Math 632

Solutions for the take-home exam

Problem 1. Let $X \subseteq \mathbf{P}^n$ be a closed subvariety of $\mathbf{P}^n$, with $\mathrm{codim}_{\mathbf{P}^n}(X) = r$, which is a (global) complete intersection: this means that the associated homogeneous ideal $I_X \subseteq S = k[x_0, \ldots, x_n]$ is generated by r homogeneous elements.

i) Show that X is Cohen-Macaulay.
ii) Show that $H^i\big(X, \mathcal{O}_X(m)\big) = 0$ for all m and all i, with $1 \leq i \leq \dim(X) - 1$.
iii) Show that if $\dim(X) > 0$, then X is connected.
iv) Show that if X is smooth and the ideal of I_X is generated by the homogeneous polynomials $f_1, \ldots, f_r$, with $\deg(f_i) = d_i$, then the canonical line bundle of X is isomorphic to $\mathcal{O}_X(d_1 + \ldots + d_r - n - 1)$.
v) Under the assumption in iv), deduce that if $\dim(X) \geq 1$ and $d_1 + \ldots + d_r \geq n+1$, then X is not a rational variety.

Solution.

There are several ways to proceed. In what follows, the key part of the argument will be based on induction.

Note that $\mathbf{P}^n$ is smooth, hence Cohen-Macaulay. In order to show that X is Cohen-Macaulay, if $\mathcal{I}$ is the radical ideal sheaf corresponding to X, it is enough to show that for every $p \in X$, the ideal $\mathcal{I}_p \subseteq \mathcal{O}_{\mathbf{P}^n,p}$ is generated by $\mathrm{codim}(\mathcal{I}_p)$ elements (indeed, we have $\mathrm{codim}(\mathcal{I}_p) = \mathrm{depth}(\mathcal{I}_p)$ since $\mathcal{O}_{\mathbf{P}^n,p}$ is Cohen-Macaulay, hence the system of generators forms a regular sequence by Corollary 12.2.13 in the Notes; then X is Cohen-Macaulay by Proposition 12.3.12 in the Notes).

Let $f_1, \ldots, f_r$ be homogeneous generators of I_X, with $d_i = \deg(f_i) > 0$. If $p \in U_i = (x_i \neq 0)$, then $\mathcal{I}_p$ is generated by $\frac{f_1}{x_i^{d_1}}, \ldots, \frac{f_r}{x_i^{d_r}}$. In particular, we have $\mathrm{codim}(\mathcal{I}_p) \leq r$ by one of the consequences to the Principal Ideal theorem. On the other hand, we clearly have

$$\mathrm{codim}(\mathcal{I}_p) \geq \mathrm{codim}_{\mathbf{P}^n}(X) = r,$$

hence this is an equality and X is Cohen-Macaulay, giving i). Moreover, this argument shows that every irreducible component of X has pure dimension $n - r$.

We next show that the following hold:

(α) If $r \leq n-1$, then the canonical morphism $S/I_X \to \bigoplus_{m \in \mathbf{Z}} \Gamma\big(X, \mathcal{O}_X(m)\big)$ is an isomorphism.
(β) For every q, with $1 \leq q \leq \dim(X) - 1$, we have

$$H^q\big(X, \mathcal{O}_X(m)\big) = 0 \quad \text{for all} \quad m \in \mathbf{Z}.$$

In order to show this, we consider $\mathcal{F}_j = \mathcal{O}_{\mathbf{P}^n}/\mathcal{I}_j$, for $1 \leq j \leq r$, where $\mathcal{I}_j$ is the ideal sheaf corresponding to the homogeneous ideal $(f_1, \ldots, f_j)$. Arguing as above, we see that each $\mathcal{F}_j$ is a Cohen-Macaulay $\mathcal{O}_{\mathbf{P}^n}$-module. Indeed, $\mathcal{I}_j$ is locally defined by j equations. In particular, every irreducible component $\mathrm{Supp}(\mathcal{F}_j)$ has codimension $\leq j$ in

$\mathbf{P}^n$. In fact, the codimension is precisely j: otherwise, the intersection of this component with $\bigcap_{i=j+1}^r V(f_i)$ would be non-empty, of codimension $< r$ by Corollary 4.2.12 in the Notes, a contradiction with the fact $\operatorname{codim}_{\mathbf{P}^n}(X) = r$.

With the convention that $\mathcal{F}_0 = \mathcal{O}_{\mathbf{P}^n}$, we show by induction on $j \geq 0$, that the following hold:

(α_j) If $j \leq n-1$, then the canonical morphism

$$S/(f_1, \ldots, f_j) \to \bigoplus_{m \in \mathbf{Z}} \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m)\big)$$

is an isomorphism.

(β_j) For every q, with $1 \leq q \leq n-j-1$, we have

$$H^q\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) = 0 \quad \text{for all} \quad m \in \mathbf{Z}.$$

For $j = r$, we obtain (α) and (β).

Both (α_0) and (β_0) hold by Theorem 11.2.2 in the Notes, hence it is enough to show that for $0 \leq j \leq r-1$, if we know the assertions for j, then we obtain them for $j+1$. Since $\mathcal{F}_j$ is Cohen-Macaulay, it follows from Proposition 12.3.13 in the Notes that the associated subvarieties of $\mathcal{F}_j$ are precisely the irreducible components of $\operatorname{Supp}(\mathcal{F}_j)$. Moreover, we have seen that each such irreducible component has codimension j, and thus can't be contained in $\operatorname{Supp}(\mathcal{F}_{j+1})$. Therefore $V(f_{j+1})$ contains no associated subvariety of $\mathcal{F}_j$, and thus we have a short exact sequence

$$0 \to \mathcal{F}_j(m - d_{j+1}) \to \mathcal{F}_j(m) \to \mathcal{F}_{j+1}(m) \to 0$$

for every $m \in \mathbf{Z}$. By taking the long exact sequence in cohomology, we obtain exact sequences

$$(1) \qquad 0 \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) \to \Gamma\big(\mathbf{P}^n, \mathcal{F}_{j+1}(m)\big) \to H^1\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big)$$

and

$$(2) \qquad H^q\big(\mathbf{P}^n, \mathcal{F}_j(m)\big) \to H^q\big(\mathbf{P}^n, \mathcal{F}_{j+1}(m)\big) \to H^{q+1}\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big)$$

for all $q \geq 0$. It is clear that using (β_j), the sequence in (**??**) implies (β_{j+1}). Moreover, if $j+1 \leq n-1$, then (β_j) implies $H^1\big(\mathbf{P}^n, \mathcal{F}_j(m - d_{j+1})\big) = 0$, and thus (**??**) and (α_j) imply (α_{j+1}). This completes the proof of the induction step. In particular, we obtain ii).

In particular, it follows from (α) that if $r \leq n-1$, then $\Gamma(X, \mathcal{O}_X) = k$. We thus conclude that in this case X is connected, giving iii).

It is easy to compute the normal bundle of X. Indeed, $s = (f_1, \ldots, f_r)$ is a regular section of $\bigoplus_{j=1}^r \mathcal{O}_{\mathbf{P}^n}(d_i)$ such that $\mathcal{I}(s) = \mathcal{I}$. We saw in a problem session (see also Example 12.2.21 in the Notes) that in this case $N_{X/\mathbf{P}^n} \simeq \bigoplus_{j=1}^r \mathcal{O}_X(d_i)$.

If X is smooth, then we have a short exact sequence

$$0 \to N_{X/\mathbf{P}^n}^\vee \to \Omega_{\mathbf{P}^n}|_X \to \Omega_X \to 0.$$

By taking determinants, we see that

$$\omega_X \simeq \omega_{\mathbf{P}^n}|_X \otimes_{\mathcal{O}_X} \det(N_{X/\mathbf{P}^n}) \simeq \mathcal{O}_X(d_1 + \ldots + d_r - n - 1).$$

This proves iv).

Finally, suppose that $r \leq n-1$ and X is smooth, hence also irreducible by iii). Recall that the geometric genus of X is a birational invariant of smooth, irreducible, projective varieties. Using iv) and (α), we see that if $m = d_1 + \ldots + d_r - n - 1 \geq 0$, then

$$p_g(X) = h^0(X, \omega_X) = h^0\big(X, \mathcal{O}_X(m)\big) = \dim_k \big(S/(f_1, \ldots, f_r)\big)_m > 0.$$

Since the geometric genus of the projective space is 0, it follows that in this case X is not rational.

Problem 2. Determine the Grothendieck group $K_0(\mathbf{P}^n)$, as follows:

i) Prove the following graded version of Nakayama's lemma: if M is a finitely generated, graded module over $S = k[x_0, \ldots, x_n]$ and $M = (x_0, \ldots, x_n)M$, then $M = 0$.
ii) Deduce that if M is a finitely generated, graded S-module, such that M is projective, then M is free, that is, it is isomorphic to $\bigoplus_{i=1}^r S(a_i)$ for some non-negative integer r and some $a_1, \ldots, a_r \in \mathbf{Z}$.
iii) Deduce that the Abelian group $K_0(\mathbf{P}^n)$ is generated by $\big[\mathcal{O}_{\mathbf{P}^n}(m)\big]$, for $m \in \mathbf{Z}$.
iv) Show that, in fact, $K_0(\mathbf{P}^n)$ is generated just by $\big[\mathcal{O}_{\mathbf{P}^n}(m)\big]$, for $-n \leq m \leq 0$.
v) Show that there is a group homomorphism $K_0(\mathbf{P}^n) \to \mathbf{Q}[t]$ that maps $[\mathcal{F}]$ to the Hilbert polynomial $P_{\mathcal{F}}(t)$.
vi) Use the morphism in v) to show that $K_0(\mathbf{P}^n)$ is freely generated by $\big[\mathcal{O}_{\mathbf{P}^n}(m)\big]$, for $-n \leq m \leq 0$, hence we have a group isomorphism $K_0(\mathbf{P}^n) \simeq \mathbf{Z}^{n+1}$.
vii) Deduce that we have a *ring isomorphism* $K^0(\mathbf{P}^n) \simeq \mathbf{Z}[x]/(x^{n+1})$.

Solution. In order to prove i), suppose that $M \neq 0$ and let $u_1, \ldots, u_r$ be non-zero homogeneous generators of M, with $\deg(u_i) = d_i$. If $d = \min_i d_i$, then using the fact that $\mathfrak{m}_i = 0$ for $i \leq 0$, it follows that $(\mathfrak{m}M)_i = 0$ for $i < d+1$. This contradicts that fact that we have a non-zero homogeneous element of degree d in $M = \mathfrak{m}M$.

The assertion in i) implies that if M is a finitely generated graded S-module and N is a graded submodule of M, such that $M = \mathfrak{m}M + N$, then $M = N$. Indeed, it is enough to apply i) for M/N. As in the local case, this allows us to talk about *minimal systems of homogeneous generators* for M. Indeed, if $u_1, \ldots, u_r \in M$ are homogeneous elements, these generate M if and only if their classes in $M/\mathfrak{m}M$ generate this k-vector space. Therefore $u_1, \ldots, u_r$ form a minimal system of generators if and only if their classes in $M/\mathfrak{m}M$ form a k-basis.

Suppose now that M is a projective finitely generated graded S-module, and consider a minimal system of homogeneous generators $u_1, \ldots, u_r$. We have a surjective morphism of graded R-modules $\phi\colon F \to M$, with F a finitely generated, free, graded R-module. Since the classes $\overline{u_1}, \ldots, \overline{u_r} \in M/\mathfrak{m}M$ form a k-basis, it follows that if

$K = \ker(\phi)$, then $K \subseteq \mathfrak{m}M$. If M is a projective R-module, then ϕ is a split surjection, and we thus have an exact sequence

$$0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \to M/\mathfrak{m}M \to 0.$$

Since $K \subseteq \mathfrak{m}F$, we conclude that $K = \mathfrak{m}K$, hence $K = 0$ by i).

We now prove the assertion in iii). Recall that for every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$, we have a finitely generated graded S-module M, such that $\mathcal{F} \simeq \widetilde{M}$. By choosing finitely many homogeneous generators for M, we construct a surjective graded homomorphism

$$\phi\colon F_0 \to M, \quad \text{where} \quad F_0 = \bigoplus_j S(-j)^{\oplus \beta_{0,j}}.$$

By taking the kernel of ϕ and repeating this construction step by step, we obtain an exact complex

$$0 \to F_{n+1} \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0,$$

where $F_0, \ldots, F_n$ are finitely generated, free, graded S-modules. Since the affine variety corresponding to S is smooth, irreducible, of dimension $n+1$, it follows from Proposition 12.2.15 that for every maximal ideal $\mathfrak{m}$ of S, we have $\mathrm{pd}_{S_\mathfrak{m}}(M_\mathfrak{m}) \leq n+1$, hence $(F_{n+1})_\mathfrak{m}$ is a projective $S_\mathfrak{m}$-module. Therefore F_{n+1} is a projective S-module, and thus it is a free graded S-module by ii). By passing to the corresponding sheaves, we see that $\mathcal{F}$ has a finite free resolution by direct sums of line bundles. This implies that $[\mathcal{F}] \in K_0(\mathbf{P}^n)$ lies in the subgroup generated by $[\mathcal{O}_{\mathbf{P}^n}(m)]$, for $m \in \mathbf{Z}$, proving iii).

Recall that we have the exact Koszul complex:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-n-1) \to \mathcal{O}_{\mathbf{P}^n}(-n)^{\oplus(n+1)} \to \ldots \to \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n} \to 0.$$

By tensoring this with $\mathcal{O}_{\mathbf{P}^n}(m)$, we see that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in the subgroup generated by $[\mathcal{O}_{\mathbf{P}^n}(m-i)]$ for $1 \leq i \leq n+1$. We thus deduce, by induction on $m \geq 0$, that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in the subgroup A of $K_0(\mathbf{P}^n)$ generated by $[\mathcal{O}_{\mathbf{P}^n}], [\mathcal{O}_{\mathbf{P}^n}(-1)], \ldots, [\mathcal{O}_{\mathbf{P}^n}(-n)]$. Similarly, by tensoring the Koszul complex with $\mathcal{O}_{\mathbf{P}^n}(m+n+1)$, we see that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in the subgroup generated by $[\mathcal{O}_{\mathbf{P}^n}(m+i)]$, for $1 \leq i \leq n+1$. Using this, we see by decreasing induction on $m \leq -n-1$ that $[\mathcal{O}_{\mathbf{P}^n}(m)]$ lies in A. We thus conclude that $A = K_0(\mathbf{P}^n)$, proving iv).

Note that for every closed subvariety X of $\mathbf{P}^n$, given an exact sequence of coherent sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

by tensoring this with $\mathcal{O}_X(m)$ and taking the Euler-Poincaré characteristic, we obtain

$$\chi\big(\mathcal{F}(m)\big) = \chi\big(\mathcal{F}'(m)\big) + \chi\big(\mathcal{F}''(m)\big) \quad \text{for all} \quad m \in \mathbf{Z}.$$

We thus have $P_\mathcal{F} = P_{\mathcal{F}'} + P_{\mathcal{F}''}$. This implies that we have a morphism of Abelian groups $K_0(X) \to \mathbf{Q}[t]$ that maps $[\mathcal{F}]$ to the Hilbert polynomial $P_\mathcal{F}$. This applies in particular when $X = \mathbf{P}^n$, proving v).

We thus have a morphism $\gamma\colon K_0(\mathbf{P}^n) \to \mathbf{Q}[t]$ that maps $[\mathcal{O}_{\mathbf{P}^n}(-i)]$ to the polynomial $P(t-i)$, where

$$P(t) = \frac{(t+1)\cdots(t+n)}{n!}.$$

We claim that $P(t), P(t-1), \ldots, P(t-n)$ are linearly independent over $\mathbf{Z}$. Indeed, if $\sum_{i=0}^{n} \lambda_i P(t-i) = 0$, where not all λ_i are 0, and if $j = \max\{i \mid \lambda_i \neq 0\}$, then by taking $t = j$, we obtain $\lambda_j = 0$, a contradiction. Since the images of $[\mathcal{O}_{\mathbf{P}^n}], [\mathcal{O}_{\mathbf{P}^n}(-1)], \ldots, [\mathcal{O}_{\mathbf{P}^n}(-n)]$ by γ are linearly independent over $\mathbf{Z}$, we conclude that these elements freely generate $K_0(\mathbf{P}^n)$, completing the proof of vi).

Recall now that by a homework problem (see Proposition 12.2.18 in the Notes), since $\mathbf{P}^n$ is smooth and carries an ample line bundle, the canonical group homomorphism $K^0(\mathbf{P}^n) \to K_0(\mathbf{P}^n)$ is an isomorphism. What we proved so far thus shows that if $h = [\mathcal{O}_{\mathbf{P}^n}(-1)]$, then $1, h, \ldots, h^n$ give a $\mathbf{Z}$-basis of $K^0(\mathbf{P}^n)$. Moreover, the Koszul complex gives the relation $\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} h^i = 0$, that is, $(1-h)^{n+1} = 0$. This implies that we have a morphism $\mathbf{Z}[x]/(x^{n+1}) \to K^0(\mathbf{P}^n)$ that maps the class of x to $1 - [\mathcal{O}_{\mathbf{P}^n}(-1)]$. Since $\{(1-x)^i \mid 0 \leq i \leq n\}$ gives a basis of $\mathbf{Z}[x]/(x^{n+1})$, we see that this morphism is an isomorphism, proving vii).

Remark. We emphasize that for $n \geq 2$, it is *not* true that every locally free sheaf on $\mathbf{P}^n$ is a direct sum of line bundles. For example, for $n \geq 2$, the cotangent bundle is not isomorphic to a direct sum of line bundles. Since $H^1\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)\big) = 0$ for every m by Theorem 11.2.2 in the Notes, it is enough to show that $H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}) \neq 0$. Recall that we have the Euler exact sequence

$$0 \to \Omega_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n} \to 0.$$

The long exact sequence in cohomology gives an exact sequence

$$0 = \Gamma\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)}\big) \to \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n})$$

$$\to H^1\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)}\big) = 0.$$

We thus obtain $H^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}) \simeq k$.

Problem 3. Let $C \subseteq \mathbf{P}^n$ be an irreducible curve (that is, $\dim(C) = 1$). We assume that C is non-degenerate (that is, it is not contained in a hyperplane).

i) Show that if $H \subseteq \mathbf{P}^n$ is a general hyperplane, then $C \cap H$ is a non-degenerate subset in H.
ii) Deduce that $\deg(C) \geq n$ (recall that the degree of a variety is the degree of its structure sheaf, defined using the Hilbert polynomial).

Solution. By assumption, for every hyperplane H in $\mathbf{P}^n$, we have $C \not\subseteq H$. Let $\mathcal{F} := \mathcal{O}_C \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_H$. Since C is irreducible, the only associated subvariety of $\mathcal{O}_C$ is C itself. Since $C \not\subseteq H$, we have a short exact sequence

$$0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathcal{O}_C \otimes \mathcal{O}_H \to 0,$$

which by tensoring with $\mathcal{O}_C(m)$ and taking Euler-Poincaré characteristics gives

$$P_{\mathcal{F}}(m) = P_C(m) - P_C(m-1),$$

where $P_{\mathcal{F}}$ is the Hilbert polynomial of $\mathcal{F}$ and P_C is the Hilbert polynomial of C. It is thus easy to deduce that $\deg(C) = \deg(\mathcal{O}_C) = \deg(\mathcal{F})$. Note that $\mathcal{F}$ has support the finite set $C \cap H$, hence

$$d = \deg(\mathcal{F}) = \sum_{p \in C \cap H} \ell_{\mathcal{O}_{\mathbf{P}^n,p}}(\mathcal{F}_p).$$

In particular, we see that $C \cap H$ consists of at most d points.

Suppose now that H is a general hyperplane. Since C has finitely many singular points, it follows from Bertini's theorem that H is transversal to C, meeting C only at smooth points of C (see Remark 6.4.2 in the Notes). In this case, it follows from Proposition 6.3.26 in the Notes that the radical ideal sheaf corresponding to $C \cap H$ is equal to $\mathcal{I}_C + \mathcal{O}_{\mathbf{P}^n}(-H)$, where $\mathcal{I}_C$ is the radical ideal sheaf corresponding to C. In other words, in this case we have $\mathcal{F} = \mathcal{O}_{C \cap H}$, and thus $C \cap H$ consists precisely of d points.

It is now easy to see that for such general H, the intersection $C \cap H$ is non-degenerate in $H \simeq \mathbf{P}^{n-1}$. Indeed, if $C \cap H$ is contained in a codimension 1 linear subspace $\Lambda \subsetneq H$, and if $p \in C \smallsetminus H$, then the linear span H' of Λ and p is a hyperplane in $\mathbf{P}^n$ that meets C in at least $(d+1)$ points. We have seen that this is not possible, and thus $C \cap H$ is non-degenerate in H. Since any $(n-1)$ points in $\mathbf{P}^{n-1}$ are contained in a hyperplane, we conclude that $d \geq n$.