Review sheet 4: Embeddings in injective modules

Let R be a ring (not necessarily commutative). Recall that a left R-module Q is *injective* if the functor $\mathrm{Hom}_R(-, Q)$ is exact. Since this functor is always left exact, Q is injective if and only if for every injective morphism of R-modules $M' \hookrightarrow M$, the induced morphism of Abelian groups

$$\mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q)$$

is surjective. In this appendix we show the basic fact that the category of left R-modules has enough injectives.

1. The Baer criterion and embeddings in injective modules

Proposition 1.1. *For every left R-module M, there is an injective morphism $M \hookrightarrow Q$, where Q is an injective left R-module.*

The proof proceeds by first treating the case when $R = \mathbb{Z}$. In this case, the key fact is the characterization of injective $\mathbb{Z}$-modules as divisible groups. This in turn follows from the following criterion for a module to be injective:

Proposition 1.2. *(Baer) A left R-module Q is injective if and only if for every left ideal I in R, the induced morphism of Abelian groups*

$$Q = \mathrm{Hom}_R(R, Q) \to \mathrm{Hom}_R(I, Q)$$

is surjective.

Proof. Of course, we only need to prove the "if" part. Suppose that M is a left R-module and M' is a submodule. We need to show that for every morphism $\phi' \colon M' \to Q$, there is a morphism $\phi \colon M \to Q$ such that $\phi|_{M'} = \phi'$. We consider the set $\mathcal{M}$ of all pairs (M_1, ϕ_1), where M_1 is a submodule of M containing M' and $\phi_1 \colon M_1 \to Q$ is a morphism such that $\phi_1|_{M'} = \phi'$. We order this set by putting $(M_1, \phi_1) \leq (M_2, \phi_2)$ if $M_1 \subseteq M_2$ and $\phi_2|_{M_1} = \phi_1$.

Since we have $(M', \phi') \in \mathcal{M}$, we see that $\mathcal{M}$ is non-empty. Moreover, given a family $(M_i, \phi_i)_{i \in I}$ of elements of $\mathcal{M}$, any two of them comparable, we can take $M'' = \bigcup_{i \in I} M_i$ and $\phi'' \colon M'' \to Q$ such that $\phi''|_{M_i} = \phi_i$ for all i; in this case $(M'', \phi'') \in \mathcal{M}$ is the supremum of the family $(M_i, \phi_i)_{i \in I}$.

We can thus apply Zorn's lemma to choose a maximal element (M_0, ϕ_0) in $\mathcal{M}$. We claim that $M_0 = M$, which would complete the proof. Suppose that this is not the case and let $u \in M \smallsetminus M_0$. We will show that there is an extension of ϕ_0 to a morphism $\phi_1 \colon M_0 + Ru \to Q$; this would contradict the maximality of (M_0, ϕ_0).

Let $I = \{a \in R \mid au \in M_0\}$. Note that I is a left ideal of R and we can define a morphism $\psi \colon I \to Q$ by $\psi(a) = \phi_0(au)$. By assumption, there is $w \in Q$ such that $\psi(a) = aw$ for every $a \in I$. It is then straightforward to see that if we put

$$\phi_1(v + au) = \phi_0(v) + aw \quad \text{for} \quad v + au \in M_0 + Ru,$$

then ϕ_1 is well-defined and gives a morphism $M_0 + Ru \to Q$ such that $\phi_1|_{M_0} = \phi_0$. This completes the proof. □

Recall that an Abelian group A is *divisible* if for every positive integer n, the multiplication map $A \xrightarrow{\cdot n} A$ is surjective.

Corollary 1.3. *A $\mathbb{Z}$-module Q is injective if and only if it is a divisible Abelian group.*

Proof. Since every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, for some non-negative integer n, it follows from the proposition that Q is injective if and only if for every such n, the induced morphism of Abelian groups

$$Q \to \mathrm{Hom}_{\mathbb{Z}}(n\mathbb{Z}, Q)$$

is surjective. This is clearly the case if $n = 0$. If $n > 0$, then this morphism gets identified to the morphism $Q \to Q$ given by multiplication by n, and we obtain the assertion in the corollary. □

We can now prove the existence of embeddings in injective modules.

Proof of Proposition 1.1. Suppose first that $R = \mathbb{Z}$. In this case, by the above corollary, we need to show that every Abelian group M can be embedded in a divisible Abelian group A. Write $M \simeq F/G$, where $F \simeq \mathbb{Z}^{(I)}$ is a free Abelian group. Since F is free, it has no torsion, and thus the canonical morphism $F \hookrightarrow F \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{(I)}$ is injective. We thus have an injective morphism $M \hookrightarrow A := (F \otimes_{\mathbb{Z}} \mathbb{Q})/G$. It is clear that $F \otimes_{\mathbb{Z}} \mathbb{Q}$ is divisible, and thus its image A is divisible, too.

Consider now the general case. By considering on M the underlying structure of $\mathbb{Z}$-module and applying what we have already proved, we get an injective morphism of $\mathbb{Z}$-modules $j\colon M \hookrightarrow A$, where A is an injective $\mathbb{Z}$-module. We claim that if we consider on $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ the left R-module structure induced by the right R-module structure of R (that is, we have

$$(\lambda \cdot \phi)(r) = \phi(r\lambda) \quad \text{for all} \quad \lambda, r \in R, \phi \in \mathrm{Hom}_{\mathbb{Z}}(R, A)),$$

then $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an injective R-module. In order to see this, it is enough to note that by the adjoint property of $R \otimes_R -$ and $\mathrm{Hom}_{\mathbb{Z}}(R, -)$, for every left R-module N, we have a canonical isomorphism

$$\mathrm{Hom}_R\big(N, \mathrm{Hom}_{\mathbb{Z}}(R, A)\big) \simeq \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R N, A) \simeq \mathrm{Hom}_{\mathbb{Z}}(N, A).$$

Finally, we note that we have an injective morphism of left R-modules given by

$$M \to \mathrm{Hom}_{\mathbb{Z}}(R, A), \quad M \ni v \to \phi_v, \quad \text{where} \quad \phi_v(r) = j(rv).$$

This completes the proof. □