Serre's normality criterion

This write-up supplements the characterization of normal domains that we gave in class, now that we also discussed the notion of depth. The general characterization of normal rings is the content of a criterion due to Serre. We first introduce Serre's conditions (R_i) and (S_i) and then prove the normality criterion.

1. Serre's conditions

Definition 1.1. Given a Noetherian ring R, we say that R satisfies Serre's condition (R_i) if for every prime ideal $\mathfrak{p}$ in R, with $\mathrm{codim}(R_{\mathfrak{p}}) \leq i$, the local ring $R_{\mathfrak{p}}$ is regular.

Example 1.2. If X is an affine variety and $A = \mathcal{O}(X)$, then A satisfies property R_i if and only if $\mathrm{codim}_X(X_{\mathrm{sing}}) \geq i+1$.

Definition 1.3. We say that a Noetherian ring R satisfies Serre's condition (S_i) if for every prime ideal $\mathfrak{p}$ in R, we have

$$\mathrm{depth}(R_{\mathfrak{p}}) \geq \min\{\dim(R_{\mathfrak{p}}), i\}.$$

Example 1.4. A Noetherian ring R satisfies (S_1) if and only if every associated prime of R is minimal. It satisfies both (R_0) and (S_1) if and only if for every associated prime $\mathfrak{p}$ of R, we have $\mathfrak{p}R_{\mathfrak{p}} = 0$. It is clear that this holds if R is reduced. The converse also holds: if $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ is a minimal primary decomposition, then conditions (R_0) and (S_1) imply that if $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, then each $\mathfrak{p}_i$ is a minimal prime ideal and $\mathfrak{q}_i R_{\mathfrak{p}_i} \subseteq \mathfrak{p}_i R_{\mathfrak{p}_i} = 0$; since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, it follows that $\mathfrak{q}_i = \mathfrak{p}_i$ for all i, hence R is reduced.

2. The normality criterion

As in the geometric setting, we say that an arbitrary Noetherian ring R is *normal* if $R_{\mathfrak{p}}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ in R (or, equivalently, for every maximal ideal $\mathfrak{p}$ in R).

Remark 2.1. We note that a normal ring is isomorphic to a product of normal domains. Indeed, if R is normal and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of R, then $\mathfrak{p}_i + \mathfrak{p}_j = R$ for every $i \neq j$ (this is due to the fact that $R_{\mathfrak{p}}$ is a domain for every maximal ideal $\mathfrak{p}$ in R). Moreover, since al localizations of R are reduced, it follows that R is reduced, hence $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r = 0$. We thus conclude from the Chinese Remainder theorem that the canonical morphism

$$R \to R/\mathfrak{p}_1 \times \ldots \times R/\mathfrak{p}_r$$

is an isomorphism. Furthermore, for every prime ideal $\mathfrak{q}$ containing $\mathfrak{p}_i$, the localization $R_{\mathfrak{q}}$ is a normal domain, hence $(R/\mathfrak{p}_i)_{\mathfrak{q}} = R_{\mathfrak{q}}$ is normal. We thus deduce that each $R/\mathfrak{p}_i$ is a normal domain.

Theorem 2.2 (Serre). *A Noetherian ring R is normal if and only if it satisfies conditions (R_1) and (S_2).*

Proof. After localizing, we may assume that $(R, \mathfrak{m})$ is a local ring. It is straightforward to see that if R is a domain, then having $(R_1) + (S_2)$ is just a reformulation of conditions i) + ii) in Proposition E.5.1 in the notes. In particular, the "only if" assertion in the theorem is clear. For the "if" part, the subtlety is that we don't know *a priori* that R is a domain.

Suppose now that R satisfies conditions (R_1) and (S_2). In particular, it satisfies $(R_0) + (S_1)$, and thus R is reduced by Example 1.4. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of R, and let $S = R \smallsetminus \bigcup_{i=1}^r \mathfrak{p}_i$ be the set of non-zero-divisors in R. Consider the inclusion map $\phi\colon R \hookrightarrow K = S^{-1}R$. The Chinese Remainder theorem gives an isomorphism $K \simeq \prod_{i=1}^r K_i$, where $K_i = \mathrm{Frac}(R/\mathfrak{p}_i) = R_{\mathfrak{p}_i}$. If we can show that $r = 1$, then R is a domain, in which case we are done. We follow the proof of Proposition E.5.1 in the notes to show that R is integrally closed in K. If we know this, and $e_i \in K$ is the idempotent corresponding to $1 \in K_i$, then $e_i^2 = e_i$ implies that e_i lies in R. Since R is local, the only idempotents it has are 0 and 1, and these are mapped by ϕ to 0 and 1, respectively, in K. We thus see that $r = 1$.

Suppose that $\frac{b}{a} \in K$ is a non-zero element that is integral over R (note that a is a non-zero-divisor). Consider a minimal primary decomposition

$$(a) = \mathfrak{q}_1 \cap \ldots \cap q_s.$$

If $\widetilde{\mathfrak{q}}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\widetilde{\mathfrak{q}}_i \in \mathrm{Ass}\big(R/(a)\big)$ by Remark E.3.13 in the notes. Condition (S_2) implies that $\mathrm{codim}(\widetilde{\mathfrak{q}}_j) = 1$, and condition (R_1) implies that $R_{\widetilde{\mathfrak{q}}_j}$ is a DVR. Let j be fixed and consider i such that $\mathfrak{p}_i \subseteq \widetilde{\mathfrak{q}}_j$. Since $\frac{b}{a}$ is integral over R, its image in K_i is integral over R, and since $R_{\widetilde{\mathfrak{q}}_j} \subseteq K_i$ is a DVR, hence integrally closed, we conclude that there is $s \in R \smallsetminus \widetilde{\mathfrak{q}}_j$ such that $sb \in (a)$. Since $\mathfrak{q}_j$ is a primary ideal, it follows that $b \in \mathfrak{q}_j$. Since this holds for every j, we conclude that $b \in (a)$ and thus $\frac{b}{a} \in R$. This completes the proof of the theorem. $\square$