal.

For the sake of simplicity, we always assume that S is generated as an S_0-algebra by S_1. This condition is equivalent with the fact that S is isomorphic, as a graded ring, to the quotient of $S_0[x_0, \ldots, x_n]$ by a homogeneous ideal, where the grading on this polynomial ring is given by the total degree of the monomials. Note that by the above exercise, our assumption implies that S_1 generates S_+ as an ideal.

Consider the affine varieties $W = \mathrm{MaxSpec}(S)$ and $W_0 = \mathrm{MaxSpec}(S_0)$ (see Exercise 2.2.17 for the notation). The inclusion $S_0 \hookrightarrow S$ corresponds to a morphism $f \colon W \to W_0$. The grading on S translates into an algebraic action of the torus k^* on W, as follows. We have a morphism

$$\alpha \colon k^* \times W \to W$$

corresponding to the k-algebra homomorphism $S \to k[t, t^{-1}] \otimes_k S$ mapping $\sum_i f_i$ to $\sum_i t^i f_i$, where $f_i \in S_i$ for all i. One can check directly that this gives an action of k^* on W, but we prefer to argue as follows: let us choose a surjective graded homomorphism of S_0-algebras $\phi \colon S_0[x_0, \ldots, x_n] \to S$, corresponding to a closed immersion $j \colon W \hookrightarrow W_0 \times \mathbf{A}^{n+1}$ such that if $p \colon W_0 \times \mathbf{A}^{n+1} \to W_0$ is the first projection, we have $p \circ j = f$. As before, we have a morphism

$$\beta \colon k^* \times W_0 \times \mathbf{A}^{n+1} \to W_0 \times \mathbf{A}^{n+1}.$$

Since ϕ is a graded homomorphism, we see that the two morphisms are compatible via j, in the sense that

$$j\big(\alpha(\lambda, w)\big) = \beta\big(\lambda, j(w)\big) \quad \text{for all} \quad \lambda \in k^*, w \in W.$$

It is straightforward to check that

$$\beta(\lambda, w_0, x_0, \ldots, x_n) = (w_0, \lambda x_0, \ldots, \lambda x_n) \quad \text{for all} \quad \lambda \in k^*, w_0 \in W_0, (x_0, \ldots, x_n) \in \mathbf{A}^{n+1}.$$

Therefore β gives an algebraic action of k^* on $W_0 \times \mathbf{A}^{n+1}$, and thus α gives an algebraic action of k^* on W. We will keep using this embedding for describing the action of k^* on W. To simplify the notation, we will write $\lambda \cdot w$ for $\alpha(\lambda, w)$.

LEMMA 4.3.3. *Given the above action of k^* on W, the following hold:*

i) *An orbit consists either of one point or it is 1-dimensional.*
ii) *A point is fixed by the k^*-action if and only if it lies in $V(S_+)$.*
iii) *If O is a 1-dimensional orbit, then O is a closed subset of $W \smallsetminus V(S_+)$, $\overline{O} \simeq \mathbf{A}^1$, and $\overline{O} \cap V(S_+)$ consists of one point.*

PROOF. By embedding W in $W_0 \times \mathbf{A}^{n+1}$ as above, we reduce the assertions in the lemma to the case when $W = W_0 \times \mathbf{A}^{n+1}$, in which case they are all clear. Note that via this embedding, we have $V(S_+) = W_0 \times \{0\}$. $\square$

REMARK 4.3.4. By arguing as in Remark 4.1.5, we see that a closed subset $Z \subseteq W$ is invariant by the k^*-action (that is, $\lambda \cdot Z = Z$ for every $\lambda \in k^*$) if and only if the corresponding ideal $I_W(Z)$ is homogeneous.

DEFINITION 4.3.5. Given S as above, we define $\mathrm{MaxProj}(S)$ to be the set of one-dimensional orbit closures for the action of k^* on W. Since every such orbit is clearly irreducible, being the image of a morphism $k^* \to W$, it follows from Lemma 4.3.3 and Remark 4.3.4 that these orbit closures are in bijection with the homogeneous prime ideals $\mathfrak{q} \subseteq S$ such that $S_+ \not\subseteq \mathfrak{q}$ and $\dim(S/\mathfrak{q}) = 1$.

We put a topology on $X = \mathrm{MaxProj}(S)$ by declaring that a subset is closed if it consists of all 1-dimensional orbit closures contained in some torus-invariant closed subset of W. Equivalently, the closed subsets are those of the form

$$V(I) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid I \subseteq \mathfrak{q}\},$$

for some homogeneous ideal $I \subseteq S$. The assertions in the next lemma, which are straightforward to prove, imply that this gives indeed a topology on $\mathrm{MaxProj}(S)$.

LEMMA 4.3.6. *With the above notation, the following hold:*

i) *We have* $V(0) = \mathrm{MaxProj}(S)$ *and* $V(S) = \emptyset$.

ii) *For every two homogeneous ideals I and J in S, we have*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

iii) *For every family $(I_\alpha)_\alpha$ of homogeneous ideals in S, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

Since every homogeneous ideal is generated by finitely many homogeneous elements, we see that every open set can be written as a finite union of sets of the form

$$D_X^+(f) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid f \not\in \mathfrak{q}\},$$

where $f \in S$ is a homogeneous element. In fact, we may take f of positive degree, since if $t_0, \ldots, t_n \in S_1$ generate S_+, we have

$$D_X^+(f) = \bigcup_{i=0}^n D_X^+(t_i f).$$

As a special case of this equality for $f = 1$, we have

$$\mathrm{MaxProj}(S) = D_X^+(t_0) \cup \ldots \cup D_X^+(t_n).$$

REMARK 4.3.7. It is clear that if I is a homogeneous ideal in S, then $V(I) = V\big(\mathrm{rad}(I)\big)$. Moreover, if

$$I' = \{f \in S \mid f \cdot S_+ \subseteq \mathrm{rad}(I)\},$$

then $V(I) = V(I')$.

For future reference, we give the following variant of graded Nullstellensatz.

PROPOSITION 4.3.8. *Let S be a graded ring as in the proposition. If I is a homogeneous, radical ideal in S, and $f \in S$ is homogeneous, such that $f \in \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{MaxProj}(S)$ with $\mathfrak{q} \supseteq I$, then $f \cdot S_+ \subseteq I$. If $\deg(f) > 0$, then $f \in I$.*

PROOF. We first prove the last assertion, assuming $\deg(f) > 0$. After writing S as a quotient of a polynomial ring over S_0, we see that we may assume that $S = A[x_0, \ldots, x_n]$, with the standard grading. Recall that we take $W_0 = \mathrm{MaxSpec}(S_0)$ and $W = \mathrm{MaxSpec}(S) = W_0 \times \mathbf{A}^{n+1}$. Let $Y \subseteq W$ be the closed subset defined by I. Note that W is k^*-invariant. Our assumption says that f vanishes on $\{w_0\} \times L$, whenever L is a line in $\mathbf{A}^{n+1}$ with $\{w_0\} \times \mathbf{A}^{n+1} \subseteq Y$. On the other hand, since $\deg(f) > 0$, we see that f automatically vanishes along $W_0 \times \{0\}$, hence f vanishes along Y (we use the fact that Y is a union of k^*-orbits). We thus conclude that $f \in I$. The first assertion in the proposition now follows by applying what we know to each product fg, with $g \in S_1$. $\square$

Given an ideal $\mathfrak{q} \in \mathrm{MaxProj}(S)$, let T denote the set of homogeneous elements in $S \smallsetminus \mathfrak{q}$. Recall that the ring of fractions $T^{-1}S$ carries a natural grading, whose degree 0 part is denoted by $S_{(\mathfrak{q})}$. This is a local ring, with maximal ideal $\mathfrak{m}_{\mathfrak{q}} := \mathfrak{q} \cdot T^{-1}S \cap S_{(\mathfrak{q})}$. Similarly, given a homogeneous element $f \in S$, the localization S_f carries a natural grading, whose degree 0 part is denoted $S_{(f)}$.

LEMMA 4.3.9. *For every $t \in S_1$, the following hold:*

i) *We have an isomorphism of graded rings $S_t \simeq S_{(t)}[x, x^{-1}]$.*
ii) *Every homogeneous ideal in S_t is of the form $\bigoplus_{m \in \mathbf{Z}} (I \cap S_{(t)}) t^m$.*
iii) *We have a homeomorphism between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$.*
iv) *For every $\mathfrak{q} \in \mathrm{MaxProj}(S)$, the residue field of $S_{(\mathfrak{q})}$ is equal to k.*

PROOF. Since the element $\frac{t}{1} \in S_t$ has degree 1 and is invertible, it follows easily that the homomorphism of graded $S_{(t)}$-algebras

$$S_{(t)}[x, x^{-1}] \to S_t$$

that maps x to $\frac{t}{1}$ is an isomorphism. This gives i) and the assertion in ii) is straightforward to check.

It is clear that localization induces a bijection between the homogeneous prime ideals in S that do not contain t and the homogeneous prime ideals in S_t. Moreover, it follows from ii) that every such prime ideal in S_t is of the form $\bigoplus_{m \in \mathbf{Z}} \mathfrak{p} t^m$, for a unique prime ideal $\mathfrak{p}$ in $S_{(t)}$. If $\mathfrak{q} \subseteq S$ corresponds to $\mathfrak{p} \subseteq S_{(t)}$, then

$$(S/\mathfrak{q})_t \simeq (S_{(t)}/\mathfrak{p})[x, x^{-1}], \tag{4.3.1}$$

hence

$$\dim(S/\mathfrak{q}) = \dim\big((S/\mathfrak{q})_t\big) = \dim(S_{(t)}/\mathfrak{p}) + 1.$$

Therefore $\mathfrak{q}$ lies in $\mathrm{MaxProj}(S)$ if and only if $\mathfrak{p}$ is a maximal ideal in $S_{(t)}$. This gives the bijection between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$ and it is straightforward to check, using the definitions of the two topologies, that this is a homeomorphism.

Finally, given any $\mathfrak{q} \in \mathrm{MaxProj}(S)$, we can find $t \in S_1$ such that $\mathfrak{q} \in D^+(t)$. If $\mathfrak{p}$ is the corresponding ideal in $S_{(t)}$, then the isomorphism (4.3.1) implies that the residue field of $S_{(\mathfrak{q})}$ is isomorphic as a k-algebra to the residue field of $(S_{(t)})_{\mathfrak{p}}$, hence it is equal to k. $\square$

We now define a sheaf of functions on $X = \mathrm{MaxProj}(S)$, with values in k, as follows. For every open subset U in X, let $\mathcal{O}_X(U)$ be the set of functions $\phi\colon U \to k$ with the following property: for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and homogeneous elements $f, g \in S$ of the same degree such that for every $\mathfrak{q} \in U_x$, we have $g \not\in \mathfrak{q}$ and $\phi(\mathfrak{q})$ is equal to the image of $\frac{f}{g}$ in the residue field of $S_{(\mathfrak{q})}$, which is equal to k by Lemma 4.3.9. It is straightforward to check that $\mathcal{O}_X(U)$ is a k-subalgebra of $\mathcal{F}un_X(U)$ and that, with respect to restriction of functions, $\mathcal{O}_X$ is a sheaf. This is the *sheaf of regular functions* on X. From now on, we denote by $\mathrm{MaxProj}(S)$ the object $(X, \mathcal{O}_X)$ in $\mathcal{T}op_k$.

Remark 4.3.10. It is clear from the definition that we have a morphism in $\mathcal{T}op_k$
$$\mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$$
that maps $\mathfrak{q}$ to $\mathfrak{q} \cap S_0$.

Proposition 4.3.11. *If we have a surjective, graded homomorphism $\phi\colon S \to T$, then we have a commutative diagram*
$$\begin{tikzcd}
\mathrm{MaxProj}(T) \arrow[r, "j"] \arrow[d] & \mathrm{MaxProj}(S) \arrow[d, "f"] \\
\mathrm{MaxSpec}(T_0) \arrow[r, "i"] & \mathrm{MaxProj}(S_0),
\end{tikzcd}$$
in which i is a closed immersion and j given an isomorphism onto $V(I)$ (with the induced sheaf from the ambient space)[2]*, where $I = \ker(\phi)$.*

Proof. Note first that since ϕ is surjective, the induced homomorphism $S_0 \to T_0$ is surjective as well, hence the induced morphism $i\colon \mathrm{MaxSpec}(T_0) \to \mathrm{MaxSpec}(S_0)$ is a closed immersion. Since ϕ is graded and surjective, we have $T_+ = \phi(S_+)$ and $S_+ = \phi^{-1}(T_+)$, hence $S_+ \subseteq \phi^{-1}(\mathfrak{p})$ if and only if $T_+ \subseteq \mathfrak{p}$. We can thus define $j\colon \mathrm{MaxProj}(T) \to \mathrm{MaxProj}(S)$ by $j(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. It is straightforward to see that the diagram in the proposition is commutative and that j gives a homeomorphism of $\mathrm{MaxProj}(T)$ onto the closed subset $V(I)$ of $\mathrm{MaxProj}(S)$. Furthermore, it is easy to see, using the definition, that if U is an open subset of $V(I)$, then a function $\phi\colon U \to k$ has the property that $\phi \circ j$ is regular on $j^{-1}(U)$ if and only if it can be locally extended to a regular function on open subsets in $\mathrm{MaxProj}(S)$. This gives the assertion in the proposition. $\square$

We now consider in detail the case when $S = A[x_0, \ldots, x_n]$, with the standard grading. As before, let $W_0 = \mathrm{MaxSpec}(A)$. We have seen that a point $\mathfrak{p}$ in $X = \mathrm{MaxProj}(S)$ corresponds to a subset in $W_0 \times \mathbf{A}^{n+1}$, of the form $\{w_0\} \times L$, where L is a 1-dimensional linear subspace in k^{n+1}, corresponding to a point in $\mathbf{P}^n$. We thus have a bijection between $\mathrm{MaxProj}(S)$ and $W_0 \times \mathbf{P}^n$. Moreover, since $x_0, \ldots, x_n$ span S_1, we see that
$$X = \bigcup_{i=0}^{n} D_X^+(x_i).$$
The above bijection induces for every i a bijection between $D_X^+(x_i)$ and $W_0 \times D_{\mathbf{P}^n}^+(x_i)$. In fact, this is the same as the homeomorphism between $D_X^+(x_i)$ and
$$\mathrm{MaxSpec}\big(A[x_0, \ldots, x_n]_{(x_i)}\big) = \mathrm{MaxSpec}\big(A[x_0/x_i, \ldots, x_n/x_i]\big)$$
given by assertion iii) in Lemma 4.3.9. Furthermore, arguing as in the proof of Proposition 4.2.2, we see that each of these homeomorphisms gives an isomorphism of objects in $\mathcal{T}op_k$. We thus obtain the following

[2]Once we will show that $\mathrm{MaxProj}(S)$ and $\mathrm{MaxProj}(T)$ are algebraic varieties, this simply says that j is a closed immersion.

PROPOSITION 4.3.12. *If $S = A[x_0, \ldots, x_n]$, with the standard grading, and $W_0 = \mathrm{MaxSpec}(A)$, then we have an isomorphism*

$$\mathrm{MaxProj}(S) \simeq W_0 \times \mathbf{P}^n$$

of varieties over W_0.

COROLLARY 4.3.13. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then $\mathrm{MaxProj}(S)$ is a quasi-projective variety.*

PROOF. By the assumption on S, we have a graded, surjective morphism of S_0-algebras

$$S_0[x_0, \ldots, x_n] \to S.$$

If $W_0 = \mathrm{MaxSpec}(S_0)$, then it follows from Propositions 4.3.11 and 4.3.12 that we have a closed immersion

$$\mathrm{MaxProj}(S) \hookrightarrow \mathrm{MaxProj}\big(S_0[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n,$$

which gives the assertion in the corollary, since a product of quasi-projective varieties is quasi-projective by Exercise 4.2.21. $\square$

REMARK 4.3.14. If X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X, then $\mathrm{MaxProj}(S_X) \simeq X$. More generally, suppose that A is a reduced, finitely generated k-algebra, $W_0 = \mathrm{MaxSpec}(A)$, and X is a closed subvariety of $W_0 \times \mathbf{P}^n$. If I is a radical, homogeneous ideal in $A[x_0, \ldots, x_n]$ such that $X = V(I)$, then

$$X \simeq \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big).$$

Indeed, the surjection

$$A[x_0, \ldots, x_n] \to A[x_0, \ldots, x_n]/I$$

induces by Proposition 4.3.11 a closed immersion

$$\iota\colon \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big) \hookrightarrow \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big).$$

It is then clear that, via the isomorphism $\mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n$ provided by Proposition 4.3.12, the image of ι is equal to X.

DEFINITION 4.3.15. Given an affine variety Y, a variety $f\colon X \to Y$ over Y is *projective* if there is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra S, generated as an S_0-algebra by S_1, such that $Y \simeq \mathrm{MaxSpec}(S_0)$, and X is isomorphic (over Y) to $\mathrm{MaxProj}(S)$. It follows from the above remark, together with Propositions 4.3.11 and 4.3.12, that X is projective over Y if and only if it admits a closed immersion (over Y) in $Y \times \mathbf{P}^n$.

PROPOSITION 4.3.16. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then for every homogeneous $f \in S$, of positive degree, the open subset $D_X^+(f) \subseteq X = \mathrm{MaxProj}(S)$ is affine.*

PROOF. By Proposition 4.3.11, it is enough to prove this when $S = S_0[x_0, \ldots, x_n]$. The argument in this case follows the one in the proof of Proposition 4.2.9. $\square$

We now give a generalization of Proposition 4.2.10 describing the regular functions on the affine open subsets $D_X^+(f)$ in $\mathrm{MaxProj}(S)$.

PROPOSITION 4.3.17. *Let S be a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, and let $X = \mathrm{MaxProj}(S)$. For every homogeneous $f \in S$, of positive degree, consider the homomorphism*

$$\Phi\colon S_{(f)} \to \mathcal{O}\big(D_X^+(f)\big)$$

that maps $\frac{g}{f^m}$ to the function taking $\mathfrak{q} \in D_X^+(f)$ to the image of $\frac{g}{f^m}$ in the residue field of $S_{(\mathfrak{q})}$, which is isomorphic to k. Then Φ is an isomorphism.

PROOF. The proof is similar to that of Proposition 1.4.7. We first show that Φ is injective. Suppose that $\frac{g}{f^m}$ lies in the kernel of Φ. In this case, for every $\mathfrak{q} \in X \smallsetminus V(f)$, we have $g \in \mathfrak{q}$. This implies that $fg \in \mathfrak{q}$ for every $\mathfrak{q} \in X$, hence $fg = 0$ by Proposition 4.3.8, hence $\frac{g}{f^m} = 0$ in $(S_X)_{(f)}$. This proves the injectivity of Φ.

In order to prove the surjectivity of Φ, consider $\phi \in \mathcal{O}\big(D_X^+(f)\big)$. By hypothesis, and using the quasi-compactness of $D_X^+(f)$, we may write

$$D_X^+(f) = V_1 \cup \ldots \cup V_r,$$

for some open subsets V_i such that for every i, there are $g_i, h_i \in S$ homogeneous of the same degree such that for every $\mathfrak{q} \in V_i$, we have $h_i \not\in \mathfrak{q}$ and $\phi(\mathfrak{q})$ is the image of $\frac{g_i}{h_i}$ in the residue field of $S_{(\mathfrak{q})}$. We may assume that $V_i = X \smallsetminus V(f_i)$ for $1 \leq i \leq r$, for some homogeneous $f_i \in S$, of positive degree. Since $h_i \not\in \mathfrak{q}$ for every $\mathfrak{q} \in X \smallsetminus V(f_i)$, it follows from Proposition 4.3.8 that $f_i \in \mathrm{rad}(h_i)$. After possibly replacing f_i by a suitable power, we may assume that $f_i \in (h_i)$ for all i. Finally, after multiplying both g_i and h_i by the same homogeneous element, we may assume that $f_i = h_i$ for all i.

We know that for $u \in X \smallsetminus V(g_i g_j)$ the two fractions $\frac{g_i(u)}{h_i(u)}$ and $\frac{g_j(u)}{h_j(u)}$ have the same image in the residue field of every $S_{(\mathfrak{q})}$. By the injectivity statement we have already proved, this implies that

$$\frac{g_i}{h_i} = \frac{g_j}{h_j} \quad \text{in} \quad S_{h_i h_j}.$$

Therefore there is a positive integer N such that

$$(h_i h_j)^N (g_i h_j - g_j h_i) = 0 \quad \text{for all} \quad i, j.$$

After replacing each g_i and h_i by $g_i h_i^N$ and h_i^{N+1}, respectively, we see that we may assume that

$$g_i h_j - g_j h_i = 0 \quad \text{for all} \quad i, j.$$

On the other hand, since

$$D_X^+(f) = \bigcup_{i=1}^r D_X^+(h_i),$$

we have

$$V(f) = V(h_1, \ldots, h_r),$$

and therefore Proposition 4.3.8 implies that $f \in \mathrm{rad}(h_1, \ldots, h_r)$. We can thus write

$$f^m = \sum_{i=1}^r a_i h_i \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a_1, \ldots, a_r \in S.$$

Moreover, by only considering the terms in $S_{m \cdot \deg(f)}$, we see that we may assume that each a_i is homogeneous, with $\deg(a_i) + \deg(h_i) = m \cdot \deg(f)$.

In order to complete the proof, it is enough to show that

$$\phi = \Phi\left(\frac{a_1g_1 + \ldots + a_rg_r}{f^m}\right).$$

Note that for $\mathfrak{q} \in D_X^+(h_j)$, we have

$$\frac{g_j}{h_j} = \frac{a_1g_1 + \ldots + a_rg_r}{f^m} \quad \text{in} \quad S_{(\mathfrak{q})}$$

since

$$h_j \cdot \sum_{i=1}^r a_ig_i = \sum_{i=1}^r a_ih_ig_j = f^mg_j.$$

This completes the proof. □

REMARK 4.3.18. Suppose that S is an $\mathbf{N}$-graded k-algebra as above and

$$f \colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$$

is the corresponding morphism. If $a \in S_0$ and we consider the $\mathbf{N}$-graded k-algebra S_a, then we have a map

$$j \colon \mathrm{MaxProj}(S_a) \to \mathrm{MaxProj}(S)$$

that maps $\mathfrak{q}$ to its inverse image in S. This gives an open immersion, whose image is $f^{-1}\big(D_Y(a)\big)$: this follows by choosing generators $t_1, \ldots, t_r \in S_1$ of S as an S_0-algebra, and by showing that for every i, the induced map

$$\mathrm{MaxSpec}\big((S_a)_{(t_i)}\big) \to \mathrm{MaxSpec}\big(S_{(t_i)}\big)$$

is an open immersion, with image equal to the principal affine open subset corresponding to $\frac{a}{1} \in S_{(t_i)}$.

REMARK 4.3.19. Suppose again that S is an $\mathbf{N}$-graded k-algebra as above and $f \colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$ is the corresponding morphism. If J is an ideal in S_0, then the inverse image $f^{-1}\big(V(J)\big)$ is the closed subset $V(J \cdot S)$. This is the image of the closed immersion

$$\mathrm{MaxProj}\big(S/\mathrm{rad}(J \cdot S)\big) \hookrightarrow \mathrm{MaxProj}(S)$$

(see Proposition 4.3.11).

REMARK 4.3.20. For every S as above, we have a surjective morphism

$$\pi \colon \mathrm{MaxSpec}(S) \smallsetminus V(S_+) \to \mathrm{MaxProj}(S).$$

Since all fibers are of dimension 1 (in fact, they are all isomorphic to $\mathbf{A}^1 \smallsetminus \{0\}$), we conclude that

$$\dim\big(\mathrm{MaxProj}(S)\big) = \dim\big(\mathrm{MaxSpec}(S) \smallsetminus V(S_+)\big) - 1 \leq \dim(S) - 1.$$

Moreover, this is an equality, unless every irreducible component of maximal dimension of $\mathrm{MaxSpec}(S)$ is contained in $V(S_+)$, in which case we have $\dim(S) = \dim(S_0)$.

EXERCISE 4.3.21. Show that if S is an $\mathbf{N}$-graded k-algebra as above and $X = \mathrm{MaxProj}(S)$, then for every $\mathfrak{q} \in X$, there is a canonical isomorphism

$$\mathcal{O}_{X,\mathfrak{q}} \simeq S_{(\mathfrak{q})}.$$

CHAPTER 5

Proper, finite, and flat morphisms

In this chapter we discuss an algebraic analogue of compactness for algebraic varieties, *completeness*, and a corresponding relative notion, *properness*. In particular, we prove Chow's lemma, which relates arbitrary complete varieties to projective varieties. As a special case of proper morphisms, we have finite morphisms, which we have already encountered in the case of morphisms of affine varieties. We prove an irreducibility criterion for varieties that admit a proper morphism onto an irreducible variety, such that all fibers are irreducible, of the same dimension; we also prove the semicontinuity of fiber dimension for proper morphisms. Finally we discuss an algebraic property, *flatness*, that is very important in the study of families of algebraic varieties.

5.1. Proper morphisms

We will define a notion that is analogous to that of *compactness* for usual topological spaces. Recall that the Zariski topology on algebraic varieties is quasi-compact, but not Hausdorff. As we have seen, *separatedness* is the algebraic counterpart to the Hausdorff property. A similar point of view allows us to define the algebraic counterpart of compactness. The key observation is the following.

REMARK 5.1.1. Let us work in the category of Hausdorff topological spaces. A topological space X is compact if and only if for every other topological space Z, the projection map $p\colon X \times Z \to Z$ is closed. More generally, a continuous map $f\colon X \to Y$ is proper (recall that this means that for every compact subspace $K \subseteq Y$, its inverse image $f^{-1}(K)$ is compact) if and only if for every continuous map $g\colon Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed.

DEFINITION 5.1.2. A morphism of varieties $f\colon X \to Y$ is *proper* if for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is closed. A variety X is *complete* if the morphism from X to a point is proper, that is, for every variety Z, the projection $X \times Z \to Z$ is closed.

REMARK 5.1.3. Note that if $f\colon X \to Y$ is a proper morphism, then it is closed (simply apply the definition to the identity map $Z = Y \to Y$.

We collect in the next proposition some basic properties of this notion.

PROPOSITION 5.1.4. *In what follows all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper morphisms, then $g \circ f$ is a proper morphism.*
ii) *If $f\colon X \to Y$ is a proper morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is proper.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is proper.*

iv) *If X is a complete variety, then any morphism $f\colon X \to Y$ is proper.*
v) *If $f\colon X \to Y$ is a morphism and Y has an open cover $Y = U_1 \cup \ldots \cup U_r$ such that each induced morphism $f^{-1}(U_i) \to U_i$ is proper, then f is proper.*

Proof. Under the assumption in i), given any morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{tikzcd} X \times_Y Y \times_Z W \arrow[r, "p"] \arrow[d] & Y \times_Z W \arrow[r, "q"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

In this case, the big rectangle is Cartesian. The assumption implies that the morphisms p and q are closed, hence the composition $q \circ p$ is closed. This gives i).

For ii), we argue similarly: given a morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{tikzcd} X \times_Y Z \times_Z W \arrow[r] \arrow[d, "p"'] & X \times_Y Z \arrow[r] \arrow[d, "q"] & X \arrow[d, "f"] \\ W \arrow[r] & Z \arrow[r, "g"] & Y. \end{tikzcd}$$

Since the big rectangle is Cartesian, it follows from the hypothesis that p is closed. This proves that q is proper.

If $i\colon X \hookrightarrow Y$ is a closed immersion, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is a closed immersion, whose image is $g^{-1}\big(i(X)\big)$ (see Example 2.4.8). Since every closed immersion is clearly closed, it follows that i is proper.

Suppose now that X is a complete variety and $f\colon X \to Y$ is an arbitrary morphism. We can factor f as

$$X \overset{i_f}{\hookrightarrow} X \times Y \xrightarrow{p} Y,$$

where i_f is the graph morphism associated to f and p is the projection. The map p is proper, by property ii), since X is complete, and i_f is proper by iii), being a closed immersion, since X and Y are separated. Therefore the composition $f = p \circ i_f$ is proper, proving iv).

Under the assumptions in v), consider a morphism $g\colon Z \to Y$ and let $p\colon X \times_Y Z \to Z$ be the induced morphism. We have an induced open cover $Z = \bigcup_{i=1}^r g^{-1}(U_i)$ and for every i, we have an induced morphism

$$p_i\colon p^{-1}\big(g^{-1}(U_i)\big) = f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \to g^{-1}(U_i).$$

Since $f^{-1}(U_i) \to U_i$ is proper, it follows that p_i is closed, which easily implies that p is closed. $\square$

Remark 5.1.5. It follows from property ii) in the proposition that if $f\colon X \to Y$ is a proper morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is a complete variety (possibly empty).

Exercise 5.1.6. Show that if X is a connected, complete variety, then $\Gamma(X, \mathcal{O}_X) = k$. Deduce that a complete variety is also affine if and only if it is a finite set of points.

EXERCISE 5.1.7. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of algebraic varieties, with $g \circ f$ proper, then f is proper. Show that the same holds if we replace "proper" by "closed immersion" or "locally closed immersion.

The following is the main result of this section.

THEOREM 5.1.8. *The projective space* $\mathbf{P}^n$ *is a complete variety.*

PROOF. We need to show that given any variety Y, the projection morphism $p\colon \mathbf{P}^n \times Y \to Y$ is closed. If we consider an affine open cover $Y = \bigcup_{i=1}^r U_i$, it is enough to show that each projection $\mathbf{P}^n \times U_i \to U_i$ is closed. Therefore we may and will assume that Y is affine, say $Y = \operatorname{MaxSpec}(A)$ and we need to show that the canonical morphism

$$f\colon X = \operatorname{MaxProj}\big(A[x_0, \ldots, x_n]\big) \to Y$$

is closed.

Let $W = V(I)$ be a closed subset of X. Recall that if

$$I' = \{f \in A[x_0, \ldots, x_n] \mid f \cdot (x_0, \ldots, x_n) \subseteq \operatorname{rad}(I)\},$$

then $V(I') = V(I)$. We need to show that if $\mathfrak{m} \not\in f(W)$, then there is $h \in A$ such that $\mathfrak{m} \in D_Y(h)$ and $D_Y(h) \cap f(W) = \emptyset$. For this, it is enough to find $h \in A$ such that $h \in I'$ and $h \not\in \mathfrak{m}$. Indeed, in this case, for every $\mathfrak{q} \in W = V(I')$, we have $h \in \mathfrak{q} \cap A$, hence $\mathfrak{q} \cap A \not\in D_Y(h)$.

For every i, with $0 \leq i \leq n$, consider the affine open subset $U_i = D_X(x_i)$ of X. Since U_i is affine, with $\mathcal{O}(U_i) = A[x_0, \ldots, x_n]_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i]$, and $W \cap U_i$ is the open subset defined by

$$I_{(x_i)} = \{g/x_i^m \mid m \geq 0, g \in I \cap A[x_0, \ldots, x_n]_m\},$$

the condition that $\mathfrak{m} \not\in f(U_i)$ is equivalent to the fact that

$$\mathfrak{m} \cdot A[x_0/x_i, \ldots, x_n/x_i] + I_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i].$$

By putting the condition that 1 lies on the left-hand side and by clearing the denominators, we conclude that

$$x_i^m \in \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I \quad \text{for some} \quad m \in \mathbf{N}.$$

Since such a condition holds for all i, we conclude that if $N \gg 0$ then

$$(x_0, \ldots, x_n)^N \subseteq \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I.$$

This implies

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq \mathfrak{m} \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N + (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N$$

and we deduce from Nakayama's lemma that

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N.$$

This implies that there is $h \in A \smallsetminus \mathfrak{m}$ such that $h \cdot (x_0, \ldots, x_n)^N \subseteq I$, hence $h \in I'$. This completes the proof of the theorem. $\square$

COROLLARY 5.1.9. *Every projective variety is complete. Moreover, every morphism of varieties* $f\colon X \to Y$*, with* X *projective, is proper; in particular, it is closed.*

Proof. This follows from the theorem, using various assertions in Proposition 5.1.4. Since X is a projective variety, there is a closed immersion $i\colon X \hookrightarrow \mathbf{P}^n$ for some n. Note that i is proper by assertion iii) in the proposition and $\mathbf{P}^n$ is complete by the theorem, hence we conclude that X is complete, using assertion i) in the proposition. The fact that every morphism $X \to Y$ is proper now follows from assertion iv) in the proposition. $\square$

Corollary 5.1.10. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then the canonical morphism $f\colon \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$ is proper.*

Proof. The morphism f factors as

$$\mathrm{MaxProj}(S) \overset{i}{\hookrightarrow} \mathrm{MaxSpec}(S_0) \times \mathbf{P}^n \xrightarrow{p} \mathrm{MaxSpec}(S_0),$$

where i is a closed immersion and p is the projection. Since $\mathbf{P}^n$ is complete, we deduce that p is proper by assertion ii) in Proposition 5.1.4 and i is a closed immersion by assertion iii) in the proposition. We thus conclude that f is proper by assertion i) in the proposition. $\square$

For the sake of completeness, we mention the following embedding theorem. Its proof is more involved (see, for example, [Con07]).

Theorem 5.1.11. *(Nagata, Deligne) For every algebraic variety X, there is an open immersion $i\colon X \hookrightarrow Y$, where Y is complete. More generally, every morphism of algebraic varieties $f\colon X \to Z$ factors as a composition*

$$X \overset{i}{\hookrightarrow} Y \xrightarrow{p} Z,$$

with i an open immersion and p a proper morphism.

The next exercise deals with an important example of a proper, birational morphism: the *blow-up* of the affine space at the origin.

Exercise 5.1.12. Thinking of $\mathbf{P}^{n-1}$ as the set of lines in $\mathbf{A}^n$, define the *blow-up of $\mathbf{A}^n$ at 0* as the set

$$\mathrm{Bl}_0(\mathbf{A}^n) := \{(P,\ell) \in \mathbf{A}^n \times \mathbf{P}^{n-1} \mid P \in \ell\}.$$

1) Show that $\mathrm{Bl}_0(\mathbf{A}^n)$ is a closed subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$.
2) Show that the restriction of the projection onto the first component gives a morphism $\pi\colon \mathrm{Bl}_0(\mathbf{A}^n) \to \mathbf{A}^n$ that is an isomorphism over $\mathbf{A}^n \smallsetminus \{0\}$.
3) Show that $\pi^{-1}(0) \simeq \mathbf{P}^{n-1}$.
4) Show that π is a proper morphism.

5.2. Chow's lemma

In this section we discuss a result that is very useful in reducing statements about complete varieties to the case of projective varieties. More generally, it allows reducing statements about proper morphisms to a special case of what we will later define as projective morphisms. In order to make things more transparent, we begin with the statement in the absolute case.

Theorem 5.2.1. *(Chow's lemma) If X is a complete variety, then there is a projective variety Y and a morphism $g\colon Y \to X$ that induces an isomorphism between dense open subsets of Y and X.*

Here is the relative version of the above result:

THEOREM 5.2.2. *(Chow's lemma, relative version) If $f\colon X \to Z$ is a proper morphism of algebraic varieties, then there is a morphism $g\colon Y \to X$ that satisfies the following conditions:*

i) *The morphism g induces an isomorphism between dense open subsets of Y and X.*
ii) *The composition $f \circ g$ factors as*

$$Y \overset{i}{\hookrightarrow} Z \times \mathbf{P}^N \overset{p}{\longrightarrow} Z,$$

where i is a closed immersion, N is a positive integer, and p is the projection onto the first factor.

Of course, it is enough to only prove the relative statement. We give the proof following [**Mum88**].

PROOF OF THEOREM 5.2.2. Note first that we may assume that X is irreducible. Indeed, if $X_1, \ldots, X_r$ are the irreducible components of X and if we can construct morphisms $Y_i \to X_i$ as in the theorem, then we have an induced morphism $Y = \bigsqcup_i Y_i \to X$ which satisfies the required conditions (note that if we have closed immersions $Y_i \hookrightarrow Z \times \mathbf{P}^{n_i}$, then we can construct a closed immersion $Y \hookrightarrow Z \times \mathbf{P}^d$, where $d+1 = \sum_{i=1}^r (n_i+1)$, by embedding the $\mathbf{P}^{n_i}$ in $\mathbf{P}^d$ as disjoint linear subspaces).

Suppose now that X is irreducible and consider an affine open cover $X = U_1 \cup \ldots \cup U_n$. Since each U_i is an affine variety, it admits a locally closed immersion in a projective space $\mathbf{P}^{m_i}$. We thus obtain a morphism $U_i \hookrightarrow Z \times \mathbf{P}^{m_i}$ which is again a locally closed immersion (see Exercise 5.1.7) and we denote its image by $\overline{U_i}$. Using the Segre embedding we see that we have a closed immersion

$$\overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \hookrightarrow Z \times \mathbf{P}^{m_1} \times \ldots \times \mathbf{P}^{m_n} \hookrightarrow Z \times \mathbf{P}^N,$$

where $N+1 = \prod_i (m_i+1)$.

Let $U^* = U_1 \cap \ldots \cap U_n$. Since X is irreducible, U^* is a nonempty open subset of X. We consider two locally closed immersions. First, we have

$$\alpha\colon U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion map. This is a locally closed immersion since it factors as the composition

$$U^* \to U^* \times_Z \times \ldots \times_Z U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n},$$

with the first map being a diagonal map (hence a closed immersion) and the second being a product of open immersions (hence an open immersion). We denote by W the closure of $\alpha(U^*)$. Since W is a closed subvariety of $\overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$, we see that the canonical morphism $W \to Z$ factors as

$$W \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

where the first morphism is a closed immersion and the second morphism is the projection onto the first component.

We also consider the map

$$\beta\colon U^* \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion. Again, this is a locally closed immersion, and we denote the closure of its image by Y. It is clear that the projection onto the last n components

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

induces a morphism $q\colon Y \to W$, while the projection onto the first component

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X$$

induces a morphism $g\colon Y \to X$. The restriction of g to U^* is the identity, hence g is birational. Note that q is a closed map, since f is proper. In particular, since its image contains the dense open subset U^*, it follows that q is surjective.

The key assertion is that q is an isomorphism. Once we know this, we see that $f \circ g$ factors as

$$Y \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

with the first map being a closed immersion, and therefore g has the required properties.

In order to show that q is an isomorphism, we consider for every i the map

$$\alpha_i\colon U_i \hookrightarrow X \times_Z \overline{U_i},$$

given by the inclusion on each component. This is again a locally closed immersion. Moreover, since the maps

$$U_i \hookrightarrow X \times_Z U_i \quad \text{and} \quad U_i \hookrightarrow U_i \times_Z \overline{U_i}$$

are closed immersions (as the graphs of the inclusion maps $U_i \hookrightarrow X$ and $U_i \hookrightarrow \overline{U_i}$, respectively), it follows that

$$\overline{\alpha_i(U_i)} \cap (X \times_Z U_i) = \{(u,u) \mid u \in U_i\} = \overline{\alpha_i(U_i)} \cap (U_i \times_Z \overline{U_i}).$$

Consider the projection map

$$\pi_{1,i}\colon X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X \times_Z \overline{U_i}.$$

Since $\pi_{1,i}(Y) \subseteq \overline{\alpha_i(U^*)} = \overline{\alpha_i(U_i)}$, we deduce that

$$V_i := Y \cap (X \times_Z \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n})$$
$$= Y \cap (U_i \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}) = Y \cap \{(u_0, u_1, \ldots, u_n) \mid u_0 = u_i \in U_i\}.$$

The first formula for V_i shows that $V_i = q^{-1}(V_i')$, where

$$V_i' = W \cap \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n}$$

is an open subset of W. From the second formula for V_i' we deduce that $Y = V_1 \cup \ldots \cup V_n$ and since q is surjective, it follows that $W = V_1' \cup \ldots \cup V_n'$.

In order to conclude the proof, it is thus enough to show that each induced morphism $V_i \to V_i'$ is an isomorphism. We define the morphism

$$\gamma_i\colon V_i' \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

by

$$\gamma_i(u_1, \ldots, u_n) = (u_i, u_1, \ldots, u_n).$$

This is well-defined, and since it maps U^* to U^*, it follows that its image lies inside Y. Moreover, we clearly have $q \circ \gamma_i(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$; in particular, the image of γ_i lies inside V_i. Finally, if $u = (u_0, u_1, \ldots, u_n) \in V_i$, then $u_0 = u_i$ lies in U_i, hence $u = \gamma_i\big(q(u)\big)$. This shows that γ_i gives an inverse of $q|_{V_i}\colon V_i \to V_i'$ and thus completes the proof of the theorem. $\square$

5.3. Finite morphisms

We discussed in Chapter 3 finite morphisms between affine varieties. We now consider the general notion.

DEFINITION 5.3.1. The morphism $f\colon X \to Y$ between algebraic varieties is *finite* if for every affine open subset $V \subseteq Y$, its inverse image $f^{-1}(V)$ is an affine variety, and the induced k-algebra homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X\big(f^{-1}(V)\big)$$

is finite.

It is not clear that in the case when X and Y are affine varieties, the above definition coincides with our old one. However, this follows from the following theorem.

PROPOSITION 5.3.2. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. If there is an affine open cover $Y = \bigcup_{i=1}^r V_i$ such that each $U_i = f^{-1}(V_i)$ is an affine variety and the induced morphism*

$$\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$$

is finite, then f is a finite morphism.

We begin with a lemma which is useful in several other situations.

LEMMA 5.3.3. *If X is an algebraic prevariety and U, $V \subseteq X$ are affine open subsets, then for every $p \in U \cap V$, there is open neighborhood $W \subseteq U \cap V$ of p that is a principal affine open subset in both U and V.*

PROOF. We first choose an open neighborhood $W_1 \subseteq U \cap V$ of p of the form $W_1 = D_U(f)$ for some $f \in \mathcal{O}(U)$. We next choose another open neighborhood $W \subseteq W_1$ of the form $W = D_V(g)$, for some $g \in \mathcal{O}(V)$. It is enough to show that W is a principal affine open subset also in U.

Since $\mathcal{O}(W_1) \simeq \mathcal{O}(U)_f$, it follows that there is $h \in \mathcal{O}(U)$ such that $g|_{W_1} = \frac{h}{f^m}$ for some non-negative integer m. In this case we have $W = D_U(fh)$, completing the proof. $\square$

PROOF OF PROPOSITION 5.3.2. Note that if W is a principal affine open subset of some of the V_i, then $f^{-1}(W)$ is affine and the induced morphism

$$\mathcal{O}_Y(W) \to \mathcal{O}_X\big(f^{-1}(W)\big) \tag{5.3.1}$$

is finite. Indeed, if $W = D_{V_i}(\phi)$, then $f^{-1}(W) = D_{U_i}(\phi \circ f)$ is affine and the morphism (5.3.1) is identified to

$$\mathcal{O}_Y(V_i)_\phi \to \mathcal{O}(U_i)_{\phi\circ f},$$

which is finite.

Let $V \subseteq Y$ be an arbitrary affine open subset. Since V is covered by the open subsets $V \cap V_i$, applying for each pair (V, V_i) Lemma 5.3.3, and using what we have already seen, we see that we can cover V by finitely many principal affine open subsets $W_1, \ldots, W_s$, such that each $f^{-1}(W_i)$ is affine and the induced morphism

$$\mathcal{O}_Y(W_i) \to \mathcal{O}_X\big(f^{-1}(W_i)\big) \tag{5.3.2}$$

is finite. Let us write $W_i = D_V(\phi_i)$, for some $\phi_i \in \mathcal{O}_Y(V)$. The condition that $V = \bigcup_{i=1}^s W_i$ is equivalent to the fact that $\phi_1, \ldots, \phi_s$ generate the unit ideal in $\mathcal{O}_Y(V)$.

This implies that the $f^{\#}(\phi_i) = \phi_i \circ f$ generate the unit ideal in $\mathcal{O}_X\big(f^{-1}(V)\big)$. Since each $D_{f^{-1}(V)}(\phi_i \circ f)$ is affine, it follows from Proposition 2.3.16 that $f^{-1}(V)$ is affine.

Moreover, the $\mathcal{O}_Y(V)$-module $\mathcal{O}_X\big(f^{-1}(V)\big)$ has the property that $\mathcal{O}_X\big(f^{-1}(V)\big)_{\phi_i}$ is a finitely generated module over $\mathcal{O}_Y(V)_{\phi_i}$ for all i. Since the ϕ_i generate the unit ideal in $\mathcal{O}_Y(V)$, we conclude using Corollary C.3.5 that $\mathcal{O}_X\big(f^{-1}(V)\big)$ is a finitely generated $\mathcal{O}_Y(V)$-module. $\square$

Remark 5.3.4. If $f\colon X \to Y$ is a finite morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is finite. Indeed, if V is an affine open neighborhood of y, then $U = f^{-1}(V)$ is affine and the induced morphism $f^{-1}(V) \to V$ is finite. Applying to this morphism Remark 3.2.7, we deduce that $f^{-1}(y)$ is finite.

In the next proposition we collect some general properties of finite morphisms.

Proposition 5.3.5. *In what follows, all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms, then $g \circ f\colon X \to Z$ is a finite morphism.*
ii) *If $f\colon X \to Y$ is a finite morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $h\colon X \times_Y Z \to Z$ is a finite morphism.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is a finite morphism.*
iv) *If $f\colon X \to Y$ is a morphism and $Y = V_1 \cup \ldots \cup V_r$ is an open cover such that each induced morphism $f^{-1}(V_i) \to V_i$ is finite, then f is finite.*

Proof. The assertions in i) and iii) are straightforward to see and the one in iv) follows by covering each V_i by affine open subsets and then using Proposition 5.3.2. We now prove the assertion in ii). Let $V = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. For every i, consider an affine open cover $g^{-1}(V_i) = \bigcup_j U_{i,j}$. Note that we have

$$h^{-1}(U_{i,j}) = f^{-1}(V_i) \times_{V_i} U_{i,j}.$$

Using Proposition 5.3.2, we thus see that it is enough to prove the assertion when X, Y, and Z are affine varieties. In this case, $X \times_Y Z$ is affine, since it is a closed subvariety of $X \times Z$ (see Proposition 2.4.7). Moreover, the morphism

$$h^{\#}\colon \mathcal{O}(Z) \to \mathcal{O}(X \times_Y Z)$$

factors as

$$\mathcal{O}(Z) = \mathcal{O}(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{f^{\#}\otimes 1} \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{p} \mathcal{O}(X \times_Y Z).$$

The homomorphism $f^{\#}\otimes 1$ is finite since $f^{\#}$ is finite and p is surjective (this follows, for example, from the fact that $X \times_Y Z$ is a closed subvariety of $X \times Z$, but see also Remark 2.4.9 for a more precise statement). This completes the proof of ii). $\square$

The next proposition extends to arbitrary morphisms some properties that we have already proved for finite morphisms between affine varieties.

Proposition 5.3.6. *Let $f\colon X \to Y$ be a finite morphism.*

1) *The map f is closed.*
2) *If $Z_1 \subsetneq Z_2$ are irreducible closed subsets of X, then $f(Z_1) \subsetneq f(Z_2)$ are irreducible, closed subsets of Y.*
3) *If f is surjective, then given any irreducible, closed subset W of Y, there is an irreducible, closed subset Z in X such that $f(Z) = W$.*

4) *If Z_1 is an irreducible, closed subset of X and $W_1 \supseteq W_2$ are irreducible, closed subsets of Y, with $W_1 = f(Z_1)$, then there is $Z_2 \subseteq Z_1$ irreducible and closed such that $f(Z_2) = W_2$.*

PROOF. We have already seen these properties when X and Y are affine varieties in Corollary 3.2.9. Let $Y = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. By definition, each $f^{-1}(V_i)$ is affine and the induced morphism $f^{-1}(V_i) \to V_i$ is finite, hence it satisfies the properties in the proposition. Since each map $f^{-1}(V_i) \to V_i$ is closed, it follows that f is closed, hence we have 1). The assertions in 2), 3), and 4) similarly follow from the corresponding ones for the morphisms $f^{-1}(V_i) \to V_i$. $\square$

COROLLARY 5.3.7. *Every finite morphism $f\colon X \to Y$ is proper.*

PROOF. Given any morphism of varieties $g\colon Z \to Y$, assertion ii) in Proposition 5.3.5 implies that the induced morphism $X \times_Y Z \to Z$ is finite. This is thus closed by assertion 1) in Proposition 5.3.6, which shows that f is proper. $\square$

We mention the following converse to Corollary 5.3.7. This is a deeper result that we will only prove later.

THEOREM 5.3.8. *If $f\colon X \to Y$ is a proper morphism with finite fibers, then f is finite.*

The following proposition gives another property of finite morphisms that we have seen for affine varieties.

PROPOSITION 5.3.9. *If $f\colon X \to Y$ is a finite, surjective morphism of algebraic varieties, then for every closed subset Z of X, we have*

$$\dim\big(f(Z)\big) = \dim(Z).$$

Moreover, if Z is irreducible, then

$$\mathrm{codim}_Y\big(f(Z)\big) = \mathrm{codim}_X(Z).$$

PROOF. This can be deduced from the properties in Proposition 5.3.6 as in the proof of Corollary 3.2.10. $\square$

EXAMPLE 5.3.10. If L_1 and L_2 are disjoint linear subspaces of $\mathbf{P}^n$, with $\dim(L_1)+\dim(L_2) = n-1$, then the *projection of* $\mathbf{P}^n$ *onto* L_2, *with center* L_1 is the morphism $\pi\colon \mathbf{P}^n \smallsetminus L_1 \longrightarrow L_2$ such that $\pi(p)$ is the intersection of L_2 with the linear span $\langle L_1, p\rangle$ of L_1 and p. In order to see that this is indeed a morphism, let's apply an element of $PGL_{n+1}(k)$ to $\mathbf{P}^n$ in order to have

$$L_1 = (x_0 = \ldots = x_r = 0) \quad \text{and} \quad L_2 = (x_{r+1} = \ldots = x_n = 0).$$

We consider the isomorphism $\mathbf{P}^r \simeq L_2$ given by

$$[u_0, \ldots, u_r] \to [u_0, \ldots, u_r, 0, \ldots, 0].$$

Note that if $p = [a_0, \ldots, a_n] \in \mathbf{P}^n \smallsetminus L_1$, then the linear span of L_1 and p is the set

$$\{[\lambda a_0, \ldots, \lambda a_r, b_{r+1}, \ldots, b_n] \mid \lambda \in k^*, b_{r+1}, \ldots, b_n \in k\}.$$

We thus see that the map $\pi\colon \mathbf{P}^n \smallsetminus L_1 \to \mathbf{P}^r$ is given by

$$\pi\big([a_0, \ldots, a_n]\big) = [a_0, \ldots, a_r]$$

and it is now straightforward to check that π is a morphism.

Let us show that if X is a closed subvariety of $\mathbf{P}^n$ such that $X \cap L_1 = \emptyset$, then the induced morphism $\pi_X \colon X \to L_2$ is finite. This is an easy consequence of Theorem 5.3.8, since the hypothesis implies that π_X has finite fibers: the fiber over a point $q \in L_2$ lies in the linear span $\langle L_1, q\rangle$ of L_1 and q, which has dimension equal to $\dim(L_1)+1$; if this is not finite, then its intersection with the hyperplane $L_1 \subseteq \langle L_1, q\rangle$ would be non-empty by Corollary 4.2.12. However, we will give a direct argument for the finiteness of π_X, since we haven't proved Theorem 5.3.8 yet.

After a linear change of coordinates as above, we may assume that

$$\pi \colon \mathbf{P}^n \smallsetminus L_1 \to \mathbf{P}^r, \quad \pi_X\big([a_0,\ldots,a_n]\big) = [a_0,\ldots,a_r].$$

Note that π is the composition of $(n-r)$ maps, each of which is the projection from a point onto a hyperplane. Indeed, if

$$\pi_i \colon \mathbf{P}^{r+i} \smallsetminus \{[0,\ldots,0,1]\} \to \mathbf{P}^{r+i-1}, \quad \pi_i\big([u_0,\ldots,u_{r+i}]\big) = [u_0,\ldots,u_{r+i-1}]$$

for $1 \le i \le n-r$, then it is clear that $\pi = \pi_1 \circ \ldots \circ \pi_{n-r}$. Since a composition of finite morphisms is finite, we see that we only need to prove our assertion when $r = n-1$.

It is enough to show that if $U_i = (x_i \neq 0) \subseteq \mathbf{P}^{n-1}$, then for each i, with $0 \le i \le n-1$, the inverse image $\pi_X^{-1}(U_i)$ is affine and the induced homomorphism

$$\mathcal{O}(U_i) \to \mathcal{O}\big(\pi_X^{-1}(U_i)\big) \tag{5.3.3}$$

is a finite homomorphism. The fact that $\pi_X^{-1}(U_i)$ is affine is clear, since this is equal to $D_X^+(x_i)$, hence it is affine by Proposition 4.2.9. Moreover, by Proposition 4.2.10, we can identify the homomorphism (5.3.3) with

$$k[x_0,\ldots,x_{n-1}]_{(x_i)} = k\left[\frac{x_0}{x_i},\ldots,\frac{x_{n-1}}{x_i}\right] \to (S_X)_{(x_i)}, \tag{5.3.4}$$

where S_X is the homogeneous coordinate ring of X. Since $(S_X)_{(x_i)}$ is generated by $\frac{x_j}{x_i}$, with $0 \le j \le n$, in order to show that (5.3.4) is a finite homomorphism, it is enough to show that each $\frac{x_j}{x_i} \in (S_X)_{(x_i)}$ is integral over $k\left[\frac{x_0}{x_i},\ldots,\frac{x_{n-1}}{x_i}\right]$. This is clear if $j \le n-1$, hence we only need to consider $\frac{x_n}{x_i}$. By hypothesis, we have $[0,\ldots,0,1] \not\in X$. Therefore there is a homogeneous polynomial f, say of degree d, in the ideal I_X corresponding to X such that x_n^d appears in f with nonzero coefficient. If $d = 0$, then X is empty, in which case the assertion to prove is trivial. If $d > 0$, we may assume that $f = x_n^d + \sum_{i=1}^d g_i(x_0,\ldots,x_{n-1})x_n^{d-i}$. Dividing by x_i^d, we thus conclude that

$$\left(\frac{x_n}{x_i}\right)^d + \sum_{i=1}^d g_i\left(\frac{x_0}{x_i},\ldots,\frac{x_{n-1}}{x_i}\right) = 0 \quad \text{in} \quad (S_X)_{(x_i)},$$

hence $\frac{x_n}{x_i}$ is integral over $k\left[\frac{x_0}{x_i},\ldots,\frac{x_{n-1}}{x_i}\right]$. This gives our assertion.

In particular, we see that if X is a projective d-dimensional variety, then there is a finite morphism $X \to \mathbf{P}^d$. Indeed, if X is a closed subvariety of $\mathbf{P}^n$ different from $\mathbf{P}^n$, by projecting from a point not in X we obtain a finite morphism $X \to Y$, where Y is a d-dimensional subvariety of $\mathbf{P}^{n-1}$. By iterating this construction we obtain a finite morphism $X \to \mathbf{P}^d$.

PROPOSITION 5.3.11. *Let $f\colon X \to Y$ be a dominant morphism of irreducible varieties. If* $\dim(X) = \dim(Y)$*, then there is a non-empty open subset V of Y such that the induced morphism $f^{-1}(V) \to V$ is finite (in this case, one says that f is* generically finite*).*

Note that the converse also holds by Proposition 5.3.9.

PROOF OF PROPOSITION 5.3.11. We may clearly replace Y by an affine open subset and X by the inverse image of this subset, in order to assume that Y is an affine variety. In fact, we may assume that X is affine as well. Indeed, let us choose an affine open subset U of X and suppose that we know the assertion in the proposition for the induced morphism $U \to Y$. In other words, we know that there is a non-empty open subset V of Y, such that the induced morphism $g\colon U \cap f^{-1}(V) \to V$ is finite. Note that if $Z = \overline{f(X \smallsetminus U)}$, then

$$\dim(Z) \leq \dim(X \smallsetminus U) < \dim(X) = \dim(Y),$$

hence Z is a proper closed subset of Y. If we take $V' = V \smallsetminus Z$, then V' is non-empty and the induced morphism $g^{-1}(V') = U \cap f^{-1}(V') \to V'$ is finite. However, it follows from the definition of X' that $f^{-1}(V') \subseteq U$, which implies that V' satisfies the requirement in the proposition.

Suppose now that both X and Y are affine varieties, and consider the homomorphism

$$f^{\#}\colon A = \mathcal{O}(Y) \to \mathcal{O}(X) = B$$

corresponding to f. Note that this is injective since f is dominant. Let $k(Y) = \mathrm{Frac}(A)$ be the field of rational functions of Y. The assumption that $\dim(X) = \dim(Y)$ implies that $\mathrm{Frac}(B)$ is algebraic, hence finite, over $\mathrm{Frac}(A)$ by Corollary 3.3.9. Noether's Normalization lemma thus implies that $B \otimes_A k(Y)$ is a finite $k(Y)$-algebra. Let $b_1, \ldots, b_r \in B$ be generators of B as a k-algebra. Since each b_i is algebraic over $k(Y)$, we see that there is $f_i \in A$ such that $\frac{b_i}{1}$ is integral over A_{f_i}. This implies that if $f = \prod_i f_i$, then each $\frac{b_i}{1}$ is integral over A_f, hence $A_f \to B_f$ is a finite homomorphism. Therefore $V = D_Y(f)$ satisfies the assertion in the proposition. $\square$

We end this section by introducing another class of morphisms.

DEFINITION 5.3.12. A morphism of algebraic varieties $f\colon X \to Y$ is *affine* if for every affine open subset $V \subseteq Y$, its inverse image $f^{-1}(V)$ is affine.

The next proposition shows that, in fact, it is enough to check the property in the definition for an affine open cover of the target. In particular, this implies that every morphism of affine varieties is affine.

PROPOSITION 5.3.13. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. If there is an open cover $Y = V_1 \cup \ldots \cup V_r$, with each V_i affine, such that all $f^{-1}(V_i)$ are affine, then f is an affine morphism.*

PROOF. The argument follows as in the proof of Proposition 5.3.2. $\square$

5.4. Semicontinuity of fiber dimension for proper morphisms

Our goal in this section is to prove the following semicontinuity result for the dimensions of the fibers of a proper morphism.

THEOREM 5.4.1. *If $f\colon X \to Y$ is a proper morphism of algebraic varieties, then for every non-negative integer m, the set*

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$

is closed in Y.

This is an immediate consequence of the following more technical statement, but which is valid for an arbitrary morphism.

PROPOSITION 5.4.2. *If $f\colon X \to Y$ is a morphism of algebraic varieties, then for every non-negative integer m, the set X_m consisting of those $x \in X$ such that the fiber $f^{-1}\big(f(x)\big)$ has an irreducible component of dimension $\geq m$ passing through x, is closed.*

PROOF. Arguing by Noetherian induction, we may assume that the assertion in the proposition holds for every $f|_Z$, where Z is a proper closed subset of X. If X is not irreducible and $X^{(1)}, \ldots, X^{(r)}$ are the irreducible components of X, we know that each $X_m^{(j)}$ is closed in $X^{(j)}$, hence in X. Since

$$X_m = \bigcup_{j=1}^{r} X_m^{(j)},$$

we conclude that X_m is closed.

Therefore we may and will assume that X is irreducible. Of course, we may replace Y by $\overline{f(X)}$ and thus assume that Y is irreducible and f is dominant. In this case, if $m \leq \dim(X) - \dim(Y)$, then $X_m = X$ by Theorem 3.4.1, hence we are done. On the other hand, it follows from Theorem 3.4.2 that there is an open subset V of Y such that if $y \in V$, then every irreducible component of $f^{-1}(y)$ has dimension equal to $\dim(X) - \dim(Y)$. We deduce that if $m > \dim(X) - \dim(Y)$ and we put $Z = X \smallsetminus f^{-1}(V)$, then Z is a proper closed subset of X such that $X_m = Z_m$. Since Z_m is closed in Z, hence in X, by the inductive assumption, we are done. $\square$

PROOF OF THEOREM 5.4.1. With the notation in the proposition, we have

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\} = f(X_m).$$

Since X_m is closed and f is proper, it follows that $f(X_m)$ is closed. $\square$

REMARK 5.4.3. If $f\colon X \to Y$ is an arbitrary morphism of algebraic varieties, we can still say that the subset

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$

is constructible in Y. Indeed, with the notation in Proposition 5.4.2, we see that this set is equal to $f(X_m)$. Since X_m is closed in X by the proposition, its image $f(X_m)$ is constructible by Theorem 3.5.3.

Note that also the set

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) = m\}$$

is constructible in Y, being the difference of two constructible subsets.

5.5. An irreducibility criterion

The following result is an useful irreducibility criterion.

PROPOSITION 5.5.1. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that Y is irreducible and that all fibers of f are irreducible, of the same dimension d (in particular, f is surjective). If either one of the following two conditions holds:*

a) *X is pure-dimensional;*
b) *f is closed,*

then X is irreducible, of dimension $d + \dim(Y)$.

We will be using the proposition for proper morphisms f, so that condition b) will be automatically satisfied.

PROOF OF PROPOSITION 5.5.1. We will show that in general–that is, without assuming a) or b)– the following assertions hold:

i) There is a unique irreducible component of X that dominates Y, and
ii) Every irreducible component Z of X is a union of fibers of f. Its dimension is equal to $\dim\big(\overline{f(Z)}\big) + d$.

Let $X = X_1 \cup \ldots \cup X_r$ be the irreducible decomposition of X. For every $y \in Y$, we put $X_y = f^{-1}(y)$, and $(X_j)_y = X_y \cap X_j$. Since $X_y = \bigcup_{j=1}^r (X_j)_y$, and since X_y is irreducible, it follows that for every y, there is j such that $X_y = (X_j)_y$.

For every i, let $U_i := X_i \smallsetminus \bigcup_{j\neq i} X_j$. This is a nonempty open subset of X. Note that if $y \in f(U_i)$, then X_y can't be contained in $(X_j)_y$ for any $j \neq i$. It follows that

$$X_y = (X_i)_y \quad \text{for all} \quad y \in f(U_i). \tag{5.5.1}$$

Note that some X_ℓ has to dominate Y: since f is surjective, we have $Y = \bigcup_j \overline{f(X_j)}$, and since Y is irreducible, we see that there is ℓ such that $Y = \overline{f(X_\ell)}$. In this case we also have $Y = \overline{f(U_\ell)}$, and Theorem 3.4.2 implies that there is an open subset V of Y contained in $f(U_\ell)$. We deduce from (5.5.1) that $X_y = (X_\ell)_y$ for every $y \in V$, hence for all $j \neq \ell$, we have $X_j \smallsetminus X_\ell \subseteq f^{-1}(Y \smallsetminus V)$. Therefore $X_j = \overline{X_j \smallsetminus X_\ell}$ is contained in $f^{-1}(Y \smallsetminus V)$ (which is closed). We conclude that X_j does not dominate Y for any $j \neq \ell$.

On the other hand, it follows from Theorems 3.4.1 and 3.4.2 that for every i, the following hold

α) $\dim(X_i)_y \geq \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$ for every $y \in f(X_i)$ and
β) There is an open subset W_i in $\overline{f(X_i)}$ such that for all $y \in W_i$ we have $\dim(X_i)_y = \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$.

Since $W_i \cap f(U_i) \neq \emptyset$, it follows from β) and (5.5.1) that $d = \dim(X_i) - \dim\big(\overline{f(X_i)}\big)$ for every i. Furthermore, for every $y \in f(X_i)$, we know by α) that $(X_i)_y$ is a closed subset of dimension d of the irreducible variety X_y of dimension d. Therefore $X_y = (X_i)_y$ for all $y \in f(X_i)$, which says that each X_i is a union of fibers of f. Therefore assertions i) and ii) hold.

In particular, it follows from i) and ii) that if $i \neq \ell$, then $\overline{f(X_i)}$ is a proper subset of Y, and

$$\dim(X_i) = d + \dim(\overline{f(X_i)}) < d + \dim(Y) = \dim(X_\ell).$$

If X is pure-dimensional, then we conclude that X is irreducible.

Suppose now that f is a closed map. Since $f(X_\ell)$ is closed, it follows that $f(X_\ell) = Y$. We have seen that X_ℓ is a union of fibers of f, hence $X_\ell = X$. Therefore X is irreducible also in this case. $\square$

EXAMPLE 5.5.2. Consider the incidence correspondence between points and hyperplanes in $\mathbf{P}^n$, defined as follows. Recall that $(\mathbf{P}^n)^*$ is the projective space parametrizing the hyperplanes in $\mathbf{P}^n$. We write $[H]$ for the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H. Consider the following subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$:

$$\mathcal{Z} = \left\{ \big(p, [H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H \right\}.$$

Note that if we take homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbf{P}^n$ and $y_0, \ldots, y_n$ on $(\mathbf{P}^n)^*$, then $\mathcal{Z}$ is defined by the condition $\sum_{i=0}^n x_i y_i = 0$. It is the straightforward to see, by considering the products of the affine charts on $\mathbf{P}^n$ and $(\mathbf{P}^n)^*$, that $\mathcal{Z}$ is a closed subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$. The projections on the two components induce morphisms $\pi_1 \colon \mathcal{Z} \to \mathbf{P}^n$ and $\pi_2 \colon \mathcal{Z} \to (\mathbf{P}^n)^*$. For every $[H] \in (\mathbf{P}^n)^*$, we have $\pi_2^{-1}([H]) \simeq H$, hence all fibers of π_2 are irreducible, of dimension $n-1$. Since $(\mathbf{P}^n)^*$ is irreducible, it follows from Proposition 5.5.1 that $\mathcal{Z}$ is irreducible, of dimension $2n-1$. Note that the picture is symmetric: for every $p \in \mathbf{P}^n$, the fiber $\pi_1^{-1}(p)$ consists of all hyperplanes in $\mathbf{P}^n$ that contain p, which is a hyperplane in $(\mathbf{P}^n)^*$.

5.6. Flat morphisms

We begin by reviewing the concept of a flat module. Recall that if M is a module over a commutative ring A, then the functor $M \otimes_A -$ from the category of A-modules to itself, is right exact. The module M is *flat* if, in fact, this is an exact functor. Given a ring homomorphism $\phi \colon A \to B$, we say that ϕ is flat (or that B is a flat A-algebra) if B is flat as an A-module.

EXAMPLE 5.6.1. The ring A is flat as an A-module, since $A \otimes_A M \simeq M$ for every A-module M.

EXAMPLE 5.6.2. A direct sum of flat A-modules is flat, since tensor product commutes with direct sums and taking a direct sum is an exact functor. It follows from the previous example that every free module is flat. In particular, every vector space over a field is flat.

EXAMPLE 5.6.3. If $(M_i)_{i \in I}$ is a filtered direct system of flat A-modules, then $M = \varinjlim_{i \in I} M_i$ is a flat A-module. Indeed, since the tensor product commutes with direct limits, for every injective morphism of A-modules $N_1 \hookrightarrow N_2$, the induced morphism

$$N_1 \otimes_A M \to N_2 \otimes_A M$$

can be identified with the direct limit of the injective morphisms

$$N_1 \otimes M_i \to N_2 \otimes_A M_i.$$

Since a filtered direct limit of injective morphisms is injective, we obtain our assertion.

EXAMPLE 5.6.4. If M is a flat A-module, then for every non-zero-divisor $a \in A$, multiplication by a is injective on A, and after tensoring with M, we see that multiplication by a is injective also on M. In particular, if A is a domain, then M is torsion-free.

The converse holds if A is a PID: every torsion-free A-module is flat. Indeed, M is the filtered direct limit of its finitely generated submodules, which are free A-modules, being finitely generated and torsion-free over a PID. Since every filtered direct limit of flat modules is flat, we conclude that M is flat.

EXAMPLE 5.6.5. For every ring A and every multiplicative system $S \subseteq A$, the A-algebra $S^{-1}A$ is flat. Indeed, for every A-module N, we have a canonical isomorphism

$$S^{-1}A \otimes_A N \simeq S^{-1}N$$

and the functor taking N to $S^{-1}N$ is exact.

We do not discuss the more subtle aspects of flatness, which we do not need at this point, and whose treatment is better handled using the Tor functors. We only collect in the next proposition some very easy properties that we need in order to define flatness for morphisms of algebraic varieties.

PROPOSITION 5.6.6. *Let M be an A-module.*

i) *If M is flat, then for every ring homomorphism $A \to B$, the B-module $M \otimes_A B$ is flat.*
ii) *If $B \to A$ is a flat homomorphism and M is flat over A, then M is flat over B.*
iii) *If $\mathfrak{p}$ is a prime ideal in A and M is an $A_{\mathfrak{p}}$-module, then M is flat over A if and only if it is flat over $A_{\mathfrak{p}}$.*
iv) *If $B \to A$ is a ring homomorphism, then M is flat over B if and only if for every prime (respectively, maximal) ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat.*

PROOF. The assertion in i) follows from the fact that for every B-module N, we have a canonical isomorphism

$$(M \otimes_A B) \otimes_B N \simeq M \otimes_A N.$$

Similarly, the assertion in ii) follows from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M \simeq (N \otimes_B A) \otimes_A M.$$

With the notation in iii), note that if M is a flat $A_{\mathfrak{p}}$-module, since $A_{\mathfrak{p}}$ is a flat A-algebra, we conclude that M is flat over A by ii). The converse follows from the fact that if N is an $A_{\mathfrak{p}}$-module, then we have a canonical isomorphism

$$N \otimes_{A_{\mathfrak{p}}} M \simeq N \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M) \simeq N \otimes_A M.$$

We now prove iv). Suppose first that M is flat over B and let $\mathfrak{p}$ be a prime ideal in A. We deduce that $M_{\mathfrak{p}}$ is flat over B from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M_{\mathfrak{p}} \simeq (N \otimes_B M) \otimes_A A_{\mathfrak{p}}.$$

Conversely, suppose that for every maximal ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat. Given an injective map of B-modules $N' \hookrightarrow N$, we see that for every maximal ideal $\mathfrak{p}$, the induced homomorphism

$$N' \otimes_B M_{\mathfrak{p}} \simeq (N' \otimes_B M)_{\mathfrak{p}} \to (N \otimes_B M)_{\mathfrak{p}} \simeq N \otimes_B M_{\mathfrak{p}}$$

is injective. This implies the injectivity of

$$N' \otimes_B M \to N \otimes_B M$$

by Corollary C.3.4. $\square$

REMARK 5.6.7. If $\phi\colon A \to B$ is a flat homomorphism of Noetherian rings and $\mathfrak{p}$ is a prime ideal in A, then for every minimal prime ideal $\mathfrak{q}$ containing $\mathfrak{p}B$, we have $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Indeed, it follows from assertion i) in Proposition 5.6.6 that the morphism $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat. It then follows from Example 5.6.4 that if $\overline{a}$ is a nonzero element in $A/\mathfrak{p}$, then its image in $B/\mathfrak{p}B$ is a non-zero-divisor, hence it can't lie in a minimal prime ideal (see Proposition E.2.1). This gives our assertion.

We now define flatness in our geometric context. We say that a morphism of varieties $f\colon X \to Y$ is *flat* if it satisfies the equivalent conditions in the next proposition.

PROPOSITION 5.6.8. *Given a morphism of varieties $f\colon X \to Y$, the following conditions are equivalent:*

i) *For every affine open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \subseteq f^{-1}(V)$, the induced homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is flat.*
ii) *There are affine open covers $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$ such that for all i, we have $U_i \subseteq f^{-1}(V_i)$ and the induced homomorphism $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ is flat.*
iii) *For every point $x \in X$, if $y = f(x)$, then the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat.*

PROOF. We begin by showing that ii)$\Rightarrow$iii). Given x and y as in iii) and covers as in ii), we choose i such that $x \in U_i$, in which case $y \in V_i$. Note that x corresponds to a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_X(U_i)$ and y corresponds to the inverse image $\mathfrak{q}$ of $\mathfrak{p}$ in $\mathcal{O}_Y(V_i)$. Since

$$B = \mathcal{O}_Y(V_i) \to A = \mathcal{O}_X(U_i)$$

is flat, we conclude that $A_{\mathfrak{q}}$ is B-flat by property iv) in Proposition 5.6.6. It follows that $A_{\mathfrak{p}}$ is flat over $B_{\mathfrak{q}}$ by property ii) in the same proposition.

Since the implication i)$\Rightarrow$ii) is trivial, in order to complete the proof it is enough to show iii)$\Rightarrow$i). Let U and V be affine open subsets as in i). Given the induced homomorphism

$$B = \mathcal{O}_Y(V) \to \mathcal{O}_X(U) = A,$$

it follows from iii) that for every maximal ideal $\mathfrak{p}$ in A, if its inverse image in B is $\mathfrak{q}$, then the induced homomorphism $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ is flat. Assertion iii) in Proposition 5.6.6 implies that $A_{\mathfrak{p}}$ is flat over B for every $\mathfrak{p}$, in which case assertion iv) in the proposition implies that A is flat over B. $\square$

REMARK 5.6.9. The argument for the implication ii)$\Rightarrow$iii) in the proof of the above proposition shows that more generally, if $f\colon X \to Y$ is a flat morphism, then for every irreducible closed subset $V \subseteq X$, if $W = \overline{f(V)}$, then the induced ring homomorphism $\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V}$ is flat.

EXAMPLE 5.6.10. Every open immersion $i\colon U \hookrightarrow X$ is flat: indeed, it is clear that property iii) in the above proposition is satisfied.

EXAMPLE 5.6.11. If X and Y are varieties, then the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are flat. Indeed, by choosing affine covers of X and Y, we reduce to the case when both X and Y are affine. In this case, since $\mathcal{O}(Y)$ is a free k-module, it follows from assertion i) in Proposition 5.6.6 that

$\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is flat over $\mathcal{O}(X)$. This shows that p is flat and the assertion about q follows similarly.

REMARK 5.6.12. A composition of flat morphisms is a flat morphism. Indeed, this follows from definition and property ii) in Proposition 5.6.6.

REMARK 5.6.13. If $f\colon X \to Y$ is flat and $W \subseteq Y$ is an irreducible, closed subset such that $f^{-1}(W) \neq \emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have $\overline{f(V)} = W$. Indeed, we may replace X and Y by suitable affine open subsets that intersect V and W, respectively, to reduce to the case when both X and Y are affine. In this case the assertion follows from Remark 5.6.7.

EXAMPLE 5.6.14. A morphism $f\colon X \to \mathbf{A}^1$ is flat if and only if every irreducible component of X dominates $\mathbf{A}^1$. The "only if" part follows from the previous remark. For the converse, note that under the hypothesis, for every affine open subset U of X, the $k[x]$-module $\mathcal{O}_X(U)$ is torsion-free: if a nonzero $u \in k[x]$ annihilates $v \in \mathcal{O}_X(U)$, it follows that every irreducible component of U on which v does not vanish is mapped by f in the zero-locus of u, a contradiction. We then deduce that f is flat using Example 5.6.4.

Our goal is to prove two geometric properties of flat morphisms. We begin with the following generalization of Proposition 1.6.6.

THEOREM 5.6.15. *If $f\colon X \to Y$ is a flat morphism between algebraic varieties, then f is open.*

The proof will make use of the following openness criterion.

LEMMA 5.6.16. *Let W be a subset of a Noetherian topological space Y. The set W is open if and only if whenever $Z \subseteq Y$ is a closed irreducible subset of Y such that $W \cap Z \neq \emptyset$, then W contains nonempty open subset of Z.*

PROOF. The "only if" part is clear, so we only need to prove the converse. Arguing by Noetherian induction, we may assume that the assertion holds for all proper closed subspaces of Y. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. We may assume that W is nonempty, and suppose that W contains a point y in some Y_i. By hypothesis, there is a nonempty open subset $U \subseteq Y_i$ such that $U \subseteq W$. After replacing U by $U \smallsetminus \bigcup_{j \neq i} Y_j$, we may assume that $U \cap Y_j = \emptyset$ for every $j \neq i$, in which case U is open in Y.

Note that $Y \smallsetminus U$ is a proper closed subset of Y. Moreover, $W \smallsetminus U \subseteq Y \smallsetminus U$ satisfies the same hypothesis as W: if $Z \subseteq Y \smallsetminus U$ is an irreducible closed subset such that $(W \smallsetminus U) \cap Z \neq \emptyset$, then W contains a nonempty open subset of Z, hence the same holds for $W \smallsetminus U$. By induction, we conclude that $W \smallsetminus U$ is open in $Y \smallsetminus U$. This implies that W is open, since

$$Y \smallsetminus W = (Y \smallsetminus U) \smallsetminus (W \smallsetminus U)$$

is closed in $Y \smallsetminus U$, hence in Y. $\square$

PROOF OF THEOREM 5.6.15. If U is an open subset of X, we may replace f by its restriction to U, which is still flat. Therefore we only need to show that $f(X)$ is open in Y and it is enough to show that $f(X)$ satisfies the condition in the lemma. Suppose that W is an irreducible closed subset of Y such that $f(X) \cap W \neq \emptyset$. If V is an irreducible component of $f^{-1}(W)$, then V dominates W by Remark 5.6.13.

In this case, the image of V in W contains an open subset of W by Theorem 3.4.2. This completes the proof. $\square$

Our second main property of flat morphisms will follow from the following

PROPOSITION 5.6.17. *(Going Down for flat homomorphisms) If $\phi\colon A\to B$ is a flat ring homomorphism, then given prime ideals $\mathfrak{p}_1\subseteq\mathfrak{p}_2$ in A and $\mathfrak{q}_2$ in B such that $\phi^{-1}(\mathfrak{q}_2)=\mathfrak{p}_2$, there is a prime ideal $\mathfrak{q}_1\subseteq\mathfrak{q}_2$ such that $\phi^{-1}(\mathfrak{q}_1)=\mathfrak{p}_1$.*

PROOF. As we have seen in the proof of Proposition 5.6.8, the fact that ϕ is flat implies that the induced homomorphism $A_{\mathfrak{p}_2}\to B_{\mathfrak{q}_2}$ is flat. After replacing ϕ by this homomorphism, we may thus assume that $(A,\mathfrak{p}_2)$ and $(B,\mathfrak{q}_2)$ are local rings and ϕ is a local homomorphism. In this case every prime ideal in B is contained in $\mathfrak{q}_2$. Since the prime ideals in B lying over $\mathfrak{p}_1$ are in bijection with the prime ideals in $(A_{\mathfrak{p}_1}/\mathfrak{p}_1A_{\mathfrak{p}_1})\otimes_A B$, it is enough to show that this ring is not the zero ring.

In fact, the following more general fact is true: under our assumptions, for every nonzero A-module M, the B-module $M\otimes_A B$. is nonzero. Indeed, if $u\in M$ is nonzero and $I=\operatorname{Ann}_A(u)$, then $I\subseteq\mathfrak{p}_2$ and $Au\simeq A/I$. We thus have an inclusion $A/I\hookrightarrow M$ and the flatness assumption implies that the induced morphism $B/IB=A/I\otimes B\to M\otimes_A B$ is injective. Since $IB\subseteq\mathfrak{q}_2$, it follows that B/IB is nonzero, hence $M\otimes_A B$ is nonzero. $\square$

PROPOSITION 5.6.18. *If $\phi\colon A\to B$ is a ring homomorphism that satisfies the Going-Down property in the previous proposition, then for every prime ideal $\mathfrak{q}$, if we put $\mathfrak{p}=\phi^{-1}(\mathfrak{q})$, then*

$$\dim(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}})\leq\dim(B_{\mathfrak{q}})-\dim(A_{\mathfrak{p}}).$$

PROOF. Let $r=\dim(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}})$ and $s=\dim(A_{\mathfrak{p}})$. We can choose prime ideals $\mathfrak{p}_s\subsetneq\ldots\subsetneq\mathfrak{p}_0=\mathfrak{p}$ in A and $\mathfrak{q}_r\subsetneq\ldots\subsetneq\mathfrak{q}_0=\mathfrak{q}$ in B, with $\mathfrak{p}B\subseteq\mathfrak{q}_r$. Applying the Going-Down property successively, we obtain a sequence of prime ideals $\mathfrak{p}'_s\subseteq\ldots\subseteq\mathfrak{p}'_0\subseteq\mathfrak{q}_r$ such that $\phi^{-1}(\mathfrak{p}'_i)=\mathfrak{p}_i$ for $0\leq i\leq s$. In particular, we have $\mathfrak{p}'_i\neq\mathfrak{p}'_{i+1}$ for $0\leq i\leq s-1$ (however, we might have $\mathfrak{p}'_0=\mathfrak{q}_s$). From the sequence of prime ideals in B

$$\mathfrak{p}'_s\subsetneq\ldots\subsetneq\mathfrak{p}'_1\subsetneq\mathfrak{q}_r\subsetneq\ldots\subsetneq\mathfrak{q}_0=\mathfrak{q},$$

we conclude that $\dim(B_{\mathfrak{q}})\geq r+s$. $\square$

By combining the above two propositions, we obtain the following consequence in our geometric setting:

THEOREM 5.6.19. *If $f\colon X\to Y$ is a flat morphism between two algebraic varieties, W is an irreducible closed subset of Y such that $f^{-1}(W)\neq\emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have*

$$\operatorname{codim}_X(V)=\operatorname{codim}_Y(W).$$

PROOF. Note first that V dominates W (see Remark 5.6.13). We apply Proposition 5.6.18 for the flat morphism

$$\mathcal{O}_{Y,W}\to\mathcal{O}_{X,V},$$

which satisfies the Going-Down property by Proposition 5.6.17. Since V is an irreducible component of $f^{-1}(W)$, we obtain the inequality

$$\operatorname{codim}_X(V)\geq\operatorname{codim}_Y(W).$$

In order to prove the opposite inequality, let X' be an irreducible component of X containing V and such that $\mathrm{codim}_X(V) = \mathrm{codim}_{X'}(V)$. If Y' is an irreducible component of Y that contains $\overline{f(X')}$, then X' dominates Y' by Remark 5.6.13. We can thus apply Theorem 3.4.1 to deduce

$$\mathrm{codim}_X(V) = \mathrm{codim}_{X'}(V) \leq \mathrm{codim}_{Y'}(W) \leq \mathrm{codim}_Y(W).$$

This completes the proof of the theorem. $\square$

EXAMPLE 5.6.20. If $f\colon X \to Y$ is a flat morphism between algebraic varieties, with X of pure dimension m and Y of pure dimension n, then for every irreducible closed subset W of Y with $W \cap f(X) \neq \emptyset$, the inverse image $f^{-1}(W)$ has pure dimension equal to $\dim(W) + m - n$. In particular, every non-empty fiber of f has pure dimension $m - n$.

CHAPTER 6

Smooth varieties

In this chapter we introduce an important local property of points on algebraic varieties: smoothness. We begin by describing a fundamental construction, the blow-up of a variety along an ideal (in the case of an affine variety). We then define the tangent space of a variety at a point and use it to define smooth points. We make use of the blow-up of the variety at a smooth point to show that the local ring of a smooth point is a domain. After discussing some general properties of smooth varieties, we prove Bertini's theorem on general hyperplane sections of smooth projective varieties and end the chapter by introducing smooth morphisms between smooth varieties.

6.1. Blow-ups

In this section we discuss the blow-up of an affine variety along an ideal. We will later globalize this construction, after having at our disposal coherent sheaves of ideals and the global MaxProj construction.

Let X be an affine variety, with $A = \mathcal{O}(X)$, and let $I \subseteq A$ be an ideal.

Definition 6.1.1. The *Rees algebra* $R(A, I)$ is the $\mathbf{N}$-graded k-subalgebra

$$R(A, I) = \bigoplus_{m \in \mathbf{N}} I^m t^m \subseteq A[t].$$

Since A is reduced, it follows that $A[t]$ is reduced, hence so is $R(A, I)$. Similarly, if X is irreducible, then $A[t]$ is a domain, hence so is $R(A, I)$.

Note that $R(A, I)$ is finitely generated and, in fact, it is generated by its degree 1 component: if $I = (a_1, \ldots, a_r)$, then $a_1 t, \ldots, a_r t$ generate $R(A, I)$. We can thus apply to $R(A, I)$ the MaxProj construction discussed in Section 4.3. Note that the degree 0 component is equal to A.

Definition 6.1.2. The blow-up of X along I is the morphism

$$f \colon \operatorname{MaxProj}\big(R(A, I)\big) \to X.$$

We will typically assume that I is nonzero, since otherwise $\operatorname{MaxProj}\big(R(A, I)\big)$ is empty. We collect in the next proposition some basic properties of this construction.

Proposition 6.1.3. *Let X be an affine variety, with $A = \mathcal{O}(X)$, and let $I \subseteq A$ be a nonzero ideal. If Z is the closed subset of X defined by I and $f \colon \widetilde{X} \to X$ is the blow-up of X along I, then the following hold:*

i) *The morphism f is an isomorphism over $X \smallsetminus Z$.*
ii) *The inverse image $f^{-1}(Z)$ is locally defined in $\widetilde{X}$ by one equation, which is a non-zero-divisor. In particular, every irreducible component of $f^{-1}(Z)$ has codimension 1 in $\widetilde{X}$.*

iii) *If X is irreducible, then $\widetilde{X}$ is irreducible and f is a birational morphism.*
iv) *More generally, if Z does not contain any irreducible component of X, by mapping X' to $f(X')$, we get a bijection between the irreducible components of $\widetilde{X}$ and those of X, such that the corresponding varieties are birational.*

Proof. In order to prove the assertion in i), it is enough to show that if $a \in A$ is such that $D_X(a) \cap V(I) = \emptyset$ (which implies $a \in \sqrt{I}$, hence $I_a = A_a$), then the induced morphism $f^{-1}\big(D_X(a)\big) \to D_X(a)$ is an isomorphism. Since $f^{-1}\big(D_X(a)\big) = \mathrm{MaxProj}\big(R(A,I)_a\big)$ (see Remark 4.3.18) and $R(A,I)_a \simeq R(A_a, I_a)$, we see that it is enough to show that if $I = A$, then f is an isomorphism. However, in this case

$$\widetilde{X} = \mathrm{MaxProj}(A[t]) = \mathrm{MaxSpec}(A) \times \mathbf{P}^0$$

by Proposition 4.3.12, with f being the projection on the first component. This is clearly an isomorphism.

In order to prove ii), note that $f^{-1}(Z) = V\big(I \cdot R(A,I)\big)$. Let us choose generators $a_1, \ldots, a_n$ of I and consider the affine open cover

$$\widetilde{X} = \bigcup_{i=1}^{n} D^+_{\widetilde{X}}(a_i t).$$

Note that by Propositions 4.3.16 and 4.3.17, we have

$$D^+_{\widetilde{X}}(a_i t) \simeq \mathrm{MaxSpec}\big(R(A,I)_{(a_i t)}\big).$$

Since the ideal $I \cdot R(A,I)_{(a_i t)}$ is generated by $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ and $\frac{a_j}{1} = \frac{a_i}{1} \cdot \frac{a_j t}{a_i t}$ for $1 \leq j \leq n$, we conclude that $I \cdot R(A,I)_{(a_i t)}$ is generated by $\frac{a_i}{1}$. Finally, note that $\frac{a_i}{1} \in R(A,I)_{(a_i t)}$ is a non-zero divisor: if $\frac{a_i}{1} \cdot \frac{h}{a_i^m t^m} = 0$ for some $h \in R(A,I)_m$, then there is $q \geq 1$ such that $h a_i^q = 0$ in A, hence $\frac{h}{a_i^m t^m} = 0$ in $R(A,I)_{(a_i t)}$. This gives the first assertion in ii) and the second one follows from the Principal Ideal theorem (see also Remark 3.3.6).

The assertion in iii) is clear: we have seen that in this case $\widetilde{X}$ is irreducible and f is an isomorphism over the nonempty closed subset $X \smallsetminus Z$.

Suppose now that $X_1, \ldots, X_r$ are the irreducible components of X and that Z does not contain any of the X_i. It follows from i) that $\widetilde{X}_i := \overline{f^{-1}(X_i \smallsetminus Z)}$ is an irreducible component of $\widetilde{X}$ such that f induces a birational morphism $\widetilde{X}_i \to X_i$. Since f is proper (see Corollary 5.1.9), the image $f(\widetilde{X}_i)$ is closed, hence $f(\widetilde{X}_i) = X_i$.

Moreover, we have

$$\widetilde{X} \subseteq f^{-1}(Z) \cup \bigcup_{i=1}^{r} \widetilde{X}_i.$$

On the other hand, no irreducible component of $f^{-1}(Z)$ can be an irreducible component of $\widetilde{X}$, since we can find, on a suitable affine chart, a non-zero-divisor that vanishes on $f^{-1}(Z)$ (see Remark 3.3.6). We thus conclude that

$$\widetilde{X} = \bigcup_{i=1}^{r} \widetilde{X}_i,$$

completing the proof of iv). $\square$

EXAMPLE 6.1.4. Suppose that $I = \mathfrak{m}$ is the maximal ideal defining a non-isolated point $x \in X = \mathrm{MaxSpec}(A)$, hence $Z = \{x\}$. It follows from Remark 4.3.19 that $f^{-1}(Z)$ is the closed subset associated to the ideal $\mathfrak{m} \cdot R(A, \mathfrak{m})$. Note that

$$R(A,\mathfrak{m})/\mathfrak{m} \cdot R(A,\mathfrak{m}) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} =: \mathrm{gr}_{\mathfrak{m}}(A).$$

Note that if $X_1, \ldots, X_r$ are the irreducible components of X that contain x, then the corresponding irreducible components of $\widetilde{X}$ are precisely those that meet $f^{-1}(Z)$. Since $f^{-1}(Z)$ is locally defined by a non-zero-divisor, we conclude that

$$\dim\big(f^{-1}(Z)\big) = \max_i\{\dim(X_i)\} - 1 = \dim(\mathcal{O}_{X,x}) - 1.$$

Since

$$f^{-1}(Z) \simeq \mathrm{MaxProj}\big(\mathrm{gr}_{\mathfrak{m}}(A)/J\big),$$

where J is the nil-radical of $\mathrm{gr}_{\mathfrak{m}}(A)$, we conclude using Proposition 4.2.11 that

$$\dim\big(\mathrm{gr}_{\mathfrak{m}}(A)\big) = \dim(\mathcal{O}_{X,x}).$$

EXAMPLE 6.1.5. With the above notation, suppose that $X = \mathbf{A}^n$, hence $A = k[x_1, \ldots, x_n]$, and $I = (x_1, \ldots, x_n)$. We thus have a surjective homomorphism

$$\phi\colon A[y_1, \ldots, y_n] \to R(A, I), \quad \phi(y_i) = x_i t \quad \text{for} \quad 1 \leq i \leq n,$$

inducing a closed immersion

$$\iota\colon \widetilde{X} \hookrightarrow X \times \mathbf{P}^{n-1}$$

of varieties over X. Note that if J is the ideal in $A[y_1, \ldots, y_n]$ generated by all differences $x_i y_j - x_j y_i$, for $i \neq j$, then $J \subseteq \ker(\phi)$, hence $\iota(\widetilde{X})$ is contained in $V(J)$. On the other hand, we have seen in Exercise 5.1.12 that $V(J)$ is an irreducible variety, of dimension n. We thus conclude that ι gives an isomorphism of $\widetilde{X}$ with $V(J)$. In particular, our old definition for the blow-up of the affine space at the origin agrees with the new one. For a generalization of this example, see Example 6.3.23 below.

DEFINITION 6.1.6. Suppose that X is an irreducible affine variety, Z is a proper closed subset of X, and $f\colon \widetilde{X} \to X$ is the blow-up of X along I. If Y is any closed subvariety of X such that no irreducible component of Y is contained in Z, then the *strict transform* (or *proper transform*) of Y in $\widetilde{X}$ is given by

$$\widetilde{Y} := \overline{f^{-1}(Y \smallsetminus Z)}.$$

REMARK 6.1.7. With the notation in the above definition, we have an induced morphism $\widetilde{Y} \to Y$ that can be identified with the blow-up of Y along the ideal $J = I \cdot \mathcal{O}(Y)$. Indeed, if $B = \mathcal{O}(Y)$, then the surjection $A \to B$ induces a graded, surjective homomorphism of k-algebras:

$$R(A, I) \to R(B, J).$$

This induces by Proposition 4.3.11 a commutative diagram

$$\begin{tikzcd} \mathrm{MaxProj}\big(R(B,J)\big) \arrow[r, "j"] \arrow[d, "g"] & \mathrm{MaxProj}\big(R(A,I)\big) \arrow[d, "f"] \\ Y \arrow[r, "i"] & X, \end{tikzcd}$$

where i and j are closed immersions. By Proposition 6.1.3, g maps birationally each irreducible component of $\widetilde{Y}$ onto a corresponding irreducible component of Y, which implies that the image of j is, indeed, the strict transform of Y.

EXAMPLE 6.1.8. In order to give some intuition about the blow-up, we discuss the strict transform of a curve in $\mathbf{A}^2$ under the blow-up at one point. Let us begin, more generally, with the computation of the strict transform of a hypersurface in $\mathbf{A}^n$ under the blow-up at one point, where $n \geq 2$. Suppose that Y is a hypersurface in $\mathbf{A}^n$, with corresponding radical ideal defined by the non-zero polynomial $f \in k[x_1, \ldots, x_n]$. Given a point $p \in Y$, the *multiplicity* $\mathrm{mult}_p(Y)$ of Y (or of f) at p is the largest $j \geq 1$ such that $f \in \mathfrak{m}_p^j$, where $\mathfrak{m}_p$ is the maximal ideal corresponding to p. Let $\pi\colon \widetilde{\mathbf{A}^n} \to \mathbf{A}^n$ be the blow-up of $\mathbf{A}^n$ at p. After a suitable translation, we may assume that $p = 0$. We can thus write

$$f = f_m + f_{m+1} + \ldots + f_d,$$

where each f_i is homogeneous of degree i, and $f_m \neq 0$, so that $\mathrm{mult}_0(Y) = m$. Recall that by Example 6.1.5, we know that $\widetilde{\mathbf{A}^n}$ is the subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$ given by the equations $x_i y_j = x_j y_i$, for $0 \leq i, j \leq n$, where $y_1, \ldots, y_n$ are the homogeneous coordinates on $\mathbf{P}^{n-1}$. Consider the chart U_i on $\widetilde{Y}$ given by $y_i \neq 0$. Note that in this chart we have $x_j = x_i \frac{y_j}{y_i}$ for $j \neq i$, hence $U_i \simeq \mathbf{A}^n$, with coordinates $u_1, \ldots, u_n$ such that $\pi^{\#}(x_i) = u_i$ and $\pi^{\#}(x_j) = u_i u_j$ for $j \neq i$. If $E = \pi^{-1}(0)$, then $E \cap U_i$ is defined by $u_i = 0$.

The inverse image $\pi^{-1}(Y)$ is defined in U_i by

$$\pi^{\#}(f) = f(u_1 u_i, \ldots, u_i, \ldots, u_n u_i)$$
$$= u_i^m \cdot \big(f_m(u_1, \ldots, 1, \ldots, u_n) + u_i \cdot f_{m+1}(u_1, \ldots, 1, \ldots, u_n) + \ldots + u_i^{m-d} f_d(u_1, \ldots, 1, \ldots, u_n)\big).$$

Since the polynomial

$$\widetilde{f} := f_m(u_1, \ldots, 1, \ldots, u_n) + u_i \cdot f_{m+1}(u_1, \ldots, 1, \ldots, u_n) + \ldots + u_i^{m-d} f_d(u_1, \ldots, 1, \ldots, u_n)$$

defines a hypersurface in U_i that does not contain $E \cap U_i$, it follows that its zero-locus defines $\widetilde{Y} \cap U_i$. In fact, since the homomorphism $k[x_1, \ldots, x_n]_{x_i} \to \mathcal{O}(U_i)_{u_i}$ is an isomorphism, it is easy to deduce that $\widetilde{f}$ is square-free, hence it generates the ideal of $\widetilde{Y} \cap U_i$.

Let us specialize now to the case $n = 2$. In this case f_m is a homogeneous polynomial of degree d, which can thus be written as $f_m = \prod_{j=1}^m \ell_j$, where each ℓ_j is a linear form (we use the fact that k is algebraically closed, hence every polynomial in one variable is the product of degree 1 polynomials). The lines through the origin defined by the factors of f_m are the *tangents* to X at 0. Note that the lines through 0 in $\mathbf{A}^2$ are parametrized by $\mathbf{P}^1 = \pi^{-1}(0)$.

We claim that after the blow-up, the points of intersection of the strict transform $\widetilde{Y}$ with E correspond precisely to the tangent lines to X at 0. Indeed, if we consider for example the chart U_1, note that the points of $\widetilde{Y} \cap E \cap U_1$ are defined by $u_1 = 0 = f_m(1, u_2)$. It follows that if $f_m = \prod_{j=1}^m (a_j x_1 + b_j x_2)$, then the points of $\widetilde{Y} \cap E \cap U_1$ are precisely the points $[b_j, -a_j] \in E$ with $b_j \neq 0$. Similarly, the points of $\widetilde{Y} \cap E \cap U_2$ are precisely the points $[b_j, -a_j] \in E$ with $a_j \neq 0$. This proves our claim. In fact, this is not just a set-theoretic correspondence: tangents that appear with multiplicity > 1 in f_m translate to tangency conditions between $\widetilde{Y}$ and E at the corresponding point. We will return to this phenomenon at a later point.

6.2. The tangent space

We begin with the following general observation. If $(R, \mathfrak{m})$ is a local Noetherian ring, then $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional vector space over the residue field $K = R/\mathfrak{m}$. It follows from Nakayama's lemma that $\dim_K \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators for the ideal $\mathfrak{m}$ (see Remark C.1.3).

In this section we are interested in the case when $(X, \mathcal{O}_X)$ is an algebraic variety, $p \in X$ is a point, and we consider the local ring $\mathcal{O}_{X,p}$, with maximal ideal $\mathfrak{m}_p$. Recall that in this case the residue field is the ground field k.

Definition 6.2.1. The *tangent space* of X at p is the k-vector space

$$T_pX := (\mathfrak{m}_p/\mathfrak{m}_p^2)^* = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k).$$

The following proposition explains the terminology in the above definition. Note that T_pX does not change if we replace X by an affine open neighborhood of p. In particular, we may assume that X is affine and choose a closed immersion $X \hookrightarrow \mathbf{A}^n$.

Proposition 6.2.2. *If X is a closed subvariety of $\mathbf{A}^n$ with corresponding radical ideal $I_X \subseteq k[x_1, \ldots, x_n]$, then T_pX is isomorphic to the linear subspace of k^n defined by the equations*

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)x_i = 0 \quad \textit{for all} \quad f \in I_X.$$

Moreover, it is enough to only consider those equations corresponding to a system of generators of I_X.

In the case of a closed subset X of $\mathbf{A}^n$, we will refer to the linear subspace in the proposition as the *embedded tangent space* in the affine space.

Proof of Proposition 6.2.2. Let $f_1, \ldots, f_r$ be a system of generators of I_X. In this case, if $p = (a_1, \ldots, a_n)$, we have

$$\mathcal{O}_{X,p} = \mathcal{O}_{\mathbf{A}^n,p}/(f_1, \ldots, f_r) \quad \text{and} \quad \mathfrak{m}_p = (x_1 - a_1, \ldots, x_n - a_n)\mathcal{O}_{\mathbf{A}^n,p}/(f_1, \ldots, f_r).$$

Therefore we have

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 = k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)^2 + (f_1, \ldots, f_r).$$

On the other hand, for every $f \in k[x_1, \ldots, x_n]$, we have

$$f \equiv f(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot (x_i - a_i) \mod (x_1 - a_1, \ldots, x_n - a_n)^2.$$

We thus see that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the quotient of the vector space over k with basis $e_i = x_i - a_i$ for $1 \leq i \leq n$, by the subspace generated by

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)e_i \quad \text{for} \quad f \in (f_1, \ldots, f_r).$$

This immediately gives the first assertion in the proposition.

Note now that if $g \in (f_1, \ldots, f_r)$ and we write $g = \sum_{j=1}^r h_j f_j$, then it follows from the product rule and the fact that $f_j(p) = 0$ for all j that

$$\sum_{i=1}^n \frac{\partial g}{\partial x_i}(p)x_i = \sum_{j=1}^r h_j(p) \cdot \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)x_i.$$

The last assertion in the proposition follows. $\square$

REMARK 6.2.3. If $f\colon X \to Y$ is a morphism of varieties and $p \in X$, we obtain a local homomorphism of local rings

$$\phi\colon \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}.$$

This induces a k-linear morphism

$$\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$$

and by taking duals, we obtain a k-linear map $df_p\colon T_pX \to T_{f(p)}Y$. It is easy to see that this definition is functorial: if $g\colon Y \to Z$ is another morphism, then

$$dg_{f(p)} \circ df_p = d(g \circ f)_p.$$

REMARK 6.2.4. If Y is a closed subvariety of the variety X and if $i\colon Y \to X$ is the inclusion, then for every $p \in Y$, the linear map $di_p\colon T_pY \to T_pX$ is injective. This follows from the fact that the homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective and therefore the induced map $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \to \mathfrak{m}_{Y,p}/\mathfrak{m}_{Y,p}^2$ is surjective, where $\mathfrak{m}_{X,p} \subseteq \mathcal{O}_{X,p}$ and $\mathfrak{m}_{Y,p} \subseteq \mathcal{O}_{Y,p}$ are the corresponding maximal ideals.

Note that if Y if if a closed subvariety of $\mathbf{A}^n$ and $i\colon Y \hookrightarrow \mathbf{A}^n$ is the inclusion, then the embedded tangent space of Y at p is the image of di_p, where we identify in the obvious way $T_p\mathbf{A}^n$ to k^n.

REMARK 6.2.5. If X and Y are closed subvarieties of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, and if $f = (f_1, \ldots, f_n)\colon X \to Y$, then via the isomorphisms given by Proposition 6.2.2, the linear map df_p is induced by the linear map $k^m \to k^n$ given with respect of the standard bases by the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)$. Indeed, by functoriality, it is enough to check this when $X = \mathbf{A}^m$ and $Y = \mathbf{A}^n$. Let $x_1, \ldots, x_m$ be the coordinate functions on $\mathbf{A}^m$ and $y_1, \ldots, y_n$ the coordinate functions on $\mathbf{A}^n$. If $p = (a_1, \ldots, a_m)$, then the maximal ideals defining p and $f(p)$ are

$$\mathfrak{m}_p = (x_1 - a_1, \ldots, x_m - a_m) \quad \text{and} \quad \mathfrak{m}_{f(p)} = \big(y_1 - f_1(p), \ldots, y_n - f_n(p)\big).$$

Moreover, the map $\mathfrak{m}_{f(p)} \to \mathfrak{m}_p$ maps $y_i - f_i(p)$ to $f_i - f_i(p)$ and Taylor's formula shows that

$$f_i - f_i(p) - \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(p)(x_j - a_j) \in \mathfrak{m}_p^2,$$

which implies, after taking duals, our assertion.

In the case of a closed subvariety of a projective space, we also have an embedded tangent space: this time, it is a linear subpace of the projective space. This is defined as follows. Suppose that X is a closed subset of $\mathbf{P}^n$, with corresponding radical homogeneous ideal I_X. Given a point $p = [u_0, \ldots, u_n] \in X$, the *projective tangent space* of X at p, that we will denote by $\mathbf{T}_pX$, is the linear subspace of $\mathbf{P}^n$ defined by the equations

$$\sum_{i=0}^n \frac{\partial f}{\partial x_i}(u_0, \ldots, u_n)x_i = 0,$$

where f varies over the homogeneous elements in I_X. Note first that since f is homogeneous, if we replace $(u_0, \ldots, u_n)$ by $(\lambda u_0, \ldots, \lambda u_n)$, for some $\lambda \in k^*$, then the equation gets multiplied by λ. Note also that it is enough to consider a system of homogeneous generators of I_X: if $f = \sum_{j=1}^r g_j f_j$, with $f_j \in I_X$, then we get using the product rule and the fact that $f_j(u_0, \ldots, u_n) = 0$ for all j

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(u_0, \ldots, u_n)x_i = \sum_{j=1}^{r} g_j(u_0, \ldots, u_n) \cdot \sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(u_0, \ldots, u_n)x_i.$$

Finally, we note that $\mathbf{T}_p X$ contains the point p: this is a consequence of Euler's identity, which says that if f is homogeneous, of degree d, then

$$\sum_{i=0} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

The terminology is justified by the following

Proposition 6.2.6. *Let X be a closed subvariety of $\mathbf{P}^n$ and $p \in X$. If i is such that $p \in U_i = (x_i \neq 0)$ and if we identify U_i with $\mathbf{A}^n$ is the usual way, then $\mathbf{T}_p X \cap U_i$ is the image of the embedded tangent space in $\mathbf{A}^n$ for $X \cap U_i$ at p by the translation mapping 0 to p.*

Proof. In order to simplify the notation, let us assume that $i = 0$. In this case, we may assume that $(u_0, \ldots, u_n) = (1, u_1, \ldots, u_n)$. Note that the ideal of $X \cap U_i$ in $k[x_1, \ldots, x_n]$ is generated by $f(1, x_1, \ldots, x_n)$, where f varies over a set of homogeneous generators of I_X (see Exercise 4.2.14). Fix such f and let $g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n)$. Therefore we have $\frac{\partial g}{\partial x_i}(u_1, \ldots, u_n) = \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n)$. On the other hand, it follows from Euler's identity that

$$\frac{\partial f}{\partial x_0}(1, u_1, \ldots, u_n) = -\sum_{i=1}^{n} u_i \cdot \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n).$$

This implies that

$$\frac{\partial f}{\partial x_0}(1, u_1, \ldots, u_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(1, u_1, \ldots, u_n)x_i = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(u_1, \ldots, u_n) \cdot (x_i - u_i).$$

This implies the assertion in the proposition. $\square$

Exercise 6.2.7. Given varieties X and Y, for every $x \in X$ and $y \in Y$, the projections $X \times Y \to X$ and $X \times Y \to Y$ induce a linear map

$$T_{(x,y)}(X \times Y) \to T_x X \times T_y Y.$$

Show that this is an isomorphism.

6.3. Smooth algebraic varieties

Let X be an algebraic variety. Given a point $p \in X$, recall that we put $\dim_p X := \dim(\mathcal{O}_{X,p})$. This is the largest dimension of an irreducible component of X that contains p (see Remark 3.3.14), and also the codimension of $\{p\}$ in X. Our first goal is to show that $\dim_k T_p X \geq \dim_p X$.

More generally, we will get a similar statement for the localization of a finite type k-algebra at a prime ideal. This applies, in particular, for the local ring $(\mathcal{O}_{X,V}, \mathfrak{m}_V)$ of X at an irreducible closed subset V. Note that in this case the residue field is the field of rational functions on V.

PROPOSITION 6.3.1. *For every local ring* $(R, \mathfrak{m})$ *that is the localization of a* k*-algebra of finite type at a prime ideal, we have*

$$\dim(R) \leq \dim_K \mathfrak{m}/\mathfrak{m}^2,$$

where $K = R/\mathfrak{m}$.

PROOF. Suppose that $R = A_{\mathfrak{p}}$, where A is a k-algebra of finite type and $\mathfrak{p}$ is a prime ideal in A. Note that if I is the nil-radical of A and $\overline{R} = R/I \cdot R$, then $\overline{R}$ is local, with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/I \cdot R$, and the same residue field. Since $\dim(R) = \dim(\overline{R})$, while

$$\dim_K \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \dim_K \mathfrak{m}/(\mathfrak{m}^2 + I \cdot R) \leq \dim_K \mathfrak{m}/\mathfrak{m}^2,$$

we see that that it is enough to prove the assertion when A is reduced. Let X be an affine variety with $\mathcal{O}(X) = A$ and let V be the irreducible closed subset defined by $\mathfrak{p}$.

Recall that by Nakayama's lemma, if $r = \dim_K \mathfrak{m}/\mathfrak{m}^2$, then $\mathfrak{m}$ is generated by r elements. This implies that there is $f \in A \smallsetminus \mathfrak{p}$ such that $\mathfrak{p}A_f$ is generated by r elements. After replacing A by A_f, we may thus assume that $\mathfrak{p}$ is generated by r elements. In this case, Corollary 3.3.7 implies $\dim(R) = \operatorname{codim}_X(V) \leq r$, giving the assertion in the proposition. $\square$

DEFINITION 6.3.2. A point $p \in X$ is *nonsingular* (or *smooth*) if $\dim_p X = \dim_k T_p X$. Otherwise, it is *singular*. The variety X is *nonsingular* (or *smooth*) if all its points are nonsingular points.

Given an irreducible, closed subset $V \subseteq X$, we say that X is *nonsingular at* V if $\dim(\mathcal{O}_{X,V}) = \dim_{k(V)} \mathfrak{m}_V/\mathfrak{m}_V^2$. We will see later that this is equivalent with the fact that some point $p \in V$ is a nonsingular point.

EXAMPLE 6.3.3. It is clear that every affine space $\mathbf{A}^n$ is a smooth variety. Since a projective space has an open cover by affine spaces, it follows that every projective space is a smooth variety.

EXAMPLE 6.3.4. Let X be a hypersurface in $\mathbf{A}^n$, defined by the radical ideal $(f) \subseteq k[x_1, \ldots, x_n]$. Since $\dim_p(X) = n - 1$ for every $p \in X$, it follows from definition and Proposition 6.2.2 that the set of singular points in X is the zero locus of the ideal

$$(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

In particular, we see that the set of smooth points is open in X. A generalization of this fact will be given in Theorem 6.3.7 below.

REMARK 6.3.5. Since Krull's Principal Ideal theorem holds in every Noetherian ring, the inequality in Proposition 6.3.1 also holds for arbitrary Noetherian local rings. A Noetherian local ring for which the inequality is an equality is a *regular* ring.

DEFINITION 6.3.6. For every regular local ring $(R, \mathfrak{m})$, a *regular system of parameters* is a minimal set of generators of $\mathfrak{m}$. Note that since R is regular, the length of such a system is equal to $\dim(R)$. If X is a variety and $p \in X$ is a smooth point, we say that some regular functions $f_1, \ldots, f_n$ defined in a neighborhood of p give a regular system of parameters at p if their images in $\mathcal{O}_{X,p}$ give a regular system of parameters.

The following is the main result of this section

THEOREM 6.3.7. *For every variety X, the set X_{sm} of smooth points $p \in X$ is a dense open subset.*

We prove the theorem, assuming the following proposition, and then give the proof of the proposition.

PROPOSITION 6.3.8. *If $p \in X$ is a nonsingular point, then the local ring $\mathcal{O}_{X,p}$ is a domain (that is, p lies on a unique irreducible component of X).*

PROOF OF THEOREM 6.3.7. In order to prove the assertion, we may assume that X is irreducible. Indeed, if $X_1, \ldots, X_r$ are the irreducible components of X, then it follows from Proposition 6.3.8 that no point on the intersection of two distinct components is nonsingular. It thus follows that if $X_i' = X_i \smallsetminus \bigcup_{j \neq i} X_j$, then

$$X_{\mathrm{sm}} = \bigcup_{i=1}^{r} (X_i')_{\mathrm{sm}}.$$

Therefore it is enough to know the assertion for irreducible varieties.

Suppose now that X is irreducible and let $r = \dim(X)$. If $X = \bigcup_i U_i$ is an affine open cover, it is enough to show that each set $X_{\mathrm{sm}} \cap U_i = (U_i)_{\mathrm{sm}}$ is open and nonempty. Therefore we may and will assume that X is a closed subset of an affine space $\mathbf{A}^n$. If $f_1, \ldots, f_m$ are generators of the ideal defining X, then it follows from definition and Proposition 6.2.2 that a point $q \in X$ is a nonsingular point if and only if the rank of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(q)\right)$ is $\geq n - r$. This is the case if and only if one of the $(n-r)$-minors of the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ does not vanish at q, condition that defines an open subset of X.

In order to prove that X_{sm} is nonempty, we may replace X by a birational variety. By Proposition 1.6.13, we may thus assume that X is an irreducible hypersurface in $\mathbf{A}^{r+1}$. Let $f \in k[x_1, \ldots, x_{r+1}]$ be the irreducible polynomial that generates the prime ideal corresponding to X. As we have seen, for a point $q \in X$, we have $q \in X_{\mathrm{sm}}$ if and only there is i such that $\frac{\partial f}{\partial x_i}(q) \neq 0$. If $X_{\mathrm{sm}} = \emptyset$, then $\frac{\partial f}{\partial x_i}$ vanishes on X for $1 \leq i \leq r+1$. Therefore $\frac{\partial f}{\partial x_i} \in (f)$ for all i. If $\deg_{x_i}(f) = d_i$, then we clearly have $\deg_{x_i}\left(\frac{\partial f}{\partial x_i}\right) < d_i$, hence $\frac{\partial f}{\partial x_i} \in (f)$ implies that $\frac{\partial f}{\partial x_i} = 0$. Since this holds for every i, we conclude that $\mathrm{char}(k) = p > 0$ and $f \in k[x_1^p, \ldots, x_n^p]$. Since k is perfect, being algebraically closed, we conclude that $f = g^p$ for some $g \in k[x_1, \ldots, x_{r+1}]$, contradicting the fact that f is irreducible. $\square$

We now turn to the proof of Proposition 6.3.8. This will be a consequence of the following useful fact about smooth points. Let X be a variety and $p \in X$ a smooth point. We put $R = \mathcal{O}_{X,p}$ and let $\mathfrak{m}$ be the maximal ideal in R. Since p is a smooth point, if $n = \dim(R)$, then we can choose generators $a_1, \ldots, a_n$ for $\mathfrak{m}$. Note that $R/\mathfrak{m} = k$ and the classes $\overline{a_1}, \ldots, \overline{\mathfrak{a}_n} \in \mathfrak{m}/\mathfrak{m}^2$ give a k-basis. Consider the graded k-algebra homomorphism

$$\phi \colon k[x_1, \ldots, x_n] \to \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

that maps each x_i to $\overline{a_i}$. Since the right-hand side is generated by $\mathfrak{m}/\mathfrak{m}^2$, it is clear that ϕ is surjective.

PROPOSITION 6.3.9. *If $p \in X$ is a smooth point, then morphism ϕ defined above is an isomorphism.*

PROOF. Let U be an affine open neighborhood of p and let $A = \mathcal{O}(U)$. If $\mathfrak{n} \subseteq A$ is the maximal ideal corresponding to x, then $R = A_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}A_{\mathfrak{n}}$. Note that

$$\bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} = \mathrm{gr}_{\mathfrak{n}}(A),$$

hence this ring has dimension n by Example 6.1.4. Since ϕ is a surjective homomorphism and $k[x_1, \ldots, x_n]$ is a domain of dimension n, it follows that ϕ is an isomorphism. $\square$

PROOF OF PROPOSITION 6.3.8. Let $R = \mathcal{O}_{X,p}$ and $\mathfrak{m}$ the maximal ideal of R. We know, by Proposition 6.3.9, that the ring $S = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a domain. We now show that this implies that R is a domain. Suppose that $a, b \in R \smallsetminus \{0\}$ are such that $ab = 0$. It follows from Krull's Intersection theorem (see Theorem C.4.1) that there are i and j such that $a \in \mathfrak{m}^i \smallsetminus \mathfrak{m}^{i+1}$ and $b \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. In this case, since S is a domain, we conclude that $ab \notin \mathfrak{m}^{i+j+1}$, a contradiction. Therefore R is a domain. $\square$

REMARK 6.3.10. It follows from Proposition 6.3.8 that every connected component of a smooth variety is irreducible. Because of this, when dealing with smooth varieties, one can easily reduce to the case when the variety is irreducible.

REMARK 6.3.11. The same line of argument can be used to prove a stronger statement: if A is a k-algebra of finite type, but non-necessarily reduced, and $\mathfrak{m}$ is a maximal ideal in A such that $A_{\mathfrak{m}}$ is a regular local ring, then $A_{\mathfrak{m}}$ is a domain. Indeed, let I be the nil-radical of A and $\overline{A} = A/I$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$. After possibly replacing A by the localization at a suitable element not in $\mathfrak{m}$, we may assume that $\mathfrak{m}$ is generated by n elements, where $n = \dim(A_{\mathfrak{m}}) = \dim(A)$. Consider the following two surjective morphisms:

$$A/\mathfrak{m}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathrm{gr}_{\mathfrak{m}}(A) \xrightarrow{\psi} \mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{A}).$$

By Example 6.1.4, we have

$$\dim\left(\mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{A})\right) = n,$$

which implies that $\psi \circ \phi$ is an isomorphism, which implies that ϕ is injective, hence an isomorphism. The argument in the proof of Proposition 6.3.8 now implies that $A_{\mathfrak{m}}$ is a domain.

REMARK 6.3.12. Suppose that $f \in k[x_1, \ldots, x_n]$ is a non-constant polynomial such that there is no point $p \in \mathbf{A}^n$, with

$$f(p) = 0 = \frac{\partial f}{\partial x_i}(p) \quad \text{for} \quad 1 \leq i \leq n.$$

In this case f generates a radical ideal and the corresponding hypersurface in $\mathbf{A}^n$ is smooth. Indeed, note that if g is a non-constant polynomial such that g^2 divides f, then for every $p \in V(g)$, we have $f(p) = 0$ and $\frac{\partial f}{\partial x_i} = 0$ for all i, a contradiction. The fact that the hypersurface defined by f is smooth now follows from Example 6.3.4.

A similar assertion holds in the projective setting, with an analogous argument: if $F \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of positive degree such that there is no point $p \in \mathbf{P}^n$ with

$$F(p) = 0 = \frac{\partial F}{\partial x_i}(p) \quad \text{for} \quad 0 \leq i \leq n,$$

then the ideal (F) is radical and it defines a smooth hypersurface in $\mathbf{P}^n$. Moreover, in this case we see that if $n \geq 2$, then this hypersurface is irreducible: indeed, two irreducible components would have non-empty intersection by Corollary 4.2.12 and any point on the intersection would be a singular point by Proposition 6.3.8.

EXERCISE 6.3.13. Show that if X and Y are algebraic varieties, the points $x \in X$ and $y \in Y$ are smooth if and only if (x, y) is a smooth point of $X \times Y$.

EXERCISE 6.3.14. Suppose that G is an algebraic group which has a transitive algebraic action on the variety X. Show that X is smooth. Deduce that every algebraic group is a smooth variety.

EXAMPLE 6.3.15. If V is an irreducible, closed subset of X, with $\operatorname{codim}_X(V) = 1$, then X is smooth at V if and only if the maximal ideal of $\mathcal{O}_{X,V}$ is principal, that is, $\mathcal{O}_{X,V}$ is a DVR (for an elementary discussion of DVRs, see Section C.5).

EXAMPLE 6.3.16. Let H be a hyperplane in $\mathbf{P}^n$ and X a closed subvariety of H. Given a point $p \in \mathbf{P}^n \smallsetminus H$, the *projective cone over X with vertex p* is the union $C_p(X)$ of the lines joining p with the points on X. Note first that this is a closed subvariety of $\mathbf{P}^n$.

In order to see this, after applying a suitable transformation in $PGL_{n+1}(k)$, we may assume that $H = (x_n = 0)$ and $p = [0, \ldots, 0, 1]$, and use the isomorphism $\mathbf{P}^{n-1} \to H$, given by $[u_0, \ldots, u_{n-1}] \to [u_0, \ldots, u_{n-1}, 0]$, to identify $\mathbf{P}^{n-1}$ and H. In this case,

$$C_p(X) = \{p\} \cup \{[u_0, \ldots, u_n] \in \mathbf{P}^n \mid [u_0, \ldots, u_{n-1}] \in X\}.$$

It is now clear that $C_p(X)$ is closed in $\mathbf{P}^n$; in fact, if $I_X \subseteq k[x_0, \ldots, x_{n-1}]$ is the homogeneous ideal corresponding to X, then the ideal of $C_p(X)$ is $I_X \cdot k[x_0, \ldots, x_n]$. Note that if U is the affine chart $U = (x_n \neq 0) \simeq \mathbf{A}^n$, then $C_p(X) \cap U$ is isomorphic to the affine cone over X.

We claim that p is a smooth point of $C_p(X)$ if and only if X is a linear subspace of H. Indeed, p is a smooth point of $C_p(X)$ if and only if 0 is a smooth point of the affine cone $C(X)$ over X. Note that the embedded tangent cone to $C(X)$ at 0 is defined by the linear polynomials in the ideal I_X of X; in other words, this is equal to the smallest vector subspace of k^n containing $C(X)$. This has the same dimension as $C(X)$ if and only if $C(X)$ is a linear space.

In the remainder of this section we give some further properties of smooth points.

PROPOSITION 6.3.17. *Let X be an algebraic variety and Y a closed subvariety, with $x \in Y_{\mathrm{sm}}$, such that there is an affine open neighborhood U of x in X, and $f_1, \ldots, f_r \in \mathcal{O}(U)$ satisfying the following conditions:*

i) *We have $I_U(Y \cap U) = (f_1, \ldots, f_r)$, and*
ii) *The subvariety $Y \cap U$ of U is irreducible, of codimension r.*

In this case x is a smooth point on X.

Note that since x is a smooth point of Y, it follows from Proposition 6.3.8 that x lies on a unique irreducible component of Y. Therefore after possibly replacing U by a smaller open subset, we can always assume that $Y \cap U$ is irreducible.

Proof of Proposition 6.3.17. Let $R = \mathcal{O}_{X,x}$ and $\overline{R}$ be the local rings at x of X and Y, respectively. If $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ are the maximal ideals in R and $\overline{R}$, then $\overline{\mathfrak{m}} = \mathfrak{m}/(f_{1,x}, \ldots, f_{r,x})$, where we denote by $f_{i,x}$ the image of f_i in R. It follows that

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \mathfrak{m}/\mathfrak{m}^2 + (f_{1,x}, \ldots, f_{r,x}),$$

hence $\dim_k T_x Y \geq \dim_k T_x X - r$. Since we clearly have

$$\dim(\mathcal{O}_{X,x}) \geq \mathrm{codim}_X(Y) + \dim(\mathcal{O}_{Y,x}) = r + \dim(\mathcal{O}_{Y,x}),$$

we conclude that $\dim(\mathcal{O}_{X,x}) \geq \dim_k T_x X$ and thus x is a smooth point of X. $\square$

Remark 6.3.18. An important special case of the above proposition is that of a hypersurface: suppose that X is an algebraic variety and Y is a closed subvariety of X such that for some point $x \in Y$ and for some affine open neighborhood $U \subseteq X$ of x, we have $I_U(Y \cap U) = (f)$, for some non-zero divisor $f \in \mathcal{O}(U)$. In this case, if x is a smooth point of Y, then x is a smooth point of X. Indeed, note that in this case the fact that $Y \cap U$ has codimension 1 in U follows from Theorem 3.3.1 and Remark 3.3.6.

Corollary 6.3.19. *If X is a variety and V is an irreducible closed subset of X such that X is nonsingular at V, then $V \cap X_{\mathrm{sm}} \neq \emptyset$.*

We will see later that the converse also holds. This is a special case of a result due to Auslander-Buchsbaum and Serre, saying that if R is a regular local ring, then for every prime ideal $\mathfrak{p}$ in R, the localization $R_{\mathfrak{p}}$ is regular (see [Eis95, Chapter 19]). We will give later a direct proof of this result in our geometric setting.

Proof of Corollary 6.3.19. Let $r = \dim(\mathcal{O}_{X,V})$. By assumption, the maximal ideal in $\mathcal{O}_{X,V}$ is generated by r elements. After possibly replacing X by a suitable affine open subset meeting V, we may assume that X is affine and that $I_X(V)$ is generated by r elements $f_1, \ldots, f_r$. By Theorem 6.3.7, we can find a point $x \in Y_{\mathrm{sm}}$. We then deduce from Proposition 6.3.17 that x is a smooth point also on X, hence $Y \cap X_{\mathrm{sm}} \neq \emptyset$. $\square$

Proposition 6.3.20. *Let p be a smooth point on a variety X. If $f_1, \ldots, f_r$ are regular functions defined in an open neighborhood of p, vanishing at p, and whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{X,p}$, then there is an affine open neighborhood U of x such that the following conditions hold:*

i) *We have $f_1, \ldots, f_r \in \mathcal{O}(U)$.*
ii) *We have a closed subvariety Y of X with $I_U(Y \cap U) = (f_1, \ldots, f_r)$.*
ii) *The subvariety Y is smooth at p and $\dim_p(Y) = \dim_p(X) - r$.*

Proof. We begin by choosing an affine open neighborhood U of p such that $f_i \in \mathcal{O}(U)$ for all i and let Y be the closure in X of the zero-locus in U of $f_1, \ldots, f_r$. Since p lies on a unique irreducible component of X by Proposition 6.3.8, we may assume, after possibly shrinking U, that X is irreducible, and let $n = \dim(X)$. Let $R = \mathcal{O}_{X,p}$ and $\overline{R} = R/(f_{1,p}, \ldots, f_{r,p})$, where $f_{i,p}$ is the image of f_i in R. If we denote by $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ the maximal ideals in R and $\overline{R}$, respectively, then by assumption, the classes of $f_{1,p}, \ldots, f_{r,p}$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, hence $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = n - r$. On the other hand, it follows from Corollaries 3.3.7 and 3.3.13 that $\dim(\overline{R}) \geq n - r$. We thus conclude that $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \leq \dim(\overline{R})$ and it follows from Proposition 6.3.1 that this is an equality, hence $\overline{R}$ is a regular local ring. We also see that $\dim(\overline{R}) = n - r$. Since $\overline{R}$ is a regular ring, it follows from Remark 6.3.11 that $\overline{R}$ is a reduced ring, hence after replacing U by a smaller neighborhood of p, we may assume that $f_1, \ldots, f_r$ generate a radical ideal in $\mathcal{O}(U)$, hence $(f_1, \ldots, f_r) = I_U(Y \cap U)$. Since $\overline{R}$ is a regular ring, it follows that Y is smooth at p, with $\dim_p(Y) = \dim_p(X) - r$. $\square$

The next result describes the behavior of smooth closed subvarieties of a smooth variety.

Proposition 6.3.21. *Let X be an algebraic variety and Y a closed subvariety of X. If $p \in Y$ is a point that is smooth on both Y and X, then after replacing X with a suitable affine open neighborhood of p, the following conditions hold:*

i) *The ideal $I = I_X(Y)$ is generated by r elements, where $r = \dim_p(X) - \dim_p(Y)$; in fact these elements can be chosen such that their images in $\mathcal{O}_{X,p}$ are part of a regular system of parameters.*
ii) *If $R = \mathcal{O}(X)$, then the generators of I induce an isomorphism*

$$R/I[x_1, \ldots, x_r] \simeq \bigoplus_{j \geq 0} I^j/I^{j+1} =: \mathrm{gr}_I(R).$$

Proof. Note first that by Proposition 6.3.8, the point p lies on unique irreducible components of X and Y, hence we may assume that both X and Y are irreducible. We may and will assume that X is affine, with $\mathcal{O}(X) = R$, and Y is defined by $I = I_X(Y)$. Let $\mathfrak{m}$ be the maximal ideal in R corresponding to p. By assumption, we can write

$$r = \dim_k T_pX - \dim_k T_pY. \tag{6.3.1}$$

It follows from (6.3.1) that

$$\dim_k (IR_{\mathfrak{m}} + \mathfrak{m}^2 R_{\mathfrak{m}})/\mathfrak{m}^2 R_{\mathfrak{m}} = r.$$

We can thus find r elements that are part of a regular system of parameters of $R_{\mathfrak{m}}$ and which lie in $IR_{\mathfrak{m}}$. After possibly replacing X by a smaller affine open neighborhood of x, we may assume, in addition, that these elements are the images $f_{1,p}, \ldots, f_{r,p}$ in $R_{\mathfrak{m}}$ of $f_1, \ldots, f_r \in I$. It follows from Proposition 6.3.20 that after possibly replacing X by a suitable open neighborhood of p, we may assume that $f_1, \ldots, f_r$ generate the ideal of a closed subvariety Z, smooth, irreducible, of dimension equal to $\dim(X) - r$. Since $Y \subseteq Z$, it follows that $Y = Z$, which gives i).

We now prove the assertion in ii). This is trivial if $I = 0$, hence we assume $r > 0$. We have a surjective homomorphism

$$\phi\colon R/I[x_1, \ldots, x_r] \longrightarrow \mathrm{gr}_I(R)$$

that maps each x_i to the class of f_i in I/I^2. Note now that

$$\dim\big(\mathrm{gr}_I(R)\big) \geq \dim(R). \tag{6.3.2}$$

Indeed, it follows from Proposition 6.1.3 that the blow-up $g\colon \widetilde{X} \to X$ of X along I is a birational morphism and $g^{-1}(Y)$ has all irreducible components of codimension

1 in $\widetilde{X}$. If J is the nil-radical of $\mathrm{gr}_I(R)$, then

$$g^{-1}(Y) \simeq \mathrm{MaxProj}\big(\mathrm{gr}_I(R)/J\big),$$

which gives by Remark 4.3.20

$$\dim\big(\mathrm{gr}_I(R)\big) = \dim\big(\mathrm{gr}_I(R)/J\big) \geq \dim\Big(\mathrm{MaxProj}\big(\mathrm{gr}_I(R)/J\big)\Big) + 1 = \dim(X).$$

Since $R/I[x_1, \ldots, x_r]$ is a domain of dimension equal to $\dim(X)$, we conclude that ϕ is an isomorphism, completing the proof of the proposition. □

COROLLARY 6.3.22. *If X is a smooth variety and $Y, Z \subseteq X$ are irreducible closed subsets, then every irreducible component W of $Y \cap Z$ satisfies*

$$\mathrm{codim}_X(W) \leq \mathrm{codim}_X(Y) + \mathrm{codim}_X(Z).$$

PROOF. The idea is similar to the one used when $X = \mathbf{A}^n$ (cf. Exercise 3.3.21). We may replace X by its unique irreducible component that contains W, and thus assume that X is irreducible. Let $n = \dim(X)$. Consider the diagonal $\Delta_X \subseteq X \times X$. Note that we have an isomorphism

$$Y \cap Z \simeq (Y \times Z) \cap \Delta_X,$$

hence we may consider W as an irreducible component of the right-hand side. Since X is smooth of dimension n and $X \times X$ is smooth of dimension $2n$, it follows from the proposition that we can find a non-empty affine open subset $U \subseteq X \times X$ such that $U \cap W \neq \emptyset$, and we have $f_1, \ldots, f_n \in \mathcal{O}(U)$ such that

$$\Delta_X \cap U = \{x \in U \mid f_1(x) = \ldots = f_n(x) = 0\}.$$

We deduce that $W \cap U$ is an irreducible component of

$$\{x \in (Y \times Z) \cap U \mid f_1(x) = \ldots = f_n(x) = 0\},$$

and therefore Corollary 3.3.7 implies that $\mathrm{codim}_{(Y\times Z)\cap U}(W \cap U) \leq n$. Using Corollary 3.3.13, this gives $\dim(W) \geq \dim(Y) + \dim(Z) - n$, and further

$$\mathrm{codim}_X(W) \leq \mathrm{codim}_X(Y) + \mathrm{codim}_X(Z).$$

□

EXAMPLE 6.3.23. If $X = \mathrm{MaxSpec}(A)$ is a smooth variety and $f\colon \widetilde{X} \to X$ is the blow-up of X along the radical ideal I, defining the smooth closed subvariety Y of X, then $\widetilde{X}$ is smooth. Indeed, note first that after covering X by suitable affine open subsets, we may assume that X and Y are irreducible and, by Proposition 6.3.21, that I is generated by $r = \mathrm{codim}_X(Y)$ elements $f_1, \ldots, f_r$. In this case, we can explicitly describe $\widetilde{X}$ by equations, as follows.

The surjection

$$\phi\colon A[y_1, \ldots, y_r] \to R(A, I), \quad \phi(y_i) = f_i t \quad \text{for} \quad 1 \leq i \leq r$$

induces a closed immersion

$$\iota\colon \widetilde{X} \hookrightarrow X \times \mathbf{P}^{r-1}$$

of varieties over X. Note that if J is the ideal generated by all differences $f_i y_j - f_j y_i$, for $i \neq j$, then $J \subseteq \ker(\phi)$, hence ι maps $\widetilde{X}$ inside $V(J)$. We will show that in fact $\iota(X) = V(J)$.

Note first that the morphism $g\colon V(J) \to X$ is an isomorphism over $X \smallsetminus Y$. Indeed, we have

$$\big(A[y_1, \ldots, y_r]/J\big)_{f_i} \simeq A_{f_i}[y_i],$$

and therefore the inverse image of $D(f_i)$ in $V(J)$ is isomorphic to

$$\mathrm{MaxProj}\big(A_{f_i}[y_i]\big) \simeq \mathrm{MaxSpec}(A_{f_i}).$$

We now show that $V(J)$ is a smooth subvariety of $X \times \mathbf{P}^{r-1}$, of codimension $r-1$. This is clear at the points lying over $X \smallsetminus Y$, so that we consider a point $q = \big(p, [u_1, \ldots, u_r]\big) \in V(J)$ lying over Y, hence $f_1(p) = \ldots = f_r(p) = 0$. Let i be such that $u_i \neq 0$ and consider the open subset $U_i = X \times D^+_{\mathbf{P}^{r-1}}(x_i) \subseteq X \times \mathbf{P}^{r-1}$. The intersection $V(J) \cap U_i$ is the zero-locus of the ideal generated by $f_j - f_i\frac{y_j}{y_i}$, for $j \neq i$. Let $\mathfrak{m}$ be the ideal defining q. Note that we can write

$$f_j - f_i\frac{y_j}{y_i} = f_j - \frac{u_j}{u_i}f_i + \left(\frac{y_j}{y_i} - \frac{u_j}{u_i}\right)f_i.$$

Since $\left(\frac{y_j}{y_i} - \frac{u_j}{u_i}\right)f_i \in \mathfrak{m}^2$ and the classes of $f_j - \frac{u_j}{u_i}f_i$ in $\mathfrak{m}/\mathfrak{m}^2$, for $j \neq i$, are linearly independent, it follows from Proposition 6.3.20 that q is a smooth point of $V(J)$, and the codimension of $\widetilde{X}$ in $X \times \mathbf{P}^{r-1}$ around q is $r-1$.

We can now see that $V(J)$ is irreducible, and thus it is equal to $\iota(\widetilde{X})$. Indeed, every irreducible component of $V(J)$ different from $\overline{g^{-1}(X \smallsetminus Y)}$ must be contained in $g^{-1}(Y) = Y \times \mathbf{P}^{r-1}$. However, we have seen that every irreducible component of $V(J)$ has dimension equal to $\dim(X) > \dim(Y) + r - 1$, hence it can't be contained in $Y \times \mathbf{P}^{r-1}$.

We thus conclude that $\widetilde{X}$ is smooth and is defined in $X \times \mathbf{P}^{r-1}$ by the ideal J.

Definition 6.3.24. Given a smooth variety X and two smooth closed subvarieties Y and Z of X, recall that for every $p \in Y \cap Z$, we may consider T_pY and T_pZ as linear subspaces of T_pX. We say that Y and Z intersect transversely if, for every $p \in Y \cap Z$, we have

$$\mathrm{codim}_{T_pX}(T_pY \cap T_pZ) = \mathrm{codim}^p_X(Y) + \mathrm{codim}^p_X(Z)$$

(note that p lies on unique irreducible components X' and Y' of X and Y, respectively, and we put $\mathrm{codim}^p_X(Y) = \mathrm{codim}_{X'}(Y')$; a similar definition applies for $\mathrm{codim}^p_X(Z)$). The condition can be equivalently formulated as follows: for every $p \in Y \cap Z$, we have

$$T_pY + T_pZ = T_pX.$$

Proposition 6.3.25. *If X is a smooth variety and Y, Z are smooth closed subvarieties of X that intersect transversely, then $Y \cap Z$ is smooth, and for every $p \in Y \cap Z$, we have*

$$\mathrm{codim}^p_X(Y \cap Z) = \mathrm{codim}^p_X(Y) + \mathrm{codim}^p_X(Z) \quad \textit{and}$$

$$T_p(Y \cap Z) = T_pY \cap T_pZ.$$

Moreover, for every affine open subset U of X, we have

$$I_U(Y \cap Z \cap U) = I_U(Y \cap U) + I_U(Z \cap U).$$

Proof. Let $r = \mathrm{codim}^p_X(Y)$ and $s = \mathrm{codim}^p_X(Z)$. It follows from Proposition 6.3.21 that if U is a suitable irreducible affine open neighborhood of p, then $I_U(Y \cap U)$ is generated by r elements and $I_U(Z \cap U)$ is generated by s elements. Consider the ideal

$$J = I_U(Y \cap U) + I_U(Z \cap U)$$

that defines the closed subset $Y \cap Z$. Since J is generated by $r + s$ elements, it follows from Corollaries 3.3.7 and 3.3.13 that every irreducible component of $Y \cap Z$

has dimension $\geq \dim_p(X) - (r+s)$. On the other hand, we have $T_p(Y \cap Z) \subseteq T_p(Y) \cap T_p(Z)$, hence by assumption

$$\dim_k T_p(Y \cap Z) \leq \dim_p(X) - (r+s).$$

This implies that p is a smooth point of $Y \cap Z$ and $T_p(Y \cap Z) = T_pY \cap T_pZ$.

In fact we can do better: it is easy to see, by translating the above argument algebraically, that if $\mathfrak{m} \subseteq \mathcal{O}(U) = R$ is the maximal ideal corresponding to p, then

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 + J \leq \dim_p(X) - (r+s) \leq \dim \big(R_{\mathfrak{m}}/JR_{\mathfrak{m}}\big).$$

This implies that $R_{\mathfrak{m}}/JR_{\mathfrak{m}}$ is a regular local ring, hence reduced by Remark 6.3.11. Therefore $J_{\mathfrak{m}} = \big(I_U(Y \cap U) + I_U(Z \cap U)\big)_{\mathfrak{m}}$ for every point in $Y \cap Z$, which implies the last assertion in the proposition (see, for example, Corollary C.3.3). $\square$

We end this section by stating one of the most useful results in algebraic geometry. Given an irreducible algebraic variety X, a *resolution of singularities* of X is a proper, birational morphism $f\colon \widetilde{X} \to X$, with $\widetilde{X}$ a smooth, irreducible variety. One can ask for more properties (for example, one can ask that f is *projective*, in a sense that we will define later, which implies in particular that if X is projective or quasi-projective, then $\widetilde{X}$ has the same property; one can also ask for f to be an isomorphism over X_{sm}). The following celebrated result is due to Hironaka.

THEOREM 6.3.26. *If* $\mathrm{char}(k) = 0$*, then every irreducible variety* X *over* k *has a resolution of singularities.*

REMARK 6.3.27. In fact, Hironaka's theorem is more precise: suppose, for simplicity, that X has a closed immersion in a smooth variety Y (for example, any quasi-projective variety satisfies this condition). In this case the theorem says that there is a sequence of morphisms

$$Y_r \xrightarrow{f_r} Y_{r-1} \longrightarrow \ldots \longrightarrow Y_1 \xrightarrow{f_1} Y_0 = Y$$

with the following properties:

i) Each f_i, with $1 \leq i \leq r$, is the blow-up along a smooth variety Z_{i-1} (hence, by induction, all Y_i are smooth).
ii) For every i, with $1 \leq i \leq r$, the strict transform X_{i-1} of X on Y_{i-1} is not contained in Z_{i-1} (so that the next strict transform X_i is defined).
iii) The strict transform X_r of X on Y_r is smooth.

EXERCISE 6.3.28. Consider the following curves in $\mathbf{A}^2$:

$$X = V(x^2 - y^3), \quad Y = V\big(y^2 - x^2(x+1)\big), \quad \text{and} \quad Z = V(x^2 - y^5).$$

Show that if $\pi\colon \widetilde{\mathbf{A}^2} \to \mathbf{A}^2$ is the blow-up of the origin, then the strict transforms $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, are smooth; the strict transform $\widetilde{Z}$ of Z has one singular point and by blowing that up, the resulting strict transform is smooth.

6.4. Bertini's theorem

Recall that the hyperplanes in $\mathbf{P}^n$ are parametrized by a projective space $(\mathbf{P}^n)^*$. We will be using the following terminology: if Z is an irreducible variety, we say that a property holds for a general point $z \in Z$ if there is an open subset U of Z such that the property holds for all $z \in U$. Note that if we have two such properties, then they both hold for a general point in Z: this follows from the fact that the intersection of two nonempty open subsets is again a nonempty open subset. This terminology is particularly convenient when the points of Z parametrize some geometric objects, as is the case with $(\mathbf{P}^n)^*$.

Given a projective variety $X \subseteq \mathbf{P}^n$, one is often interested in the following type of statement: if X has a certain property, then for a general hyperplane H in $\mathbf{P}^n$, the intersection $X \cap H$ still has the same property. In this section we prove such a result for smoothness.

THEOREM 6.4.1 (Bertini). *If $X \subseteq \mathbf{P}^n$ is a smooth variety, then for a general hyperplane H in $\mathbf{P}^n$, the subvarieties X and H of $\mathbf{P}^n$ intersect transversely; in particular, the intersection $X \cap H$ is smooth, and if X has pure dimension d, then $X \cap H$ has pure dimension $d - 1$.*

PROOF. We may assume that X is irreducible: indeed, if we know this, then for every connected component of X, we find a corresponding open subset of $(\mathbf{P}^n)^*$. The intersection of these open subsets then satisfies the conclusion in the theorem. From now on we assume that X is irreducible, and let $d = \dim(X)$.

Note that for every hyperplane H in $\mathbf{P}^n$ and every $p \in H$, we have $\mathbf{T}_pH = H$. It follows from Proposition 6.2.6 that H and X do not intersect transversely if and only if there is $p \in X \cap H$ such that $\mathbf{T}_pX \subseteq H$. Consider the set

$$Z := \big\{\big(p, [H]\big) \in X \times (\mathbf{P}^n)^* \mid \mathbf{T}_p(X) \subseteq H\big\}.$$

We claim that Z is closed in $X \times (\mathbf{P}^n)^*$. In order to check this, let $f_1, \ldots, f_r$ be homogeneous generators for the ideal I_X of X in $\mathbf{P}^n$. The linear subspace $\mathbf{T}_pX$ at a point $p \in X$ is defined by the linear equations

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(p)x_j = 0 \quad \text{for} \quad 1 \leq i \leq r.$$

By assumption, for every $p \in X$, the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$ is $n - d$. The hyperplane H defined by $\sum_{j=0}^{n} a_jx_j = 0$ contains $\mathbf{T}_pX$ if and only if the rank of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ \frac{\partial f_1}{\partial x_0}(p) & \frac{\partial f_1}{\partial x_1}(p) & \ldots & \frac{\partial f_1}{\partial x_n}(p) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_r}{\partial x_0}(p) & \frac{\partial f_r}{\partial x_1}(p) & \ldots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}$$

is $\leq n - d$. Equivalently, all $(n - d + 1)$-minors of this matrix must be 0, and it is clear that it is enough to only consider those minors involving the first row. Each of these conditions is of the form

$$\sum_{j=0}^{n} a_jg_j(p) = 0$$

for some homogeneous polynomials $g_0, \ldots, g_n$, all of the same degree. It is now straightforward to check (for example, by covering each of X and $(\mathbf{P}^n)^*$ by the standard affine charts) that the subset Z is closed in $X \times (\mathbf{P}^n)^*$. In particular, Z is a projective variety.

The projections onto the two components induce two morphisms $\pi_1 \colon Z \to X$ and $\pi_2 \colon Z \to (\mathbf{P}^n)^*$. For every $p \in X$, consider the fiber $\pi_1^{-1}(p)$. This is identified with the subset of $(\mathbf{P}^n)^*$ consisting of all hyperplanes containing $\mathbf{T}_p(X)$. This is a linear subspace of dimension $n - d - 1$. Indeed, since X is smooth, of dimension

d, the linear subspace $\mathbf{T}_p(X)$ of $\mathbf{P}^n$ has dimension d. After choosing suitable coordinates, we may assume that this is given by $x_{d+1} = \ldots = x_n = 0$. In this case, the hyperplane with equation $\sum_{i=0}^n a_i x_i = 0$ contains $\mathbf{T}_p(X)$ if and only if $a_0 = \ldots = a_d = 0$; this is thus a linear subspace in $(\mathbf{P}^n)^*$ of codimension $d+1$.

Therefore we conclude from Corollary 3.4.3 that

$$\dim(Z) = \dim(X) + (n - d - 1) = n - 1.$$

In this case, the morphism $\pi_2\colon Z \to (\mathbf{P}^n)^*$ can't be dominant. Its image is thus a proper closed subset of $(\mathbf{P}^n)^*$ and if U is the complement of this image, we see that for every hyperplane H in $\mathbf{P}^n$ with $[H] \in U$, X and H intersect transversely, and therefore Proposition 6.3.25 implies that $X \cap H$ is a smooth variety of pure dimension $d-1$ (of course, if $d = 0$, this simply means that $X \cap H$ is empty). $\square$

REMARK 6.4.2. It follows from the above proof that even if $X \subseteq \mathbf{P}^n$ is a subvariety with finitely many singular points, for a general hyperplane H in $\mathbf{P}^n$, the intersection $X \cap H$ is still smooth. Indeed, with the notation in the proof, we still have that the fiber $\pi_1^{-1}(p)$, for $p \in X$, has dimension $\leq n - d - 1$ (in fact, one can get a better bound at the singular points). We thus still have the bound $\dim(Z) \leq n-1$, which implies that Z does not dominate $(\mathbf{P}^N)^*$. Since a general hyperplane does not contain any of the singular points of X, we deduce that such a hyperplane intersects $X_{\rm sm}$ transversally, and therefore $X \cap H$ is smooth.

REMARK 6.4.3. There are several other versions of Bertini's theorem. One which is often useful says that if $X \subseteq \mathbf{P}^n$ is an irreducible closed subvariety, with $\dim(X) \geq 2$, then for a general hyperplane $H \subseteq \mathbf{P}^n$, the intersection $X \cap H$ is again irreducible (see [**Jou83**] for this and related results). Another useful version, due to Kleiman, concerns smoothness in the case when instead of a closed subvariety of $\mathbf{P}^n$ one deals with an arbitrary morphism $X \to \mathbf{P}^n$ (this, however, works only over a ground field of characteristic 0). We will give a proof of this result at some later point.

6.5. Smooth morphisms between smooth varieties

In this section we discuss the notion of smooth morphism between smooth varieties. We will later return to this concept, to consider the case of arbitrary varieties.

DEFINITION 6.5.1. A morphism $f\colon X \to Y$ between smooth algebraic varieties is *smooth* at a point $x \in X$ if the linear map

$$df_x\colon T_x X \to T_{f(x)} Y$$

is surjective. The morphism f is *smooth* if it is smooth at every point.

Given a morphism of smooth varieties $f\colon X \to Y$, for every irreducible component X' of X there is a unique irreducible component Y' of Y such that $f(X') \subseteq Y'$. We can thus easily reduce to the case of morphisms between smooth, irreducible varieties.

PROPOSITION 6.5.2. *If $f\colon X \to Y$ is a smooth morphism between the smooth, irreducible varieties X and Y, then f is dominant and for every $y \in f(X)$, the* *fiber* $f^{-1}(y)$ *is smooth, of pure dimension* $\dim(X) - \dim(Y)$*. Moreover, for every* $x \in f^{-1}(y)$*, we have*

$$T_x\big(f^{-1}(y)\big) = \ker(df_x \colon T_xX \to T_{f(x)}Y).$$

PROOF. By Theorem 3.4.1, we know that every irreducible component of $f^{-1}(y)$ has dimension $\geq \dim(X) - \dim(Y)$. Moreover, the inequality is strict if f is not dominant.

On the other hand, the composition

$$f^{-1}(y) \overset{i}{\hookrightarrow} X \xrightarrow{f} Y$$

where i is the inclusion map can also be factored as

$$f^{-1}(y) \longrightarrow \{y\} \hookrightarrow Y.$$

This implies that the restriction of df_x to $T_x\big(f^{-1}(y)\big) \subseteq T_xX$ is zero, hence $T_x\big(f^{-1}(y)\big)$ is contained in the kernel of df_x. Since df_x is surjective, it follows that

$$\dim_k T_x\big(f^{-1}(y)\big) \leq \dim_k \ker(df_x) = \dim_k T_xX - \dim_k T_{f(x)}Y = \dim(X) - \dim(Y).$$

Since $\dim_x \big(f^{-1}(y)\big) \leq \dim_k T_x\big(f^{-1}(y)\big)$, we thus conclude that this is, in fact, an equality. This implies that f is dominant, $T_x\big(f^{-1}(y)\big) = \ker(df_x)$, and $f^{-1}(y)$ is smooth at x, of dimension $\dim(X) - \dim(Y)$. □

EXAMPLE 6.5.3. Consider the morphism $f \colon \mathbf{A}^1 \to \mathbf{A}^1$ given by $f(t) = t^2$, where we assume that $\mathrm{char}(k) \neq 2$. For every point $t \in \mathbf{A}^1$, the map

$$T_t\mathbf{A}^1 = k \to k = T_{f(t)}\mathbf{A}^1$$

is given by multiplication by $2t$ (see Remark 6.2.5). It follows that f is smooth at every point $t \neq 0$, but it is not smooth at 0.

DEFINITION 6.5.4. A morphism of smooth varieties $f \colon X \to Y$ is *étale* at $x \in X$ if it is smooth at x and $\dim_x X = \dim_{f(x)} Y$. The morphism is *étale* if it is étale at every point.

The following theorem is the Generic Smoothness theorem. We will prove this later.

THEOREM 6.5.5. *If* $\mathrm{char}(k) = 0$*, then for every dominant morphism of smooth varieties* $f \colon X \to Y$*, there is a non-empty open subset* $U \subseteq Y$ *such that the induced morphism* $f^{-1}(U) \to U$ *is smooth.*

REMARK 6.5.6. The hypothesis on the characteristic in the above theorem is essential. If $\mathrm{char}(k) = p$, note that the morphism $f \colon \mathbf{A}^1 \to \mathbf{A}^1$ given by $f(t) = t^p$ is not smooth at any point.

REMARK 6.5.7. The Generic Smoothness theorem is the analogue of Sard's theorem in differential topology. Note that by combining it with Proposition 6.5.2, we conclude that if $f \colon X \to Y$ is a dominant morphism of smooth, irreducible algebraic varieties over an algebraically closed field of characteristic 0, then there is a non-empty open subset U of Y such that for every $y \in Y$, the fiber $f^{-1}(y)$ is smooth.

CHAPTER 7

The Grassmann variety and other examples

In this chapter we discuss various geometric examples related to the Grassmann variety. In the first section we construct this variety and discuss several related constructions, such as the Plücker embedding and the incidence correspondence. In the second section we discuss flag varieties, while in the third section we give a resolution of singularities for the generic determinantal variety. We next consider the parameter space for projective hypersurfaces and discuss linear subspaces on such hypersurfaces. In the last section we treat the variety of nilpotent matrices.

7.1. The Grassmann variety

Let $V = k^n$ and let r be an integer with $0 \leq r \leq n$. In this section we describe the structure of algebraic variety on the set $G(r, n)$ parametrizing the r-dimensional linear subspaces of V. These are the *Grassmann varieties*. Given an r-dimensional linear subspace W of V, we denote by $[L]$ the corresponding point of $G(r, n)$.

This is trivial for $r = 0$ or $r = n$: in this case $G(r, n)$ is just a point. The first non-trivial case that we have already encountered is for $r = 1$: in this case $G(r, n) = \mathbf{P}^{n-1}$. A similar description holds for $r = n-1$: hyperplanes in k^n are in bijection with lines in $(k^n)^* \simeq k^n$, hence these are again parametrized by a $\mathbf{P}^{n-1}$ (cf. Exercise 4.2.18).

We now proceed with the description in the general case. Given an r-dimensional linear subspace W of k^n, choose a basis $u_1, \ldots, u_r$ of W. By writing $u_i = (a_{i,1}, \ldots, a_{i,n})$ for $1 \leq i \leq r$, we obtain a matrix $A = (a_{i,j}) \in M_{r,n}(k)$. Note that we have an action of $GL_r(k)$ on $M_{r,n}(k)$ given by left multiplication. Choosing a different basis of W corresponds to multiplying the matrix on the left by an element of $GL_r(k)$. Moreover a matrix in $M_{r,n}(k)$ corresponds to some r-dimensional linear subspace in k^n if and only if it has maximal rank r. We can thus identify $G(r, n)$ with the quotient set $U/GL_r(k)$, where U is the open subset of $M_{r,n}(k)$ consisting of matrices of rank r.

For every subset $I \subseteq \{1, \ldots, n\}$ with r elements, let U_I be the open subset of U given by the non-vanishing of the r-minor on the columns indexed by the elements of I. Note that this subset is preserved by the $GL_r(k)$-action and let V_I be the corresponding subset of $G(r, n)$. We now construct a bijection

$$\phi_I \colon V_I \to M_{r,n-r}(k) = \mathbf{A}^{r(n-r)}.$$

In order to simplify the notation, say $I = \{1, \ldots, r\}$. Given any matrix $A \in U_I$, let us write it as $A = (A', A'')$ for matrices $A' \in M_{r,r}(k)$ and $A'' \in M_{r,n-r}(k)$. Note that by assumption $\det(A') \neq 0$. In this case there is a unique matrix $B \in GL_r(k)$ such that $B \cdot A = (I_r, C)$, for some matrix $C \in M_{r,n-r}(k)$ (namely $B = (A')^{-1}$, in which case $C = (A')^{-1} \cdot A''$). Therefore every matrix class in V_I is the class of a unique matrix of the form (I_r, C), with $C \in M_{r,n-r}(k)$. This gives the desired bijection between $V_{\{1,\ldots,r\}} \to \mathbf{A}^{r(n-r)}$, and a similar argument works for every V_I.

We put on each V_I the topology and the sheaf of functions that make the above bijection an isomorphism in $\mathcal{T}op_k$. We need to show that these glue to give on $G(r,n)$ a structure of a prevariety: we need to show that for every subsets I and J as above, the subset $\phi_I(V_I \cap V_J)$ is an open subset of $\mathbf{A}^{r(n-r)}$ and the map

$$\phi_J \circ \phi_I^{-1} \colon \phi_I(V_I \cap V_J) \to \phi_J(V_I \cap V_J) \tag{7.1.1}$$

is a morphism of algebraic varieties (in which case, by symmetry, it is an isomorphism). In order to simplify the notation, suppose that $I = \{1,\ldots,r\}$. It is then easy to see that if $\#(I \cap J) = \ell$, then $\phi_I(V_I \cap V_J) \subseteq \mathbf{A}^{r(n-r)}$ is the principal affine open subset defined by the non-vanishing of the $(r-\ell)$-minor on the rows indexed by those $i \in I \smallsetminus J$ and on the columns indexed by those $j \in J \smallsetminus I$. Moreover, the map (7.1.1) is given by associating to a matrix C the $r \times n$ matrix $M = (I_r, C)$, multiplying it on the left with the inverse of the $r \times r$-submatrix of M on the columns in J to get M', and then keeping the $r \times (n-r)$ submatrix of M' on the columns in $\{1,\ldots,n\} \smallsetminus J$. It is clear that this is a morphism.

We thus conclude that $G(r,n)$ is an object in $\mathcal{T}op_k$. In fact, it is a prevariety, since it is covered by open subsets isomorphic to affine varieties. In fact, since each V_I is isomorphic to an affine space, it is smooth and irreducible, and since we have seen that any two V_I intersect, we conclude that $G(r,n)$ is irreducible by Exercise 1.3.17. Furthermore, since each V_I has dimension $r(n-r)$, we conclude that $\dim\big(G(r,n)\big) = r(n-r)$. We collect these facts in the following proposition.

PROPOSITION 7.1.1. *The Grassmann variety $G(r,n)$ is a smooth, irreducible prevariety of dimension $r(n-r)$, that has a cover by open subsets isomorphic to $\mathbf{A}^{r(n-r)}$.*

EXAMPLE 7.1.2. If $r = 1$, the algebraic variety $G(1,n)$ is just $\mathbf{P}^{n-1}$, described via the charts $U_i = (x_i \neq 0) \simeq \mathbf{A}^{n-1}$.

EXAMPLE 7.1.3. If $r = n-1$, the algebraic variety $G(n-1,n)$ has an open cover

$$G(n-1,n) = U_1 \cup \ldots \cup U_n.$$

For every i, we have an isomorphism $\mathbf{A}^{n-1} \simeq U_i$ such that $(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots\lambda_n)$ is mapped to the hyperplane generated by $\{e_j + \lambda_j e_i \mid j \neq i\}$. This is the hyperplane defined by the equation $e_i^* - \sum_{j\neq i}\lambda_j e_j^* = 0$. We thus see that the variety structure on $G(n-1,n)$ is the same one as on $(\mathbf{P}^{n-1})^*$, which is isomorphic to $\mathbf{P}^{n-1}$ (cf. Exercise 4.2.18).

Our next goal is to show that, in fact, $G(r,n)$ is a projective variety. Note that if W is an r-dimensional linear subspace of $V = k^n$, then $\wedge^r W$ is a 1-dimensional linear subspace of $\wedge^r V \simeq k^d$, where $d = \binom{n}{r}$. If $e_1,\ldots,e_n$ is the standard basis of k^n, then we have a basis of $\wedge^r V$ given by the $e_I = e_{i_1} \wedge \ldots \wedge e_{i_r}$, where $I = \{i_1,\ldots,i_r\}$ is a subset of $\{1,\ldots,n\}$ with r-elements (and where, in order to write e_I, we order the elements $i_1 < \ldots < i_r$). We correspondingly denote the coordinates on the projective space of lines in $\wedge^r V$ by x_I.

PROPOSITION 7.1.4. *The map $f \colon G(r,n) \to \mathbf{P}^{d-1}$ that maps $[W]$ to $[\wedge^r W]$ is a closed immersion. In particular, $G(r,n)$ is a projective variety.*

The embedding in the above proposition is the *Plücker embedding* of the Grassmann variety.

Proof of Proposition 7.1.4. If $W \subseteq V$ is an r-dimensional linear subspace described by the matrix A, then $f\big([W]\big) \in \mathbf{P}^{d-1}$ is given in the above homogeneous coordinates by the r-minors of A. In particular, we see that the inverse image of the affine chart $W_I = (x_I \neq 0)$ is the affine open subset $V_I \subseteq G(r,n)$.

In order to complete the proof, it is enough to show that for every I, the induced map $V_I \to W_I$ is a morphism and the corresponding ring homomorphism

$$\mathcal{O}(W_I) \to \mathcal{O}(V_I) \tag{7.1.2}$$

is surjective. The argument is the same for all I, but in order to simplify the notation, we assume $I = \{1, \ldots, r\}$. Note that the map $V_I \to W_I$ gets identified to $M_{r,n-r}(k) \to \mathbf{A}^{\binom{n}{r}-1}$, than maps a matrix B to all r-minors of (I_r, B), with the exception of the one on the first r columns. In particular, we see that this map is a morphism. By choosing $r-1$ columns of the first r ones and an additional column of the last $(n-r)$ ones, we obtain every entry of B as an r-minor as above. This implies that the homomorphism (7.1.2) is surjective. $\square$

Remark 7.1.5. The algebraic group $GL_n(k)$ acts on k^n and thus acts on $G(r,n)$ by $g \cdot [W] = [g \cdot W]$. Note that if W is described by the matrix $A \in M_{r,n}(k)$, then $g \cdot W$ is described by $A \cdot g^t$. It is straightforward to see that this is an algebraic action. Since any two linear subspaces can by mapped one to the other by a linear automorphism of k^n, we see that the $GL_n(k)$-action on $G(r,n)$ is transitive.

Remark 7.1.6. If W is an r-dimensional linear subspace of $V = k^r$, then we have an induced surjection $V^* \to W^*$, whose kernel is an $(n-r)$-dimensional linear subspace of $(k^n)^* \simeq k^n$. In this way we get a bijection $G(r,n) \to G(n-r,n)$ and it is not hard to check that this is, in fact, an isomorphism of algebraic varieties.

Remark 7.1.7. Given an arbitrary n-dimensional vector space V over k, let $G(r,V)$ be the set of r-dimensional linear subspace of V. By choosing an isomorphism $V \simeq k^n$, we obtain a bijection $G(r,V) \simeq G(r,n)$ and we put on $G(r,V)$ the structure of an algebraic variety that makes this an isomorphism. Note that this is independent of the choice of isomorphism $V \simeq k^r$: for a different isomorphism, we have to compose the map $G(r,V) \to G(r,n)$ with the action on $G(r,n)$ of a suitable element in $GL_n(k)$.

Remark 7.1.8. It is sometimes convenient to identify $G(r,n)$ with the set of $(r-1)$-dimensional linear subspaces in $\mathbf{P}^{n-1}$.

Notation 7.1.9. Given a finite-dimensional k-vector space V, we denote by $\mathbf{P}(V)$ the projective space parametrizing *hyperplanes* in V. Therefore the homogeneous coordinate ring of $\mathbf{P}(V)$ is given by the symmetric algebra $\mathrm{Sym}^\bullet(V)$. With this notation, the projective space parametrizing the lines in V is given by $\mathbf{P}(V^*)$.

We end this section by discussing the incidence correspondence for the Grassmann variety and by giving some applications. More applications will be given in the next sections.

Consider the set of r-dimensional linear subspaces in $\mathbf{P}^n$, parametrized by $G = G(r+1, n+1)$. The *incidence correspondence* is the subset

$$\mathcal{Z} = \big\{\big(q, [V]\big) \in \mathbf{P}^n \times G \mid q \in V\big\}.$$

Note that this is a closed subset of $\mathbf{P}^n \times G$. Indeed, if we represent $[W]$ by the matrix $A = (a_{i,j})_{0\leq i\leq r+1, 0\leq j\leq n}$, then $\big([b_0,\ldots,b_n],[W]\big)$ lies in $\mathcal{Z}$ if and only if the rank of the matrix

$$B = \begin{pmatrix} b_0 & b_1 & \ldots & b_n \\ a_{0,0} & a_{0,1} & \ldots & a_{0,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{r,0} & a_{r,1} & \ldots & a_{r,n} \end{pmatrix}$$

is $\leq r+1$. This is the case if and only if all $(r+2)$-minors of B vanish. By expanding along the first row, we can write each such minor as $\sum_{j\in I} b_j\delta_j$, where $I \subseteq \{0,\ldots,n\}$ is the subset with $r+2$ elements determining the minor and each δ_j is a suitable minor of A. Consider the closed immersion

$$\mathbf{P}^n \times G \overset{i}{\hookrightarrow} \mathbf{P}^n \times \mathbf{P}^N \overset{j}{\hookrightarrow} \mathbf{P}^M,$$

where i is given by $i(u,v) = \big(u,\phi(v)\big)$, with ϕ being the Plücker embedding, and j is the Segre embedding. It follows from the above discussion that via this embedding, $\mathcal{Z}$ is the inverse image of a suitable linear subspace of $\mathbf{P}^M$, and therefore it is closed in $\mathbf{P}^n \times G$. Since both $\mathbf{P}^n$ and G are projective varieties, we conclude that $\mathcal{Z}$ is a projective variety.

The projections onto the two components induce the morphisms $\pi_1\colon \mathcal{Z} \to \mathbf{P}^n$ and $\pi_2\colon \mathcal{Z} \to G$. It follows from the definition that for every $[W] \in G$, we have $\pi_2^{-1}\big([W]\big) \simeq W$.

EXERCISE 7.1.10. Show that the morphism $\pi_2\colon \mathcal{Z} \to G$ is locally trivial, with fiber[1] $\mathbf{P}^r$.

Since all fibers of π_2 are irreducible, of dimension r, we conclude from Proposition 5.5.1 that $\mathcal{Z}$ is irreducible, of dimension

$$\dim(\mathcal{Z}) = r + \dim(G) = r + (r+1)(n-r).$$

(we use here the fact that G is irreducible and $\mathcal{Z}$ is a projective variety).

Given a point $q \in \mathbf{P}^n$, the fiber $\pi^{-1}(q) \subseteq G$ consists of all r-dimensional linear subspaces of $\mathbf{P}^n$ containing q (equivalently, these are the $(r+1)$-dimensional linear subspaces of k^{n+1} containing a given line). These are in bijection with the Grassmann variety $G(r,n)$.

EXERCISE 7.1.11. Show that the morphism $\pi_1\colon \mathcal{Z} \to \mathbf{P}^n$ is locally trivial, with fiber $G(r,n)$.

We use the incidence correspondence to prove the following

PROPOSITION 7.1.12. *Let $X \subseteq \mathbf{P}^n$ be a closed subvariety of dimension d and let $G = G(r+1,n+1)$. If we put*

$$M_r(X) = \big\{[W] \in G \mid W \cap X \neq \emptyset\big\},$$

then the following hold:

i) *The set $M_r(X)$ is a closed subset of G, which is irreducible if X is irreducible.*
ii) *We have* $\dim\big(M_r(X)\big) = \dim(G) - (n-r-d)$ *for* $0 \leq r \leq n-d$.

[1]Given a variety F, we say that a morphism $f\colon X \to Y$ is locally trivial, with fiber F, if there is an open cover $Y = U_1 \cup \ldots \cup U_r$ such that for every i, we have an isomorphism $f^{-1}(U_i) \simeq U_i \times F$ of varieties over U_i.

Proof. Using the previous notation, note that $M_r(X) = \pi_2\big(\pi_1^{-1}(X)\big)$, hence $M_r(X)$ is closed, since X is closed and π_2 is a closed map (recall that $\mathcal{Z}$ is a projective variety). Consider now the morphism $\pi_1^{-1}(X) \to X$ induced by π_1. Since all fibers are irreducible, of dimension $r(n-r)$ (being isomorphic to $G(r,n)$), and $\pi_1^{-1}(X)$ is a projective variety, we deduce from Proposition 5.5.1 that if X is irreducible, then $\pi_1^{-1}(X)$ is irreducible, with

$$\dim\left(\pi_1^{-1}(X)\right) = \dim(X) + r(n-r).$$

The irreducibility of $\pi^{-1}(X)$ implies the irreducibility of $\pi_2\big(\pi_1^{-1}(X)\big)$.

It is clear that if $X = X \cup \ldots \cup X_s$ is the irreducible decomposition of X, then we have $M_r(X) = M_r(X_1) \cup \ldots \cup M_r(X_s)$. Therefore, in order to prove ii), we may assume that X is irreducible. We claim that the morphism $\pi_1^{-1}(X) \to M_r(X)$ has at least one finite, non-empty fiber. Using Theorems 3.4.1 and 3.4.1, this implies that

$$\dim\left(M_r(X)\right) = \dim\left(\pi_1^{-1}(X)\right) = d + r(n-r).$$

hence

$$\operatorname{codim}_G\left(M_r(X)\right) = (r+1)(n-r) - d - r(n-r) = n - r - d.$$

We thus only need to find an r-dimensional linear subspace that intersects X in a nonempty, finite set. This is easy to see and we leave the argument as an exercise for the reader. $\square$

Exercise 7.1.13. Consider the Grassmann variety $G = G(r+1, n+1)$ parametrizing the r-dimensional linear subspaces in $\mathbf{P}^n$. Show that if Z is a closed subset of G, then the set

$$\widetilde{Z} := \bigcup_{[V]\in Z} V \subseteq \mathbf{P}^n$$

is a closed subset of $\mathbf{P}^n$, with $\dim(\widetilde{Z}) \leq \dim(Z) + r$.

Exercise 7.1.14. Show that if X and Y are disjoint closed subvarieties of $\mathbf{P}^n$, then the *join* $J(X,Y) \subseteq \mathbf{P}^n$, defined as the union of all lines in $\mathbf{P}^n$ joining a point in X and a point in Y, is a closed subset of $\mathbf{P}^n$, with

$$\dim\left(J(X,Y)\right) \leq \dim(X) + \dim(Y) + 1.$$

7.2. Flag varieties

In this section we define flag varieties and prove some basic properties. Let V be a vector space over k, with $\dim_k V = n$ and let $1 \leq \ell_1 < \ldots < \ell_r \leq n$. A *flag of type* $(\ell_1, \ldots, \ell_r)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq V$, where $\dim_k(V_i) = \ell_i$. A *complete flag* is a flag of type $(1, 2, \ldots, n)$.

The *flag variety* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ parametrizes flags in V. In other words, this is the set

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) := \{(V_1, \ldots, V_r) \in G(\ell_1, V) \times \cdots \times G(\ell_r, V) \mid V_1 \subseteq \cdots \subseteq V_r\}.$$

In particular, the *complete flag variety* $\mathrm{Fl}(V) = \mathrm{Fl}_{1,\ldots,n}(V)$ parametrizes complete flags in V.

Proposition 7.2.1. *The subset* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ *of* $G(\ell_1, V) \times \cdots \times G(\ell_r, V)$ *is closed, hence* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ *is a projective variety.*

PROOF. The assertion is trivial for $r = 1$, hence we may assume $r \geq 2$. For i with $1 \leq i \leq r-1$, consider the map

$$\pi_{i,i+1} \colon G(\ell_1, V) \times \cdots \times G(\ell_r, V) \to G(\ell_i, V) \times G(\ell_{i+1}, V)$$

given by the projection on the factors i and $i+1$. It is clear that

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) = \bigcap_{i=1}^{r-1} \pi_{i,i+1}^{-1}\big(\mathrm{Fl}_{\ell_i,\ell_{i+1}}(V)\big),$$

hence it s enough to prove the assertion in the proposition when $r = 2$.

Let us fix a basis $e_1, \ldots, e_n$ on V. Consider now the set $M_1 \times M_2 \subseteq M_{\ell_1,n}(k) \times M_{\ell_2,n}(k)$ consisting of pairs of matrices of maximal rank. Let Z be the subset of $M_1 \times M_2$ consisting of matrices (A, B) with the property that the linear span of the rows of A is contained in the linear span of the rows of B. Recall that we have morphisms

$$M_1 \to G(\ell_1, V) \quad \text{and} \quad M_2 \to G(\ell_2, V)$$

such that the product map $M_1 \times M_2 \to G(\ell_1, V) \times G(\ell_2, V)$ maps Z onto $\mathrm{Fl}_{\ell_1,\ell_2}(V)$.

Note that Z is closed in $M_1 \times M_2$. Indeed, a pair $\big((a_{i,j}, (b_{i,j})\big)$ lies in Z if and only if the rank of the matrix $(c_{i,j})_{1 \leq i \leq \ell_1+\ell_2, 1 \leq j \leq n}$ given by

$$c_{i,j} = a_{i,j} \quad \text{for} \quad i \leq \ell_1 \quad \text{and} \quad c_{i,j} = b_{i-\ell_1,j} \quad \text{for} \quad \ell_1 + 1 \leq i \leq \ell_1$$

has rank $\leq \ell_2$. Using now the description of $G(\ell_1, V)$ and $G(\ell_2, V)$ in terms of charts arising by covering M_1 and M_2 by corresponding open subsets, it is now easy to see that $\mathrm{Fl}_{\ell_1,\ell_2}(V)$ is closed in $G(\ell_1, V) \times G(\ell_2, V)$. $\square$

Recall that the group $GL(V)$ of linear automorphisms of V has an induced action on each $G(\ell, V)$ and it is clear that the product action on $G(\ell_1, V) \times \cdots \times G(\ell_r, V)$ induces an algebraic action of $GL(V)$ on $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$. This action is clearly transitive: given any two flags of type $(\ell_1, \ldots, \ell_r)$, we can find an invertible linear automorphism of $GL(V)$ that maps one to the other (for example, choose for each flag a basis of V such that the i^{th} element of the flag is generated by the first ℓ_i elements of the basis, and then choose a linear transformation that maps one basis to the other). By Exercise 6.3.14, we conclude that $\mathrm{Fl}(\ell_1, \ldots, \ell_r)(V)$ is a smooth variety.

EXAMPLE 7.2.2. If $e_1, \ldots, e_n$ is a basis of n and V_i is the linear span of $e_1, \ldots, e_i$, then the stabilizer of the point on the complete flag variety corresponding to $V_1 \subseteq \ldots \subseteq V_n$ is the subgroup $B \subseteq GL(V) \simeq GL_n(k)$ of upper-triangular matrices.

It is clear that if $r = 1$, then $\mathrm{Fl}_{\ell_1}(V) = G(\ell_1, V)$. Suppose now that $r \geq 2$. For every $(\ell_1, \ldots, \ell_r)$ as above the projection

$$G(\ell_1, V) \times \cdots \times G(\ell_r, V) \longrightarrow G(\ell_1, V) \times \cdots \times G(\ell_{r-1}, V)$$

onto the first $(r-1)$ components induces a morphism

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) \longrightarrow \mathrm{Fl}_{\ell_1,\ldots,\ell_{r-1}}(V).$$

The fiber over a point corresponding to the flag $(V_1, \ldots, V_{r-1})$ is isomorphic to the Grassmann variety $G(\ell_r - \ell_{r-1}, V/V_{r-1})$, hence it is irreducible, of dimension $(\ell_r - \ell_{r-1})(n - \ell_r)$. Arguing by induction on r and using Proposition 5.5.1, we obtain the following:

PROPOSITION 7.2.3. *If V is an n-dimensional vector space over k, then for every $(\ell_1, \ldots, \ell_r)$, the flag variety $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is an irreducible variety, of dimension $\sum_{i=1}^r \ell_i(\ell_{i+1} - \ell_i)$, where $\ell_{r+1} = n$. In particular, the complete flag variety $\mathrm{Fl}(V)$ is an irreducible variety of dimension $\frac{n(n-1)}{2}$.*

7.3. A resolution of the generic determinantal variety

Fix positive integers m and n and a non-negative integer $r \leq \min\{m,n\}$. Recall that if we identify the space $M_{m,n}(k)$ of $m \times n$ matrices with entries in k with $\mathbf{A}^{mn}$ in the obvious way, we have a closed subset $M^r_{m,n}(k)$ of $\mathbf{A}^{mn}$ consisting of those matrices of rank $\leq r$. Two cases are trivial: if $r = 0$, then $M^r_{m,n}(k) = \{0\}$, and if $r = \min\{m,n\}$, then $M^r_{m,n}(k) = M_{m,n}(k)$.

If we denote the coordinates on $\mathbf{A}^{mn}$ by $x_{i,j}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, then $M^r_{m,n}(k)$ is defined by the vanishing of all $(r+1)$-minors of the matrix $(x_{i,j})$. We have already seen that $M^r_{m,n}(k)$ is irreducible in Exercise 1.4.27. We will give another argument for this, that allows us to also compute the dimension of this variety. In fact, we will give a resolution of singularities for $M^r_{m,n}(k)$.

As usual, we identify $M_{m,n}(k)$ with $\mathrm{Hom}_k(k^n, k^m)$. Consider the following subset of $\mathbf{A}^{mn} \times G(n-r,n)$:

$$\mathcal{Y} = \{(A, [W]) \in \mathbf{A}^{mn} \times G(n-r,n) \mid W \subseteq \ker(A)\}.$$

We first show that $\mathcal{Y}$ is a closed subset of $\mathbf{A}^{mn} \times G(n-r,n)$. Consider the affine open cover $G(n-r,n)$ by subsets $V_I \simeq \mathbf{A}^{(n-r)r}$ described in Section 7.1. Suppose, as usual, that $I = \{1, \ldots, r\}$. If $B \in M_{(n-r)r}(k)$ represents the linear subspace W and if $M = (I_{n-r}, B)$, then $(A, [W]) \in \mathcal{Y}$ if and only if $A \cdot M^t = 0$. We thus see that $\mathcal{Y} \cap (\mathbf{A}^{mn} \times V_I)$ is the zero-locus of the homogeneous degree 2 polynomials given by writing the entries of $A \cdot M^t$ in terms of the entries of A and M. We thus conclude that $\mathcal{Y}$ is a closed subset of $\mathbf{A}^{mn} \times G(n-r,n)$

The projections onto the two components induce maps $\pi_1 \colon \mathcal{Y} \to \mathbf{A}^{mn}$ and $\pi_2 \colon \mathcal{Y} \to G(n-r,n)$. Note that since $G(n-r,n)$ is a projective variety, π_1 is a proper morphism. Its image consists of that $A \in M_{m,n}(k)$ such that $\dim_k \ker(A) \geq n-r$: this is precisely $M^r_{m,n}(k)$.

Let us consider the fiber of π_2 over a point $[W] \in G(n-r,n)$. This is identified to the set of all $A \in M_{m,n}(k)$ that vanish of W, which is isomorphic to $\mathrm{Hom}(k^n/W, k^m) \simeq \mathbf{A}^{rm}$. In fact we can say more: π_1 is locally trivial, with fiber $\mathbf{A}^{rm}$. Indeed, for every subset with r elements $I \subseteq \{1, \ldots, n\}$, we have an isomorphism of varieties over V_I:

$$\pi_1^{-1}(V_I) \simeq V_I \times \mathbf{A}^{rm}.$$

In order to see this, let us assume that $I = \{1, \ldots, r\}$. Via the identification $V_I \simeq M_{n-r,r}(k)$, the intersection $\mathcal{Y} \cap (M_{m,n}(k) \times V_I)$ consists of pairs of matrices $A = (a_{i,j})$ (of size $m \times n$) and $B = (b_{p,q})$ (of size $(n-r) \times r$) such that

$$a_{i,\ell} + \sum_{j=1}^{r} a_{i,n-r+j} b_{\ell,j} = 0 \quad \text{for} \quad 1 \leq i \leq m, 1 \leq \ell \leq n-r.$$

It is then clear that by mapping the pair

$$\big((a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}, (b_{p,q})\big) \quad \text{to} \quad \big((a_{i,j})_{1 \leq i \leq m, n-r+1 \leq j \leq n}, (b_{p,q})\big)$$

we obtain the desired isomorphism. Since $G(n-r,n)$ is smooth, this clearly implies that $\mathcal{Y}$ is smooth. We also see that $\mathcal{Y}$ is irreducible via the general lemma below. Finally, it follows from Theorem 3.4.2 that

$$\dim(\mathcal{Y}) = \dim\big(G(n-r,n)\big) + mr = (n-r)r + mr = (m+n)r - r^2.$$

LEMMA 7.3.1. *If F is an irreducible algebraic variety and $f\colon X \to Y$ is a morphism of algebraic varieties that is locally trivial with fiber F, and such that Y is irreducible, then X is irreducible.*

PROOF. Consider a cover $Y = V_1 \cup \ldots \cup V_i$, with each V_i a nonempty open subset of Y such that $f^{-1}(V_i)$ is isomorphic to $V_i \times F$ as a variety over V_i. In particular, since Y is irreducible, each V_i is irreducible, and therefore $V_i \times F$ is irreducible. Moreover, using again the irreducibility of Y we see that $V_i \cap V_j \neq \emptyset$ for every i and j. Therefore

$$f^{-1}(V_i) \cap f^{-1}(V_j) \simeq (V_i \cap V_j) \times F$$

is nonempty, and we conclude that X is irreducible using Exercise 1.3.17. $\square$

Since $M^r_{m,n}(k)$ is the image of $\mathcal{Y}$, we get another proof for the fact that $M^r_{m,n}(k)$ is irreducible. Note that if $U = M^r_{m,n}(k) \smallsetminus M^{r-1}_{m,n}(k)$, then for every $A \in U$, there is a unique point in $\mathcal{Y}$ mapping to A, namely $\big(A, [\ker(A)]\big)$. By Theorem 3.4.2, we conclude that $\dim\big(M^r_{m,n}(k)\big) = \dim(\mathcal{Y})$, hence the codimension of $M^r_{m,n}(k)$ in $M_{m,n}(k)$ is

$$mn - (m+n)r + r^2 = (m-r)(n-r).$$

In fact, we will show that π_2 is an isomorphism over U; in particular, it is birational. We need to show that the inverse map $U \to f^{-1}(U)$ is a morphism. Of course, since $f^{-1}(U)$ is a locally closed subvariety of $\mathbf{A}^{mn} \times G(n-r,n)$ it is enough to show that the map taking $A \in U$ to $\ker(A) \in G(n-r,n)$ is a morphism. We cover U by the subsets $U_{\Lambda,\Gamma}$, where $\Lambda \subseteq \{1,\ldots,m\}$ and $\Gamma \subseteq \{1,\ldots,n\}$ are subsets with r elements, and where $U_{\Lambda,\Gamma}$ is the subset of $M^r_{m,n}(k)$ consisting of those matrices A such that the minor on the rows in Λ and on the columns in Γ is nonzero. We will show that each map $U_{\Lambda,\Gamma} \to G(n-r,n)$ is a morphism.

In order to simplify the notation, let us assume that $\Lambda = \{1,\ldots,r\}$ and $\Gamma = \{n-r+1,\ldots,n\}$. Let $A \in U_{\Lambda,\Gamma}$. Note that in this case, if $e_1,\ldots,e_n$ is the standard basis of k^n, then $A(e_{n-r+1}),\ldots,A(e_n)$ are linearly independent, hence

$$\ker(A) + \langle e_{n-r+1},\ldots,e_n\rangle = k^n.$$

This implies that $\ker(A) \in V_{\{1,\ldots,n-r\}}$. Moreover, if $\ker(A)$ is described by the matrix $(b_{p,q})_{1\leq p\leq n-r, 1\leq q\leq r}$, then the $b_{p,q}$ are determined by the condition

$$A(e_p) = -\sum_{q=n-r+1}^{n} b_{p,q}A(e_q).$$

It thus follows easily from Cramer's rule that if $A = (a_{i,j}) \in U_{\Lambda,\Gamma}$, then we can write each $b_{p,q}$ as

$$b_{p,q} = \frac{R_{p,q}(A)}{\delta(A)},$$

where $R_{p,q}$ is a polynomial in the $a_{i,j}$, while $\delta(A) = \det(a_{i,j})_{1\leq i\leq r, n-r+1\leq j\leq n}$. This completes the proof of the fact that π_2 is birational. We collect the results we proved in this section in the following proposition

PROPOSITION 7.3.2. *The closed subset $M^r_{m,n}(k)$ of $M_{m,n}(k)$ is irreducible, of codimension $(m-r)(n-r)$, and the morphism $\pi_2\colon \mathcal{Y} \to M^r_{m,n}(k)$ is a resolution of singularities.*

7.4. Linear subspaces on projective hypersurfaces

We consider a projective space $\mathbf{P}^n$ and let S be its homogeneous coordinate ring. Recall that a hypersurface in $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ whose corresponding radical homogeneous ideal is of the form (F), for some nonzero homogeneous polynomial of positive degree. If $\deg(F) = d$, then the hypersurface has *degree* d.

We begin by constructing a parameter space for hypersurfaces of degree d. Note that two polynomials F and G define the same hypersurface if and only if there is $\lambda \in k^*$ such that $F = \lambda G$. Let $\mathbf{P}^{N_d}$ be the projective space parametrizing lines in the vector space S_d, hence $N_d = \binom{n+d}{n} - 1$. We consider on $\mathbf{P}^{N_d}$ the coordinates y_α, where $\alpha = (\alpha_0, \ldots, \alpha_n)$ has $|\alpha| := \sum_i \alpha_i = d$; therefore the point $[c_\alpha]_\alpha$ corresponds to the hypersurface defined by $\sum_\alpha c_\alpha x^\alpha$, where $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Therefore the set $\mathcal{H}_d$ is parametrized by a subset of the projective space $\mathbf{P}^{N_d}$ consisting of classes of homogeneous polynomials $F \in S_d$ such that the ideal (F) is radical. We will denote by $[H]$ the point of $\mathcal{H}_d$ corresponding to the hypersurface $H \subseteq \mathbf{P}^n$.

LEMMA 7.4.1. *The subset $\mathcal{H}_d \subseteq \mathbf{P}^{N_d}$ is a non-empty open subset.*

PROOF. Note that given $F \in S_d$, the ideal (F) is not reduced if and only if there is a positive integer e and a homogeneous polynomial $G \in S_e$ such that G^2 divides F. For every e such that $0 < 2e \leq d$, consider the map

$$\alpha_e\colon \mathbf{P}^{N_e} \times \mathbf{P}^{N_{d-2e}} \to \mathbf{P}^{N_d}$$

that maps $\big([G],[H]\big)$ to $[G^2H]$. It is straightforward to see that this is a morphism. Since the source is a projective variety, it follows that the image of α_e is closed. Since $\mathcal{H}_d$ is equal to

$$\mathbf{P}^{N_d} \smallsetminus \bigcup_{1 \leq e \leq \lfloor d/2 \rfloor} \mathrm{Im}(\alpha_e),$$

we see that this set is open in $\mathbf{P}^{N_d}$. In order to see that it is non-empty, it is enough to consider $f \in S_d$ which is the product of d distinct linear forms. $\square$

REMARK 7.4.2. We have seen in Theorem 6.4.1 that if $X \subseteq \mathbf{P}^n$ is a smooth variety of pure dimension r, then for a general hyperplene $H \subseteq \mathbf{P}^n$, the intersection $X \cap H$ is smooth, of pure dimension $r-1$. The same assertion holds if we take H a general hypersurface in $\mathbf{P}^n$, of degree d. Indeed, if $\nu_d\colon \mathbf{P}^n \hookrightarrow \mathbf{P}^{N_d}$ is the d^{th} Veronese embeddings, then the intersections $X \cap H$ is isomorphic to the intersection $\nu_d(X) \cap L$, where $L \subseteq \mathbf{P}^{N_d}$ is the hyperplane corresponding to H. We thus conclude by applying Bertini's theorem to $\nu_d(X)$.

By applying the above remark to the case $X = \mathbf{P}^n$, we see that a general hypersurface $H \subseteq \mathbf{P}^n$ of degree d is smooth. The following proposition makes this more precise.

PROPOSITION 7.4.3. *The subset $\mathcal{S}ing_d \subseteq \mathcal{H}_d$ consisting of singular hypersurfaces is an irreducible closed subset, of codimension* 1.

PROOF. Let $\mathcal{Y}$ be the subset of $\mathbf{P}^{N_d}$ consisting of pairs $\big(p,[F]\big)$ such that

$$F(p)=0 \quad\text{and}\quad \frac{\partial F}{\partial x_i}(p)=0 \quad\text{for}\quad 0\le i\le n. \tag{7.4.1}$$

It is straightforward to see that $\mathcal{Y}$ is a closed subset of $\mathbf{P}^n\times\mathbf{P}^{N_d}$; in particular, it is a projective variety. Let $\alpha\colon\mathcal{Y}\to\mathbf{P}^n$ and $\beta\colon\mathcal{Y}\to\mathbf{P}^{N_d}$ be the maps induced by the two projections.

We claim that for every $p\in\mathbf{P}^n$, the fiber $\alpha^{-1}(p)\hookrightarrow\mathbf{P}^{N_d}$ is a linear subspace, of codimension $n+1$. Indeed, we may choose coordinates on $\mathbf{P}^n$ such that $p=[1,0,\ldots,0]$. In this case, the conditions in (7.4.1) are equivalent with the fact that the coefficients of $x_0^d, x_0^{d-1}x_1,\ldots,x_0^{d-1}x_n$ are equal to 0, which gives our claim.

In particular, all fibers of α are irreducible, of the same dimension. Since α is proper, we deduce using Proposition 5.5.1 that $\mathcal{Y}$ is irreducible, and Proposition 3.4.2 gives

$$\dim(\mathcal{Y})=N_d-1.$$

Since β is a closed map, it follows that its image is a closed, irreducible subset of $\mathbf{P}^{N_d}$. In order to conclude the proof of the proposition, it is enough to find a singular hypersurface, with only finitely many singular points. Indeed, this implies via Theorem 3.4.1 that $\dim\big(\beta(\mathcal{Y})\big)=\dim(\mathcal{Y})=N_d-1$. Since

$$\mathcal{S}ing_d=\beta(\mathcal{Y})\cap\mathcal{H}_d,$$

it follows that $\mathcal{S}ing_d$ is closed in $\mathcal{H}_d$, and being a non-empty open subset of $\beta(\mathcal{Y})$, it is irreducible, of dimension N_d-1.

In order to construct a hypersurface that satisfies the required condition, it is enough to consider $g\in k[x_0,\ldots,x_{n-1}]$ homogeneous, of degree d, defining a smooth hypersurface in $\mathbf{P}^{n-1}$. Such g exists by Remark 7.4.2. For an explicit example, when $\mathrm{char}(k)\not|\,d$, one can take

$$g=\sum_{i=0}^{n-1}x_i^d.$$

For any such example, if we consider g as a polynomial in $k[x_0,\ldots,x_n]$, it defines a hypersurface in $\mathbf{P}^n$ that has precisely one singular point, namely $[0,\ldots,0,1]$. This completes the proof of the proposition. $\square$

EXAMPLE 7.4.4. Let us describe the hypersurfaces of degree 2 (the *quadrics*) in $\mathbf{P}^n$. For simplicity, let us assume that $\mathrm{char}(k)\neq 2$. Any non-zero homogeneous polynomial $F\in k[x_0,\ldots,x_n]$ of degree 2 can be written as

$$F=\sum_{i,j}a_{i,j}x_ix_j, \quad\text{with}\quad a_{i,j}=a_{j,i}\quad\text{for all}\quad i,j.$$

The *rank* of F is the rank of the symmetric matrix $(a_{i,j})$ (note that if we do a linear change of variables, this rank does not change).

Since k is algebraically closed, it follows that after a suitable linear change of variables, we can write

$$F=\sum_{i=0}^{r}x_i^2, \tag{7.4.2}$$

in which case $\mathrm{rank}(F) = r+1 \geq 1$. This can be deduced from the structure theorem for symmetric bilinear forms over a field, but one can also give a direct argument: we leave this as an exercise for the reader.

Given the expression in (7.4.2), note that (F) is radical if and only if $r \geq 1$ and (F) is prime if and only if $r \geq 2$. It follows from the above description that a quadric is either smooth (precisely when $r = n$) or the projective cone over a quadric of lower dimension.

For example, a quadric in $\mathbf{P}^3$ is either a smooth quadric, or a cone over a smooth conic (quadric in $\mathbf{P}^2$) or a union of 2 planes. After a suitable change of variables, a smooth quadric in $\mathbf{P}^3$ has equation $x_0x_3 + x_1x_2 = 0$. This is the image of the Segre embedding

$$\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3, \quad \big([u_0, u_1], [v_0, v_1]\big) \to [u_0u_1, u_0v_1, u_1v_0, u_1v_1].$$

We next construct the *universal hypersurface* over $\mathcal{H}_d$. In fact, for many purposes, it is more convenient to work with the whole space $\mathbf{P}^{N_d}$ instead of restricting to $\mathcal{H}_d$ (this is due to the fact that $\mathbf{P}^{N_d}$ is complete, while $\mathcal{H}_d$ is not). Define

$$\mathcal{Z}_d := \left\{\big(p, [F]\big) \in \mathbf{P}^n \times \mathbf{P}^{N_d} \mid F(p) = 0\right\}.$$

It is easy to see that via the composition of closed embeddings

$$\mathbf{P}^n \times \mathbf{P}^{N_d} \overset{\nu_d \times 1}{\hookrightarrow} \mathbf{P}^{N_d} \times \mathbf{P}^{N_d} \overset{\beta}{\hookrightarrow} \mathbf{P}^M,$$

where ν_d is the d^{th} Veronese embedding and β is the Segre embedding, $\mathcal{Z}_d$ is the inverse image of a hyperplane, hence it is a closed subset of $\mathbf{P}^n \times \mathbf{P}^{N_d}$.

Note that the projections onto the two components induce two morphisms

$$\phi\colon \mathcal{Z}_d \to \mathbf{P}^n \quad \text{and} \quad \psi\colon \mathcal{Z}_d \to \mathbf{P}^{N_d}.$$

Since $\mathbf{P}^n$ and $\mathbf{P}^{N_d}$ are projective varieties, we deduce that both ϕ and ψ are proper morphisms. It follows from definition that for every $[H] \in \mathcal{H}_d$, we have $\psi^{-1}\big([H]\big) = H$.

On the other hand, for every $p \in \mathbf{P}^n$, the fiber $\phi^{-1}(p)$ consists of the classes of those $F \in S_d$ such that $F(p) = 0$. This is a hyperplane in $\mathbf{P}^{N_d}$. We deduce from Proposition 5.5.1 that $\mathcal{Z}_d$ is irreducible, of dimension $N_d + n - 1$.

We now turn to linear subspaces on projective hypersurfaces. Given $r < n$, let $G = G(r+1, n+1)$ be the Grassmann variety parametrizing the r-dimensional linear subspaces in $\mathbf{P}^n$. Consider the incidence correspondence $I \subseteq \mathbf{P}^{N_d} \times G$ consisting of pairs $\big([F], [\Lambda]\big)$ such that F vanishes on Λ.

We first show that I is closed in $\mathbf{P}^{N_d} \times G$. Suppose that we are over the open subset $V = V_{\{1,\ldots,r\}} \simeq \mathbf{A}^{(r+1)(n-r)}$ of G, where a subspace Λ is described by the linear span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{0,r+1} & \ldots & a_{0,n} \\ 0 & 1 & \ldots & 0 & a_{1,r+1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a_{r,r+1} & \ldots & a_{r,n} \end{pmatrix}.$$

The hypersurface corresponding to $c = (c_\alpha)$, which is defined by $f_c = \sum_\alpha c_\alpha x^\alpha$ contains the subspace corresponding to the above matrix if and only if

$$f_c\left(x_0, \ldots, x_r, \sum_{0 \leq i \leq r} a_{i,r+1}x_i, \ldots, \sum_{0 \leq i \leq r} a_{i,n}x_i\right) = 0 \quad \text{in} \quad k[x_0, \ldots, x_r].$$

We can write

$$f_c\left(x_0, \ldots, x_r, \sum_{0 \leq i \leq r} a_{i,r+1}x_i, \ldots, \sum_{0 \leq i \leq r} a_{i,n}x_i\right) = \sum_\beta F_\beta(a, c)x^\beta, \tag{7.4.3}$$

where the sum is running over those $\beta = (\beta_0, \ldots, \beta_r)$ with $\sum_i \beta_i = d$. Note that each F_β is a polynomial in the $a_{i,j}$ and c_α variables, homogeneous of degree 1 in the c_α's. With this notation, $I \cap (\mathbf{P}^{N_d} \times V)$ is the zero-locus in $\mathbf{P}^{N_d} \times V$ of the ideal generated by all F_β; in particular, it is a closed subset. The equations over the other charts in G are similar.

In particular, we see that I is a projective variety. Let $\pi_1 \colon I \to \mathbf{P}^{N_d}$ and $\pi_2 \colon I \to G$ be the morphisms induced by the projections onto the two factors.

Definition 7.4.5. For every hypersurface H of degree d in $\mathbf{P}^n$, the *Fano variety* of r-planes in H, denoted $F_r(H)$, is the fiber $\pi_1^{-1}\big([H]\big)$ of π_1, parametrizing the r-dimensional linear subspaces contained in H. .

Proposition 7.4.6. *The projective variety I is irreducible, of dimension*

$$(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1.$$

Proof. Consider the morphism $\pi_2 \colon I \to G$. By Proposition 5.5.1, it is enough to show that every fiber $\pi^{-1}\big([\Lambda]\big)$ is isomorphic to a linear subspace of $\mathbf{P}^{N_d}$, of codimension $\binom{r+d}{d}$. In order to see this, we may assume that Λ is defined by $x_{r+1} = \ldots = x_n = 0$. It is clear that a polynomial f vanishes on Λ if and only if all coefficients of the monomials in $x_0, \ldots, x_r$ in f vanish; this gives a linear subspace of codimension $\binom{r+d}{d}$. $\square$

Exercise 7.4.7. Given a smooth quadric X in $\mathbf{P}^3$, we have 2 families of lines on X: choose coordinates such that X is given by $x_0x_3 - x_1x_2 = 0$, hence X is the image of the Segre embedding $\iota \colon \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. One family of lines is given by $\big(\iota(\mathbf{P}^1 \times \{q\})\big)_{q \in \mathbf{P}^1}$ and the other one is given by $\big(\iota(\{p\} \times \mathbf{P}^1)\big)_{p \in \mathbf{P}^1}$. Show that these are all the lines on X; deduce that the Fano variety of lines on X has two connected components, each of them isomorphic to $\mathbf{P}^1$.

Example 7.4.8. Consider lines on cubic surfaces: that is, we specialize to the case when $n = 3 = d$ and $r = 1$. Note that in this case I is an irreducible variety of dimension 19, the same as the dimension of the projective space parametrizing homogeneous polynomials of degree 3 in $S = k[x_0, x_1, x_2, x_3]$. We claim that the morphism $\pi_1 \colon I \to \mathbf{P}^{19}$ is surjective; in other words, every hypersurface in $\mathbf{P}^3$ which is the zero-locus of a degree 3 homogeneous polynomial contains at least one line. In order to see this, it is enough to exhibit such a hypersurface that only contains finitely many lines (this follows from Theorem 3.4.1). At least for $\mathrm{char}(k) \neq 3$, such an example is given by the Fermat cubic surface below.

EXAMPLE 7.4.9. Suppose that $\mathrm{char}(k) \neq 3$ and let X be the Fermat surface in $\mathbf{P}^3$ defined by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

Of course, if $\mathrm{char}(k) = 3$, then the zero locus of this polynomial is the hyperplane $x_0 + x_1 + x_2 + x_3 = 0$, which contains infinitely many lines.

Up to reordering the variables, every line $L \subseteq X$ can be given by equations of the form

$$x_0 = \alpha x_2 + \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2 + \delta x_3,$$

for some $\alpha, \beta, \gamma, \delta \in k$. This line lies on X if and only if

$$(\alpha x_2 + \beta x_3)^3 + (\gamma x_2 + \delta x_3)^3 + x_2^3 + x_3^3 = 0 \quad \text{in} \quad k[x_2, x_3].$$

This is equivalent to the following system of equations:

$$\alpha^3 + \gamma^3 = -1, \alpha^2\beta + \gamma^2\delta = 0, \alpha\beta^2 + \gamma\delta^2 = 0, \text{ and } \beta^3 + \delta^3 = -1.$$

If α, β, γ, δ are all nonzero, then it follows from the third equation that

$$\gamma = -\alpha\beta^2\delta^{-2},$$

and plugging in the second equation, we get

$$\alpha^2\beta + \alpha^2\beta^4\delta^{-4} = 0,$$

which implies $\beta^3 = -\delta^3$, contradicting the fourth equation.

Suppose now, for example, that $\alpha = 0$. We deduce from the second equation that $\gamma\delta = 0$. Moreover, $\gamma^3 = -1$ by the first equation, hence $\delta = 0$ and $\beta^3 = -1$ by the fourth equation. We thus get in this way the 9 lines with the equations

$$x_0 = \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2,$$

where $\beta, \gamma \in k$ are such that $\beta^3 = -1 = \gamma^3$. After permuting the variables, we obtain 2 more sets of lines on X, hence in total we have 27 lines.

We next discuss hypersurfaces that contain linear spaces of small codimension.

PROPOSITION 7.4.10. *We consider hypersurfaces in $\mathbf{P}^n$ of degree $d \geq 2$.*

i) *If X is a smooth such hypersurface containing a linear subspace $\Lambda \subseteq \mathbf{P}^n$ of dimension r, then $r \leq \frac{n-1}{2}$.*
ii) *If $\Lambda \subseteq \mathbf{P}^n$ is a linear subspace of dimension $r \leq \frac{n-1}{2}$, then a general hypersurface containing Λ is smooth.*

PROOF. After a suitable choice of coordinates on $\mathbf{P}^n$, we may assume that Λ is the linear subspace defined by

$$x_{r+1} = \ldots = x_n = 0.$$

Suppose that X is the hypersurface defined by a homogeneous polynomial F, of degree d. If X contains Λ, then we can write

$$F = \sum_{i=1}^{n-r} x_{r+i} f_i, \tag{7.4.4}$$

for some $f_i \in k[x_0, \ldots, x_n]$, homogeneous of degree $d-1$. For every i, with $1 \leq i \leq n-r$, consider the homogeneous polynomials of degree $d-1$

$$g_i(x_0, \ldots, x_r) = f_i(x_0, \ldots, x_r, 0, \ldots, 0).$$

If $n-r\leq r$, then a repeated application of Corollary 4.2.12 implies that there is a point $[u_0,\ldots,u_r]\in\mathbf{P}^r$ such that

$$g_i(u_0,\ldots,u_r)=0 \quad \text{for} \quad 1\leq i\leq n-r.$$

In other words, there is a point $p\in\Lambda$ such that $f_i(p)=0$ for all $1\leq i\leq n-r$. In this case, it follows from (7.4.4) that $F(p)=0$ and $\frac{\partial F}{\partial x_j}(p)=0$ for $0\leq j\leq n$, hence p is a singular point of X. We thus deduce that if X is smooth, then $n-r\geq r+1$, giving i).

Suppose now that $r\leq\frac{n-1}{2}$ and consider the subset W of $\mathbf{P}^{N_d}$ consisting of those $[F]$ such that Λ is contained in the zero-locus $(F=0)$. This consists of those $[F]$ such that $F\in(x_{r+1},\ldots,x_n)$, which is a linear subspace in $\mathbf{P}^{N_d}$, of codimension $\binom{r+d}{d}$. Let U be the subset of W consisting of those $[F]$ such that there is no $p\in\mathbf{P}^n$, with

$$F(p)=0=\frac{\partial F}{\partial x_i}(p) \quad \text{for} \quad 0\leq i\leq n. \tag{7.4.5}$$

Note that such F generates a radical ideal (see Remark 6.3.12) and the corresponding degree d hypersurface contains Λ and is smooth. We need to show that U is open and non-empty.

As in Proposition 7.4.3, we consider the set $\mathcal{Y}_W$ of pairs $(p,[F])\in\mathbf{P}^n\times W$ such that (7.4.5) holds. This is a closed subset of $\mathbf{P}^n\times W$, hence it is a projective variety. Let $\alpha\colon\mathcal{Y}_W\to\mathbf{P}^n$ and $\beta\colon\mathcal{Y}_W\to W$ be the morphisms induced by the two projections. Since $U=W\smallsetminus\beta(\mathcal{Y}_W)$, it follows that U is open in W, and it is enough to show that $\beta(\mathcal{Y}_W)\neq W$.

We now describe the fiber $\alpha^{-1}(p)$ for $p\in\mathbf{P}^n$. Suppose first that $p\in\Lambda$. We may choose coordinates such that $p=[1,0,\ldots,0]$. The conditions in (7.4.5) are equivalent with the fact that the coefficients of $x_0^d,x_0^{d-1}x_1,\ldots,x_0^{d-1}x_n$ in F are 0. Since $F\in(x_{r+1},\ldots,x_n)$, we see that $\alpha^{-1}(p)\hookrightarrow W$ is a linear subspace of codimension $n-r$. Suppose now that $p\not\in\Lambda$, in which case we may choose coordinates such that $p=[0,\ldots,0,1]$, in which case the conditions in (7.4.5) are equivalent with the fact that the coefficients of $x_n^d,x_n^{d-1}x_{n-1},\ldots,x_n^{d-1}x_0$ are 0. We thus see that in this case $\alpha^{-1}(p)\hookrightarrow W$ is a linear subspace of codimension $n+1$. We deduce from Corollary 3.4.3 that

$$\dim\big(\alpha^{-1}(\Lambda)\big)=\dim(\Lambda)+\dim(W)-(n-r)=\dim(W)+(2r-n)$$

and

$$\dim\big(\alpha^{-1}(\mathbf{P}^n\smallsetminus\Lambda)\big)=\dim(\mathbf{P}^n\smallsetminus\Lambda)+\dim(W)-(n+1)=\dim(W)-1.$$

Since by assumption we have $2r-n\leq -1$, we deduce that $\dim(\mathcal{Y}_W)=\dim(W)-1$, hence $\dim\big(\beta(\mathcal{Y}_W)\big)\leq\dim(\mathcal{Y}_W)<\dim(W)$. This completes the proof of the proposition. $\square$

7.5. The variety of nilpotent matrices

Fix a positive integer n and let

$$\mathcal{N}_n=\{A\in M_n(k)\mid A\,\text{is nilpotent}\}.$$

The case $n=1$ is trivial ($\mathcal{N}_1$ consists of one point), hence from now on we will assume that $n\geq 2$.

Recall that a matrix $A\in M_n(k)$ is nilpotent if and only if $A^n=0$. Since the entries of A^n are homogeneous polynomials of degree n in the entries of A, it follows that $\mathcal{N}_n$ is a closed subset of $M_n(k)$, preserved by the standard k^*-action on $M_n(k)$. Note that there are nonzero nilpotent matrices (we use here the fact that $n \geq 2$). It follows that we have a non-empty projective variety $\mathcal{N}_n^{\rm proj}$ in the projective space $\mathbf{P} \simeq \mathbf{P}^{n^2-1}$ of lines in $M_n(k)$, such that $\mathcal{N}_n$ is the affine cone over $\mathcal{N}_n^{\rm proj}$.

In fact, we can define $\mathcal{N}_n$ by only n equations. Indeed, a matrix A is nilpotent if and only if its characteristic polynomial $\det(A-\lambda I)$ is equal to $(-\lambda)^n$. If we write

$$\det(A-\lambda I) = \sum_{i=0}^{n}(-1)^i p_i(A)\lambda^i,$$

then $p_n(A) = 1$ and for each i, with $0 \leq i \leq n-1$, $p_i(A)$ is a homogeneous polynomial of degree $n-i$ in the entries of A. We thus see that $\mathcal{N}_n$ is the zero-locus of the ideal $(p_0, \ldots, p_{n-1})$.

Our next goal is to show that $\mathcal{N}_n$ is irreducible and compute its dimension. For this, it is a bit more convenient to work with the corresponding projective variety $\mathcal{N}_n^{\rm proj}$.

The key observation is the following: a matrix $A \in M_n(k)$ is nilpotent if and only if there is a complete flag of subspaces

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n = V,$$

with $\dim_k(V_i) = i$ and $A(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq n$ (where we put $V_0 = 0$). Indeed, it is clear that if we have such a flag, then $A^n = 0$. Conversely, if $A^n = 0$, let $W_i = A^{n-i}(k^n)$. It follows from definition that

$$W_0 = 0 \subseteq W_1 \subseteq \ldots \subseteq W_n = k^n$$

and $A(W_i) \subseteq W_{i-1}$ for $1 \leq i \leq n$. If we refine this sequence of subspaces to a complete flag, this flag will satisfy the required conditions.

Motivated by this, we define

$$Z = \big\{\big([A], (V_1, \ldots, V_n)\big) \in \mathbf{P} \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1}\,\text{for}\,1 \leq i \leq n\big\}$$

(where in the above formula we make the convention that $V_0 = \{0\}$). We leave it as an exercise for the reader to check that Z is a closed subset of $\mathbf{P} \times \mathrm{Fl}(k^n)$. In particular, we see that Z is a projective variety. The projections of $\mathbf{P} \times \mathrm{Fl}(k^n)$ onto the two components induce proper morphisms

$$\pi_1 \colon Z \to \mathbf{P} \quad \text{and} \quad \pi_2 \colon Z \to \mathrm{Fl}(k^n).$$

Let us consider the fiber of π_2 over a flag $V_\bullet = (V_1, \ldots, V_n)$. If we choose a basis $e_1, \ldots, e_n$ such that each V_i is generated by $e_1, \ldots, e_i$, it follows that $\pi_2^{-1}(V_\bullet)$ is isomorphic to the the subvariety of $\mathbf{P}$ consisting of classes of nonzero strictly upper-triangular matrices, hence it is isomorphic to $\mathbf{P}^{\frac{n(n-1)}{2}-1}$. Since $\mathrm{Fl}(k^n)$ is irreducible, of dimension $\frac{n(n-1)}{2}$, it follows from Proposition 5.5.1 that Z is an irreducible variety, of dimension $n^2 - n - 1$.

Consider now the morphism $\pi_1 \colon Z \to \mathbf{P}$, whose image is $\mathcal{N}_n^{\rm proj}$. This implies that $\mathcal{N}_n^{\rm proj}$ is irreducible. We next show that over a non-empty open subset of $\mathcal{N}_n^{\rm proj}$, each fiber of π_1 consists of just one point. Note that if $A \in M_n(k)$ is a nilpotent matrix, then its rank is $\leq n-1$. Let $\mathcal{U}_n^{\rm proj}$ be the open subset of $\mathcal{N}_n^{\rm proj}$ consisting of matrices of rank $n-1$. Note that this is a non-empty subset: for example, the nilpotent matrix $(a_{i,j})$ with $a_{\ell,\ell-1} = 1$ for $2 \leq \ell \leq n$ and all other $a_{i,j}$ equal to 0 has rank $n-1$. We note that if $[A] \in \mathcal{U}_n^{\rm proj}$, then $\pi^{-1}\big([A]\big)$ has only one element: if $(V_1, \ldots, V_n)$ is a flag in k^n such that $A(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq n$, then $V_i = A^{n-i}(V)$ for all i. Indeed, the condition on the flag implies that $A^{n-i}(k^n) \subseteq V_i$ and the condition on the rank of A implies easily, by descending induction on i, that $\dim_k A^{n-i}(k^n) = i$. Therefore $A^{n-i}(k^n) = V_i$ for $1 \leq i \leq n$.

Since π_1 has finite fibers over $\mathcal{U}_n$, we deduce from Theorem 3.4.2 that

$$\dim(\mathcal{N}_n^{\mathrm{proj}}) = \dim(Z) = n^2 - n - 1.$$

We thus conclude that $\mathcal{N}_n$ is an irreducible variety of dimension $n^2 - n$.

Remark 7.5.1. In fact, the above construction, but done for the affine cone $\mathcal{N}_n$, gives a resolution of singularities of $\mathcal{N}_n$. Indeed, let

$$W = \left\{ \big(A, (V_1, \ldots, V_{n-1})\big) \in M_n(k) \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1} \,\mathrm{for}\, 1 \leq i \leq n \right\}.$$

One can check that the projection onto the second component induces a morphism $\pi_2 \colon W \to \mathrm{Fl}(k^n)$ that is locally trivial, with fiber $\mathbf{A}^{\frac{n(n-1)}{2}}$. In particular, it follows that W is smooth, irreducible, of dimension $n^2 - n$. The projection onto the first component induces a proper, surjective morphism $\pi_1 \colon W \to \mathcal{N}_n$. In order to see that this is birational, note that if

$$\mathcal{U}_n = \{A \in \mathcal{N}_n \mid \mathrm{rk}(A) = n - 1\},$$

then the induced morphism $\pi_1^{-1}(\mathcal{U}_n) \to \mathcal{U}_n$ is an isomorphism, whose inverse maps A to $\big(A, (A^{n-1}(k^n), \ldots, A(k^n), k^n)\big)$.

Remark 7.5.2. One can see that the ideal $(p_0, \ldots, p_{n-1}) \subseteq \mathcal{O}\big(M_n(k)\big)$ is a radical ideal, but we do not pursue this here, since the argument involves some deeper facts of commutative algebra than we have used so far.

CHAPTER 8

Coherent sheaves on algebraic varieties

In algebra, when one is interested in the study of rings, modules naturally appear: for example, as ideals and quotient rings. Because of this, it is more natural to study the whole category of modules over the given ring. This method becomes even more powerful with the introduction of cohomological techniques, since by working in the category of modules over a given ring, we can construct derived functors of familiar functors like Hom and the tensor product. Our goal in this chapter is to introduce objects that in the context of arbitrary varieties extend what (finitely generated) modules over a ring are in the case of an affine variety: these are the quasi-coherent (respectively, the coherent) sheaves. This will provide us with the language to treat in later chapters global objects, such as divisors, vector bundles, and projective morphisms. We begin with some general constructions for sheaves of R-modules, then discuss sheaves of $\mathcal{O}_X$-modules, and then introduce quasi-coherent and coherent sheaves. In particular, we use these to globalize the MaxSpec and MaxProj constructions. In the last section of this chapter we describe coherent sheaves on varieties of the form $\mathrm{MaxProj}(S)$.

8.1. General constructions with sheaves

In this section we discuss several general constructions involving sheaves. We fix a commutative ring R and consider presheaves and sheaves of R-modules. Important examples are the cases when $R = \mathbf{Z}$ or R is a field. Given a topological space X, we denote by $\mathcal{P}sh_X^R$ and $\mathcal{S}h_X^R$ the categories of presheaves, respectively sheaves, of R-modules on X. However, when R is understood, we simply write $\mathcal{P}sh_X$ and $\mathcal{S}h_X$.

8.1.1. The sheaf associated to a presheaf. Let R be a fixed commutative ring and consider a topological space X. We show that the inclusion functor $\mathcal{P}sh_X \hookrightarrow \mathcal{S}h_X$ has a left adjoint. Explicitly, this means that for every presheaf $\mathcal{F}$ on X, we have a sheaf $\mathcal{F}^+$, together with a morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{F}^+$ that satisfies the following universal property: given any morphism of presheaves $\psi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism of sheaves $\alpha\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\alpha \circ \phi = \psi$. In other words, ϕ induces a bijection

$$\mathrm{Hom}_{\mathcal{S}h_X}(\mathcal{F}^+, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{P}sh_X}(\mathcal{F}, \mathcal{G}).$$

Note that the universal property implies that given any morphism of presheaves $u\colon \mathcal{F} \to \mathcal{G}$, we obtain a unique morphism of sheaves $u^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "u"'] & \mathcal{F}^+ \arrow[d, "u^+"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd}$$

is commutative.

Given a presheaf $\mathcal{F}$, for every open subset $U \subseteq X$ we define $\mathcal{F}^+(U)$ to consist of all maps $t\colon U \to \bigsqcup_{x\in U} \mathcal{F}_x$ that satisfy the following properties:

i) We have $t(x) \in \mathcal{F}_x$ for all $x \in U$.
ii) For every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and $s \in \mathcal{F}(U_x)$, such that $t(y) = s_y$ for all $y \in U_x$.

Note that since each $\mathcal{F}_x$ is an R-module, addition and scalar multiplication of functions makes each $\mathcal{F}^+(U)$ an R-module. We also see that restriction of functions induces for every open subsets $U \subseteq V$ a map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$ that make $\mathcal{F}^+$ a presheaf of R-modules. In fact, it is straightforward to check that $\mathcal{F}^+$ is a sheaf: this is a consequence of the local characterization of the sections of $\mathcal{F}^+$. We have a morphism of presheaves of R-modules $\phi\colon \mathcal{F} \to \mathcal{F}^+$ that maps $s \in \mathcal{F}(U)$ to the map $U \to \bigsqcup_{x\in U} \mathcal{F}_x$ that takes x to s_x.

Let's check the universal property: consider a morphism of presheaves $\psi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf. Given $t \in \mathcal{F}^+(U)$, it follows from definition that we can cover U by open subsets U_i and we have $s_i \in \mathcal{F}(U_i)$ such that for every i and every $y \in U_i$, we have $t(y) = (s_i)_y \in \mathcal{F}_y$. This implies that the sections $t'_i := \psi(s_i) \in \mathcal{G}(U_i)$ have the property that $(t'_i)_y = (t'_j)_y$ for all $y \in U_i \cap U_j$. Using the fact that $\mathcal{G}$ is a sheaf, we first see that $t'_i|_{U_i\cap U_j} = t'_j|_{U_i\cap U_j}$ for all i and j, and then that there is a unique $t' \in \mathcal{G}(U)$ such that $t'|_{U_i} = t'_i$ for all i. We then define $\alpha(t) = t'$. It is straightforward to see that this gives a morphism of sheaves $\alpha\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\alpha \circ \phi = \psi$ and that in fact α is the unique morphism of sheaves with this property.

Remark 8.1.1. It is straightforward to check, using the definition, that if $\mathcal{F}$ is a sheaf, then the canonical morphism $\phi\colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

Remark 8.1.2. For every presheaf $\mathcal{F}$ and every $x \in X$, the morphism $\phi\colon \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism $\phi_x\colon \mathcal{F}_x \to \mathcal{F}^+_x$. The inverse map is defined as follows. Given an element $u \in \mathcal{F}^+_x$ represented by $\big(U, t \in \mathcal{F}^+(U)\big)$, by hypothesis we have an open neighborhood U_x of x and $s \in \mathcal{F}(U_x)$ such that $t(y) = s_y$ for all $y \in U_x$. We define $\tau(u) = s_x \in \mathcal{F}_x$ and leave it as an exercise for the reader to check that this is well-defined and that τ gives an inverse of ϕ_x.

Remark 8.1.3. Wherever we mention stalks in this section, the same results hold, with analogous proofs, for the stalks at irreducible closed subsets of the given topological space. For simplicity, we only give the statements at points of X, since this is sufficient for the study of sheaves on topological spaces; however, in the setting of algebraic varieties it is sometimes convenient to also consider more general stalks (corresponding to localizing a ring to a possibly non-maximal prime ideal).

Remark 8.1.4. It is clear from definition that if U is an open subset of X, then we have a canonical isomorphism

$$(\mathcal{F}|_U)^+ \simeq \mathcal{F}^+|_U.$$

EXAMPLE 8.1.5. If $\mathcal{F}$ is a sheaf and $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, then the inclusion morphism $\imath\colon \mathcal{G} \hookrightarrow \mathcal{F}$ induces a morphism of sheaves $j\colon \mathcal{G}^+ \to \mathcal{F}$. This gives an isomorphism of $\mathcal{G}^+$ with the subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that for an open subset U of X, $\mathcal{F}'(U)$ consists of those $s \in \mathcal{F}(U)$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x such that $s|_{U_x}$ lies in $\mathcal{G}(U_x)$. Indeed, it is easy to see that $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ and j induces a morphism of sheaves $\alpha\colon \mathcal{G}^+ \to \mathcal{F}'$ such that for all $x \in X$, the induced morphism $\mathcal{G}^+_x \to \mathcal{F}'_x$ is an isomorphism; therefore α is an isomorphism (see Exercise 2.1.20).

EXAMPLE 8.1.6. If M is any R-module, then we have the *constant presheaf* on X that associates M to every open subset of X, the restriction maps being the identity maps. The associated sheaf is the *constant sheaf* $\underline{M}$ associated to M. If X has the property that every open subset is a union of open connected subsets (for example, this is the case for an algebraic variety), then $\Gamma(U, \underline{M})$ can be identified with the set of maps $U \to M$ that are constant on every connected open subset of U.

8.1.2. Kernels and cokernels. Let R be a fixed commutative ring and X a fixed topological space. We first note that for every two sheaves $\mathcal{F}$ and $\mathcal{G}$, the set of morphisms $\mathrm{Hom}_{\mathcal{S}h_X}(\mathcal{F}, \mathcal{G})$ is an R-module. In particular, we have a zero morphism. We also note that composition of morphisms of sheaves is bilinear.

Given finitely many sheaves $\mathcal{F}_1, \ldots \mathcal{F}_n$ on X, we define $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ by

$$(\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n)(U) := \mathcal{F}_1(U) \oplus \ldots \oplus \mathcal{F}_n(U),$$

with the restriction maps being induced by those for each $\mathcal{F}_i$. It is straightforward to see that this is a sheaf. We have canonical sheaf morphisms $\mathcal{F}_i \to \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ that make $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ the coproduct of $\mathcal{F}_1, \ldots \mathcal{F}_n$ and we have sheaf morphisms $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n \to \mathcal{F}_i$ that make $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ the product of $\mathcal{F}_1, \ldots, \mathcal{F}_n$. Note that for every $x \in X$ we have a canonical isomorphism

$$(\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n)_x \simeq (\mathcal{F}_1)_x \oplus \ldots \oplus (\mathcal{F}_n)_x,$$

due to the fact that filtered direct limits commute with finite direct sums.

We now show that the category $\mathcal{S}h_X$ has kernels. Given a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, define for an open subset U of X

$$\ker(\phi)(U) := \ker\big(\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big).$$

The restriction maps of $\mathcal{F}$ induce restriction maps for $\ker(\phi)$ that make $\ker(\phi)$ a presheaf and it is straightforward to see that it is a sheaf (in fact, a subsheaf of $\mathcal{F}$). It is an easy exercise to see that the inclusion morphism $i\colon \ker(\phi) \hookrightarrow \mathcal{F}$ is a kernel of ϕ: this means that $\phi \circ i = 0$ and for every morphism of sheaves $u\colon \mathcal{F}' \to \mathcal{F}$ such that $\phi \circ u = 0$, there is a unique morphism of sheaves $v\colon \mathcal{F}' \to \ker(\phi)$ such that $u = i \circ v$. Note that since filtered inductive limits are exact functors, it follows that for every $x \in X$, we have

$$\ker(\phi)_x \simeq \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

We now define the cokernel of a morphism of sheaves of R-modules $\phi\colon \mathcal{F} \to \mathcal{G}$. For every open subset U of X, define

$$\widetilde{\mathrm{coker}}(\phi)(U) := \mathrm{coker}\big(\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big).$$

It is straightforward to see that the restriction maps of $\mathcal{G}$ induce restriction maps for $\widetilde{\operatorname{coker}}(\phi)$ which make it a presheaf. We define

$$\operatorname{coker}(\phi) := \widetilde{\operatorname{coker}}(\phi)^{+}.$$

Note that the composition map p

$$\mathcal{G} \to \widetilde{\operatorname{coker}}(\phi) \to \operatorname{coker}(\phi)$$

is a cokernel of ϕ; this means that $p \circ \phi = 0$ and for every morphism of sheaves $u \colon \mathcal{G} \to \mathcal{G}'$ such that $u \circ \phi = 0$, there is a unique morphism of sheaves $v \colon \operatorname{coker}(\phi) \to \mathcal{G}'$ such that $v \circ p = u$ (this follows using the corresponding property of cokernels of morphsms of R-modules and the universal property of the sheaf associated to a presheaf). Finally, we note that since filtering direct limits are exact and since passing to the associated sheaf preserves the stalks, for every $x \in X$ we have a canonical isomorphism

$$\operatorname{coker}(\phi)_x \simeq \operatorname{coker}(\mathcal{F}_x \to \mathcal{G}_x).$$

If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, we define $\mathcal{F}/\mathcal{F}'$ as the cokernel of the inclusion morphism $\mathcal{F}' \hookrightarrow \mathcal{F}$. It follows that for every $x \in X$, we have a short exact sequence

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to (\mathcal{F}/\mathcal{F}')_x \to 0.$$

The image $\operatorname{Im}(\phi)$ of a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ is defined as the kernel of

$$\mathcal{G} \to \operatorname{coker}(\phi).$$

Using the universal property of the kernel and of the cokernel, we obtain a canonical morphism

$$\mathcal{F}/\ker(\phi) \to \operatorname{Im}(\phi). \tag{8.1.1}$$

This is an isomorphism: this follows by considering the induced morphisms at the levels of stalks, using the fact that a morphism of sheaves $\alpha \colon \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\alpha_x \colon \mathcal{A}_x \to \mathcal{B}_x$ is an isomorphism for every $x \in X$ (see Exercise 2.1.20). The existence of kernels and cokernels, together with the fact that the canonical morphism (8.1.1) is an isomorphism mean that Sh_X^R is an *Abelian category*.

EXAMPLE 8.1.7. Given a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$, the image $\operatorname{Im}(\phi)$ is the subsheaf of $\mathcal{G}$ described as follows: for every open subset $U \subseteq X$, the subset $\operatorname{Im}(\phi)(U) \subseteq \mathcal{G}(U)$ consists of those $s \in \mathcal{G}(U)$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x, such that $s|_{U_x}$ lies in the image of $\mathcal{F}(U_x) \to \mathcal{G}(U_x)$. This follows from Example 8.1.5.

A morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ is *injective* if $\ker(\phi) = 0$. Equivalently, for every open subset U of X, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective; moreover, this holds if and only if $\phi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for every $x \in X$. In this case, ϕ gives an isomorphism of $\mathcal{F}$ with a subsheaf of $\mathcal{G}$.

The morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ is *surjective* if $\operatorname{coker}(\phi) = 0$, or equivalently, $\operatorname{Im}(\phi) = \mathcal{G}$ (in this case we say that $\mathcal{G}$ is a *quotient* of $\mathcal{F}$). Equivalently, for every $x \in X$, the morphism $\mathcal{F}_x \to \mathcal{G}_x$ is surjective. However, this does *not* imply that for an open subset U of X, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. What we can say in this case is that for every $s \in \mathcal{G}(U)$ and every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x such that $s|_{U_x}$ lies in the image of $\mathcal{F}(U_x) \to \mathcal{G}(U_x)$.

As in any Abelian category, we can consider exact sequences: given morphisms

$$\mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'',$$

this is exact if $\mathrm{Im}(u) = \ker(v)$; equivalently, for every $x \in X$, the sequence of R-modules

$$\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$$

is exact.

In particular, the sequence

$$0 \longrightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \longrightarrow 0$$

is exact if v is surjective and u gives an isomorphism $\mathcal{F}' \simeq \ker(v)$; equivalently, u is injective and v induces an isomorphism $\mathrm{coker}(u) \simeq \mathcal{F}''$. Moreover, this is equivalent with the fact that for every $x \in X$, the sequence of R-modules

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

is exact. Note that in this case, for every open subset U of X, the induced sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

is exact. In other words, the functor $\Gamma(U, -)$ is left exact. However, in general this is not an exact functor.

8.1.3. The sheaf $\mathcal{H}om$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of R-modules on X, then for every open subset U of X, we may consider the R-module $\mathrm{Hom}_{\mathcal{S}h_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. If $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a morphism of sheaves and $V \subseteq U$ is an open subset, then we clearly get an induced morphism $\phi|_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$. We thus get a presheaf of R-modules denoted $\mathcal{H}om_R(\mathcal{F}, \mathcal{G})$. In fact, this is a sheaf: this follows from the fact that morphisms of sheaves can be uniquely patched together (see Exercise 2.1.22).

8.1.4. The functor f^{-1}. Recall that if $f\colon X \to Y$ is a continuous map, then we have the functor $f_*\colon \mathcal{S}h_X^R \to \mathcal{S}h_Y^R$ such that

$$\Gamma(V, f_*\mathcal{F}) = \Gamma\big(f^{-1}(V), \mathcal{F}\big) \quad \text{for every open subset} \quad V \subseteq Y.$$

A special case is that when Y is a point, in which case this functor gets identified with $\Gamma(X, -)$.

Like the special case of the functor $\Gamma(X, -)$, the functor f_* is left-exact. Indeed, given an exact sequence of sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and an open subset V in Y, the corresponding sequence

$$0 \to \mathcal{F}'\big(f^{-1}(V)\big) \to \mathcal{F}\big(f^{-1}(V)\big) \to \mathcal{F}''\big(f^{-1}(V)\big)$$

is exact.

We now construct a left adjoint of this functor. Given a sheaf of R-modules $\mathcal{G}$ on Y, consider the presheaf $\widetilde{G}$ on X given by

$$\widetilde{G}(U) := \varinjlim_{f(U) \subseteq V} \mathcal{G}(V),$$

where the direct limit is over the open subsets V of Y containing $f(U)$, ordered by reverse inclusion. Note that if $U_1 \subseteq U_2$, then for every open subset V in Y such that $f(U_2) \subseteq V$, we also have $f(U_1) \subseteq V$, which induces a restriction map

$$\widetilde{G}(U_2) \to \widetilde{G}(U_1)$$

and it is easy to see that these maps make $\widetilde{G}$ a presheaf. We define $f^{-1}(\mathcal{G}) := \widetilde{G}^{+}$.

If $\phi\colon \mathcal{G} \to \mathcal{G}'$ is a morphism of sheaves on Y, then for every open subset U of X, we have a morphism of R-modules

$$\varinjlim_{f(U)\subseteq V} \phi_V\colon \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) \to \varinjlim_{f(U)\subseteq V} \mathcal{G}'(V)$$

and these give a morphism of presheaves $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$. This in turn induces a morphism of sheaves $f^{-1}(\mathcal{G}) \to f^{-1}(\mathcal{G}')$. This is compatible with composition of morphisms, hence we get a functor

$$f^{-1}\colon \mathcal{S}h_Y^R \to \mathcal{S}h_X^R.$$

Note that for every sheaf $\mathcal{G}$ on Y and every $x \in X$, we have canonical isomorphisms

$$f^{-1}(\mathcal{G})_x \simeq \widetilde{G}_x \simeq \varinjlim_{x\in U} \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) \simeq \varinjlim_{f(x)\in V} \mathcal{G}(V) \simeq \mathcal{G}_{f(x)}.$$

This immediately implies that f^{-1} is an exact functor.

EXAMPLE 8.1.8. Note that if U is an open subset of X and $i\colon U \hookrightarrow X$ is the inclusion, then we have a canonical isomorphism $i^{-1}(\mathcal{F}) \simeq \mathcal{F}|_U$.

An important property is that the pair (f^{-1}, f_*) is an adjoint pair of functors. This means that for every sheaves of R-modules $\mathcal{F}$ on X and $\mathcal{G}$ on Y, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{S}h_X}\big(f^{-1}(\mathcal{G}), \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{S}h_Y}\big(\mathcal{G}, f_*(\mathcal{F})\big).$$

Indeed, giving a morphism of sheaves $f^{-1}(\mathcal{G}) \to \mathcal{F}$ is equivalent to giving a morphism of presheaves $\widetilde{\mathcal{G}} \to \mathcal{F}$, which is equivalent to giving for every open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$ morphisms of R-modules

$$\mathcal{G}(V) \to \mathcal{F}(U)$$

that are compatible with the maps induced by restriction. Because of this compatibility, it is enough to give such maps when $U = f^{-1}(V)$, and such a family of maps compatible with the restriction maps is precisely a morphism of sheaves $\mathcal{G} \to f_*(\mathcal{F})$.

8.2. Sheaves of $\mathcal{O}_X$-modules

Suppose now that $(X, \mathcal{O}_X)$ is a *ringed space*, that is, X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. Our main example will be that when X is an algebraic variety and $\mathcal{O}_X$ is the sheaf of regular functions on X, but it is more natural to develop the notions that we need here in the general framework.

DEFINITION 8.2.1. A *sheaf of $\mathcal{O}_X$-modules* (or, simply, *$\mathcal{O}_X$-module*) is a sheaf of Abelian groups $\mathcal{F}$ such that for every open subset U of X we have an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$, and these structures are compatible with restriction maps, in the sense that for every open sets $V \subseteq U$, we have

$$(a \cdot s)|_V = a|_V \cdot s|_V \quad \text{for all} \quad a \in \mathcal{O}_X(U) \quad \text{and} \quad s \in \mathcal{F}(U).$$

If $\mathcal{F}$ is a presheaf, instead of a sheaf, we call it a *presheaf of $\mathcal{O}_X$-modules*.

A morphism of sheaves (or presheaves) of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (respectively, presheaves) of Abelian groups such that for every open subset U of X, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. We write

$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ for the set of such morphisms. It is clear that the $\mathcal{O}_X$-modules form a category that we will denote $\mathcal{O}_X$-mod.

EXAMPLE 8.2.2. The sheaf $\mathcal{O}_X$ has an obvious structure of $\mathcal{O}_X$-module.

EXAMPLE 8.2.3. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and U is an open subset of X, with $\mathcal{O}_U = \mathcal{O}_X|_U$, then $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module.

REMARK 8.2.4. It is easy to see that a sheaf (presheaf) of $\mathcal{O}_X$-modules is the same as a sheaf (respectively, presheaf) of Abelian groups $\mathcal{F}$, together with a morphism of sheaves (respectively, presheaves)

$$\mathcal{O}_X \to \mathcal{H}om_{\mathbf{Z}}(\mathcal{F},\mathcal{F}).$$

This easily implies that if $\mathcal{O}_X = \underline{R}$, for a ring R, then giving a sheaf of $\mathcal{O}_X$-modules is equivalent to giving a sheaf of R-modules.

REMARK 8.2.5. Note that every $\mathcal{O}_X$-module $\mathcal{F}$ is in particular an $\mathcal{O}_X(X)$-module. Indeed, for every open subset U of X, the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ induces an $\mathcal{O}_X(X)$-module structure on $\mathcal{F}(U)$. We get in this way a functor from $\mathcal{O}_X$-mod to $\mathcal{S}h_X^{\mathcal{O}_X(X)}$.

REMARK 8.2.6. It follows easily from definition that if $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then for every $x \in X$, the stalk $\mathcal{F}_x$ has a canonical structure of $\mathcal{O}_{X,x}$-module. More generally, if V is an irreducible, closed subset of X, then $\mathcal{F}_V$ has a canonical structure of $\mathcal{O}_{X,V}$-module.

REMARK 8.2.7. Note that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \subseteq \mathrm{Hom}_{\mathbf{Z}}(\mathcal{F},\mathcal{G})$$

is a subgroup. In fact, it follows from Remark 8.2.5 that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ has a natural $\mathcal{O}_X(X)$-module structure.

Moreover, we have a subsheaf

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \subseteq \mathcal{H}om_{\mathbf{Z}}(\mathcal{F},\mathcal{G}),$$

whose sections over an open subset $U \subseteq X$ consist of the morphisms of $\mathcal{O}_U$-modules $\mathcal{F}|_U \to \mathcal{G}|_U$. Since each $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U)$ is an $\mathcal{O}_X(U)$-module, we see that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ becomes naturally an $\mathcal{O}_X$-module.

Note that for every $\mathcal{O}_X$-module $\mathcal{G}$, we have a canonical isomorphism of $\mathcal{O}_X(X)$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{F}) \simeq \mathcal{F}(X), \quad \phi \to \phi_X(1)$$

and therefore an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{F}) \simeq \mathcal{F}.$$

REMARK 8.2.8. It is clear that if $\mathcal{F}_1,\ldots,\mathcal{F}_n$ are sheaves of $\mathcal{O}_X$-modules, then $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n$ has a natural structure of $\mathcal{O}_X$-module such that with respect to the obvious maps, it is both the coproduct and the product of the $\mathcal{F}_i$.

REMARK 8.2.9. It follows immediately from Remark 8.2.6 that if $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}^+$ has an induced structure of sheaf of $\mathcal{O}_X$-modules such that the canonical map $\mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves of $\mathcal{O}_X$-modules. Moreover, this satisfies an obvious universal property with respect to morphisms to sheaves of $\mathcal{O}_X$-modules.

REMARK 8.2.10. It follows from definitions and the previous remark that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then $\ker(\phi)$, $\operatorname{coker}(\phi)$, and $\operatorname{Im}(\phi)$ carry natural $\mathcal{O}_X$-module structures. In particular, $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are the kernel, respectively the cokernel, of ϕ in the category of $\mathcal{O}_X$-modules. Moreover, the isomorphism of sheaf of Abelian groups

$$\mathcal{F}/\ker(\phi) \to \operatorname{Im}(\phi)$$

is now an isomorphism in the category of $\mathcal{O}_X$-modules. Therefore $\mathcal{O}_X$-mod is an Abelian category.

The notions of *injective* and *surjective* morphisms of $\mathcal{O}_X$-modules are defined as in the case of sheaves of R-modules. We also have a notion of *$\mathcal{O}_X$-submodule*, which is an $\mathcal{O}_X$-module that is also a subsheaf. In particular, a *sheaf of ideals* is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

EXAMPLE 8.2.11. The following notion will play an important role later: an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free (of finite rank)* if for every $x \in X$, there is an open neighborhood U of x such that we have an isomorphism

$$\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus n}.$$

If the integer n does not depend on x, then we say that $\mathcal{F}$ *has rank* n.

EXERCISE 8.2.12. Show that if $(\mathcal{M}_i)_{i\in I}$ in an inverse system of $\mathcal{O}_X$-modules, then the inverse limit $\varprojlim_{i\in I} \mathcal{M}_i$ can be constructed as follows. For every open subset U of X, consider the $\mathcal{O}_X(U)$-module

$$\mathcal{M}(U) := \varprojlim_{i\in I} \mathcal{M}_i(U).$$

If $V \subseteq U$, then the inverse limit of the restriction maps induce a restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ and these maps make $\mathcal{M}$ an $\mathcal{O}_X$-module. Moreover, for every $j \in I$, the projection

$$\varprojlim_{i\in I} \mathcal{M}_i(U) \to \mathcal{M}_j(U)$$

defines a morphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{M}_j$ and $\mathcal{M}$, together with these morphisms, is the inverse limit of $(\mathcal{M}_i)_{i\in I}$.

EXERCISE 8.2.13. Show that if $(\mathcal{M}_i)_{i\in I}$ is a direct system of $\mathcal{O}_X$-modules, then the direct limit $\varinjlim_{i\in I} \mathcal{M}_i$ can be constructed as follows. For every open subset $U \subseteq X$, consider the $\mathcal{O}_X(U)$-module

$$\mathcal{M}(U) := \varinjlim_{i\in I} \mathcal{M}_i(U).$$

If V is an open subset of U, then the direct limit of the restriction maps induces a restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ and these maps make $\mathcal{M}$ a presheaf of $\mathcal{O}_X$-modules. Moreover, for every $j \in I$, the canonical morphisms $\mathcal{M}_j(U) \to \varinjlim_{i\in I} \mathcal{M}_i(U)$ give a morphism of presheaves $\mathcal{M}_j \to \mathcal{M}$.

i) Show that the compositions $\mathcal{M}_j \to \mathcal{M} \to \mathcal{M}^+$ make $\mathcal{M}^+$ the direct limit of the direct system $(\mathcal{M}_i)_{i\in I}$.

ii) Deduce that for every $x \in X$, we have a canonical isomorphism

$$(\varinjlim_{i\in I} \mathcal{M}_i)_x \simeq \varinjlim_{i\in I} \mathcal{M}_{i,x}.$$

8.2.1. Multilinear algebra for $\mathcal{O}_X$-modules. Operations like tensor product, exterior, and symmetric products have analogues for $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, the we can consider the presheaf that associates to an open subset U of X, the $\mathcal{O}_X(U)$-module

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

If V is an open subset of U, the restriction map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V)$$

is the tensor product of the restriction maps of $\mathcal{F}$ and $\mathcal{G}$. The associated sheaf is the *tensor product* of $\mathcal{F}$ and $\mathcal{G}$, and it is denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. It is easy to see that we have a bilinear map of sheaves

$$\mathcal{F} \oplus \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$$

that satisfies the same universal property in $\mathcal{O}_X$-mod as the usual tensor product in the category of R-modules.

While the sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ over some U are not vert explicit, the stalks of this sheaf are easier to understand. In fact, using the fact that a presheaf and its associated sheaf have the same stalks, and the fact that tensor product commutes with direct limits, we obtain for every $x \in X$ a canonical isomorphsim

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \varinjlim_{U\ni x} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x. \tag{8.2.1}$$

Similarly, given an $\mathcal{O}_X$-module $\mathcal{F}$ and a non-negative integer m, we define $\mathcal{O}_X$-modules $\wedge^m\mathcal{F}$ and $\mathrm{Sym}^m(\mathcal{F})$ by taking the sheaf associated to the presheaf that maps an open subset U to $\wedge^m_{\mathcal{O}_X(U)}\mathcal{F}(U)$, respectively to $\mathrm{Sym}^m_{\mathcal{O}_X(U)}\mathcal{F}(U)$. Again, for every $x \in X$, we have canonical isomorphisms

$$(\wedge^m\mathcal{F})_x \simeq \wedge^m_{\mathcal{O}_{X,x}}\mathcal{F}_x \quad \text{and} \quad \big(\mathrm{Sym}^m(\mathcal{F})\big)_x \simeq \mathrm{Sym}^m_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$

Similar isomorphisms hold for the stalks at irreducible closed subsets of X.

8.2.2. Push-forward and pull-back for $\mathcal{O}_X$-modules. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a pair $(f, f^\#)$, where $f\colon X \to Y$ is a continuous map and $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. By a slight abuse, $f^\#$ is sometimes dropped from the notation and the morphism is simply denoted by f. The main example for us is that given by a morphism of algebraic varieties. A special feature in this case is that $f^\#$ is determined by the continuous map f.

Note that morphisms of ringed spaces can be composed: if $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces, with associated morphisms of sheaves of rings

$$f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X \quad \text{and} \quad g^\#\colon \mathcal{O}_Z \to g_*\mathcal{O}_Y,$$

then the composition $(X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is given by the continuous map $g \circ f$ and the morphism of sheaves of rings

$$\mathcal{O}_Z \xrightarrow{g^\#} g_*\mathcal{O}_Y \xrightarrow{g_*(f^\#)} g_*(f_*\mathcal{O}_X).$$

It is easy to see that in this way the ringed spaces form a category.

Let $f\colon X \to Y$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we see that for every open subset V of Y, the Abelian group

$$\Gamma\big(V, f_*(\mathcal{F})\big) = \Gamma\big(f^{-1}(V), \mathcal{F}\big)$$

is a module over $\Gamma\big(f^{-1}(V), \mathcal{O}_X\big)$, hence via the given homomorphism $\Gamma(V, \mathcal{O}_Y) \to \Gamma\big(f^{-1}(V), \mathcal{O}_X\big)$ it becomes a module over $\Gamma(V, \mathcal{O}_V)$. This makes $f_*(\mathcal{F})$ an $\mathcal{O}_Y$-module. We thus obtain a left exact functor, the *push-forward* functor

$$f_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}.$$

We now construct a left adjoint of this functor, the *pull-back*. Recall that we have a left adjoint f^{-1} for the corresponding functor between the categories of sheaves of Abelian groups. Note also that by the adjointness of (f^{-1}, f_*) the structure morphism $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponds to a morphism of sheaves of rings $\psi\colon f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. It is straightforward to see that if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then $f^{-1}(\mathcal{G})$ has a natural structure of $f^{-1}(\mathcal{O}_Y)$-module. We put

$$f^*(\mathcal{G}) := f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

and this has a natural structure of $\mathcal{O}$-module. Again, it is not easy to describe the sections of $f^*(\mathcal{G})$ over an open subset of X, but for every $x \in X$, we have a homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induced by $f^{\#}$ and a canonical isomorphism

$$f^*(\mathcal{G})_x \simeq \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}. \tag{8.2.2}$$

Since the functor $-\otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ is right-exact, it follows that the functor f^* is right exact. More generally, if V is an irreducible, closed subset of X and $W = \overline{f(V)}$, then for every $\mathcal{O}_Y$-module $\mathcal{G}$, we have a canonical isomorphism

$$f^*(\mathcal{G})_V \simeq \mathcal{G}_W \otimes_{\mathcal{O}_{Y,W}} \mathcal{O}_{X,V}.$$

EXAMPLE 8.2.14. It follows from definition that, with the above notation, we have $f^*(\mathcal{O}_Y) = \mathcal{O}_X$.

EXAMPLE 8.2.15. If U is an open subset of X and $\mathcal{O}_U = \mathcal{O}_X|_U$, then we have a morphism of ringed spaces $i\colon (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$, where $i\colon U \to X$ is the inclusion and the morphism of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_U$ maps $\phi \in \mathcal{O}_X(V)$ to $\phi|_{U\cap V}$. The corresponding morphism $i^{-1}\mathcal{O}_X = \mathcal{O}_U \to \mathcal{O}_U$ is the identity, so that we have a canonical isomorphism $i^*(\mathcal{F}) \simeq \mathcal{F}|_U$ for every $\mathcal{O}_X$-module $\mathcal{F}$. In particular, in this case the functor i^* is exact.

EXAMPLE 8.2.16. If $f\colon X \to Y$ is a flat morphism of algebraic varieties, then the functor f^* is exact. This follows from the fact that for every $\mathcal{O}_Y$-module $\mathcal{G}$ and every $x \in X$ we have the isomorphism (8.2.2) and $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

PROPOSITION 8.2.17. *The pair of functors (f^*, f_*) is an adjoint pair, that is, for every $\mathcal{O}_X$-module $\mathcal{F}$ and every $\mathcal{O}_Y$-module $\mathcal{G}$, we have a natural isomorphism of Abelian groups*

$$\mathrm{Hom}_{\mathcal{O}_X}\big(f^*(\mathcal{G}), \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{G}, f_*(\mathcal{F})\big).$$

PROOF. The assertion follows easily from the fact that (f^{-1}, f_*) is an adjoint pair of functors between the corresponding categories of sheaves of Abelian groups, together with the universal property of the tensor product. $\square$

REMARK 8.2.18. The push-forward and pull-back functors are compatible with compositions of morphisms of ringed spaces: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of ringed spaces, then for every $\mathcal{O}_X$-modules $\mathcal{F}$ and every $\mathcal{O}_Z$-module $\mathcal{G}$, we have

$$(g \circ f)_*(\mathcal{F}) = g_*\big(f_*(\mathcal{F})\big)$$

and a natural isomorphism

$$(g \circ f)^*(\mathcal{G}) \simeq f^*\big(g^*(\mathcal{G})\big).$$

Indeed, the first assertion follows directly from definition, and the second one follows from the fact that both functors $(g \circ f)^*$ and $f^* \circ g^*$ are left adjoints of $(g \circ f)_*$.

We end this section by showing that the pull-back is compatible with multilinear operations. For example, we have the following:

PROPOSITION 8.2.19. *If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_Y$-modules, then we have a natural isomorphism*

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \simeq f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} f^*(\mathcal{G}).$$

PROOF. Note first that if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_X$-modules, we have a canonical morphism of $\mathcal{O}_Y$-modules

$$f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{N}) \to f_*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \tag{8.2.3}$$

defined as follows. Let $\mathcal{S}$ be the presheaf of $\mathcal{O}_Y$-modules such that for an open subset V of Y, we have

$$\mathcal{S}(V) = f_*(\mathcal{M})(V) \otimes_{\mathcal{O}_Y(V)} f_*(\mathcal{N})(V) = \mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}\big(f^{-1}(V)\big)$$

and $\mathcal{T}$ the presheaf of $\mathcal{O}_X$-modules such that for an open subset U of X, we have

$$\mathcal{T}(U) = \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U).$$

It thus follows from definition that

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \mathcal{T}^+ \quad \text{and} \quad f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{N}) = \mathcal{S}^+.$$

It is clear that we have a morphism of $\mathcal{O}_Y$-modules

$$\mathcal{S} \to f_*(\mathcal{T})$$

which for an open subset $V \subseteq Y$ is given by the canonical morphism

$$\mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}\big(f^{-1}(V)\big) \to \mathcal{M}\big(f^{-1}(V)\big) \otimes_{\mathcal{O}_X(f^{-1}(V))} \mathcal{N}\big(f^{-1}(V)\big).$$

mapping $u \otimes_{\mathcal{O}_Y(V)} v \to u \otimes_{\mathcal{O}_X(f^{-1}(V))} v$. By composing this with the morphism $f_*(\mathcal{T}) \to f_*(\mathcal{T}^+)$, we obtain a morphism $\mathcal{S} \to f_*(\mathcal{T}^+)$ and since the target is a sheaf, this corresponds to a unique morphism of $\mathcal{O}_Y$-modules

$$\mathcal{S}^+ \to f_*(\mathcal{T}^+)$$

which is the morphism in (8.2.3).

Note now that the adjoint property of (f^*, f_*) gives canonical morphisms $\alpha\colon \mathcal{F} \to f_*\big(f^*(\mathcal{F})\big)$ and $\beta\colon \mathcal{G} \to f_*\big(f^*(\mathcal{G})\big)$. We thus obtain the following composition

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to f_*\big(f^*(\mathcal{F})\big) \otimes_{\mathcal{O}_X} f_*\big(f^*(\mathcal{G})\big) \to f_*\big(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{G})\big),$$

where the first morphism is $\alpha \otimes \beta$ and the second morphism is given by (8.2.3). Using the fact that (f^*, f_*) is an adjoint pair, this corresponds to a morphism of $\mathcal{O}_X$-modules

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \longrightarrow f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} f^*(\mathcal{G}). \tag{8.2.4}$$

In order to complete the proof, it is enough to show that this is an isomorphism and this follows if we show that it induces an isomorphism at the level of stalks (see Exercise 2.1.20). This is a consequence of the formulas in (8.2.1) and (8.2.2). $\square$

Remark 8.2.20. A similar argument shows that if $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then for every non-negative integer m, we have canonical isomorphisms

$$f^*\big(\mathrm{Sym}^m(\mathcal{F})\big) \simeq \mathrm{Sym}^m\big(f^*(\mathcal{F})\big) \quad \text{and} \quad f^*(\wedge^m \mathcal{F}) \simeq \wedge^m f^*(\mathcal{F}).$$

8.3. Quasi-coherent sheaves on affine varieties

We now introduce quasi-coherent sheaves in the setting of affine varieties. We will see that these correspond to modules over the coordinate ring of the affine variety.

We begin with a general proposition about constructing sheaves in the presence of a suitable basis of open subsets. We will use it for the principal affine open subsets of an affine variety and later, for the principal affine open subsets of varieties of the form $\mathrm{MaxProj}(S)$. We state it for $\mathcal{O}_X$-modules, but the reader will see that a similar statement holds in other settings (for example, for sheaves of R-algebras).

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{P}$ a family of open subsets of X that satisfies the following two properties:

i) Every open subset of X is a union of subsets in $\mathcal{P}$ (that is, $\mathcal{P}$ gives a basis of open subsets), and
ii) For every $U, V \in \mathcal{P}$, we have $U \cap V \in \mathcal{P}$.

We define a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules on X to be a map α that associates to every $U \in \mathcal{P}$ an $\mathcal{O}_X(U)$-module $\alpha(U)$ and to every inclusion $U \subseteq V$ a map $\alpha(V) \to \alpha(U)$, $s \to s|_U$, such that

$$(a \cdot s)|_U = a|_U \cdot s|_U \quad \text{for every} \quad a \in \mathcal{O}_X(V), s \in \alpha(V).$$

These restriction maps are supposed to satisfy the usual compatibility conditions. Furthermore, the map α should satisfy the following gluing condition: for every cover $U = \bigcup_{i \in I} U_i$, with U and U_i in $\mathcal{P}$, and for every family $(s_i)_{i \in I}$, with $s_i \in \alpha(U_i)$ for all i, such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, there is a unique $s \in \alpha(U)$ such that $s|_{U_i} = s_i$ for all i. If α and β are $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules, a *morphism* $g \colon \alpha \to \beta$ associates to every $U \in \mathcal{P}$ a morphism of $\mathcal{O}_X(U)$-modules $g_U \colon \alpha(U) \to \beta(U)$ and these are compatible with the restriction maps in the obvious sense. It is clear that $\mathcal{P}$-sheaves form a category.

Proposition 8.3.1. *The functor from the category of sheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules, given by only recording the information for the open subsets in $\mathcal{P}$, is an equivalence of categories.*

Proof. Given a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules α, we define a corresponding sheaf $\mathcal{F}_\alpha$, such that for an open subset $W \subseteq X$, we let $\mathcal{F}_\alpha(W)$ be the kernel of the map

$$\prod_{U \in \mathcal{P}; U \subseteq W} \alpha(U) \to \prod_{U,V \in \mathcal{P}; U,V \subseteq W} \alpha(U \cap V), \quad (s_U)_U \to (s_U|_{U \cap V} - s_V|_{U \cap V})_{U,V}.$$

Component-wise addition makes this an Abelian group and we get a structure of $\mathcal{O}_X(U)$-module by putting

$$a \cdot (s_U)_U = (a|_U \cdot s_U)_U \quad \text{for all} \quad a \in \mathcal{O}_X(W), (s_U)_U \in \mathcal{F}_\alpha(U).$$

Note that if $W' \subseteq W$, then we have a restriction map given by

$$(s_U)_U \to (s_V)_V,$$

where the second tuple is indexed by those V that lie inside W'. It is clear that this is compatible with scalar multiplication and makes $\mathcal{F}_\alpha$ a presheaf of $\mathcal{O}_X$-modules. Moreover, it is a straightforward (though somewhat tedious) to check that the glueing condition on α implies that $\mathcal{F}_\alpha$ is a sheaf.

Suppose now that $g\colon \alpha \to \beta$ is a morphism of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules. Given any open subset W of X, we have a commutative diagram

$$\begin{tikzcd}
\prod_{U\in\mathcal{P};U\subseteq W} \alpha(U) \arrow[r] \arrow[d, "\prod_U g_U"'] & \prod_{U,V\in\mathcal{P};U,V\subseteq W} \alpha(U\cap V) \arrow[d, "\prod_{U,V} g_{U\cap V}"] \\
\prod_{U\in\mathcal{P};U\subseteq W} \beta(U) \arrow[r] & \prod_{U,V\in\mathcal{P};U,V\subseteq W} \beta(U\cap V),
\end{tikzcd}$$

which induces a morphism of $\mathcal{O}_X(U)$-modules $\mathcal{F}_\alpha(U) \to \mathcal{F}_\beta(U)$. It is straightforward to check that these maps are compatible with the restriction maps and that in this way we get a functor from the category of $\mathcal{P}$-sheaves of $\mathcal{O}_X$-modules to the category of sheaves of $\mathcal{O}_X$-modules. Checking that this is an inverse of the functor in the statement of the proposition is an easy exercise that we leave for the enthusiastic reader. $\square$

Suppose now that $(X, \mathcal{O}_X)$ is an affine variety and $A = \mathcal{O}_X(X)$. We consider the set $\mathcal{P}$ consisting of the principal affine open subsets of X. Recall that $D_X(f) \cap D_X(g) = D_X(fg)$. Let M be an A-module. Given any $U \in \mathcal{P}$, say $U = D_X(f)$, we put

$$\alpha(U) := M_f.$$

Note that if $D_X(f) \supseteq D_X(g)$, then $V(f) \subseteq V(g)$, hence $\sqrt{(f)} \supseteq \sqrt{(g)}$. We thus have a localization morphism $A_f \to A_g$ and a corresponding canonical morphism of A_f-modules $M_f \to M_g$. In particular, we see that $\alpha(U)$ only depends on U (up to a canonical isomorphism) and that we have restriction maps that satisfy the usual compatibility relations. The next lemma allows us to apply Proposition 8.3.1 to conclude that we have a sheaf of $\mathcal{O}_X$-modules on X, that we denote $\widetilde{M}$, such that for every $f \in A$, we have a canonical isomorphism

$$\Gamma\big(D_X(f), \widetilde{M}\big) \simeq M_f.$$

LEMMA 8.3.2. *If X is an affine variety, with $A = \mathcal{O}(X)$, and M is an A-module, then for every cover*

$$D_X(f) = \bigcup_{i\in I} D_X(g_i),$$

the sequence

$$0 \longrightarrow M_f \longrightarrow \prod_{i\in I} M_{g_i} \longrightarrow M_{g_i g_j}$$

is exact.

PROOF. The proof is similar to the proof of Proposition 1.4.7. After replacing X by $D_X(f)$ and M by M_f, we may assume that $f = 1$. The condition $X = \bigcup_{i\in I} D_X(g_i)$ is equivalent to the fact that the ideal $(g_i \mid i \in I)$ is the unit ideal. The injectivity of the map $M \to \prod_{i\in I} M_{g_i}$ is clear: if $u \in M$ is such that $\frac{u}{1} = 0$ in M_{g_i} for all i, then there is m_i such that $g_i^{m_i} \in \mathrm{Ann}_A(u)$. Since the elements $g_i^{m_i}$ generate the unit ideal, it follows that $\mathrm{Ann}_A(u) = A$, hence $u = 0$.

Suppose now that we have $u_i \in M_{g_i}$ for all $i \in I$, such that for all $i, j \in I$, the images of u_i and u_j in $M_{g_ig_j}$ coincide. Note first that we may assume that I is finite. Indeed, we may choose a finite subset $J \subseteq I$ such that $(g_i \mid i \in J) = A$. If we can find $u \in M$ such that $\frac{u}{1} = u_i \in M_{g_i}$ for all $i \in J$, then it follows that $\frac{u}{1} = u_i \in M_{g_i}$ also for all $i \in I$. Indeed, $D_X(g_i) = \bigcup_{j\in J} D_X(g_ig_j)$, and we deduce using the first part of the proof that it is enough to show that $\frac{u}{1}$ and u_i have the same image in $M_{g_ig_j}$ for all $j \in J$. This is a consequence of the fact that $\frac{u}{1} = u_j \in M_{g_j}$ and the fact that by hypothesis, u_i and u_j have the same image in $M_{g_ig_j}$.

Suppose now that I is finite and let us write

$$u_i = \frac{v_i}{g_i^{n_i}} \quad \text{for all} \quad i \in I.$$

After replacing each g_i by a suitable power, we may assume that $n_i = 1$ for all i. The condition that u_i and u_j have the same image in $M_{g_ig_j}$ implies that

$$(g_ig_j)^{q_{i,j}}(g_jv_i - g_iv_j) = 0 \quad \text{for some} \quad q_{i,j}.$$

After replacing one more time each g_i by a suitable power, we may assume that $g_iv_j = g_jv_i$ for all i and j. In this case, if we write $1 = \sum_{i\in I} a_ig_i$ and take $u = \sum_{i\in I} a_iv_i \in M$, we have $\frac{u}{1} = u_i \in M_{g_i}$ for all i. Indeed, we have

$$g_iu = \sum_{j\in I} a_jg_iu_j = \sum_{j\in I} a_jg_ju_i = u_i.$$

This completes the proof of the lemma. □

EXAMPLE 8.3.3. With the above notation, the sheaf $\widetilde{A}$ is the structure sheaf $\mathcal{O}_X$. This follows from the fact that for every $f \in A$, the canonical morphism $\mathcal{O}_X(X)_f \to \mathcal{O}_X\big(D_X(f)\big)$ is an isomorphism.

REMARK 8.3.4. If $\mathcal{F} = \widetilde{M}$, then for every irreducible, closed subset $V \subseteq X$, we have a canonical isomorphism

$$\mathcal{F}_V \simeq M_{\mathfrak{p}},$$

where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to V. Indeed, it follows from definition that

$$\mathcal{F}_V = \varinjlim_{V\cap D_X(f)\neq\emptyset} \mathcal{F}\big(D_X(f)\big) = \varinjlim_{f\not\in\mathfrak{p}} M_f \simeq M_{\mathfrak{p}}.$$

Given a morphism of A-modules $\phi\colon M \to N$, for every $f \in A$, we have an induced morphism of A_f-modules $M_f \to N_f$ and these satisfy the obvious compatibility conditions with respect to inclusions of principal affine open subsets. By Proposition 8.3.1, we thus get a morphism of sheaves $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$ such that over every $D_X(f)$, this is given by $M_f \to N_f$. It is clear that in this way we get a functor from the category of A-modules to the category of $\mathcal{O}_X$-modules.

DEFINITION 8.3.5. Let X be an affine variety and $A = \mathcal{O}_X(X)$. A *quasi-coherent* sheaf on X is an $\mathcal{O}_X$-module isomorphic to $\widetilde{M}$, for some A-module M. The sheaf is *coherent* if, in addition, M is a finitely generated A-module. The category of quasi-coherent (or coherent) sheaves on X is a full subcategory of the category of $\mathcal{O}_X$-modules on X.

TO BE CONTINUED

APPENDIX A

Finite and integral homomorphisms

A running assumption for all the appendices is that all rings are commutative, unital (that is, they have multiplicative identity), and all homomorphisms are of unital rings (that is, they map the identity to the identity). In this appendix we discuss the definition and basic properties of integral and finite ring homomorphisms.

A.1. Definitions

Let $\varphi\colon R \to S$ be a ring homomorphism. One says that φ is of *finite type* if S becomes, via φ, a finitely generated R-algebra. One says that φ is *finite* if S becomes, via φ, a finitely generated R-module. One says that φ is *integral* if every element $y \in S$ is integral over R, that is, there is a positive integer n, and elements $a_1, \ldots, a_n \in R$, such that

$$y^n + a_1 y^{n-1} + \ldots + a_n = 0 \quad \text{in} \quad S.$$

Remark A.1.1. It is clear that if φ is finite, then it is of finite type: if $y_1, \ldots, y_m \in S$ generate S as an R-module, then they also generate it as an R-algebra. The converse is of course false: for example, the inclusion $R \hookrightarrow R[x]$ is finitely generated, but not finite (the R-submodule of $R[x]$ generated by finitely many polynomials consists of polynomials of bounded degree).

Remark A.1.2. If φ is of finite type and integral, then it is finite. Indeed, if $y_1, \ldots, y_r$ generate S as an R-algebra, and we can write

$$y_i^{d_i} + a_{i,1} y_i^{d_i - 1} + \ldots + a_{i,d_i} = 0$$

for some positive integers d_i and some $a_{i,j} \in R$, then it is easy to see that

$$\{y_1^{a_1} \cdots y_r^{a_r} \mid 0 \leq a_i \leq d_i - 1\}$$

generate S as an R-module.

Proposition A.1.3. *If φ is finite, then it is integral.*

Proof. The assertion follows from the *Determinant Trick*: suppose that $b_1, \ldots, b_n$ generate S as an R-module. For every $y \in S$, we can write for each $1 \leq i \leq n$:

$$y b_i = \sum_{j=1}^{n} a_{i,j} b_j \quad \text{for some} \quad a_{i,j} \in R.$$

If A is the matrix $(a_{i,j})_{1 \leq i,j \leq n}$ and I is the identity matrix, then we see that

$$(yI - A) \cdot \begin{pmatrix} b_1 \\ \ldots \\ b_n \end{pmatrix} = 0.$$

By multiplying with the classical adjoint of $yI - A$, we see that if $D = \det(yI - A)$, then $Db_i = 0$ for all i. This implies $D \cdot S = 0$, and in particular $D \cdot 1_S = 0$. However, it is clear that we can write

$$D = y^n + c_1 y^{n-1} + \ldots + c_n \quad \text{for some} \quad c_1, \ldots, c_n \in R.$$

We thus see that y is integral over R. □

REMARK A.1.4. We will almost always consider homomorphisms of finite type. For such a homomorphism φ, it follows from Remark A.1.2 and Proposition A.1.3 that φ is finite if and only if it is integral.

A.2. Easy properties

The following property of integral morphisms is very useful.

PROPOSITION A.2.1. *If $\varphi\colon R \hookrightarrow S$ is an integral injective homomorphism of integral domains, then R is a field if and only if S is a field.*

PROOF. Suppose first that R is a field, and let $u \in S \smallsetminus \{0\}$. Since u is integral over R, it follows that we can write

$$u^n + a_1 u^{n-1} + \ldots + a_n = 0$$

for some positive integer n, and some $a_1, \ldots, a_n \in R$. We may assume that n is chosen to be minimal; in this case, since $u \neq 0$, we have $a_n \neq 0$. We see that we have $uv = 1$, where

$$v = (-a_n)^{-1} \cdot (u^{n-1} + \ldots + a_{n-2}u + a_{n-1}),$$

hence u is invertible. Since this holds for every nonzero u, it follows that S is a field.

Conversely, suppose that S is a field and let $a \in R \smallsetminus \{0\}$. Let $b = \frac{1}{a} \in S$. Since b is integral over R, we can write

$$b^r + \alpha_1 b^{r-1} + \ldots + \alpha_r = 0$$

for some positive integer r and some $\alpha_1, \ldots, \alpha_r \in R$. Since

$$\frac{1}{a} = -\alpha_1 - \alpha_2 a - \ldots - \alpha_r a^{r-1} \in A,$$

we conclude that a in invertible in R. Since this holds for every nonzero a, it follows that R is a field. □

PROPOSITION A.2.2. *Given a ring homomorphism $\varphi\colon R \to S$, the subset*

$$S' := \{y \in S \mid y \textit{ integral over } R\}$$

is a subring of S. This is the integral closure of R in S.

PROOF. Since it is clear that $1_S \in S'$, we only need to check that for every $y_1, y_2 \in S'$, we have $y_1 - y_2, y_1 y_2 \in S'$. Since y_1 and y_2 are integral over R, the subring $R[y_1, y_2]$ of S is finite over R (the argument is the same as in Remark A.1.2). In particular, it is integral over R by Proposition A.1.3. This implies that $y_1 - y_2$ and $y_1 y_2$, which lie in $R[y_1, y_2]$, are integral over R. □

PROPOSITION A.2.3. *Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be two ring homomorphisms. If both φ and ψ are of finite type (respectively finite, integral), then $\psi \circ \varphi$ has the same property.*

PROOF. The assertion is straightforward for finite and finite type morphisms. Suppose now that φ and ψ are integral. Given $u \in T$, we can write

$$u^n + b_1 u^{n-1} + \ldots + b_n = 0$$

for some positive integer n and $b_1, \ldots, b_n \in S$. Since $b_1, \ldots, b_n$ are integral over R, it follows that $R' := R[b_1, \ldots, b_n]$ is finite over R (see Remark A.1.2). Since u is integral over R', it follows that $R'[u]$ is finite over R', and therefors it is finite over R. By Proposition A.1.3, we conclude that u is integral over R. $\square$

APPENDIX B

Noetherian rings and modules

In this appendix we discuss the definition and basic properties of Noetherian rings and modules. The main result is Hilbert's basis theorem.

B.1. Definitions

PROPOSITION B.1.1. *Given a ring R and an R-module M, the following are equivalent:*

i) *Every submodule N of M is finitely generated.*
ii) *There is no infinite strictly increasing chain of submodules of M:*

$$N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \ldots.$$

iii) *Every nonempty family of submodules of M contains a maximal element.*

An R-module M is *Noetherian* if it satisfies the equivalent conditions in the proposition. The ring R is Noetherian if it is Noetherian as an R-module.

PROOF OF PROPOSITION B.1.1. Suppose first that i) holds. If there is an infinite strictly increasing sequence of submodules of M as in ii), consider $N := \bigcup_{i\geq 1} N_i$. This is a submodule of M, hence it is finitely generated by i). If $u_1, \ldots, u_r$ generate N, then we can find m such that $u_i \in N_m$ for all m. In this case we have $N = N_m$, contradicting the fact that the sequence is strictly increasing.

The implication ii)$\Rightarrow$iii) is clear: if a nonempty family $\mathcal{F}$ has no maximal element, let us choose $N_1 \in \mathcal{F}$. Since this is not maximal, there is $N_2 \in \mathcal{F}$ such that $N_1 \subsetneq N_2$, and we continue in this way to construct an infinite strictly increasing sequence of submodules of M.

In order to prove the implication iii)$\Rightarrow$i), let N be a submodule of M and consider the family $\mathcal{F}$ of all finitely generated submodules of N. This is nonempty, since it contains the zero submodule. By iii), $\mathcal{F}$ has a maximal element N''. If $N'' \neq N$, then there is $u \in N \smallsetminus N''$ and the submodule $N'' + Ru$ is a finitely generated submodule of N strictly containing N'', a contradiction. Therefore $N'' = N$ and thus N is finitely generated. □

PROPOSITION B.1.2. *Given a short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of R-modules, M is Noetherian if and only if both M' and M'' are Noetherian.

PROOF. Suppose first that M is Noetherian. Since every submodule of M' is a submodule of M, hence finitely generated, it follows that M' is Noetherian. Since every submodule of $M'' \simeq M/M'$ is isomorphic toN/M', for a submodule N of M that contains M', and since N being finitely generated implies that N/M' is finitely generated, we conclude that M'' is Noetherian.

Conversely, suppose that both M' and $M'' \simeq M/M'$ are Noetherian, and let N be a submodule of M. Since $N \cap M'$ is a submodule of M', it is finitely generated, and since $N/(N \cap M')$ is isomorphic to a submodule of M/M', we have that $N/(N \cap M')$ is finitely generated. Finally, since both $N \cap M'$ and $N/(N \cap M')$ are finitely generated, it is clear that N is finitely generated. $\square$

COROLLARY B.1.3. *If R is a Noetherian ring, then an R-module M is Noetherian if and only if it is finitely generated.*

PROOF. We only need to show that if M is finitely generated, then it is Noetherian, since the converse follows from definition. Since M is finitely generated, we have a surjective morphism $R^{\oplus n} \to M$, and it follows from the proposition that it is enough to show that $R^{\oplus n}$ is Noetherian. This follows again from the proposition by induction on n. $\square$

REMARK B.1.4. If R is a Noetherian ring and I is an ideal in R, then R/I is a Noetherian ring. This is an immediate application of Corollary B.1.3.

REMARK B.1.5. If R is a Noetherian ring and $S \subseteq R$ is a multiplicative system, then the fraction ring $S^{-1}R$ is Noetherian. Indeed, every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R. If I is generated by $a_1, \ldots, a_r$, then $S^{-1}I$ is generated as an ideal of $S^{-1}R$ by $\frac{a_1}{1}, \ldots, \frac{a_r}{r}$.

B.2. Hilbert's basis theorem

The following theorem is one of the basic results in commutative algebra.

THEOREM B.2.1 (Hilbert). *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

PROOF. Let I be an ideal in $R[x]$. We consider the following recursive construction. If $I \neq 0$, let $f_1 \in I$ be a polynomial of minimal degree. If $I \neq (f_1)$, then let $f_2 \in I \smallsetminus (f_1)$ be a polynomial of minimal degree. Suppose now that $f_1, \ldots, f_n$ have been chosen. If $I \neq (f_1, \ldots, f_n)$, let $f_{n+1} \in I \smallsetminus (f_1, \ldots, f_n)$ be a polynomial of minimal degree.

If this process stops, then I is finitely generated. Let us assume that this is not the case, aiming for a contradiction. We write

$$f_i = a_i x^{d_i} + \text{lower degree terms}, \quad \text{with} \quad a_i \neq 0.$$

By our minimality assumption, we have

$$d_1 \leq d_2 \leq \ldots.$$

Let J be the ideal of R generated by the a_i, with $i \geq 1$. Since R is Noetherian, J is a finitely generated ideal, hence there is m such that J is generated by $a_1, \ldots, a_m$. In particular, we can find $u_1, \ldots, u_m \in R$ such that

$$a_{m+1} = \sum_{i=1}^{m} a_i u_i.$$

In this case, we have

$$h := f_{m+1} - \sum_{i=1}^{m} u_i x^{d_{m+1}-d_i} f_i \in I \smallsetminus (f_1, \ldots, f_m)$$

and $\deg(h) < d_{m+1}$, a contradiction. This completes the proof of the theorem. $\square$

By applying Theorem B.2.1 several times, we obtain

COROLLARY B.2.2. *If R is a Noetherian ring, then the polynomial ring $R[x_1, \ldots, x_n]$ is Noetherian for every positive integer n.*

In particular, since a field is clearly Noetherian, we obtain

COROLLARY B.2.3. *For every field k and every positive integer n, the polynomial ring $k[x_1, \ldots, x_n]$ is Noetherian.*

APPENDIX C

Nakayama's lemma and Krull's intersection theorem

In this appendix we collect a few basic results on local rings and localization. We begin with Nakayama's lemma and an application to finitely generated projective modules over local rings. We then overview some general results concerning the behavior of certain properties of modules under localization. We prove the Artin-Rees lemma and deduce Krull's Intersection theorem. In the last section we introduce discrete valuation rings (we will return to this topic in a later appendix).

C.1. Nakayama's lemma

The following is one of the most basic results in commutative algebra, known as Nakayama's lemma.

PROPOSITION C.1.1. *If $(A, \mathfrak{m})$ is a local ring and M is a finitely generated module over A such that $M = \mathfrak{m}M$, then $M = 0$.*

PROOF. The proof is another application of the determinant trick. Let $u_1, \ldots, u_n$ be generators of M over A. Since $M = \mathfrak{m}M$, for every i we can write

$$u_i = \sum_{j=1}^{n} a_{i,j} u_j \quad \text{for some} \quad a_{i,j} \in \mathfrak{m}.$$

If A is the matrix $(a_{i,j})_{1 \leq i,j \leq n}$ and I is the identity matrix, then we can rewrite the above relations as

$$(I - A) \cdot \begin{pmatrix} u_1 \\ \ldots \\ u_n \end{pmatrix} = 0.$$

By multiplying with the classical adjoint of $I - A$, we conclude that $\det(I - A) \cdot u_i = 0$ for all i. Since all entries of A lie in $\mathfrak{m}$, it is clear that

$$\det(I - A) \equiv 1 \,(\operatorname{mod} \mathfrak{m}).$$

Since A is local, it follows that $\det(I - A)$ is invertible, and therefore we conclude that $u_i = 0$ for all i, hence $M = 0$. □

This is sometimes applied in the following form.

COROLLARY C.1.2. *If $(A, \mathfrak{m})$ is a local ring, M is a finitely generated module over A, and N is a submodule of M such that $M = N + \mathfrak{m}M$, then $N = M$.*

PROOF. The assertion follows by applying the proposition to M/N. □

REMARK C.1.3. The above corollary implies, in particular, that given elements $u_1, \ldots, u_r$ of M, they generate M if and only if their classes $\overline{u_1}, \ldots, \overline{u_r} \in M/\mathfrak{m}M$ generate $M/\mathfrak{m}M$ over $k = A/\mathfrak{m}$. We thus see that the cardinality of every minimal system of generators of M is equal to $\dim_k M/\mathfrak{m}M$.

C.2. Projective modules over local rings

PROPOSITION C.2.1. *If $(A, \mathfrak{m})$ is a local Noetherian ring and M is a finitely generated A-module, then M is projective if and only if M is free.*

PROOF. Consider a minimal system of generators $u_1, \ldots, u_n$ for M and the surjective morphism of A-modules

$$\phi\colon F = A^{\oplus n} \to M, \quad \phi(e_i) = u_i \quad \text{for} \quad 1 \leq i \leq n.$$

If $N = \ker(\phi)$, since A is Noetherian and F is a finitely generated A-module, it follows that N is a finitely generated A-module. Since M is projective, the exact sequence

$$0 \to N \to F \to M \to 0$$

is split, hence tensoring with $k = A/\mathfrak{m}$ gives an exact sequence

$$0 \to N/\mathfrak{m}N \to k^{\oplus n} \to M/\mathfrak{m}M \to 0.$$

However, we have seen in Remark C.1.3 that the elements $\overline{u_1}, \ldots, \overline{u_n} \in M/\mathfrak{m}M$ form a basis, so that we deduce from the above exact sequence that $N/\mathfrak{m}N = 0$. Since N is finitely generated, it follows from Nakayama's lemma that $N = 0$, hence $M \simeq F$ is free. $\square$

REMARK C.2.2. It is a result of Kaplansky (see [**Kap58**]) that if M is any projective module over a local ring, then M is free.

C.3. Modules and localization

We collect in this section some easy properties relating statements about modules to corresponding statements about certain localizations.

PROPOSITION C.3.1. *Given an A-module M, the following are equivalent:*

i) $M = 0$.
ii) $M_{\mathfrak{p}} = 0$ *for all maximal ideals* $\mathfrak{p}$ *in* A.
iii) $M_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$ *in* A.
iv) *There are elements* $f_1, \ldots, f_r \in A$ *such that* $(f_1, \ldots, f_r) = A$ *and* $M_{f_i} = 0$ *for all* i.

PROOF. The implication iv)$\Rightarrow$iii) follows from the fact that if $f_1, \ldots, f_r$ generate the unit ideal, then for every prime ideal $\mathfrak{p}$ in A, there is i such that $f_i \not\in \mathfrak{p}$, in which case $M_{\mathfrak{p}}$ is a localization of M_{f_i}. Since the implications i)$\Rightarrow$iv) and iii)$\Rightarrow$ii) are trivial, in order to complete the proof it is enough to prove the implication ii)$\Rightarrow$i). Let $u \in M$ and consider $\mathrm{Ann}_A(u)$. For every maximal ideal $\mathfrak{p}$ in A, we have $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$, hence $\mathrm{Ann}_A(u) \not\subseteq \mathfrak{p}$. This implies that $\mathrm{Ann}_A(u) = A$, hence $u = 0$. $\square$

REMARK C.3.2. The same argument in the proof of the above proposition shows that if M is an A-module and $u \in M$, then the following assertions are equivalent:

i) $u = 0$.
ii) $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ in A.

iii) $\frac{u}{1} = 0$ in $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in A.
iv) There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and $\frac{u}{1} = 0$ in M_{f_i} for all i.

COROLLARY C.3.3. *If M is an A-module and M', M'' are submodules of M, then the following are equivalent:*

i) *$M' \subseteq M''$.*
ii) *$M'_{\mathfrak{p}} \subseteq M''_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ in A.*
iii) *$M'_{\mathfrak{p}} \subseteq M''_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in A.*
iv) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and $M'_{f_i} \subseteq M''_{f_i}$ for all i.*

PROOF. We can simply apply Proposition C.3.1 for the A-module $(M'+M'')/M''$. □

COROLLARY C.3.4. *Given two morphisms of A-modules*

$$M' \xrightarrow{\phi} M \xrightarrow{\psi} M'',$$

the following are equivalent:

i) *The above sequence is exact.*
ii) *The induced sequence*

$$M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$$

is exact for every prime (maximal) ideal $\mathfrak{p}$ in A.

iii) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and each induced sequence*

$$M'_{f_i} \to M_{f_i} \to M''_{f_i}$$

is exact

PROOF. The exactness of the sequence in the statement is equivalent to the two inclusions

$$\mathrm{Im}(\phi) \subseteq \ker(\phi) \quad \text{and} \quad \ker(\phi) \subseteq \mathrm{Im}(\phi).$$

The equivalence in the statement now follows by applying Corollary C.3.3 for the submodules $\mathrm{Im}(\phi)$ and $\ker(\psi)$ of M (note that localization is an exact functor, hence it commutes with taking the image and kernel). □

COROLLARY C.3.5. *Given an A-module M, the following are equivalent:*

i) *M is a finitely generated A-module.*
ii) *There are elements $f_1, \ldots, f_r \in A$ such that $(f_1, \ldots, f_r) = A$ and each M_{f_i} is a finitely generated A_{f_i}-module.*

PROOF. For every i, we may choose finitely many $u_{i,j} \in M$ such that $\left\{\frac{u_{i,j}}{1} \mid j\right\}$ generate M_{f_i} as an A_{f_i}-module. It follows that if N is the A-submodule of M generated by all $u_{i,j}$, then N is finitely generated and $(M/N)_{f_i} = 0$ for all i. We then deduce from Proposition C.3.1 that $M = N$. □

C.4. Krull's Intersection theorem

THEOREM C.4.1. *If I is an ideal in a Noetherian ring A, M is a finitely generated A-module, and $N = \bigcap_{m\geq 1} I^m M$, then $IN = N$. In particular, if $(A, \mathfrak{m})$ is a local ring and $I \subseteq \mathfrak{m}$, then $N = 0$.*

We will deduce the theorem from the following result, known as the Artin-Rees lemma.

LEMMA C.4.2. *Let A be a Noetherian ring and I an ideal in A. If M is a finitely generated A-module and N is a submodule of M, then for every $n \geq 0$, there is $m \geq 0$ such that*

$$I^m M \cap N \subseteq I^n N.$$

PROOF. Consider the **N**-graded ring

$$R(A, I) := \bigoplus_{j\geq 0} I^n t^n \subseteq A[t].$$

Note that if I is generated by $a_1, \ldots, a_r$, then $R(A, I)$ is generated over A by $a_1 t, \ldots, a_r t$. In particular, $R(A, I)$ is a Noetherian ring.

Consider now the **N**-graded $R(A, I)$-module

$$T = \bigoplus_{j\geq 0} I^j M t^j \subseteq M[t] = M \otimes_A A[t].$$

Since M is finitely generated over A, it is clear that T is a finitely generated $R(A, I)$-module. Consider the $R(A, I)$-submodule of T given by

$$\bigoplus_{j\geq 0} (N \cap I^j M) t^j.$$

Since M is a finitely generated module over a Noetherian ring, it follows that M is Noetherian, hence N is finitely generated. Choose generators of N of the form $u_j t^{d_j}$ for some $u_j \in N \cap I^{d_j} M$, with $1 \leq j \leq r$. Given any $u \in N \cap I^m M$ we can thus write

$$ut^m = \sum_{j=1}^{r} (a_j t^{b_j}) \cdot (u_j t^{d_j})$$

for some $a_j \in I^{b_j}$, where $b_j = m - d_j$. We thus see that if $m \geq n + d_j$ for all j, then

$$N \cap I^m M \subseteq I^n N.$$

This completes the proof of the lemma. $\square$

PROOF OF THEOREM C.4.1. Of course, we only need to show that $N \subseteq IN$. We apply the lemma for the submodule N of M to get a non-negative integer m such that $I^m M \cap N \subseteq IN$. However, since $N \subseteq I^m M$, this implies $N \subseteq IN$. The last assertion in the theorem is a consequence of Nakayama's lemma. $\square$

C.5. Discrete Valuation Rings

Recall that a *discrete valuation* on a field K is a surjective map $v \colon K \to \mathbf{Z} \cup \{\infty\}$ that satisfies the following properties:

i) $v(a) = \infty$ if and only if $a = 0$.
ii) $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.
iii) $v(ab) = v(a) + v(b)$ for all $a, b \in K$.

PROPOSITION C.5.1. *Given an integral domain R, with fraction field K, the following are equivalent:*

i) *There is a discrete valuation v on K such that $R = \{a \in K \mid v(a) \geq 0\}$.*
ii) *R is a local PID.*
iii) *R is local and the maximal ideal is principal.*

A ring that satisfies the above equivalent properties is a *discrete valuation ring* (or *DVR*, for short).

PROOF. Let us show first that i)$\Rightarrow$ii). Let $\mathfrak{m} = \{a \in K \mid v(a) > 0\}$. It follows from the definition of a discrete valuation that $\mathfrak{m}$ is an ideal in R and that for every $u \in R \smallsetminus \mathfrak{m}$, we have $u^{-1} \in R$. Therefore R is local and $\mathfrak{m}$ is the maximal ideal of R. Given any non-zero ideal I in R, consider $a \in I$ such that $v(a)$ is minimal. Given any other $b \in I$, we have $v(b) \geq v(a)$, hence $v(ba^{-1}) \geq 0$, and therefore $b \in (a)$. This shows that $I = (a)$ and therefore R is a PID.

Since the implication ii)$\Rightarrow$iii) is trivial, in order to complete the proof, it is enough to prove iii)$\Rightarrow$i). Suppose that $(R, \mathfrak{m})$ is a local domain and $\mathfrak{m} = (\pi)$, for some $\pi \neq 0$. Given any non-zero element α, it follows from Theorem C.4.1 that there is $j \geq 0$ such that $\alpha \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. Therefore we can write $\alpha = u\pi^j$, with u invertible. Since K is the fraction ring of R, it follows that every non-zero element β in K can be written as $\beta = u\pi^j$ for some $j \in \mathbf{Z}$ and $u \in R \smallsetminus \mathfrak{m}$. If we put $v(\beta) = j$, then it is straightforward to check that v is a discrete valuation and $R = \{a \in K \mid v(a) \geq 0\}$. $\square$

APPENDIX D

The norm map for finite field extensions

In this appendix we define and prove some basic properties of the norm map for a finite field extension.

D.1. Definition and basic properties

Let K/L be a finite field extension. Given an element $u \in K$, we define $N_{K/L}(u) \in L$ as the determinant of the L-linear map

$$\varphi_u : K \to K, \quad v \to uv.$$

This is the *norm* of u with respect to K/L.

We collect in the first proposition some easy properties of this map.

PROPOSITION D.1.1. *Let K/L be a finite field extension.*

i) *We have $N_{K/L}(0) = 0$ and $N_{K/L}(u) \neq 0$ for every nonzero $u \in K$.*
ii) *We have*

$$N_{K/L}(u_1 u_2) = N_{K/L}(u_1) \cdot N_{K/L}(u_2) \quad \textit{for every} \quad u_1, u_2 \in K.$$

iii) *For every $u \in L$, we have*

$$N_{K/L}(u) = u^{[K:L]}.$$

PROOF. The first assertion in i) is clear and the second one follows from the fact that φ_u is invertible for every nonzero u. The assertion in ii) follows from the fact that

$$\varphi_{u_1} \circ \varphi_{u_2} = \varphi_{u_1 u_2} \quad \text{for every} \quad u_1, u_2 \in K$$

and the multiplicative behavior of determinants. Finally, iii) follows from the fact that for $u \in L$, the map φ_u is given by scalar multiplication. $\square$

PROPOSITION D.1.2. *Let K/L be a finite field extension and $u \in K$. If $f \in L[x]$ is the minimal polynomial of u over L and $\operatorname{char}(\varphi_u)$ is the characteristic polynomial of φ_u:*

$$\operatorname{char}(\varphi_u) = \det(x \cdot \mathrm{Id} - \varphi_u),$$

then $\operatorname{char}(\varphi_u) = f^r$, where $r = [K : L(u)]$. In particular, we have

$$N_{K/L}(u) = (-1)^{[K:L]} \cdot f(0)^r.$$

PROOF. Let $\operatorname{char}'(\varphi_u)$ be the characteristic polynomial of $\varphi'_u = \varphi_u|_{L(u)}$. We write

$$f = x^m + a_1 x^{m-1} + \ldots + a_m,$$

where $m = [L(u) : L]$. By writing the linear map φ'_u in the basis $1, u, \ldots, u^{m-1}$ of $L(u)$ over L, we see that $x \cdot \mathrm{Id} - \varphi'_u$ is given by the matrix

$$A = \begin{pmatrix} x & 0 & \ldots & 0 & a_m \\ 1 & x & \ldots & 0 & a_{m-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & x & a_2 \\ 0 & 0 & \ldots & 1 & a_1 \end{pmatrix},$$

whose determinant can be easily computed to be equal to f. If $e_1, \ldots, e_r$ is a basis of K over $L(u)$ and we write $x \cdot \mathrm{Id} - \varphi_u$ with respect to the basis given by $u^i e_j$, for $0 \leq i \leq m-1$ and $1 \leq j \leq r$ (suitably ordered), this is the block diagonal matrix

$$\begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ 0 & 0 & \ldots & A \end{pmatrix}.$$

The first assertion in the proposition follows. The last assertion is a consequence of the fact that the constant term in $\mathrm{char}(\varphi_u)$ is $(-1)^{[K:L]} \cdot \det(\varphi_u)$. □

D.2. A property of the norm for integrally closed domains

Recall that an integral domain A with fraction field K is *integrally closed* if every element of K that is integral over A lies in A.

PROPOSITION D.2.1. *Let $B \hookrightarrow A$ be an integral ring extension of integral domains such that the corresponding field extension $L \hookrightarrow K$ between the two fraction fields is finite. If B is integrally closed, then for every element $u \in A$, we have $v := N_{K/L}(u) \in B$. Moreover, if $u \in J$, where J is an ideal in A, then $v \in J \cap B$.*

PROOF. Let $f = x^m + a_1 x^{m-1} + \ldots + a_m \in L[x]$ be the minimal polynomial of u over L. Since u is integral over B, there is a monic polynomial $g \in B[x]$ such that $g(u) = 0$. Note that f divides g in $L[x]$. Every other root of f (in some algebraic closure $\overline{K}$ of K) is automatically a root of g, and therefore it is again integral over B. Since the set of elements of $\overline{K}$ integral over B is a ring (see Proposition 2.2. in Review Sheet 1), and every a_i is (up to sign) a symmetric function of the roots of f, we conclude that a_i is integral over B. Finally, since B is integrally closed in L and the a_i lie in L, we conclude that the a_i lie in B. By Proposition D.1.2, we can write $N_{K/L}(u)$, up to sign, as a power of a_m, hence $N_{K/L}(u) \in B$.

Suppose now that $u \in J$, for an ideal J in A. Since

$$a_m = -u(u^{m-1} + a_1 u^{m-2} + \ldots + a_{m-1}),$$

and $a_i \in B \subseteq A$ for all i, we deduce that $a_m \in J$. Arguing as before, we conclude that $N_{K/L}(u) \in J \cap B$. □

APPENDIX E

Zero-divisors in Noetherian rings

In the first section we prove a basic result about prime ideals, the prime avoidance lemma. In the second section we give a direct proof for the fact that minimal prime ideals consist of zero-divisors. Finally, in the last section we discuss more generally zero-divisors on finitely generated modules over a Noetherian ring and primary decomposition.

E.1. The prime avoidance lemma

The following result, known as the Prime Avoidance lemma, is often useful.

Lemma E.1.1. *Let R be a commutative ring, r a positive integer, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ ideals in R such that $\mathfrak{p}_i$ is prime for all $i \geq 3$. If I is an ideal in R such that $I \subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$, then $I \subseteq \mathfrak{p}_i$ for some $i \geq 1$.*

Proof. The assertion is trivial for $r = 1$. We prove it by induction on $r \geq 2$. If $r = 2$ and $I \not\subseteq \mathfrak{p}_1$ and $I \not\subseteq \mathfrak{p}_2$, then we may choose $a \in I \smallsetminus \mathfrak{p}_1$ and $b \in I \smallsetminus \mathfrak{p}_2$. Note that since $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$, we have $a \in \mathfrak{p}_2$ and $b \in \mathfrak{p}_1$. Note that $a + b \in I$, hence $a + b \in \mathfrak{p}_1$ or $a + b \in \mathfrak{p}_2$. In the first case, we see that $a = (a + b) - b \in \mathfrak{p}_1$, a contradiction and in the second case, we see that $b = (a+b) - a \in \mathfrak{p}_2$, leading again to a contradiction. This settles the case $r = 2$.

Suppose now that $r \geq 3$ and that we know the assertion for $r - 1$ ideals. If $I \not\subseteq \mathfrak{p}_i$ for every i, it follows from the induction hypothesis that given any i, we have $I \not\subseteq \bigcup_{j \neq i} \mathfrak{p}_j$. Let us choose

$$a_i \in I \smallsetminus \bigcup_{j \neq i} \mathfrak{p}_j.$$

By hypothesis, we must have $a_i \in \mathfrak{p}_i$ for all i.

Since $\mathfrak{p}_r$ is a prime ideal and $a_i \not\in \mathfrak{p}_r$ for $i \neq r$, it follows that $\prod_{1 \leq j \leq r-1} a_j \not\in \mathfrak{p}_r$. Consider now the element

$$u = a_r + \prod_{1 \leq j \leq r-1} a_j \in I.$$

By assumption, we have $u \in \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$. If $u \in \mathfrak{p}_r$, since $a_r \in \mathfrak{p}_r$, we deduce that $\prod_{1 \leq j \leq r-1} a_j \in \mathfrak{p}_r$, a contradiction. On the other hand, if $u \in \mathfrak{p}_i$ for some $i \leq r - 1$, since $\prod_{1 \leq j \leq r-1} a_j \in \mathfrak{p}_i$, we conclude that $a_r \in \mathfrak{p}_i$, a contradiction. We thus conclude that $I \subseteq \mathfrak{p}_i$ for some i, completing the proof of the induction step. $\square$

E.2. Minimal primes and zero-divisors

Let R be a Noetherian ring. We refer to Exercise 3.1.4 for the definition of the topological space $\mathrm{Spec}(R)$. Since R is a Noetherian ring, $\mathrm{Spec}(R)$ is a Noetherian topological space, hence we can apply Proposition 1.3.12 to write it as the union of finitely many irreducible components. Since the irreducible closed subsets of $\mathrm{Spec}(R)$ are those of the form $V(\mathfrak{p})$, with $\mathfrak{p}$ a prime ideal in R, we conclude that there are finitely many minimal primes $\mathfrak{p}_1, \ldots \mathfrak{p}_r$ in $\mathrm{Spec}(R)$. The decomposition

$$\mathrm{Spec}(R) = V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r)$$

says that

$$\mathrm{rad}(0) = \bigcap_{i=1}^{r} \mathfrak{p}_i.$$

PROPOSITION E.2.1. *With the above notation, every minimal prime ideal* $\mathfrak{p}_i$ *is contained in the set of zero-divisors of* R.

PROOF. Given $a \in \mathfrak{p}_i$, we choose for every $j \neq i$ an element $b_j \in \mathfrak{p}_j \smallsetminus \mathfrak{p}_i$. If $b = \prod_{j \neq i} b_j$, then $b \not\in \mathfrak{p}_i$, but $b \in \mathfrak{p}_j$ for all $j \neq i$. We thus have

$$ab \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r = \mathrm{rad}(0),$$

hence $(ab)^N = 0$ for some positive integer N. If a is a non-zero-divisor, we would get that $b^N = 0$, hence $b \in \mathfrak{p}_i$, a contradiction. $\square$

REMARK E.2.2. If R is reduced, then the set of zero-divisors of R is precisely the union of the minimal prime ideals. Indeed, in this case we have $\bigcap_{i=1}^{r} \mathfrak{p}_i = 0$. It follows that if $ab = 0$ and $a \not\in \mathfrak{p}_i$ for all i, then $b \in \mathfrak{p}_i$ for all i, hence $b = 0$. In the next section we will discuss the set of zero-divisors for an arbitrary Noetherian ring (and, more generally, for a finitely generated module over such a ring).

E.3. Associated primes and zero-divisors

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