The local flatness criterion

A finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ is flat if and only if it is free, which holds if and only if $\mathrm{Tor}_R(R/\mathfrak{m}, M) = 0$ (see Corollary 12.1.17 in the notes). It turns out that the latter characterization of flatness over R holds under much more general conditions: it is enough for M to be finitely generated over a Noetherian local ring S such that we have a local homomorphism $R \to S$. The goal of this write-up is to explain the proof of this result.

We begin with the following general criterion for the flatness of a module.

Lemma 0.1. *Given a commutative ring R and an R-module M, the following are equivalent:*

i) *M is flat.*

ii) *For every ideal $\mathfrak{a}$ in R, the canonical map*

$$\mathfrak{a} \otimes_R M \to M$$

is injective.

iii) *For every ideal $\mathfrak{a}$ in R, we have $\mathrm{Tor}_1^R(R/\mathfrak{a}, M) = 0$.*

Proof. The implication i)⇒ii) is obvious, while the equivalence of ii) and iii) follows from the long exact sequence of Tor modules associated to

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0,$$

namely

$$0 = \mathrm{Tor}_1^R(R, M) \to \mathrm{Tor}_1^R(R/\mathfrak{a}, M) \to \mathfrak{a} \otimes_R M \to M \to R/\mathfrak{a} \otimes_R M \to 0.$$

In order to complete the proof, it is enough to show that iii)⇒i).

By Proposition 10.7.20 in the notes, it is enough to show that $\mathrm{Tor}^1_R(N, M) = 0$ for every R-module N. Since N is the filtering direct limit of its finitely generated R-submodules, using Lemma 0.2 below, we see that it is enough to consider the case when N is finitely generated. Suppose that N can be generated by r elements and let us argue by induction on r. If $r = 0$, then $N = 0$ and the assertion is trivial. For the induction step, if N is generated by $u_1, \ldots, u_r$ and N' is the R-submodule generated by $u_1, \ldots, u_{r-1}$, then we have an exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

where N'' is generated by one element, hence $N'' \simeq R/\mathfrak{a}$, for some ideal $\mathfrak{a}$ in R. The long exact sequence for Tor modules gives an exact sequence

$$\mathrm{Tor}_1^R(N', M) \to \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}^1_R(N'', M).$$

The left term vanishes by induction and the right term vanishes by hypothesis, hence the middle one vanishes, too. This completes the proof. □

Lemma 0.2. *If* $N = \varinjlim_{i\in I} N_i$*, where* $(I,\leq)$ *is a filtering ordered set, then for every* R*-module* M*, we have a functorial isomorphism*

$$\mathrm{Tor}_j^R(N,M) \simeq \varinjlim_{i\in I} \mathrm{Tor}_j^R(N_i,M).$$

Proof. If $F_\bullet \to M$ is a free resolution of M, then the assertion follows from the isomorphisms

$$\mathrm{Tor}_j^R(N,M) \simeq \mathcal{H}_j(N\otimes_R F_\bullet) \simeq \mathcal{H}_j\big(\varinjlim_{i\in I}(N_i\otimes_R F_\bullet)\big)$$
$$\simeq \varinjlim_{i\in I} \mathcal{H}_j(N_i\otimes_R F_\bullet) \simeq \varinjlim_{i\in I} \mathrm{Tor}_j^R(N_i,M),$$

where we used the fact that the tensor product commutes with direct limits and that filtering direct limits give an exact functor. □

The following result is known as the Local Flatness criterion.

Proposition 0.3. *Let* $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ *be a local homomorphism of Noetherian local rings. If* M *is a finitely generated* S*-module, then* M *is flat over* R *if and only if* $\mathrm{Tor}^1_R(R/\mathfrak{m},M) = 0$.

Proof. The "only if" part is obvious, so we only need to prove the "if" part. By Lemma 0.1, it is enough to show that the canonical map $\phi\colon \mathfrak{a}\otimes_R M \to M$ is injective for every ideal $\mathfrak{a}$ in R. Let $\mathfrak{a}$ be such an ideal and denote by Q the kernel of ϕ.

Note first that $\mathrm{Tor}_1^R(N,M) = 0$ if N is an R-module of finite length. Indeed, we argue by induction on $\ell(N)$, the case $\ell(N) = 1$ being taken care of by hypothesis. By considering a composition series of N, we obtain a short exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

with $\ell(N') = \ell(N) - 1$ and $\ell(N'') = 1$. By considering the long exact sequence for Tor modules and the inductive hypothesis, we obtain our assertion.

In particular, we see that $\mathrm{Tor}_1^R\big(R/(\mathfrak{a}+\mathfrak{m}^j),M\big) = 0$ for all $j\geq 1$, or equivalently, the canonical morphism

$$\phi_j\colon (\mathfrak{a}+\mathfrak{m}^j)\otimes_R M \to M$$

is injective. Since the map ϕ factors as

$$\mathfrak{a}\otimes_R M \to (\mathfrak{a}+\mathfrak{m}^j)\otimes_R M \xrightarrow{\phi_j} M,$$

it follows that

$$Q \subseteq \ker\big(\mathfrak{a}\otimes_R M \to (\mathfrak{a}+\mathfrak{m}^j)\otimes_R M\big) \subseteq \ker\big(\mathfrak{a}\otimes_R M \to ((\mathfrak{a}+\mathfrak{m}^j)/\mathfrak{m}^j)\otimes_R M\big).$$

Using the isomorphism $(\mathfrak{a}+\mathfrak{m}^j)/\mathfrak{m}^j \simeq \mathfrak{a}/(\mathfrak{a}\cap\mathfrak{m}^j)$, we thus obtain the inclusion

$$Q \subseteq \mathrm{Im}\big((\mathfrak{a}\cap\mathfrak{m}^j)\otimes_R M \to \mathfrak{a}\otimes_R M\big).$$

By the Artin-Rees lemma (see Lemma C.4.2 in the notes), for every $\ell\geq 1$, we have $\mathfrak{a}\cap\mathfrak{m}^j \subseteq \mathfrak{m}^\ell\mathfrak{a}$ for $j\gg 0$. Given $\ell\geq 1$, by taking $j\gg 0$, we conclude that

$$Q \subseteq \mathfrak{m}^\ell\cdot(\mathfrak{a}\otimes_R M).$$

Note that $\mathfrak{a} \otimes_R M$ is in fact a finitely generated S module and we see that

$$Q \subseteq \bigcap_{\ell \geq 1} \mathfrak{n}^\ell \cdot (\mathfrak{a} \otimes_R M).$$

Since the right-hand side is 0 by Krull's Intersection theorem (see Theorem C.4.1 in the notes), we conclude that $Q = 0$, completing the proof of the proposition.

□

The above proposition is often used via the following corollary.

Corollary 0.4. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings and M a finitely generated S-module. If $x_1, \ldots, x_n \in \mathfrak{m}$ form a regular sequence on both R and M, then M is flat over R if and only if $M/(x_1, \ldots, x_n)M$ is flat over $R/(x_1, \ldots, x_n)$.*

Proof. By an obvious induction on n, it is enough to treat the case $n = 1$, when we have $x \in \mathfrak{m}$ which is a non-zero-divisor on both A and M. The "only if" part is clear: it follows from assertion i) in Proposition 5.6.6 in the notes. In fact, for both implications, by the proposition, it is enough to show that we have an isomorphism

$$\mathrm{Tor}_1^R(R/\mathfrak{m}, M) \simeq \mathrm{Tor}_1^{R/(x)}(R/\mathfrak{m}, M/xM). \tag{1}$$

Note first that

$$\mathrm{Tor}_i^R\big((R/(x), M\big) = 0 \quad \text{for all} \quad i \geq 1. \tag{2}$$

Indeed, since x is a non-zero-divisor on R, we have a free resolution of $R/(x)$ given by

$$0 \to R \xrightarrow{\cdot x} R \to 0.$$

By tensoring with M and using the fact that x is a non-zero-divisor on M, we obtain (2).

Consider now a free resolution $F_\bullet$ of M over R. Note that because of (2), the complex $R/(x) \otimes_R F_\bullet$ gives a free resolution of M/xM over $R/(x)$. We thus have

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) \simeq \mathcal{H}_i(R/\mathfrak{m} \otimes_R F_\bullet)$$
$$\simeq \mathcal{H}_i\big((R/\mathfrak{m} \otimes_{R/(x)} (R/(x) \otimes_R F_\bullet)\big) \simeq \mathrm{Tor}_i^{R/(x)}(R/\mathfrak{m}, M/xM).$$

In particular, for $i = 1$, we obtain (1). □